Master of Mathematics ACSYON/SIC
Polynomial Systems

Université de Limoges
Faculté des Sciences et Techniques

Homework / Exercices session 4

Monomials and monomial order

Exercise 1. Let $\prec$ be a monomial order, $\boldsymbol{x}^{\boldsymbol{\alpha}}$ and $\boldsymbol{x}^{\boldsymbol{\beta}}$ be two monomials in $\mathbb{M}_n$. Show that $\boldsymbol{x}^{\boldsymbol{\alpha}}|\boldsymbol{x}^{\boldsymbol{\beta}}$ implies $\boldsymbol{x}^{\boldsymbol{\alpha}} \prec \boldsymbol{x}^{\boldsymbol{\beta}}$.

Exercise 2. Let I be an ideal of $\mathbb{K}[x_1, ., x_n]$, show that $\mathrm{in}(I) := < \mathrm{LT}(I) > = \{\mathrm{LT}(p) | p \in I\}$ is an ideal (of course a monomial one).

Exercise 3. Find an instance of two monomial such that they do not form a Gröbner basis of the ideal they generate, i.e. find $I = < \boldsymbol{x}^{\boldsymbol{\alpha}}, \boldsymbol{x}^{\boldsymbol{\beta}} >$ such that $\{\boldsymbol{x}^{\boldsymbol{\alpha}}, \boldsymbol{x}^{\boldsymbol{\beta}}\}$ is not a Gröbner of I.

Exercise 4. Let I_1, I_2 and I_3 be three monomial ideals. Prove that $(I_1 + I_2) \cap I_3 = (I_1 \cap I_3) + (I_2 \cap I_3)$.

Exercise 5. Let I_1 and I_2 be two monomial ideals, show that $I_1 \cap I_2$ is a monomial ideal (you can use preecedings exercices).

Basics on Gröbner basis

Exercise 6. Let $f_1 = x^2 y + z$ and $f_2 = x z + y \in \mathbb{K}[x, y, z]$.

1. Compute a Gröbner basis of $< f_1, f_2 >$ for lex order with $x > y > z$.
2. Dividing $f = x^2 z^3 - x y^2 - z y^2 + z^2$ by the computed Gröbner basis, show that $f \in < f_1, f_2 >$. Would you be able to conclude without computing the Gröbner basis ?
3. Determines q_1 and $q_2 \in \mathbb{K}[x, y, z]$ such that $f = q_1 f_1 + q_2 f_2$.

Exercise 7. Let I be an ideal of $\mathbb{K}[x_1, ..., x_n]$ and G be a finite subset of I. Show that G is a Gröbner basis of I with respect to a given monomial order $\prec$ if and only if for each $f \in I$ the remainder of the division of f by G is 0.