Stability Issues in Non-regular Electrical Circuits

Samir ADLY
University of Limoges, France
A joint work with R. Cibulka

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Given matrices $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{m \times n}$, and mappings $f : \mathbb{R}^n \to \mathbb{R}^n$, $F : \mathbb{R}^m \rightrightarrows \mathbb{R}^m$ with $m \leq n$, consider the problem of finding for a vector $p \in \mathbb{R}^n$ the solution $z \in \mathbb{R}^n$ to the inclusion

$$p \in f(z) + BF(Cz).$$

Standing assumptions:

(A1) $B$ is injective;

(A2) $f$ is continuously differentiable on $\mathbb{R}^n$;

(A3) $F$ has closed graph; and

(A4) $C$ is surjective;
Let \((\bar{z}, \bar{p}) \in \text{gph } \Phi\). The solution mapping \(\Psi := \Phi^{-1}\) has

- **calmness property** at \((\bar{p}, \bar{z})\) if there is a constant \(\kappa \geq 0\) along with neighborhoods \(U\) of \(\bar{p}\) and \(V\) of \(\bar{z}\) such that

\[
\Psi(p) \cap V \subset \Psi(\bar{p}) + \kappa \|p - \bar{p}\|B \text{ whenever } p \in U,
\]

*Calmness modulus:* \(\text{clm}(\Psi; (\bar{p}, \bar{z})) := \inf \kappa\).

- **isolated calmness property** at \((\bar{p}, \bar{z})\) provided that it has calmness property and \(\bar{z}\) is an isolated point of \(\Psi(\bar{p})\), i.e.
\(\Psi(\bar{p}) \cap B(\bar{z}, r) = \{\bar{z}\}\) for some \(r > 0\). This amounts to the existence of a constant \(\kappa \geq 0\) along with neighborhoods \(U\) of \(\bar{p}\) and \(V\) of \(\bar{z}\) such that

\[
\Psi(p) \cap V \subset \bar{z} + \kappa \|p - \bar{p}\|B \text{ for each } p \in U.
\]
Theorem
Suppose that the assumptions (A1) – (A4) hold true. Put
\( \bar{v} := (B^T B)^{-1} B^T (\bar{p} - f(\bar{z})) \). Then \( \Psi \) has the isolated calmness property at \( (\bar{p}, \bar{z}) \) if and only if

\[
\begin{align*}
(Cb, -(B^T B)^{-1} B^T \nabla f(\bar{z})b) &\in T((C\bar{z}, \bar{v}); \text{gph } F) \\
\nabla f(\bar{z})b &\in \text{rge } B
\end{align*}
\]  \implies b = 0.

\( T(\bar{x}; \Omega) \) is a \textit{Bouligand-Severi tangent cone} to a non-empty subset \( \Omega \) of \( \mathbb{R}^d \) at \( \bar{x} \in \Omega \) which contains those \( v \in \mathbb{R}^d \) for which there are sequences \( (t^k)_{k \in \mathbb{N}} \) in \( (0, \infty) \) and \( (v^k)_{k \in \mathbb{N}} \) in \( \mathbb{R}^d \) converging to 0 and \( v \), respectively, such that \( \bar{x} + t^k v^k \in \Omega \) whenever \( k \in \mathbb{N} \).
Theorem
Suppose that the assumptions \((A1) \rightarrow (A4)\) hold true. Put
\[
\bar{v} := (B^T B)^{-1} B^T (\bar{p} - f(\bar{z})).
\]
Then \(\Psi\) has the isolated calmness property at \((\bar{p}, \bar{z})\) if and only if
\[
\left\{ \begin{array}{l}
(Cb, -(B^T B)^{-1} B^T \nabla f(\bar{z}) b) \in T((C\bar{z}, \bar{v}); \text{gph } F) \\
\nabla f(\bar{z}) b \in \text{rge } B
\end{array} \right\} \implies b = 0.
\]

Moreover, the calmness modulus is given by
\[
\text{clm} (\Psi; (\bar{p}, \bar{z})) = \sup \{ \|b\| : (\nabla f(\bar{z}) b + BDF(C\bar{z}, \bar{v})(Cb)) \cap \mathbb{B} \neq \emptyset \}.
\]

\(DH(\bar{x}, \bar{y})\) denotes the contingent (graphical) derivative of a mapping \(H : \mathbb{R}^d \rightrightarrows \mathbb{R}^l\) at \((\bar{x}, \bar{y}) \in \text{gph } H\) defined by
\[
DH(\bar{x}, \bar{y})(u) = \{ v \in \mathbb{R}^l : (u, v) \in T((\bar{x}, \bar{y}); \text{gph } H) \}, \quad u \in \mathbb{R}^d.
\]
Ingredients of the proof:

- Calculus rules to compute the graphical derivative of $\Phi$ at the reference point (injectivity of $B$ seems to be essential);
- Apply the criterion by T. Rockafellar.

An analogues condition ensuring Aubin continuity of $\Psi$ at the reference point exits (using the coderivative criterion by B. Mordukhovich and the limiting normal cone);

The isolated calmness and the Aubin property are stable with respect to a small perturbation of the function $f$ as well as the reference point. This is not the case of the calmness property!
Example ($m = n = 1, B = C = 1$):

\[ V_{\text{ref}} + i(t) + u(t) = E - V_D - V_R \]

Kirchhoff’s laws reveal that

\[ u - E \in R_i + F_1(i), \]

where $F_1$ is the subdifferential mapping (in the sense of Clarke) of a locally Lipschitz-continuous function $j_1 : \mathbb{R} \to \mathbb{R}$ which is called Moreau-Panagiotopoulos super-potential.
Assume that $R = 1$ and $F_1 : \mathbb{R} \rightarrow \mathbb{R}$ is such that

- $F_1(0) = [-1, 1]$;
- it is single-valued and continuously differentiable on $\mathbb{R} \setminus \{0\}$;
- its graph is symmetric with respect to the origin;
- $F_1(0-) = -1$ and $F_1'(0-) = -a$ for some $a > 0$. 
One gets an equivalent circuit

\[ V_{\text{ref}} + u(t) \]

\[ V_D \]

to which corresponds an equivalent generalized equation (with \( z := i \) and \( p := u - E \))

\[ p \in f(z) + F(z), \]

where \( F := \partial | \cdot | \) and \( f : \mathbb{R} \to \mathbb{R} \) is a continuously differentiable odd function with \( f'(0) = 1 - a \).
Let \((\bar{z}, \bar{p}) = (0, -1)\), thus \(\bar{v} = (B^T B)^{-1} B^T (\bar{p} - f(\bar{z}))\).

\(\Psi\) has the isolated calmness property at \((-1, 0)\) if and only if

\[
(b, -(1 - a)b) \in T((0, -1); \text{gph } F) \implies b = 0,
\]

where

\[
T((0, -1); \text{gph } F) = \mathbb{R}_+ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cup \mathbb{R}_+ \begin{pmatrix} -1 \\ 0 \end{pmatrix}.
\]
Let \((\bar{z}, \bar{p}) = (0, -1)\), thus \(\bar{v} = -1\).

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\(\Psi\) has the isolated calmness property at \((-1, 0)\) if and only if

\[
\left( Cb, -\left(B^T B\right)^{-1}B^T \nabla f(\bar{z})b \right) \in T((C\bar{z}, \bar{v}); \text{gph } F) \nabla f(\bar{z})b \in \text{rge } B \implies b = 0.
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where

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T((0, -1); \text{gph } F) = \mathbb{R}_+ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cup \mathbb{R}_+ \begin{pmatrix} -1 \\ 0 \end{pmatrix}.
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\]
Let $(\bar{z}, \bar{p}) = (0, -1)$, thus $\bar{v} = -1$. 

$\Psi$ has the isolated calmness property at $(-1, 0)$ if and only if 

$$(b, -(1 - a)b) \in T((0, -1); \text{gph } F) \implies b = 0,$$

where 

$$T((0, -1); \text{gph } F) = \mathbb{R}_+ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cup \mathbb{R}_+ \begin{pmatrix} -1 \\ 0 \end{pmatrix}.$$ 

So, $\Psi$ has the isolated calmness property at $(-1, 0)$ if and only if $a \neq 1$. 
clm (Ψ; (¯p, ¯z)) = sup \{∥b∥ : (\nabla f(¯z)b + BDF(C¯z, ¯v)(Cb)) \cap \mathbb{B} \neq \emptyset\}.

One has to find a maximum of |x| subject to

\[-1 \leq (1 - a)x + y \leq 1 \quad \text{with} \quad (x, y) \in T((0, -1); \text{gph} F)\].

\[
\clm (\psi; (\bar{p}, \bar{z})) = \begin{cases} 
\frac{1}{|a-1|} & \text{if} \quad a \neq 1; \\
\infty & \text{otherwise}.
\end{cases}
\]
\text{clm}(\Psi; (-1, 0)) = \sup \{|x| : ((1 - a)x + DF(0, -1)(x)) \cap [-1, 1] \neq \emptyset \}.

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\infty & \text{otherwise}.
\end{cases}
\end{align*}
\]
Given a lower semi-continuous function $g : \mathbb{R}^d \to \overline{\mathbb{R}} := \mathbb{R} \cup \{\pm \infty\}$ and a point $x \in \mathbb{R}^d$ with $g(x) \in \mathbb{R}$, the

- **Fréchet subdifferential** of $g$ at $x$ is the set

$$\partial_F g(x) := \left\{ \xi \in \mathbb{R}^d : \liminf_{0 \neq h \to 0} \frac{g(x + h) - g(x) - \langle \xi, h \rangle}{\|h\|} \geq 0 \right\};$$

- **outer subdifferential** of $g$ at $x$ is the set $\partial > g(x)$ which contains those $\xi \in \mathbb{R}^d$ for which there are sequences $(x^k)_{k \in \mathbb{N}}$ and $(\xi^k)_{k \in \mathbb{N}}$ converging to $x$ and $\xi$, respectively, with

$$g(x^k) \downarrow g(x) \quad \text{as} \quad k \to \infty \quad \text{and} \quad \xi^k \in \partial_F g(x^k) \quad \text{for each} \quad k \in \mathbb{N}.$$
Given a lower semi-continuous function $g : \mathbb{R}^d \rightarrow \overline{\mathbb{R}} := \mathbb{R} \cup \{\pm \infty\}$ and a point $x \in \mathbb{R}^d$ with $g(x) \in \mathbb{R}$, the

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Replacing $g(x^k) \downarrow g(x)$ by $g(x^k) \to g(x)$ one gets the *limiting (Mordukhovich) subdifferential* $\partial g(x)$ of $g$ at $x$. 


Given a lower semi-continuous function \( g : \mathbb{R}^d \to \overline{\mathbb{R}} := \mathbb{R} \cup \{ \pm \infty \} \) and a point \( x \in \mathbb{R}^d \) with \( g(x) \in \mathbb{R} \), the

► **Fréchet subdifferential** of \( g \) at \( x \) is the set

\[
\partial_F g(x) := \left\{ \xi \in \mathbb{R}^d : \liminf_{0 \neq h \to 0} \frac{g(x + h) - g(x) - \langle \xi, h \rangle}{\| h \|} \geq 0 \right\};
\]

► **outer subdifferential** of \( g \) at \( x \) is the set \( \partial > g(x) \) which contains those \( \xi \in \mathbb{R}^d \) for which there are sequences \( (x^k)_{k \in \mathbb{N}} \) and \( (\xi^k)_{k \in \mathbb{N}} \) converging to \( x \) and \( \xi \), respectively, with

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g(x^k) \downarrow g(x) \quad \text{as} \quad k \to \infty \quad \text{and} \quad \xi^k \in \partial_F g(x^k) \quad \text{for each} \quad k \in \mathbb{N}.
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Replacing \( g(x^k) \downarrow g(x) \) by \( g(x^k) \to g(x) \) one gets the **limiting (Mordukhovich) subdifferential** \( \partial g(x) \) of \( g \) at \( x \).

If \( g(x) \) is infinite then the above subdifferentials of \( g \) at \( x \) are defined to be empty set.
Theorem

Suppose that the assumptions (A1) – (A3) are satisfied. Put $\Lambda = \text{gph} F \times \text{rge} B$ and define the functions $g : \mathbb{R}^n \to \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^n$ and $h : \mathbb{R}^n \to \mathbb{R}_+$ for each $z \in \mathbb{R}^n$ by $g(z) = (Cz, (B^T B)^{-1} B^T (\bar{p} - f(z)), \bar{p} - f(z))$ and $h(z) = d(g(z), \Lambda)$. Then $\Psi$ has the calmness property at $(\bar{p}, \bar{z})$ provided that

$$0 \notin \partial h(\bar{z}).$$
Theorem

Suppose that the assumptions (A1) – (A3) are satisfied. Put \( \Lambda = \text{gph } F \times \text{rge } B \) and define the functions \( g : \mathbb{R}^n \to \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^n \) and \( h : \mathbb{R}^n \to \mathbb{R}_+ \) for each \( z \in \mathbb{R}^n \) by 
\[
g(z) = (Cz, (B^T B)^{-1} B^T (\bar{p} - f(z)), \bar{p} - f(z))
\]
and 
\[
h(z) = d(g(z), \Lambda).
\]
Then \( \Psi \) has the calmness property at \((\bar{p}, \bar{z})\) provided that 
\[
0 \notin \partial> h(\bar{z}).
\]

If there is \( \gamma > 0 \) such that \( \|\xi\| \geq \gamma \) for each \( \xi \in \partial> h(\bar{z}) \) then 
\[
\text{clm} (\Psi; (\bar{p}, \bar{z})) \leq 1/\gamma.
\]
Example \((n = 2, \ m = 1)\):

\[
\begin{align*}
\left( \frac{dz_1}{dt} \right) &= \left( \begin{array}{cc}
0 & 1 \\
-\frac{1}{LC} & -\frac{R}{L}
\end{array} \right) \left( \begin{array}{c}
z_1 \\
z_2
\end{array} \right) - \left( \begin{array}{cc}
0 & 1 \\
-\frac{1}{L}
\end{array} \right) y_L + \left( \begin{array}{c}
0 \\
\frac{1}{L}
\end{array} \right) u, \\
y &= \left( \begin{array}{c}
0 \\
-1
\end{array} \right) \left( \begin{array}{c}
z_1 \\
z_2
\end{array} \right)
\end{align*}
\]

and

\[y_L \in F(y).\]
Assume that $R = L = C = 1$. Steady states of the previous dynamic system correspond to the following generalized equation

$$p \in Az + BF(Cz), \quad z = (z_1, z_2)^T \in \mathbb{R}^2,$$

with

$$A = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad C = (0 \quad -1),$$

and

$$F(x) := \begin{cases} -100 & x < 0, \\ [-100, 1], & x = 0, \\ 1, & x > 0. \end{cases}$$
Let $\bar{z} = (0, 0)^T$ and $\bar{p} = (0, 0)^T$.

As $B^T B = 1$, one has $\bar{v} = (B^T B)^{-1} B^T (\bar{p} - f(\bar{z}))$.

$\Psi$ has the isolated calmness property at $(\bar{p}, \bar{z})$ if and only if

$$
\begin{cases}
(-b_2, b_1 + b_2) \in T((0, 0); gph F) \\
(b_2, -b_1 - b_2) \in \{0\} \times \mathbb{R}
\end{cases} \implies b_1 = 0 \text{ and } b_2 = 0.
$$

where

$$
T((0, 0); gph F) = \mathbb{R} \begin{pmatrix} 0 \\ 1 \end{pmatrix}
$$

Thus $\Psi$ does not have the isolated calmness property at $(\bar{p}, \bar{z})$. 
Let $\bar{z} = (0, 0)^T$ and $\bar{p} = (0, 0)^T$.

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where

\[
T\left((0, 0); \text{gph} \ F\right) = \mathbb{R}\begin{pmatrix} 0 \\ 1 \end{pmatrix}
\]

Thus $\Psi$ does not have the isolated calmness property at $(\bar{p}, \bar{z})$. 
For each \( z = (z_1, z_2)^T \in \mathbb{R}^2 \), one has

\[
g(z) = (Cz, (B^T B)^{-1}B^T(\bar{p} - f(z)), \bar{p} - f(z))
\]

and

\[
h(z) = d(g(z), \Lambda).
\]

Then, choosing a suitable equivalent norm on \( \mathbb{R}^2 \times \mathbb{R}^2 \), one gets

\[
h(z) = d((-z_2, z_1 + z_2), \text{gph } F) + d((-z_2, z_1 + z_2), \{0\} \times \mathbb{R})
\]

\[
= d((-z_2, z_1 + z_2), \text{gph } F) + |z_2| = d(-Az, \text{gph } F) + |z_2|.
\]

Hence

\[
\partial h(z) \subset \partial d(-A(\cdot), \text{gph } F)(z) + \{0\} \times \partial |(z_2)
\]

\[
= -A^T \partial d(\cdot, \text{gph } F)(-z_2, z_1 + z_2) + \{0\} \times \partial |(z_2).
\]

Find \( r \in (0, 1) \) such that \( \partial d(\cdot, \text{gph } F)(u) = \left\{ \left( \frac{x}{|x|}, 0 \right)^T \right\} \) whenever

\[
u = (x, y)^T \in rB_2.
\]
For each $z = (z_1, z_2)^T \in \mathbb{R}^2$, one has
\[ g(z) = (-z_2, z_1 + z_2, -z_2, z_1 + z_2) \]
and
\[ h(z) = d(g(z), \Lambda). \]

Then, choosing a suitable equivalent norm on $\mathbb{R}^2 \times \mathbb{R}^2$, one gets
\[
\begin{align*}
h(z) &= d((-z_2, z_1 + z_2), \text{gph } F) + d((-z_2, z_1 + z_2), \{0\} \times \mathbb{R}) \\
&= d((-z_2, z_1 + z_2), \text{gph } F) + |z_2| = d(-Az, \text{gph } F) + |z_2|.
\end{align*}
\]

Hence
\[
\begin{align*}
\partial h(z) &\subset \partial d(-A(\cdot), \text{gph } F)(z) + \{0\} \times \partial |\cdot|(z_2) \\
&= -A^T \partial d(\cdot, \text{gph } F)(-z_2, z_1 + z_2) + \{0\} \times \partial |\cdot|(z_2).
\end{align*}
\]

Find $r \in (0, 1)$ such that $\partial d(\cdot, \text{gph } F)(u) = \left\{ \left( \frac{x}{|x|}, 0 \right)^T \right\}$ whenever
$u = (x, y)^T \in r \mathbb{B}_2$. 
For each \( z = (z_1, z_2)^T \in \mathbb{R}^2 \), one has

\[
g(z) = (-z_2, z_1 + z_2, -z_2, z_1 + z_2)
\]

and

\[
h(z) = d(g(z), \Lambda).
\]

Then, choosing a suitable equivalent norm on \( \mathbb{R}^2 \times \mathbb{R}^2 \), one gets

\[
h(z) = d((-z_2, z_1 + z_2), \text{gph} \ F) + d((-z_2, z_1 + z_2), \{0\} \times \mathbb{R})
\]

\[
= d((-z_2, z_1 + z_2), \text{gph} \ F) + |z_2| = d(-Az, \text{gph} \ F) + |z_2|.
\]

Hence

\[
\partial h(z) \subset \partial d(-A(\cdot), \text{gph} \ F)(z) + \{0\} \times \partial |\cdot|(z_2)
\]

\[
= -A^T \partial d(\cdot, \text{gph} \ F)(-z_2, z_1 + z_2) + \{0\} \times \partial |\cdot|(z_2).
\]

Find \( r \in (0, 1) \) such that \( \partial d(\cdot, \text{gph} \ F)(u) = \left\{ \left( \frac{x}{|x|}, 0 \right)^T \right\} \) whenever

\[
u = (x, y)^T \in rB_2.
\]
For each $z = (z_1, z_2)^T \in \mathbb{R}^2$, one has
\[ g(z) = (-z_2, z_1 + z_2, -z_2, z_1 + z_2) \]
and
\[ h(z) = d((-z_2, z_1 + z_2, -z_2, z_1 + z_2), \text{gph } F \times \{0\} \times \mathbb{R}). \]
Then, choosing a suitable equivalent norm on $\mathbb{R}^2 \times \mathbb{R}^2$, one gets
\[
\begin{align*}
h(z) &= d((-z_2, z_1 + z_2), \text{gph } F) + d((-z_2, z_1 + z_2), \{0\} \times \mathbb{R}) \\
&= d((-z_2, z_1 + z_2), \text{gph } F) + |z_2| = d(-Az, \text{gph } F) + |z_2|.
\end{align*}
\]
Hence
\[
\partial h(z) \subset \partial d(-A(\cdot), \text{gph } F)(z) + \{0\} \times \partial |(z_2)|
\]
\[
= -A^T \partial d(\cdot, \text{gph } F)(-z_2, z_1 + z_2) + \{0\} \times \partial |(z_2)|.
\]
Find $r \in (0, 1)$ such that $\partial d(\cdot, \text{gph } F)(u) = \left\{ \left( \frac{x}{|x|}, 0 \right)^T \right\}$ whenever $u = (x, y)^T \in r \mathbb{B}_2$. 
Fix any \( z = (z_1, z_2)^T \in \mathbb{R}^2 \) with \( h(z) > 0 \) and \( -Az \in rB_2 \).

Then \( z_2 \neq 0 \). If not, as \( h(z) = d((−z_2, z_1 + z_2), \text{gph } F) + |z_2| > 0, \ z_1 \notin F(0) \), so either \( z_1 > 1 \) or \( z_1 < −100 \). Both the cases are impossible because \( |z_1| \leq r < 1 \).

As \( ∂h(z) \subset −A^T \left( \frac{-z_2}{|z_2|}, 0 \right)^T + \{0\} \times ∂|⋅|(z_2) \), one has either

\[
∂h(z) \subset \left( \begin{array}{cc} 0 & 1 \\ -1 & 1 \end{array} \right) \left( \begin{array}{c} -1 \\ 0 \end{array} \right) + \left( \begin{array}{c} 0 \\ 1 \end{array} \right) = \left( \begin{array}{c} 0 \\ 2 \end{array} \right) \quad \text{when } z_2 > 0,
\]

or

\[
∂h(z) \subset \left( \begin{array}{cc} 0 & 1 \\ -1 & 1 \end{array} \right) \left( \begin{array}{c} 1 \\ 0 \end{array} \right) + \left( \begin{array}{c} 0 \\ -1 \end{array} \right) = \left( \begin{array}{c} 0 \\ -2 \end{array} \right) \quad \text{when } z_2 < 0.
\]

In any case, \( ∥η∥ \geq 2 \) for each \( η \in ∂h(z) \).

Hence \( ∥ξ∥ \geq 2 \) whenever \( ξ \in ∂h(\bar{z}) \). Thus \( Ψ \) is calm at \((\bar{p}, \bar{z})\) with the calmness modulus not exceeding \( 1/2 \).
Similar computation yields that
Do all the electrical devices on board have the Aubin or calmness properties?
Do all the electrical devices on board have the Aubin or calmness properties?

I don't know but they fly!


