Computing the topology of implicit space curves with certainty

See the wrong curve with the right topology

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The problem

Let P_1 and $P_2 \in \mathbb{Q}[x, y, z]$ such that :

- P_1 and P_2 are squarefree;
- $gcd(P_1, P_2) = 1$.

We want to compute the topology of $\mathcal{C} = \left\{(x, y, z) \in \mathbb{R}^3 \| P_1(x, y, z) = P_2(x, y, z) = 0\right\}$, i.e. compute a piecewise linear structure isotopic to $\mathcal{C}$.

Definition *Let E be a $\mathbb{R}$-vector space and let U and $V \subset E$, an isotopy from U on V is a continuous map $\varphi : [0, 1] \times E \longrightarrow E$ such that for all $t \in [0, 1]$, $\varphi(t, \cdot) : E \longrightarrow E$ is an homeomorphism, $\varphi(0, U) = U$ and $\varphi(1, U) = V$.*

Algebraic tools

Subresultants

Definition *Let f et $g \in \mathbb{Q}[x]$, the sequence $S_0 = f, S_1 = g, \ldots, S_l$ is said to be a Sturm sequence if :*

- *$S_l | S_i$ for all $i \in \{1, \ldots, l-1\}$ and in this case we denote $\sigma_i = \frac{S_i}{S_l}$;*
- *if $z \in \mathbb{R}$ is such that $\sigma_j(z) = 0$, for $j \in \{0, \ldots, l\}$ then $\sigma_{i-1}(z) * \sigma_{i+1}(z) < 0$;*
- *if $z \in \mathbb{R}$ is such that $\sigma_0(z) = 0$ then $\sigma_1(x) * \sigma_0(x)$ has the sign of $(x - z)$ near z.*

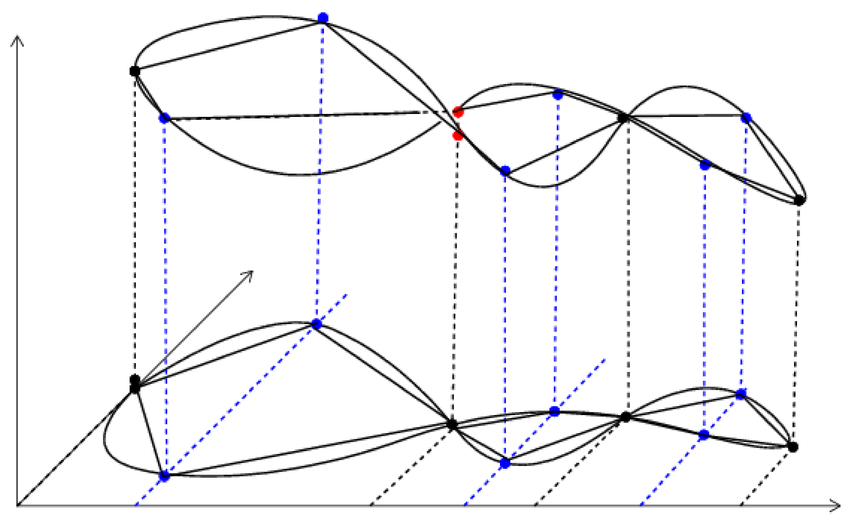
Definition *Let M be a $k \times l$ matrix with $k > l$, we define the determinantal polynomial associated to M :*
$detpol(M) = \det(M_k) z^{k-l} + \cdots + \det(M_k)$ where M_j is the submatrix of M obtained in taking the $l-1$ first rows of M that follow the j^{th}.

Definition *Let f and $g \in \mathbb{Q}[x]$ two polynomials of degree, resp. , d and e. The subresultant polynomial of order i associated to f and g is :*
$Sr_i(x) = detpol(x^{e-i-1} f, \ldots, f, x^{d-i-1} g, \ldots, g) = \sum_{k=0}^{i} sr_{i,k} x^k$.

Theorem *The first polynomial $Sr_k(x)$ such that $sr_{k,k} \neq 0$ is the gcd(f, g).*

Strategy

The basic idea is to compute the topology of the projection of the curve on a plane and to lift the computed topology on the space curve. Our technic allows to use a new sweeping algorithm using only one projection of the space curve.



To allow this lifting, we introduce a new notion of generic position for space curves.

Generic position

Let us denote $\Pi_z : (x, y, z) \in \mathbb{R}^3 \mapsto (x, y) \in \mathbb{R}^2$ and $\mathcal{D} = \Pi_z(\mathcal{C}) \subset \mathbb{R}^2$ the curve obtained by projection of $\mathcal{C}$. We want $\mathcal{C}$ and $\mathcal{D}$ to have some good geometric properties with respect to a projection. For instance, we want that for almost all point of $\mathcal{D}$ have only one pre-image. This notion is the one of pseudo-generic position:

Definition [Pseudo-generic position] *The curve $\mathcal{C}$ is in pseudo-generic position with respect to the (x, y)-plane if and only if almost every point of $\Pi_z(\mathcal{C})$ has only one pre-image, i.e. generically, if $(\alpha, \beta) \in \Pi_z(\mathcal{C})$, then $\Pi_z^{-1}(\alpha, \beta)$ consists in one point possibly multiple.*

Definition [Generic position] *The curve $\mathcal{C}$ is in generic position with respect to the (x, y)-plane if and only if:*

i. $\mathcal{C}$ is in pseudo-generic position with respect to the (x, y)-plane,

ii. $\mathcal{D} = \Pi_z(\mathcal{C})$ is in generic position (as a plane algebraic curve),

iii. any x-critical fiber of $\mathcal{C}$ contain only one point of $\mathcal{C}$.

Computing the topology of a plane algebraic curve

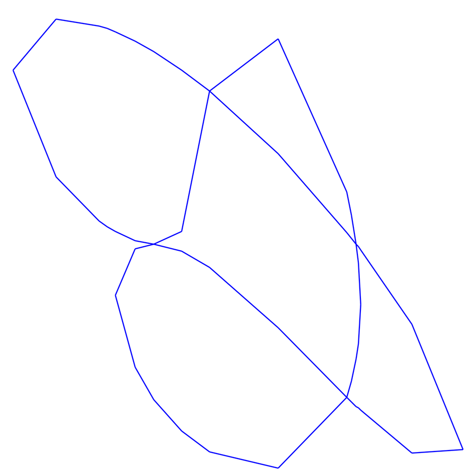
Definition *Let $f \in \mathbb{Q}[x, y]$ be a squarefree polynomial and $\mathcal{D} = \left\{(\alpha, \beta) \in \mathbb{R}^2 : f(\alpha, \beta) = 0\right\}$ be the curve defined by f. Let $\mathcal{N}_x(\alpha) = \#\{\beta \in \mathbb{R}$, such that (α, β) is a x-critical point of $\mathcal{D}\}$. $\mathcal{D}$ is in generic position for the x-direction, if:*

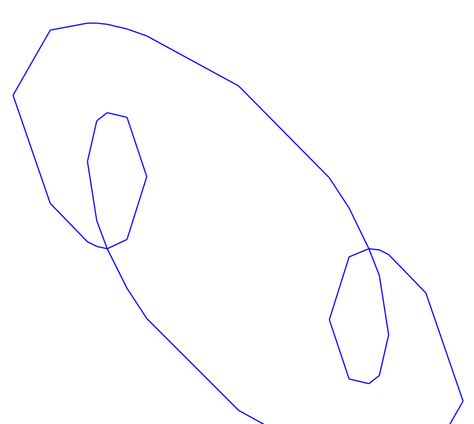
- *$\forall \alpha \in \mathbb{R}, \mathcal{N}_x(\alpha) \leqslant 1$,*
- *there is no asymtoptic direction of $\mathcal{D}$ parallel to the y-axis.*

An algebraic certificate of generic position for a plane curve

Theorem *Let $f \in \mathbb{Q}[x, y]$ be a squarefree polynomial and $d = \deg_y(f)$. Let $Sr_i(x, y)$ be the i^{th} subresultant polynomial and $sr_{i,j}(x)$ be the subresultant coefficient polynomial of order (i, j) associated to $f(x, y)$ and $\partial_y f(x, y)$.We define inductively the following polynomials:*
$\Phi_0(x) = \frac{sr_{0,0}(x)}{\gcd(sr_{0,0}(x), sr'_{0,0}(x))}, \forall i = 1 \ldots d, \Phi_i(x) = \gcd(\Phi_{i-1}(x), sr_{i,i}(x)); \Gamma_i(x) = \frac{\Phi_{i-1}(x)}{\Phi_i(x)}$.
Then $\mathcal{D} = \Pi_z(\mathcal{C})$ is in generic position if and only if:
$\forall k \in \{1, \ldots, d-1\}, \forall i \in \{0, \ldots, k-1\}$, $k * (k - i) * sr_{k,i}(x) * sr_{k,k}(x) - (i + 1) * sr_{k,k-1}(x) * sr_{k,i+1}(x) = 0 \mod \Gamma_k(x)$.

Projections of two first examples (see last column on the right of the poster):





Computing the topology of a space algebraic curve

An algebraic certificate of pseudo-generic position for a space curve

Theorem *Let $P_1, P_2 \in \mathbb{Q}[x, y, z]$ be two squarefree polynomials and $\Theta_0(x, y)$ be the squarefree part of $Res_z(P_1, P_2)$. Let $(Sr_j(x, y, z))_{j \in \{0,\ldots,m\}}$ be the subresultant sequence and $(sr_j(x, y))_{j \in \{0,\ldots,m\}}$ be the principal subresultant coefficient sequence associated to P_1 and P_2. Let $(\Delta_i(x, y))_{i \in \{1,m\}}$ the sequence of $\mathbb{Q}[x, y]$ defined by the following relations: $\forall i = 1 \ldots m$,*

$$\Theta_i(x, y) = \gcd(\Theta_{i-1}(x, y), sr_i(x, y)); \Delta_i(x, y) = \frac{\Theta_{i-1}(x, y)}{\Theta_i(x, y)}$$

The curve $\mathcal{C}$ is in pseudo-generic position with respect to the (x, y)-plane if and only if the following equalities hold:
$\forall i \in \{1, \ldots, m-1\}, \forall j \in \{0, \ldots, i-1\}$, $i * (i - j) sr_{i,j}(x, y) * sr_{i,i}(x, y) - (j + 1) * sr_{i,i-1}(x, y) * sr_{i,j+1}(x, y) = 0 \mod \Delta_i(x, y)$.

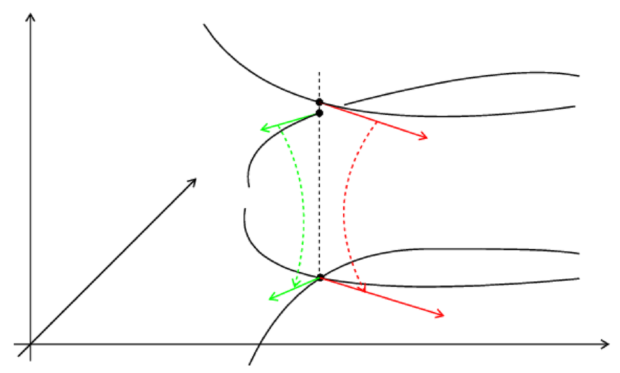
The Lifting up : detect apparent singularities

Definition *An apparent singularity is a the singularity of $\mathcal{D}$ corresponding to regular points of $\mathcal{C}$ by the considered projection.*

After certifing the genericity of the position of our space curve, we need to distinguish the singularities from the apparent singularities.
If the fiber of a singular point of $\mathcal{D}$ contains more than one point it is an apparent singularity.

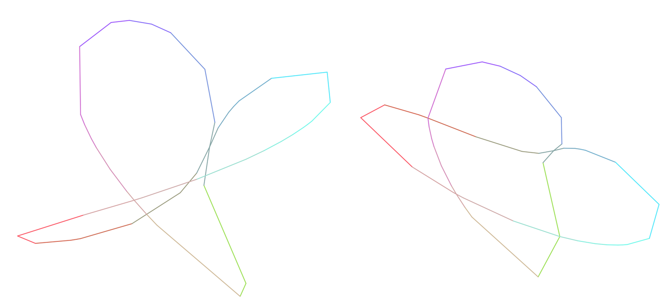
The lifting up near an apparent singularity

To lift up the topology near an apparent singularity we study the tangent space on regular points of the fiber of the apparent singularity.



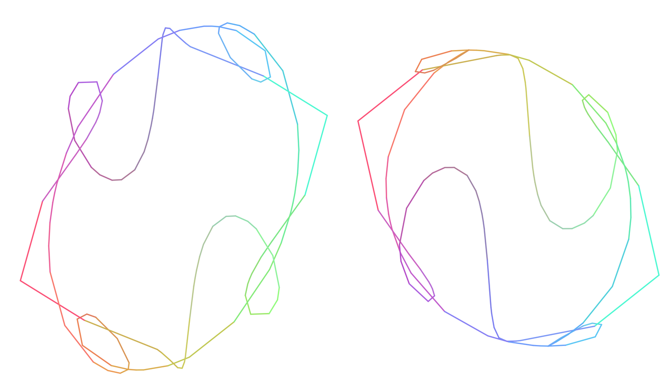
Examples

$(x^2 + y^2 + z^2)^2 - 4(x^2 + y^2)$
$x^2 + y^2 - 2x$



$x^2 + y^2 + z^2 - 4$
$x^3 + y^3 - xyz$

$\left(z^3 - z - x^3 + 3xy^2\right)\left(x^3 + y^3 - xyz\right)$
$x^2 + y^2 + z^2 - 1$



References

[1] Gatellier, G. and Labrouzy, A. and Mourrain, B. and Técourt, J. P. **Computing the topology of three-dimensional algebraic curves**, in *Computational methods for algebraic spline surfaces*, p. 27–43, Springer, Berlin, 2005.

[2] Gonzalez-Vega, Laureano; Necula, Ioana **Efficient topology determination of implicitly defined algebraic plane curves**. *Comput. Aided Geom. Design* 19 (2002), no. 9, 719–743.

[3] Owen, John C. and Rockwood, Alyn P. **Intersection of general implicit surfaces**. In *Geometric modeling, SIAM*, 335–345, 1987.

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