```
> restart:
```

We load our package:

```
> with(IntegrableConnections);
```

$$[Eigenring, Evalm, G\_matrix, G\_matrix0, HyperexponentialSolutions, Id, Mapply, MatrixColumnPrimpart, MatrixColumnPrimpartRat, Mexpsolde, Mpolsolde, Mratsolde, Msylvester, Mylinsolve, NNIexponents, PolynomialSolutions, RationalSolutions, Reduction, TestIntegrabilityConditions, Trigonalize, backsubs, candidates, col\_pval, colechelon, colval, complement, convert\_back, denomatrix, denomvector, direct\_ratsol, eval\_block\_reduced, evalam, exp\_parts, formal\_reduce, gen\_sylvester, getargs, good\_form, intDiff, int\_diff\_split, l\_colechelon, l\_column\_rank, l\_moser\_reduce, l\_qtcd, l\_super, l\_super\_reduce, lc\_evala, lc\_evalam, localize, log\_collect, log\_free, mat\_block\_reduce, mat\_change\_exp\_part, mat\_check, mat\_coeff, mat\_colvaluation, mat\_convert, mat\_copy, mat\_difference, mat\_eval, mat\_get\_dim, mat\_get\_exp\_part, mat\_get\_ext, mat\_get\_indicial\_polynomial, mat\_get\_inv\_transformation, mat\_get\_order, mat\_get\_ramification, mat\_get\_terms, mat\_get\_transformation, mat\_get\_type, mat\_lc, mat\_lead\_mon, mat\_product, mat\_pval, mat\_subs, mat\_sum, mat\_swap, mat\_to\_simple, mat\_tools, mat\_transform, mat\_val, mat\_valuation, matrix\_series, matval, minimalIntegerRoot, mult, my\_linsolve, mydegree, myequaind, new\_matrix, newton, newton\_polygon, nonzero, normalm, pencil\_block\_reduce, pencil\_solve, pgaussUnimod, poly\_pval, poly\_sols, poly\_val, prank, pval, qtcd, ramified\_case, ramified\_reduction, randomMatrix, randomPoly, randomRat, ratsuper, reduceColRank4, reg\_sols, regular\_series, remove, rowechelon, seq\_evala, sim\_sylvester, simple\_eval, simple\_sols, sortColumns, super\_form, system\_expsolde, tensor\_prod, transform, truncate, utils, val, val1, val\_ldegree, valuation, vectdegree] \tag{1}$$

The library linalg is also needed.

```
> with(linalg):
```

We load OreModules in order to use the procedure to write a D-finite partial differential system to an integrable connection.

```
> with(OreModules):
```

BrycLetac system in dimension 2 - Generic Case

We define the OreAlgebra (needed for OreModules):

```
> Alg2:=DefineOreAlgebra(diff=[d1,x1],diff=[d2,x2],polynom=[x1,x2],
  comm=[d,a,b]):
```

We give the equations of the system:

```
> R2 := matrix(2,1,[-(1/2)*d*d2+d1^2-x2*d2^2-(a^2-4*b),2*d1*d2+x1*
  d2^2]);
```

$$R2 := \begin{bmatrix} -\frac{1}{2}\, d\, d2 + d1^2 - x2\, d2^2 - a^2 + 4\, b \\ 2\, d1\, d2 + x1\, d2^2 \end{bmatrix} \tag{2}$$

We write the system as an integrable connection:

```
> C2:=OreModules[Connection](R2,Alg2);
```

$$C2 := \left[\left[\begin{array}{cccc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -\frac{1}{2}\,x1 \\ a^2-4\,b & \frac{1}{2}\,d & 0 & x2 \\ 0 & \frac{\left(8\,b-2\,a^2\right)x1}{-4\,x2+x1^2} & 0 & \frac{\left(-3-d\right)x1}{-4\,x2+x1^2} \end{array}\right], \left[\begin{array}{cccc} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -\frac{1}{2}\,x1 \\ 0 & \frac{-16\,b+4\,a^2}{-4\,x2+x1^2} & 0 & \frac{6+2\,d}{-4\,x2+x1^2} \end{array}\right]\right] \quad (3)$$

If we want to know the basis of the associated module in which the connection is written, we can use the procedure KBasis of OreModules:

```
> OreModules[KBasis](R2,Alg2);
```

$$\left[\lambda_1, \lambda_1\,d2, d1\,\lambda_1, \lambda_1\,d2^2\right] \quad (4)$$

We use our procedures for computing rational solutions of the integrable connection:

```
> RatSols:=RationalSolutions(C2,[x1,x2],['param',[d,a,b]]);
```

$$RatSols := \{\ \} \quad (5)$$

```
> HyperExpSols:=HyperexponentialSolutions(C2,[x1,x2],['param',[d,a,b]
  ]);
Warning, unable to find a provably non-zero pivot
```

$$HyperExpSols := \left[\begin{array}{c} e^{-RootOf\left(\_Z^2-a^2+4\,b\right)x1} \\ 0 \\ -e^{-RootOf\left(\_Z^2-a^2+4\,b\right)x1}\,RootOf\left(\_Z^2-a^2+4\,b\right) \\ 0 \end{array}\right] \quad (6)$$

BrycLetac system in dimension 2 - Poisson Case

We define the OreAlgebra (needed for OreModules):

```
> Alg2:=DefineOreAlgebra(diff=[d1,x1],diff=[d2,x2],polynom=[x1,x2],
  comm=[d,a]):
```

We give the equations of the system:

```
> R2 := matrix(2,1,[-(1/2)*d*d2+d1^2-x2*d2^2-2*a*d1,2*d1*d2+x1*d2^2
  -2*a*d2]);
```

(7)

$$R2 := \begin{bmatrix} -\frac{1}{2}\, d\, d2 + d1^2 - x2\, d2^2 - 2\, a\, d1 \\ 2\, d2\, d1 + x1\, d2^2 - 2\, a\, d2 \end{bmatrix} \quad (7)$$

We write the system as an integrable connection:

```
> C2:=OreModules[Connection](R2,Alg2);
```

$$C2 := \left[\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & a & 0 & -\frac{1}{2}\, x1 \\ 0 & \frac{1}{2}\, d & 2\, a & x2 \\ 0 & -\frac{2\, x1\, a^2}{-4\, x2 + x1^2} & 0 & \frac{x1^2\, a + (-3 - d)\, x1 - 4\, x2\, a}{-4\, x2 + x1^2} \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & a & 0 & -\frac{1}{2}\, x1 \\ 0 & \frac{4\, a^2}{-4\, x2 + x1^2} & 0 & \frac{6 + 2\, d}{-4\, x2 + x1^2} \end{bmatrix} \right] \quad (8)$$

If we want to know the basis of the associated module in which the connection is written, we can use the procedure KBasis of OreModules:

```
> OreModules[KBasis](R2,Alg2);
```

$$\left[\lambda_1, \lambda_1\, d2, d1\, \lambda_1, \lambda_1\, d2^2\right] \quad (9)$$

We use our procedure for computing rational solutions of the integrable connection:

```
> RatSols:=RationalSolutions(C2,[x1,x2],['param',[d,a]]);
```

$$RatSols := \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (10)$$

We use our procedure for computing hyperexponential solutions of the integrable connection:

```
> HyperExpSols:=HyperexponentialSolutions(C2,[x1,x2],['param',[d,a]])
;
```

(11)

$$HyperExpSols := \begin{bmatrix} 1 & \frac{1}{2} & \frac{e^{2\,a\,x1}}{a} \\ 0 & 0 \\ 0 & e^{2\,a\,x1} \\ 0 & 0 \end{bmatrix} \quad \mathbf{(11)}$$

BrycLetac system in dimension 3 - Generic Case

We define the OreAlgebra (needed for OreModules):

```
> Alg3:=DefineOreAlgebra(diff=[d1,x1],diff=[d2,x2],diff=[d3,x3],
  polynom=[x1,x2,x3],comm=[d,a,b]):
```

We give the equations of the system:

```
> R3 := matrix(3,1,[-d*d2+d1^2-x2*d2^2-2*x3*d2*d3-(a^2-4*b),-(1/2)*d*
  d3+2*d1*d2+x1*d2^2-x3*d3^2,2*d1*d3+d2^2+2*x1*d2*d3+x2*d3^2]);
```

$$R3 := \begin{bmatrix} -d\,d2 + d1^2 - x2\,d2^2 - 2\,x3\,d2\,d3 - a^2 + 4\,b \\ -\frac{1}{2}\,d\,d3 + 2\,d2\,d1 + x1\,d2^2 - x3\,d3^2 \\ 2\,d1\,d3 + d2^2 + 2\,x1\,d2\,d3 + x2\,d3^2 \end{bmatrix} \quad \mathbf{(12)}$$

We write the system as an integrable connection:

```
> C3:=OreModules[Connection](R3,Alg3):
```

If we want to know the basis of the associated module in which the connection is written, we can use the procedure KBasis of OreModules:

```
> OreModules[KBasis](R3,Alg3);
```

$$\left[\lambda_1, \lambda_1\,d3, \lambda_1\,d2, \lambda_1\,d1, \lambda_1\,d3^2, \lambda_1\,d2\,d3, \lambda_1\,d1\,d3, d3^3\,\lambda_1\right] \quad \mathbf{(13)}$$

We use our procedures for computing rational solutions of the integrable connection:

```
> RatSols:=RationalSolutions(C3,[x1,x2,x3],['param',[d,a,b]]);
```

$$RatSols := \{\,\} \quad \mathbf{(14)}$$

```
> HyperExpSols:=HyperexponentialSolutions(C3,[x1,x2,x3],['param',[d,
  a,b]]);
```

Warning, unable to find a provably non-zero pivot

$$HyperExpSols := \begin{bmatrix} e^{-RootOf(\_Z^2 - a^2 + 4\,b)\,x1} \\ 0 \\ 0 \\ -e^{-RootOf(\_Z^2 - a^2 + 4\,b)\,x1}\,RootOf(\_Z^2 - a^2 + 4\,b) \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \mathbf{(15)}$$

Note that the latter computation takes some time.

BrycLetac system in dimension 3 - Poisson Case

We define the OreAlgebra (needed for OreModules):

```
> Alg3:=DefineOreAlgebra(diff=[d1,x1],diff=[d2,x2],diff=[d3,x3],
  polynom=[x1,x2,x3],comm=[d,a]):
```

We give the equations of the system:

```
> R3 := matrix(3,1,[-d*d2+d1^2-x2*d2^2-2*x3*d2*d3-2*a*d1,-(1/2)*d*
  d3+2*d1*d2+x1*d2^2-x3*d3^2-2*a*d2,2*d1*d3+d2^2+2*x1*d2*d3+x2*d3^2
  -2*a*d3]);
```

$$R3 := \begin{bmatrix} -d\, d2 + d1^2 - x2\, d2^2 - 2\, x3\, d2\, d3 - 2\, a\, d1 \\ -\frac{1}{2}\, d\, d3 + 2\, d2\, d1 + x1\, d2^2 - x3\, d3^2 - 2\, a\, d2 \\ 2\, d1\, d3 + d2^2 + 2\, x1\, d2\, d3 + x2\, d3^2 - 2\, a\, d3 \end{bmatrix} \quad \textbf{(16)}$$

We write the system as an integrable connection:

```
> C3:=OreModules[Connection](R3,Alg3):
```

If we want to know the basis of the associated module in which the connection is written, we can use the procedure KBasis of OreModules:

```
> OreModules[KBasis](R3,Alg3);
```

$$\left[\lambda_1, \lambda_1\, d3, \lambda_1\, d2, \lambda_1\, d1, \lambda_1\, d3^2, \lambda_1\, d2\, d3, \lambda_1\, d1\, d3, d3^3\, \lambda_1\right] \quad \textbf{(17)}$$

We use our procedure for computing rational solutions of the integrable connection:

```
> RatSols:=RationalSolutions(C3,[x1,x2,x3],['param',[d,a]]);
```

$$RatSols := \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \textbf{(18)}$$

We use our procedure for computing hyperexponential solutions of the integrable connection:

```
> HyperExpSols:=HyperexponentialSolutions(C3,[x1,x2,x3],['param',[d,
  a]]);
```

(19)

$$HyperExpSols := \begin{bmatrix} 1 & \frac{1}{2}\,\frac{e^{2\,a\,x1}}{a} \\ 0 & 0 \\ 0 & 0 \\ 0 & e^{2\,a\,x1} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{19}$$

The latter computation takes a long time.