Program of the 5th edition of the colloquium

Functional Equations in LIMoges (FELIM) 2012

Analytic and algebraic approaches to integrability

March 5-7, 2012

organised by

Moulay Barkatou, Thomas Cluzeau, Carole El Bacha, Eckhard Pflügel and Jacques-Arthur Weil

taking place at "Salle de Conférences", XLIM 3

sponsored by

"Département Mathématiques et Informatique d'XLIM",

FELIM 2012 Functional Equations in LIMoges

Analytic and algebraic approaches to integrability

Monday, March 5

9:30-9:45 **Welcome**

9:45-10:45 **C. Koutschan**: The face-centered cubic lattice.

10:45-11:15 **Coffee break, Discussions**

11:15-11:50 **S. Abramov**: On valuations of meromorphic solutions of arbitrary-order linear difference systems with polynomial coefficients.

11:50-14:30 **Lunch break**

14:30-15:30 **A. Ferragut**: Local behavior of planar analytic vector fields via integrability.

15:35-16:10 **G. Chèze**: An efficient algorithm for computing rational first integrals of polynomial vector fields.

16:10-16h40 **Coffee break, Discussions**

16:40-17:15 **A. Aparicio-Monforte**: Fields of formal Laurent series in several variables.

17:20-17:55 **T. Dreyfus**: Kovacic's algorithm for parameterized linear differential equations of order 2.

FELIM 2012 Functional Equations in LIMoges

Analytic and algebraic approaches to integrability

Tuesday, March 6

9:15-10:15 **P. Vanhaecke**: Non-commutative integrable systems on Poisson manifolds and the existence of action-angle coordinates.

10:15-10:45 **Coffee break, Discussions**

10:45-11:20 **T. Combot**: Third order integrability conditions for homogeneous potentials of degree -1.

11:25-12:00 **F. Lemaire**: Solving a chemical reaction system by a PDE.

12:00-14:00 **Lunch break**

14:00-15:00 **W. Balser**: Multisummation and its application to integrability of Hamiltonian systems.

15:00-15:30 **Coffee break, Discussions**

15:30- **Excursion**

FELIM 2012 Functional Equations in LIMoges

Analytic and algebraic approaches to integrability

Wednesday, March 7

9:00-9:35 **D. Robertz**: Implicitization of parametrized families of analytic functions.

09:40-10:15 **G. Regensburger**: Polynomial solutions and index of linear ordinary integro-differential operators.

10:15-10:45 **Coffee break, Discussions**

10:45-11:20 **M. Barkatou**: On the exponential parts of linear differential equations with meromorphic coefficients and their computation.

11:25-12:00 **F. Stan**: Bivariate rank-reduction techniques and their applications to linear functional systems.

12:00-14:00 **Lunch break**

14:00-14:35 **F. Monfreda**: On an index reduction method by deflation for quasilinear differential algebraic equations.

14:40-15:15 **E. Corel**: Gérard-Levelt Membranes: tropical computation of invariants of meromorphic connections.

15:20-15:55 **E. Pflügel**: ISOLDE - The comeback.

15:55-16:25 **Coffee break, Discussions**

Abstracts of the talks

- **Christoph Koutschan** (Centre de Recherche Commun INRIA-Microsoft Research, Saclay)
  The face-centered cubic lattice

  A lattice in $\mathbb{R}^d$ is given as an infinite set of points

  $$\left\{ \sum_{i=1}^{d} n_i \mathbf{a}_i : n_1, \ldots, n_d \in \mathbb{Z} \right\} \subseteq \mathbb{R}^d$$

  for some linearly independent vectors $\mathbf{a}_1, \ldots, \mathbf{a}_d \in \mathbb{R}^d$. The simplest instance of such a lattice is obtained by choosing $\mathbf{a}_i = \mathbf{e}_i$, the i-th unit vector; the result is the integer lattice $\mathbb{Z}^d$. This talk deals with the family of *face-centered cubic (fcc) lattices*, which are obtained from the lattice $\mathbb{Z}^d$ by adding the center point of each (two-dimensional) face to the set of lattice points. The three-dimensional fcc lattice is regularly encountered in nature, for example in the atomic structure of aluminium, copper, silver, and gold. We want to study random walks on the fcc lattice in several dimensions, namely $d = 3, 4, 5, 6$.

  We consider walks that allow only steps to the nearest neighbors and assume that all steps are taken with the same probability. Let $p_n(\mathbf{x})$ denote the probability that a random walk which started at the origin $\mathbf{0}$ ends at point $\mathbf{x}$ after n steps. The object of interest is the probability generating function

  $$P(\mathbf{x}; z) = \sum_{n=0}^{\infty} p_n(\mathbf{x}) z^n.$$

  which also is called the *lattice Green's function*. It can be expressed as a d-dimensional integral

  $$P(\mathbf{x}; z) = \frac{1}{\pi^d} \int_0^{\pi} \cdots \int_0^{\pi} \frac{e^{i\mathbf{x}\cdot\mathbf{k}}}{1 - z\lambda(\mathbf{k})} \, \mathrm{d}k_1 \ldots \mathrm{d}k_d.$$

  where

  $$\lambda(\mathbf{k}) = \lambda(k_1, \ldots, k_d) = \sum_{\mathbf{x}\in\mathbb{R}^d} p_1(\mathbf{x}) e^{i\mathbf{x}\cdot\mathbf{k}}$$

  is the discrete Fourier transform of the single-step probability function $p_1(\mathbf{x})$.

  We will discuss several computer algebra approaches how to obtain a differential equation for $P(\mathbf{0}; z)$, the probability generating function for excursions. Our work is mainly based on two methodologies: the first is *guessing* of linear recurrences and differential equations, the second is *creative telescoping* in the spirit of Zeilberger's holonomic systems approach. With these tools we are able to produce rigorous proofs of results that were conjectured by Broadhurst and Guttmann in the cases $d = 4$ and $d = 5$. Additionally we derive a differential equation for the lattice Green's function of the six-dimensional fcc lattice, a result that was not believed to be achievable with current computer hardware.

- **Sergei Abramov** (Computing Centre of the Russian Academy of Science)
  On valuations of meromorphic solutions of arbitrary-order linear difference systems with polynomial coefficients

  Algorithms for computing lower bounds on valuations (e.g., orders of the poles) of the components of meromorphic solutions of arbitrary-order linear difference systems with polynomial coefficients are considered. In addition to algorithms based on ideas which have been already utilized in computer algebra for treating normal first-order systems, a new algorithm using "tropical" calculations is proposed. It is shown that the latter algorithm is rather fast, and produces the bounds with good accuracy. (A joint work with D. Khmelnov).

- **Antoni Ferragut** (Universitat Politècnica de Catalunya)
  Local behavior of planar analytic vector fields via integrability

  We present an algorithm to study the local behavior of singular points of planar analytic vector fields having a generalized rational first integral. The algorithm is based on the blow-up method. It emphasizes the curves passing through the singular points and avoids the computation of the desingularized systems. Vector fields having a rational first integral are a particular case. Joint work with M. J. Alvarez (UIB).

- **Guillaume Chèze** (Institut de Mathématiques de Toulouse)
  An efficient algorithm for computing rational first integrals of polynomial vector fields

  During this talk we will consider a planar polynomial vector field

  $$(S)\ :\quad \begin{cases} \dot{x} &= A(x,y), \\ \dot{y} &= B(x,y), \end{cases} \qquad A,\, B \in \mathbb{K}[x,y],$$

  where $\mathbb{K}$ is a field of characteristic zero, and discuss the problem of computing rational first integrals of (S), *i.e.*, rational functions $F \in \mathbb{K}(x,y)$ that are constant on the solutions $(x(t), y(t))$ of (S). More precisely, the present paper is concerned with the following algorithmic problem:

$(\mathcal{P}_N)$: given a degree bound $N \in \mathbb{N}$, either compute a non-trivial rational first integral $F \in \mathbb{K}(x, y)$ of (S) of total degree at most N if it exists, or prove that no such F exists.

This old problem was already studied by Darboux in 1878 ([?]). It has been the subject of numerous works leading either to quadratic equations in the coefficients of F, or methods using what we called nowadays Darboux polynomials in the spirit of the celebrated Prelle-Singer's method. Recently, Chèze has shown in [?] that we can solve problem $(\mathcal{P}_N)$ in polynomial time. Unfortunately, this result is theoretical since the exponent of the polynomial in the complexity estimate is bigger than 10.

Our starting point was the article [?] of Ferragut and Giacomini. Their observation is that, (S) has a rational first integral if and only if all power series solutions of the first order non-linear differential equation

$$(E) \ : \frac{dy}{dx} = \frac{B(x, y)}{A(x, y)},$$

are algebraic. Furthermore, minimal polynomials of these algebraic functions leads to non-trivial rational first integrals. However, they still need to solve quadratic equations.

The main contributions of our work are the following. We push the observation of Ferragut and Giacomini further so as to give fast algorithms solving Problem $(\mathcal{P}_N)$. In particular, we prove that this can be done by considering only *linear* systems instead of quadratic systems. We develop a probabilistic algorithm using at most $\tilde{\mathcal{O}}(N^{2\omega})$ arithmetic operations, where d is the maximum of the degree of A and B, $N \geq d$ and $2 \leq \omega \leq 3$. This probabilistic algorithm is then turned into a deterministic one solving Problem $(\mathcal{P}_N)$ in at most $\tilde{\mathcal{O}}(N^{2\omega+3})$ arithmetic operations. This is to compare with the previous polynomial time algorithm that was given in [?] which uses at least $\tilde{\mathcal{O}}(d^{\omega+1}N^{4\omega+4})$ arithmetic operations. We illustrate the results of our Maple implementation on some examples.

A joint work with A. Bostan, T. Cluzeau and J.-A. Weil.

References

[1] G. Chèze, *Computation of Darboux polynomials and rational first integrals with bounded degree in polynomial time*, J. Complexity, 27(2):246-262, 2011.

[2] G. Darboux, *Mémoire sur les équations différentielles du premier ordre et du premier degré*, Bull. Sci. Math., 32:60–96, 123–144, 151-200, 1878.

[3] A. Ferragut and H. Giacomini, *A new algorithm for finding rational first integrals of polynomial vector fields*, Qual. Theory Dyn. Syst., 9(1-2):89-99, 2010.

- **Ainhoa Aparicio-Monforte** (RISC)
  Fields of formal Laurent series in several variables

  Let $K((x))$ be the set of all formal Laurent power series in the indeterminate x with coefficients over a commutative field K. It is a classic result that $K((x))$ with the term-wise sum and the Cauchy product forms a commutative field. But what about the multivariate case? To begin with, it is easy to see that in principle the notions of multiplicative inverse or leading term are not well defined (consider for instance $x+y \in K((x,y))$ and try to compute "the" inverse). In this talk, taking as a starting point the construction introduced by McDonald, we introduce the refinements necessary to accurately define the notion of leading term and inverse. We are able thus to construct accurately fields of formal Laurent series in several variables. As applications we obtain results on composition and implicit function analogous to those already known for the case of power series in one and several variables (Bousquet-Mélou, Petkovsek, Kauers, Sokal et al.). This is common work with Manuel Kauers (RISC-Linz).

- **Thomas Dreyfus** (Institut de Mathématiques de Jussieu)
  Kovacic's algorithm for parameterized linear differential equations of order 2

  Let us consider a linear differential equation of the form $y''(X) = a(X)y(X)$, where $a(X)$ is a complex rational function. Kovacic's algorithm find the Liouvillian solutions and compute the differential Galois group. Recently, it has been developped a Galois theory for parameterized linear differential equations. We will see how to adapt the original Kovacic's algorithm in the parameterized case.

- **Pol Vanhaecke** (Université de Poitiers)
  Non-commutative integrable systems on Poisson manifolds and the existence of action-angle coordinates

  Liouville integrable systems on symplectic manifolds admit two natural generalizations, to wit Liouville integrable systems on Poisson manifolds and non-commutative integrable systems on symplectic manifolds. Combining the two generalizations leads to the notion of a non-commutative integrable system on a Poisson manifold. The geometry of these systems is naturally encoded in a pair of foliations, where one of them is a torus fibration, under some topological assumption. The torus fibration is actually locally trivial, leading to the existence of action-angle coordinates at least on the regular part of the non-commutative integrable system (still assuming compactness of the fibers). This result will be explained in detail, with some hints about ongoing work about obstructions to the existence of global action-angle coordinates.

- **Thierry Combot** (Observatoire de Paris, IMCCE)
  Third order integrability conditions for homogeneous potentials of degree -1

  We prove an integrability criterion of order 3 for a homogeneous potential of degree -1 in the plane. Still, this criterion depends on some integer and it is impossible to apply it directly except for families of potentials whose eigenvalues at Darboux points are bounded. To address this issue, we use holonomic and asymptotic computations with error control of this criterion and as an example we apply it to the potential of the form $V(r,\theta) = 1/r\, h(\exp(i\theta))$ with h a polynomial of degree less than 3, for which no other method was succesful. We find then all meromorphically integrable potentials of this form. (In collaboration with C. Koutschan).

- **François Lemaire** (LIFL)
  Solving a chemical reaction system by a PDE

  Chemical reactions systems can be simulated by a stochastic approach, which naturally yields statistical moments such as mean values, standard deviations, ... Those statistical moments can be encoded by a formal power series in several variables, which is solution of boundary value problem, involving a single linear partial differential equation. We will see on several examples, the difficulties arising during the solving the PDE, caused by the initial conditions, the singularities, ...

  The talk is mostly prospective and will present more problems than results! Focus will be made on examples. Feedback from the audience will be welcome.

- **Werner Balser** (University of Ulm)
  Multisummation and its application to integrability of Hamiltonian systems

  Hamiltonian systems, or even general non-linear systems of ODE, have what we shall here call *complete semi-formal solutions.* To explain this notion, we consider a ν-dimensional non-linear system of ordinary differential equations of the form

  $$z^{r+1}\, x' \;=\; \check{g}(z,x)\,, \tag{1}$$

  where the *Poincaré rank* r is a positive integer, $x = (x_1, \ldots, x_\nu)^T$ is a vector of dimension $\nu \geq 1$, and

  $$\check{g}(z,x) \;=\; \sum_{|p|\geq 1} g_p(z) x^p \;=\; G(z)\, x \;+\; \sum_{|p|\geq 2} g_p(z) x^p \tag{2}$$

  is a *ν-dimensional formal power series* in the variables $x_1, \ldots, x_\nu$. While the power series (**??**) may diverge for every $x \neq 0$, we require the coefficients $g_p(z)$ to be holomorphic functions in a fixed disc $\mathbb{D}_\rho$ of radius $\rho > 0$ about the origin. Systems (**??**) satisfying these requirements will henceforth be called *semi-formal systems.*

Even more general systems have been shown in [?, ?] to admit a *complete formal solution*. This is a (formal) power series in ν free parameters $c = (c_1, \ldots, c_\nu)^T \in \mathbb{C}^\nu$ having invertible linear part. In [?] it has been shown that all the formal power series occuring in such a complete formal solution are multi-summable in the sense of *J. Ecalle* – also compare [?] for a convenient reference to the theory of multi-summability and its application to linear systems. Here, we are concerned with *semi-formal solutions*, of which some are obtained from a formal one by replacing all formal power series by their sums. Following Ecalle's terminology, these semi-formal solutions may be called *transseries*, but in the author's opinion they are much better understood as formal power series in c. For more details on this matter, refer to a manuscript of the author's [?].

In recent work of the author and *M. Yoshino* [?, ?], very particular Hamiltonian systems have been investigated. While they are not analytically integrable, is has been shown that some solutions exist having the form of a transseries. As we shall see, these transseries may be obtained from the corresponding semi-formal solution by a special choice of the parameter vector c.

References

[1] W. Balser, *Formal solutions of non-linear systems of ordinary differential equations*, in The Stokes phenomenon and Hilbert's 16th problem (Groningen, 1995), World Sci. Publ., River Edge, NJ, 1996, pp. 25–49.

[2] ——, *Existence and structure of complete formal solutions of non-linear meromorphic systems of ordinary differential equations*, Asymptot. Anal., 15 (1997), pp. 261–282.

[3] ——, *Multisummability of complete formal solutions for non-linear systems of meromorphic ordinary differential equations*, Complex Variables Theory Appl., 34 (1997), pp. 19–24.

[4] ——, *Formal power series and linear systems of meromorphic ordinary differential equations*, Universitext, Springer-Verlag, New York, 2000.

[5] ——, *Semi-formal theory and stokes' phenomenon of nonlinear meromorphic systems of ordinary differential equations.* Submitted, 2012.

[6] W. Balser and M. Yoshino, *Integrability of Hamiltonian systems and transseries expansions*, Math. Z., 268 (2011), pp. 257–280.

[7] M. Yoshino, *Liouville nonintegrability and transseries expansions.* Manuscript, 2011.

- **Eduardo Corel** (Universität Göttingen, Institut für Mikrobiologie und Genetik)
  Gérard-Levelt Membranes: tropical computation of invariants of meromor-

phic connections

We present an unexpected application of methods from tropical geometry to the computation of invariants of differential systems at a singular point. This involves a reinterpretation of Gérard and Levelt's classical results on saturated lattices as a projection map on a tropically convex subspace.

- **Daniel Robertz** (RWTH Aachen University, Lehrstuhl B für Mathematik)
Implicitization of parametrized families of analytic functions

The correspondence between solution sets of systems of algebraic equations and radical ideals of the affine coordinate ring is fundamental for algebraic geometry. This talk discusses aspects of an analogous correspondence between systems of polynomial differential equations and their analytic solutions. Implicitization problems for certain families of analytic functions are approached in different generality. While the linear case is understood to a large extent, the non-linear case requires new algorithmic methods, e.g., the use of differential inequations, as proposed by J. M. Thomas in the 1930s.

- **Georg Regensburger** (INRIA Saclay-Île de France, L2S, Supélec)
Polynomial solutions and index of linear ordinary integro-differential operators

Joint work with Alban Quadrat.
We present ongoing work on algebraic properties and algorithmic aspects of the algebra of linear ordinary integro-differential operators with polynomial coefficients. Even though this algebra is not Noetherian, Bavula recently proved that it is coherent algebra, that is, syzygy modules of finitely generated left/right ideals are finitely generated. The computation of syzygies is a central task for developing an algorithmic approach to linear systems of integro-differential equations with boundary conditions based module theory and homological algebra. This task, in turn, leads to computing polynomial solutions of inhomogeneous ordinary integro-differential equations with boundary conditions.The problem of computing polynomial solutions of (systems) of linear ordinary differential and (q-)difference equations is well-studied in Symbolic Computation. It appears as a subproblem of many important algorithms, and several implementations are nowadays available. Combining and reinterpreting some of the underlying ideas with tools from homological algebra and Fredholm operators, we discuss an algebraic setting and algorithmic approach for computing polynomial solutions (kernel), compatibility conditions (cokernel) and the "polynomial" index for a class of linear operators including integro-differential operators. We illustrate our approach with computations obtained with an implementation using the IntDiffOp Maple package developed by

Anja Korporal. G. Regensburger was supported by the Austrian Science Fund (FWF): J 3030-N18.

- **Moulay Barkatou** (Université de Limoges, CNRS, XLIM, Département de Mathématiques et Informatique)
  On the exponential parts of linear differential equations with meromorphic coefficients and their computation

  In this talk we shall study the relationship between the exponential parts of formal solutions of a differential system $dY/dz = A(z)Y$ having a singularity of pole type at $z = 0$ and the singular parts of the eigenvalues of the matrix $A(z)$ considered as Puiseux series in z.

  Given a linear differential system $dY/dz = A(z)Y$ of size n with meromorphic coefficients of order p at the origin, we first show that the exponential parts of the system and the eigenvalues of its matrix $A(z)$ do agree up to a certain order that depends on n and p. We then show that under suitable conditions on the matrix $A(z)$, some formal invariants of the differential system $dY/dz = A(z)Y$ can be computed from the characteristic polynomial of $A(z)$. We conclude by showing how these results can be used to compute efficiently the formal solutions of the system.

- **Flavia Stan** (INRIA Paris-Rocquencourt)
  Bivariate rank-reduction techniques and their applications to linear functional systems

  The notion of Moser- and super-irreducible forms has been stated for various types of linear functional systems in the past. In this talk, we show how to extend these concepts to a bivariate setting. Our approach is based on applying consecutive univariate algebraic similarity transformations until we arrive at a Moser-reduced bivariate system. For the employed transformations, we define an appropriate compatibility condition. We present a few example computations using the latest version of the ISOLDE package. Additionally, we give an outlook on future applications for various linear functional systems in several variables. This is joint work with Moulay Barkatou and Eckhard Pfluegel.

- **Fabien Monfreda** (Institut de Mathématiques de Toulouse)
  On an index reduction method by deflation for quasilinear differential algebraic equations

In this talk, we present a deflation method for solving quasilinear differential algebraic equations (DAEs). These equations are nonlinear DAEs of type

$$E(x(t))\dot{x}(t) = A(x(t)), \tag{3}$$

where $E \in \mathcal{A}(\mathcal{O}_n, \mathbb{R}^{n\times n})$ has constant rank on $\mathcal{O}_n$ and $A \in \mathcal{A}(\mathcal{O}_n, \mathbb{R}^n)$ where $\mathcal{O}_n$ is an open set of $\mathbb{R}^n$. Under an assumption of regularity, we prove that the quasilinear DAE (**??**) has a general solution which satisfies both at most an ODE and a set of algebraic constraints. The proof of this result is based on a deflation algorithm. More precisely, we explain how to construct a sequence of quasilinear DAEs of strictly decreasing size and also a sequence of algebraic constraints from the initial DAE. The last equation is either an ODE or an algebraic constraint. Then the general solution satisfies the last equation and the previous algebraic constraints. The number of steps of this deflation algorithm is called the index of this quasilinear DAE. Finally, we use the deflation algorithm to reduce the DAE satisfied by the Cartesian coordinates of the n-dimensional pendulum, and we show that it is an index three problem. (A joint work with G. Chèze and J.-C. Yakoubsohn)

- **Eckhard Pflügel** (Faculty of SEC, Kingston University)
  ISOLDE - The comeback

In this talk, we will give an overview of the latest release of the Maple package ISOLDE. After several years of research, a new milestone in the development has been reached, with new functionality for linear difference and q-difference systems and improved modularity of the code. We will summarise the mathematical problems that can be solved, and give a demonstration of the package. We will also give some ideas about future directions.