On Taylor polynomials of rational functions

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Introduction

$$\frac{1+x+x^2}{1-x-x^2} = 1+2x+4x^2+6x^3+10x^4+16x^5+O(x^6)$$

Indeed:

$$(1-x-x^2)(1+2x+4x^2+6x^3+10x^4+16x^5) =$$
$$= 1+x+x^2-26x^6-16x^7.$$

$$\frac{1+p_1x+p_2x^2}{1+q_1x+q_2x^2} = 1+c_1x+c_2x^2+c_3x^3+c_4x^4+c_5x^5+O(x^6)$$

where c_1, c_2, c_3, c_4, c_5 are (polynomial) functions of p_1, p_2, q_1, q_2.

4 parameters, 5 target coefficients $\Rightarrow$ 1 relation ?

Introduction

From the equality modulo x^6 one gets

$$c_1 = p_1 - q_1$$
$$c_2 = p_1 q_1 - q_1^2 - p_2 + q_2$$
$$c_3 = p_1 q_1^2 - q_1^3 - p_1 q_2 - p_2 q_1 + 2 q_1 q_2$$
$$c_4 = p_1 q_1^3 - q_1^4 - 2 p_1 q_1 q_2 - p_2 q_1^2 + 3 q_1^2 q_2 + p_2 q_2 - q_2^2$$
$$c_5 = p_1 q_1^4 - q_1^5 - 3 p_1 q_1^2 q_2 - p_2 q_1^3 + 4 q_1^3 q_2 + p_1 q_2^2 + 2 p_2 q_1 q_2 - 3 q_1 q_2^2$$

eliminating the parameters, one gets the *cubic* equation:

$$c_1 c_3 c_5 - c_1 c_4^2 - c_2^2 c_5 + 2 c_2 c_3 c_4 - c_3^3 = 0$$

$$\det \begin{bmatrix} c_5 & c_4 & c_3 \\ c_4 & c_3 & c_2 \\ c_3 & c_2 & c_1 \end{bmatrix} = c_1 c_3 c_5 - c_1 c_4^2 - c_2^2 c_5 + 2 c_2 c_3 c_4 - c_3^3 = 0$$

Taylor varieties

Fix n variables $x = (x_1, \ldots, x_n)$, and degrees (d, e) and m:

$$\mathbb{C}(x) \ni \frac{P(x)}{Q(x)} = \sum_{|\gamma| \leq m} c_\gamma x^\gamma + \langle x_1, \ldots, x_n \rangle^{m+1}$$

with $Q(0, \ldots, 0) = 1$.

The (Zariski) closure in $\mathbb{P}^{\binom{n+m}{m}-1}$ of the set of all Taylor polynomials of degree $\leq m$ of rational functions of degree $\leq (d, e)$ in n variables is called the *Taylor variety* and denoted $\mathcal{T}^n_{d,e,m}$.

Main questions: dimension, defining equations, hypersurfaces...

Main motivation: Padé Approximation

Central question of Approximation Theory and Computer Algebra.

Given a function f, known through its Taylor approximation up to some order m, find two polynomials p, q of degree d, e, such that

$$\frac{p}{q} = f \mod x^{m+1}$$

More generally given $f_1, \ldots, f_s$, find $(p_1, \ldots, p_s)$ such that

$$p_1 f_1 + \cdots + p_s f_s = 0 \mod I$$

$\rightsquigarrow$ syzygy modules

The geometry of Taylor varieties is related to

- **existence** of solutions: is there such a rational function approximation?
- **uniqueness/identifiability**: how many rational function approximations?

Dimension

The Taylor variety $\mathcal{T}^n_{d,e,m} \subset \mathbb{P}^{\binom{n+m}{m}-1}$ is *irreducible*, as (closure of the) image of the following morphism:

$$\begin{array}{rccc} \psi & : & \mathbb{C}^{\binom{n+d}{d}} \times \mathbb{C}^{\binom{n+e}{e}} & \to & \mathbb{C}^{\binom{n+m}{m}} \\ & & (P, Q) & \mapsto & T := \sum_{|\gamma| \leq m} c_\gamma x^\gamma \end{array}$$

Its *dimension* is bounded by the *expected dimension*:

$$\dim(\mathcal{T}^n_{d,e,m}) \ \leq \ \exp\dim(\mathcal{T}^n_{d,e,m}) := \min\left\{ \underbrace{\binom{n+m}{m} - 1}_{\text{ambient dimension}}, \underbrace{\binom{n+d}{d} + \binom{n+e}{e} - 2}_{\text{number of parameters}} \right\}$$

$\downarrow$

this inequality **can** be **strict** when $n \geq 2$.

Univariate case ($n = 1$)

Fix $d, e \in \mathbb{N}$, and let $T = 1 + c_1 x + c_2 x^2 + \cdots + c_m x^m \in \mathbb{C}[x]$.

We define the $(m-d) \times (e+1)$ *(univariate) Padé Matrix*:

$$M_T = \begin{bmatrix} c_m & c_{m-1} & \cdots & c_{m-e} \\ c_{m-1} & c_{m-2} & \cdots & c_{m-e-1} \\ \vdots & \vdots & \ddots & \vdots \\ c_{d+2} & c_{d+1} & \cdots & c_{d-e+2} \\ c_{d+1} & c_d & \cdots & c_{d-e+1} \end{bmatrix}$$

- The Taylor variety $\mathcal{T}^1_{d,e,m} \subset \mathbb{P}^m$ has the expected dimension $\min\{d+e, m\}$
- If $d + e \geq m$, then $\mathcal{T}^1_{d,e,m} = \mathbb{P}^m$ (in words, every degree-m polynomial is a Taylor polynomial of a rational function of degree $\leq (d, e)$)
- If $d + e < m$, its defining ideal is the (prime) ideal of $e + 1$ minors of M_T, and it equals some *secant variety* of the *rational normal (moment) curve* of degree m

Multivariate Padé Matrix

Define $M_{d+1,m} :=$ monomials[1] in x of degree between $d+1$ and m.

The *Padé Matrix* M_T constructed before is the matrix of the linear map

$$\begin{array}{ccccc} \mathbb{C}[x]_{\le e} & \to & \mathbb{C}[x]_{\le e+m} & \to & \mathbb{C}\{M_{d+1,m}\}, \\ Q & \mapsto & QT & \mapsto & QT \text{ restricted to } M_{d+1,m}. \end{array}$$

For instance for $(n,d,e,m) = (2,1,1,3)$ the linear map is

$$Q = \begin{bmatrix} 1 \\ q_{01} \\ q_{10} \end{bmatrix} \mapsto \begin{bmatrix} c_{30} & 0 & c_{20} \\ c_{21} & c_{20} & c_{11} \\ c_{12} & c_{11} & c_{02} \\ c_{03} & c_{02} & 0 \\ c_{20} & 0 & c_{10} \\ c_{11} & c_{10} & c_{01} \\ c_{02} & c_{01} & 0 \end{bmatrix} \begin{bmatrix} 1 \\ q_{01} \\ q_{10} \end{bmatrix} \quad \begin{array}{l} \leftarrow \text{ coeff. of } x^3 \text{ of } P := QT \\ \leftarrow \text{ coeff. of } x^2y \\ \leftarrow \text{ coeff. of } xy^2 \\ \leftarrow \text{ coeff. of } y^3 \\ \leftarrow \text{ coeff. of } x^2 \\ \leftarrow \text{ coeff. of } xy \\ \leftarrow \text{ coeff. of } y^2 \end{array}$$

[1] These coefficients of QT must vanish if T is the Taylor polynomial of some P/Q.

Example $(n,d,e,m) = (2,1,1,3)$ (continued)

The variety $\mathcal{T}^2_{1,1,3} \subset \mathbb{P}^9$ contains ternary cubics that are (order 3) Taylor polynomials of rational functions of degree $(1,1)$:

$$\frac{1 + p_{10}x + p_{01}y}{1 + q_{10}x + q_{01}y} \quad = \quad 1 + c_{10}x + c_{01}y + \cdots + c_{12}xy^2 + c_{03}y^3 \quad + \langle x, y \rangle^4$$

The ideal $I_3(M_T)$ of maximal minors of M_T has the expected codimension 5, but it is **not prime**: it has two components, one of which is the (prime) ideal of $\mathcal{T}^2_{1,1,3}$.

Conclusion: For $n \geq 2$, taking maximal minors of the Padé Matrix is not sufficient to get the equations of $\mathcal{T}^n_{d,e,m}$.

Defective cases

The variety $\mathcal{T}^3_{2,2,3} \subset \mathbb{P}^{19}$ has codimension 2 (expected to be a hypersurface)

$$\det \begin{bmatrix} c_{300} & 0 & 0 & c_{200} & 0 & 0 & 0 & 0 & 0 & c_{100} \\ c_{210} & 0 & c_{200} & c_{110} & 0 & 0 & 0 & 0 & c_{100} & c_{010} \\ c_{201} & c_{200} & 0 & c_{101} & 0 & 0 & 0 & c_{100} & 0 & c_{001} \\ c_{120} & 0 & c_{110} & c_{020} & 0 & 0 & c_{100} & 0 & c_{010} & 0 \\ c_{111} & c_{110} & c_{101} & c_{011} & 0 & c_{100} & 0 & c_{010} & c_{001} & 0 \\ c_{102} & c_{101} & 0 & c_{002} & c_{100} & 0 & 0 & c_{001} & 0 & 0 \\ c_{030} & 0 & c_{020} & 0 & 0 & 0 & c_{010} & 0 & 0 & 0 \\ c_{021} & c_{020} & c_{011} & 0 & 0 & c_{010} & c_{001} & 0 & 0 & 0 \\ c_{012} & c_{011} & c_{002} & 0 & c_{010} & c_{001} & 0 & 0 & 0 & 0 \\ c_{003} & c_{002} & 0 & 0 & c_{001} & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \equiv 0$$

The variety $\mathcal{T}^3_{8,5,9} \subset \mathbb{P}^{219}$ is a hypersurface (expected to fill the whole $\mathbb{P}^{219}$)

Indeed, the maximal minors of the Padé Matrix have one common factor.

Conjecture

- For $n = 2$, all Taylor varieties $\mathcal{T}^2_{d,e,m}$ are non-defective.
- For $n \geq 3$ there are only finitely-many defective cases.

Hessians

The variety $\mathcal{T}^2_{1,1,2}$ is known as the *Perazzo*[2] *cubic surface*. It has *vanishing Hessian*: the determinant of Hessian matrix is identically zero.

$$M_T = \begin{bmatrix} c_{20} & c_{10} & 0 \\ c_{11} & c_{01} & c_{10} \\ c_{02} & 0 & c_{01} \end{bmatrix} \quad \text{and} \quad H = \begin{bmatrix} 2c_{20} & -c_{11} & 0 & -c_{10} & 2c_{01} \\ -c_{11} & 2c_{02} & 2c_{10} & -c_{01} & 0 \\ 0 & 2c_{10} & 0 & 0 & 0 \\ -c_{10} & -c_{01} & 0 & 0 & 0 \\ 2c_{01} & 0 & 0 & 0 & 0 \end{bmatrix}$$

thus $\det(H) \equiv 0$.

Let $n \geq 2$. If $\mathcal{T}^n_{d,e,m}$ is a hypersurface, then it has vanishing Hessian.

Conjecture.
$\mathcal{T}^1_{d,e,d+e+1}$ has *zero Gaussian curvature*, for every $d \geq 1, e \geq 2$, that is, its Hessian determinant is a multiple of the defining polynomial of $\mathcal{T}^1_{d,e,d+e+1}$.

[2] U. Perazzo, *Sulle varietà cubiche la cui hessiana svanisce identicamente.* G. Mat. Battaglini 38 (1900), 337-354

Taylor varieties are defined as set of Taylor polynomials of rational functions

Classical well-known varieties for $n = 1$ (secants to rational normal curve)

Interesting phenomena for $n \geq 2$: defectivity, vanishing Hessians...

Available on **arXiv/2304.00712**