Program of the conference

Functional Equations in LIMoges (FELIM) 2011

Constructive Algebra for Systems Theory

March 14-16, 2011

organised by

Moulay Barkatou, Thomas Cluzeau, Carole El Bacha
and Jacques-Arthur Weil

taking place at

"la salle de réunion XR203"

sponsored by

"Département de Mathématiques et Informatique de XLIM" and

FELIM 2011 Functional Equations in LIMoges

Constructive Algebra for Systems Theory

Monday, March 14

8:45-9:00 **Welcome**

9:00-9:40 **Hugues Mounier**: Propriétés structurelles pour la commande des systèmes de dimension infinie (I)

9:50-10:30 **Hugues Mounier**: Propriétés structurelles pour la commande des systèmes de dimension infinie (II)

10:30-11:00 **Coffee break, Discussion**

11:00-12:00 **Alban Quadrat**: Introduction to constructive algebraic analysis and its applications to linear systems theory and mathematical physics

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12:00-14:00 **Lunch**

14:00-14:50 **Viktor Levandovskyy**: Diagonalization of a matrix over Euclidean Ore domain using Gröbner bases

15:00-15:30 **Markus Lange-Hegermann**: Thomas decomposition of differential systems

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15:30-16:15 **Coffee break, Discussion**

16:15-16:45 **Driss Boularas**: Intégrabilité formelle et nouvelles conditions nécessaires d'existence d'un centre

16:50-17:20 **Yongjae Cha**: Finding closed form solutions of linear difference equations using local invariants

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FELIM 2011 Functional Equations in LIMoges

Constructive Algebra for Systems Theory

Tuesday, March 15

9:00-9:50 **Daniel Robertz**: Differential elimination for analytic functions

10:00-10:30 **Georg Regensburger**: Integro-differential operators and boundary problems for LODEs

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10:30-11:00 **Coffee break, Discussion**

11:00-11:30 **Eckhard Pflügel**: On the irregular part of formal solutions of systems of pseudo-linear equations

11:35-12:05 **Abdelkarim Chakhar**: A new differential ring approach to quasilinear equations

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12:05-14:00 **Lunch**

14:00- **Discussion**

FELIM 2011 Functional Equations in LIMoges

Constructive Algebra for Systems Theory

Wednesday, March 16

9:00-9:30 **Sergei Abramov**: Singularities of solutions of linear differential systems with polynomial coefficients

9:35-10:05 **Alexandre Benoit**: Rigorous uniform approximation of D-finite functions

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10:05-10:35 **Coffee break, Discussion**

10:35-11:05 **Maria Przybylska**: Non-integrability of the generalized three body problem

11:10-11:40 **Thierry Combot**: Integrability of higher variational equations and holonomic recurrences

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12:00- **Lunch**

Abstracts of the talks

- **Hugues Mounier** (Université Paris Sud 11, L2S, Supélec)

  Propriétés structurelles pour la commande des systèmes de dimension infinie

  Une théorie algébrique pour la commande des systèmes à paramètres répartis commandés aux bords est présentée et illustrée au travers de divers exemples à caractère pratique issus soit de la physique (équation de la chaleur ou des ondes) soit de procédés industriels (tel le forage). Un système y est représenté par un module et les notions de commandabilités développées étendent des notions classiques de la littérature. Le point de vue ici adopté trouve ses origines dans l'extension du cadre élaboré pour les systèmes à retards. Notre philosophie est guidée par deux attentions majeures : la première (attention pratique) est de dégager des propriétés structurelles qui surviennent fréquemment dans les applications. La deuxième (attention de simplicité), reliée à la première, consiste en l'obtention des propriétés les plus simples pour chaque classe d'applications. Ceci nous a conduit à la découverte d'une nouvelle notion, nommée pi-liberté, qui permet de résoudre le suivi d'une trajectoire de référence selon le même cheminement que celui emprunté pour les systèmes non linéaires de dimension finie différentiellement plats que nous rappellerons également.

- **Alban Quadrat** (INRIA Saclay-Île de France, L2S, Supélec)

  Introduction to constructive algebraic analysis and its applications to linear systems theory and mathematical physics

  The purpose of this talk is to give a short introduction to recent developments of constructive algebraic analysis and their applications to linear systems theory and mathematical physics.

  Algebraic analysis is a mathematical theory, developed in the sixties by Malgrange, Bernstein, Sato, Kashiwara..., which aims at studying linear systems of partial differential equations and, more generally, linear functional equations (e.g., systems of difference equations, systems of differential time-delay equations), by means of techniques of module theory, homological algebra and sheaf theory.

  Certain aspects of algebraic analysis have recently been made effective using constructive algebra techniques such as Gröbner or Janet bases, which has led to the development of dedicated packages developed in computer algebra systems such as Maple (OreModules, OreMorphisms), Singular: Plural and GAP 4 (homalg).

In this talk, we shall show how a dictionary between certain properties of linear functional systems (e.g., invariants, symmetries, first integrals, conservation laws, autonomous elements, parametrizability, controllability, observability, flatness, decomposability) and module theory can be developed and effectively checked using constructive algebraic analysis.

This talk will be based on Hugues Mounier's lectures.

- **Viktor Levandovskyy** (RWTH Aachen University, Lehrstuhl B für Mathematik)

Diagonalization of a matrix over Euclidean Ore domain using Gröbner bases

In this talk we discuss a general algorithm, computing a diagonal form of a given matrix M over an Euclidean Ore domain A. We view such domains as Ore localizations of polynomial non-commutative algebras. In particular, we introduce an Ore Localization of a G-Algebra (OLGA) for the class of polynomial non-commutative G-algebras. If one recognizes an OLGA structure on a given Euclidean Ore domain A, we propose a diagonalization algorithm, which uses Gröbner bases. Moreover, we propose its fraction-free version together with its implementation in computer algebra system SINGULAR. We stress the usefullness of diagonal forms to the question whether two given finitely presented A-modules are isomorphic. In particular, we propose to split the computation of a normal form of a matrix (like Jacobson form over a simple Euclidean domain) into two steps, where after the first diagonalization step, the normalization step follows. We prove some statements about richness of normal forms over non-simple Euclidean domains.

We pay special attention to square transformation matrices U, V, which satisfy the relation $U * M * V = \mathrm{diag}(d_1, \ldots, d_n, 0)$ and analyze their meaning for the solving of an original system of equations. Also, we show, that by using fraction-free approach, one can easily compute a much smaller algebra $A' \subset A$, such that the isomorphism of A-modules, presented by M resp. by $U * M * V = D$ still holds over A'. Thus, one can lift an isomorphism from a big localization into a much smaller one.

If the time allows, some live computations will be demonstrated. (A joint work with K. Schindelar)

- **Markus Lange-Hegermann** (RWTH Aachen University, Lehrstuhl B für Mathematik)

Thomas Decomposition of Differential Systems

In this talk, we consider systems of non-linear (partial) differential equations and inequations. We decompose these system into so-called simple subsystems and thereby partition the set of solutions. Simplicity of systems consists of three algebraic properties and one differential property. The algebraic properties are triangularity, square-freeness and non-vanishing initials. The differential property is the involutivity of the system, which guarantees the inclusion of differential consequences in a simple system. We build upon the constructive ideas of J. M. Thomas and develop them into a new algorithm for a disjoint decomposition of solutions of a differential system. We apply the decomposition into simple systems and refine it to "count" its solutions.

- **Driss Boularas** (Université de Limoges, CNRS, XLIM, Département de Mathématiques et Informatique)

  Intégrabilité formelle et nouvelles conditions nécessaires d'existence d'un centre

  Depuis Poincaré, on sait que le problème du centre-foyer pour les systèmes différentiels plans est équivalent à l'existence d'une intégrale première analytique. Cet exposé comportera deux parties. Dans la première, on explicitera les écritures matricielles qui permettent d'exprimer les conditions d'existence d'une intégrale formelle. Ces écritures seront utilisées pour "révisiter" le théorème de Lyapunov-Poincaré.
  Dans la seconde partie, considérant les systèmes différentiels

$$\begin{aligned} x' &= -y + P_m(x,y), \\ y' &= x + Q_m(x,y) \end{aligned}$$

  où P_m et Q_m sont deux polynômes homogènes de même degré $m \geq 2$, on donnera une (dans le cas où m est pair) et deux (dans le cas où m est impair) conditions nécessaires d'existence d'un centre.

- **Yongjae Cha** (J. Kepler University, RISC)

  Finding closed form solutions of linear difference equations using local invariants

  In this talk we will present an algorithm that finds closed form solutions for homogeneous linear difference equations. The key idea is transforming an input operator L_{inp} to an operator L_g with known solutions. The main problem of this idea is how to find a solved equation L_g to which L_{inp} can be reduced. To solve this problem, we use local data of a difference operator, that is invariant under the transformation.

  The set of valuation growth is the local data at each finite singularity in $\mathbb{C}/\mathbb{Z}$ and generalized exponent is the local data at the point of infinity. Another topic of this talk will be how to compute generalized exponent which involves computing valuation, Newton polygon and indicial equation of a linear difference operator.

- **Daniel Robertz** (RWTH Aachen University, Lehrstuhl B für Mathematik)

  Differential elimination for analytic functions

  This talk presents first results of investigating the following program:

  Given a set S of analytic functions in several complex variables described by parametrizations of rather general type.

  1. (Recognition) Decide whether a given function lies in S.
  2. (Explicit recognition) Find one/all sets of parameters in the affirmative case.
  3. (Description by PDEs) Decide whether the set can be characterized as the set of solutions of a PDE system; find such a system in the positive case.

  The case of parametrizations in terms of certain linear expressions is joint work with Wilhelm Plesken. After giving a survey of the linear case, new results for the case of bilinear parametrizations are presented. Methodically, apart from Janet's algorithm and differential elimination, we develop a calculus for composite functions, of course based on the chain rule, which seems to be novel in differential algebra.

- **Georg Regensburger** (INRIA Saclay-Île de France, L2S, Supélec)

  Integro-differential operators and boundary problems for LODEs

  An integro-differential algebra combines a differential algebra with a suitable notion of an integral operator. By the fundamental theorem of calculus, the integral should be a right inverse of the derivation. Additionally, we require a version of integration by parts that allows us to define an "evaluation" in any integro-differential algebra. This is also the starting point for treating initial and boundary conditions in an algebraic setting.

  After reviewing basic properties and examples of integro-differential algebras, we discuss the construction and some algebraic and algorithmic aspects of the associated ring of integro-differential operators. Our Maple implementation is based on the fact that every integro-differential operator can be written uniquely as a sum of a differential, an integral, and a boundary operator. Integro-differential operators provide in particular an algebraic structure for computing with boundary problems for linear ODEs as well as their solution operators (Green's operators), and we illustrate this with some sample computations. We discuss also how solving a boundary problem can be interpreted as a special case of dealing with general linear functional systems over integro-differential operators. (A joint work with M. Rosenkranz and A. Korporal)

- **Eckhard Pflügel** (Kingston University)

  On the irregular part of formal solutions of systems of pseudo-linear equations

  In last year's talk, we have investigated the structure of regular solutions of systems of pseudo-linear equations with power series coefficients over a field of characteristic zero, and also have given an efficient algorithm for their computation. The next step for further research that immediately follows from these results is the study of the case of an irregular singularity. This is the subject of the present talk. We define a unifying notion of irregular parts and construct a full basis of formal solutions of arbitrary pseudo-linear systems in the neighbourhood of a singularity. We give an efficient algorithm for computing unramified irregular parts and apply this to systems of linear differential, difference and q-difference equations.

- **Abdelkarim Chakhar** (Université de Limoges, CNRS, XLIM, Département de Mathématiques et Informatique)

  A new differential ring approach to quasilinear equations

  Les méthodes de l'analyse algébrique (D-modules et algèbre homologique) permettent d'étudier de façon systématique les systèmes linéaires d'équations aux dérivées partielles. Il n'existent pas de théorie similaire pour les systèmes d'équations aux dérivées partielles non linéaires. Dans cet exposé, je montrerai comment on peut, en travaillant sur des anneaux différentiels, se servir de l'analyse algébrique constructive pour déduire certaines propriétés (e.g. symétries, lois de conservations, etc) sur une classe particulière d'équations non linéaires, dites quasi-linéaires

- **Sergei Abramov** (Computing Centre of the Russian Academy of Science)

  Singularities of solutions of linear differential systems with polynomial coefficients

  We consider systems of linear ordinary differential equations containing m unknown functions of a single variable x. Coefficients of the systems are polynomials over a field k of characteristic 0. Each of the systems consists of m equations independent over

$k\left[x, \frac{d}{dx}\right]$. The equations are of arbitrary orders. We propose an algorithm which, given a system S of this form, constructs a polynomial $d(x) \in k[x] \setminus \{0\}$ such that if S possesses a solution in $\bar{k}((x-\alpha))^m$ for some $\alpha \in \bar{k}$, and a component of this solution has a nonzero polar part then $d(\alpha) = 0$. In the case when k is a numeric field and S possesses an analytic solution having a singularity of an arbitrary type (not necessary a pole) at α the equality $d(\alpha) = 0$ is also satisfied. (A joint work with D. E. Khmelnov)

- **Alexandre Benoit** (INRIA Paris-Rocquencourt)

  Rigorous uniform approximation of D-finite functions

  A wide range of numerical methods exists for computing approximate truncated Chebyshev series solutions of linear differential equations with polynomial coefficients. However, in the context of rigorous computing, we need a certified way for obtaining guarantees on the accuracy of the result obtained, with respect to both the truncation and rounding errors. Besides, it is well-known that the coefficients of exact Chebyshev series solutions of linear differential equations with polynomial coefficients (also known as D-finite functions) obey linear recurrence relations with polynomial coefficients; and that the order-n truncation of the Chebyshev series expansion of a function over a given interval is a near-best uniform polynomial approximation of the function over the given interval. However, the coefficients of this expansion are not easy to compute in general, due to the lack of initial conditions for the recursion. In this talk, I will show that this recurrence can nonetheless be used, as part of a validated process, to compute good uniform approximations of D-finite functions, together with rigorous approximation error bounds. (A joint work with M. Joldes and M. Mezzarobba)

- **Maria Przybylska** (University of Zielona Gora, Institute of Physics)

  Non-integrability of the generalized three body problem

  We consider the planar problem of three bodies which attract mutually with the force proportional to a certain negative integer power of the distance between the bodies. We show that such generalisation of the gravitational three body problem is not integrable in the Liouville sense. (A joint work with A. J. Maciejewski)

- **Thierry Combot** (Observatoire de Paris, IMCCE)

  Intégrabilité des équations variationnelles supérieures et récurrences holonomes

Nous démontrons une condition d'intégrabilité à l'ordre 2 pour les potentiels homogènes de degré -1 puis à l'ordre 3 pour les potentiels du plan. Pour cela, nous montrons qu'après réduction, les équations variationnelles d'ordre 2 se ramènent à une équation différentielle dépendant de 3 indices entiers dont la monodromie satisfait une récurrence holonome. Dans le cas du plan, on étudie l'ordre 3 et nous montrons que les cas restants correspondent aux cas intégrables déjà connus.