On the BCRE type models for the dynamics of granular materials (2)

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Outline

1. The table problem for growing sandpiles

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2. Numerical approximation and simulations

The table problem for growing sandpiles

The table problem: main notations

- $\Omega \subseteq \mathbb{R}^2$: bounded *table* , $\Omega_T = \Omega \times (0, T)$
- $\partial\Omega = \Gamma_0 \cup \Gamma_w$: Γ_0 *open boundary*, Γ_w *wall*
- $f(x) \geq 0$: vertical *source* , $D_f \subset \Omega$ its *support*
- $u(x, t)$, *pile height* in $x \in \Omega$ at time t
- $u(x, 0) = u_0(x)$: *initial pile profile* at $t = 0$
- $|\nabla u| \leq \alpha$: *critical slope* (here $\boxed{\alpha = 1}$)
- $d(x) = \mathrm{dist}(x, \Gamma_0)$: *distance* from the open bdry Γ_0
- S : *cut locus* of Ω (singular set of d)

The extended model of Hadeler and Kuttler

[Hadeler-Kuttler, '99]

$$(HK)\begin{cases} v_t = \beta\nabla\cdot(v\nabla u) - \gamma(\alpha - |\nabla u|)v + f & \text{in } \Omega_T \\ u_t = \gamma(\alpha - |\nabla u|)v & \text{in } \Omega_T \\ u(x,0) = u_0(x) & \text{in } \Omega \\ +\textit{boundary conditions on } \partial\Omega \end{cases}$$

where u : *standing layer*, v : *rolling layer*, $u+v$: *pile height*.

[By appropriate rescaling $\alpha = \beta = \gamma = 1$ ('dimensionless' model)]

Describes granular surface flow, neglecting avalanche phenomena (realistic for small source intensities). No existence and uniqueness results known for the complete system, even in one dimension!

Boundary conditions for the table problem

Boundary conditions

1. open table problem ($\partial\Omega = \Gamma_0$) $\Rightarrow$ Dirichlet
2. partially open table problem ($\Gamma_0, \Gamma_w \neq \emptyset$) $\Rightarrow$ Mixed
3. closed table problem (silo) ($\partial\Omega = \Gamma_w$) $\Rightarrow$ Neumann

The problem changes very much according to boundary conditions.
In cases (1) and (2), when $f = f(x)$, as time grows, we expect

$$u(x,t) \to \overline{u}(x) \qquad (\textit{equilibria}).$$

For the silo problem (3) (see e.g. [Hadeler-Kuttler, '99 and '01]):

$$u(x,t) \to \overline{u}(x) + ct \qquad (\textit{similarity solutions}).$$

Boundary conditions for the open table problem

In the 1D case ($\Omega = [0, L]$), near $x = 0$ the equations give

$$v_t = u_x v_x + ..., \qquad u_t = -v u_x + ...$$

Since we can suppose $u_x > 0$ and $v > 0$, we deduce a v-wave running to the left and an u-wave running to the right $\Rightarrow$ at $x = 0$ we need to impose a boundary condition on u, but not on v. The same is true at $x = L$. Then we simply impose the Dirichlet boundary condition

$$u(0, t) = u(L, t) = 0, \quad \forall t > 0$$

(no matter can accumulate on the boundary)

The same is true in 2D.

Stationary solutions for (HK)

When $f = 0$ any couple $(u, 0)$ is an equilibrium solution for the system. Of course if $|u_x| > 1$ such solutions are not stable under perturbations. Therefore the interesting configurations for the standing layer are the functions of the set

$$M = \{u \in C(\overline{\Omega}) : Lip(u) \leq 1, u \geq 0, u = 0 \text{ on } \partial\Omega\}$$

The mass (volume) of each function in M is given by

$$V(u) = \int_\Omega u(x)\, dx$$

Then

Proposition

Given an open set Ω:

1. $\exists!\ \overline{u} \in M : V(\overline{u}) = \max\{V(u) : u \in M\}$;
2. For any $u \in M$: $u(x) \leq \overline{u}(x) \quad \forall x \in \Omega$;
3. $\overline{u}(x) = d(x) = dist(x, \partial\Omega)$.

Proof

1. M is convex and compact, V is continuous, then the maximum is atteint. It is unique: if there exist two distinct elements in M, $\overline{u}$ and $\tilde{u}$ such that $V(\overline{u}) = V(\tilde{u})$ then $w = max(\overline{u}, \tilde{u}) \in M$ and $V(w) > V(\overline{u})$!

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2. If $u \in M$ and $u(x) \geq \overline{u}(x)$ for some $x \in \Omega$, then by continuity there exists an interval where the same inequality holds true and the same contradiction follows for the function $w = max(\overline{u}, u) \in M$.

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3. $\tilde{u}(x) = d(x) \in M$, then $\tilde{u} \leq \overline{u}$. If for some point x the inequality is strict, then the Lipschitz bound should be violated in $(0, x)$.

The distance function from the boundary

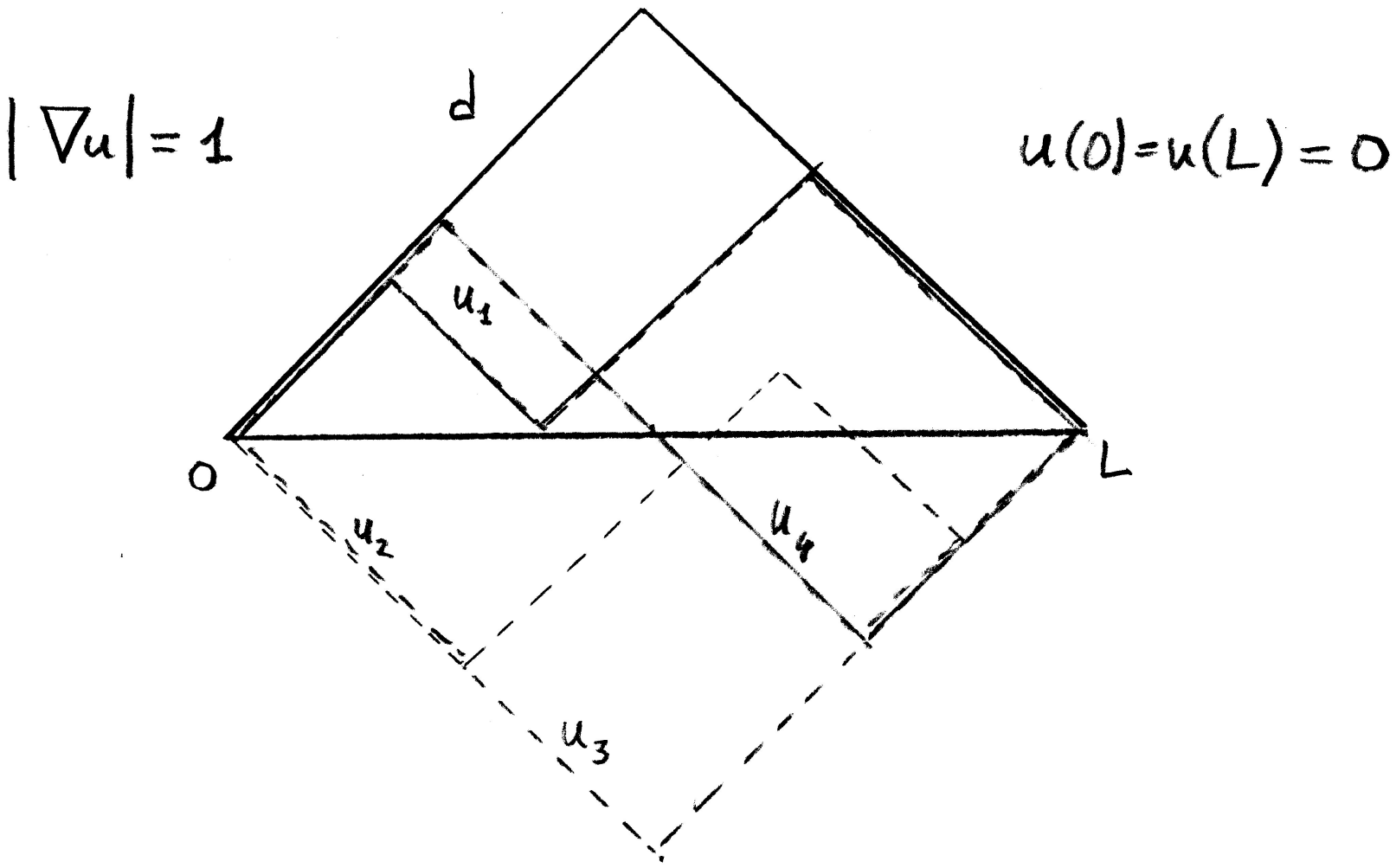
- $d : \overline{\Omega} \to \mathbb{R}$ *distance function* to the boundary $\partial\Omega$

$$d(x) = \min_{y \in \partial\Omega} |y - x|$$

- d is a Lipschitz continuous function, the unique viscosity solution ▸ DEF of the Eikonal equation:

$$|\nabla u| = 1 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega \ ,$$

that is the largest function such that: $\|\nabla u\|_\infty \leq 1, \ u = 0$ on $\partial\Omega$.


WEAK SOLUTIONS TO EIKONAL EQUATION
$|\nabla u| = 1$
d
$u(0) = u(L) = 0$
u_1
0
L
u_2
u_4
u_3

The singular set of the distance function

- Let $\Pi(x) : \overline{\Omega} \to \partial\Omega$ be the projection of x to the boundary, i.e.

$$\Pi(x) = \{y \in \partial\Omega : d(x) = |y - x|\}$$

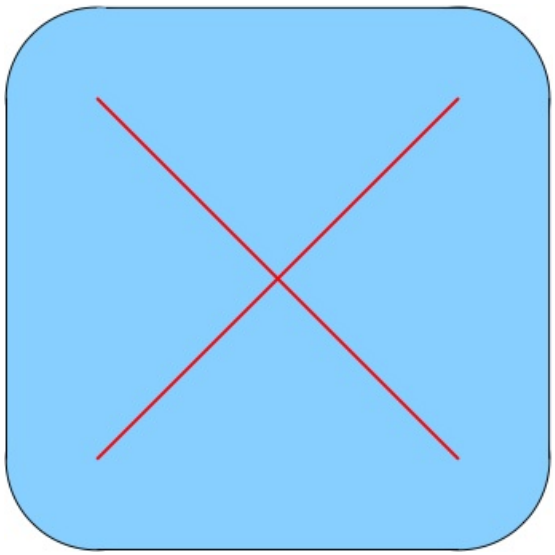
- The singular set of d is the set S of points of Ω where d is not differentiable; equivalently

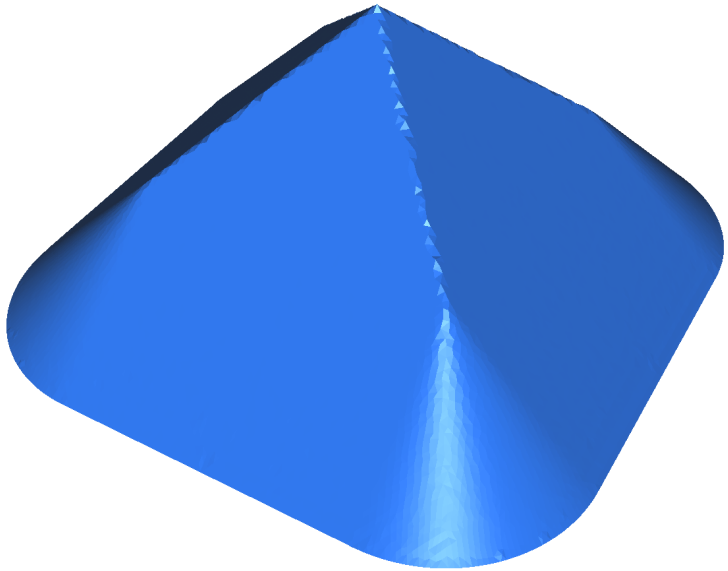
$$S = \{x \in \Omega : \Pi(x) \text{ is not a singleton}\}$$

- $\overline{S}$ is called the *cut locus* of Ω
- Since d is Lipschitz continuous then $|S| = 0$
- $\partial\Omega$ smooth $\Rightarrow \overline{S} \subset \Omega$
- If $\partial\Omega$ has corners $\Rightarrow S$ has an arc running into each corner;
- In 1D, $\Omega = (0, L)$ then $S(\Omega) = \{L/2\}$, with

$$d(x) = \begin{cases} x & 0 \le x \le L/2 \\ L - x & L/2 \le x \le L \end{cases}$$

Cut locus and distance functions (examples)





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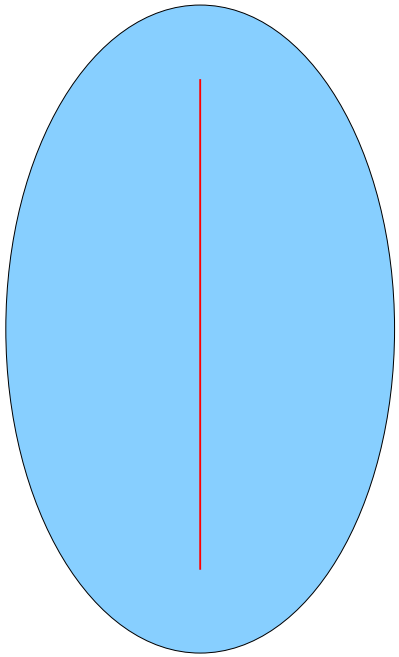

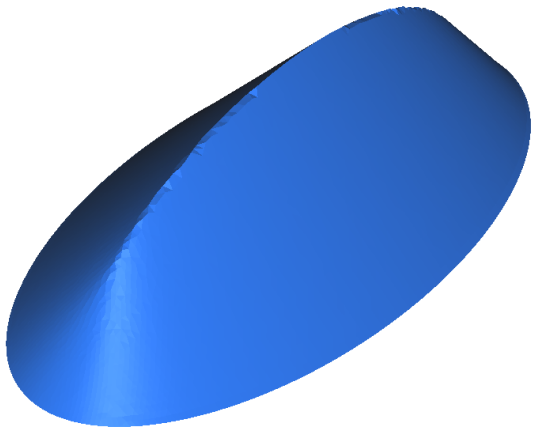

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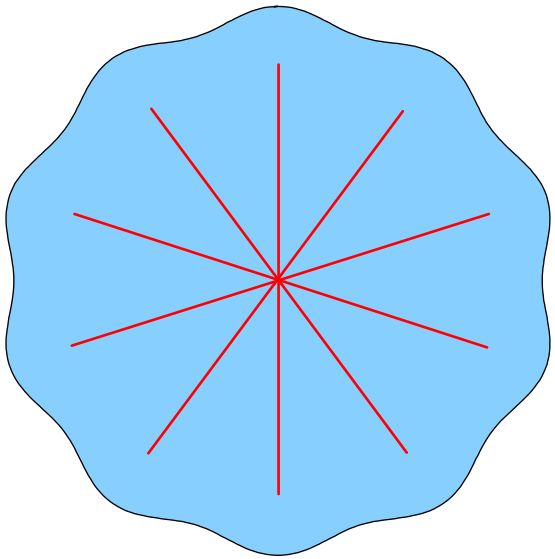

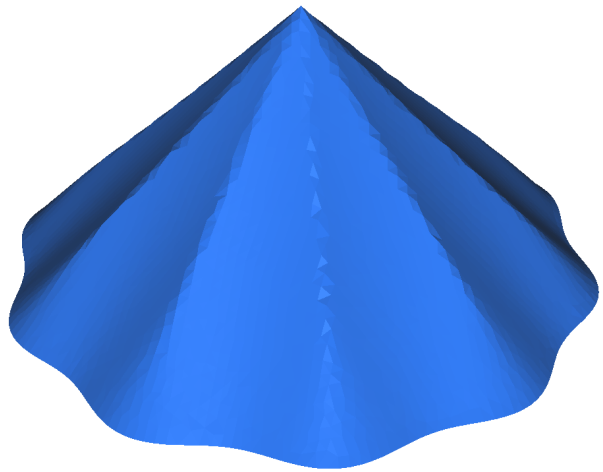

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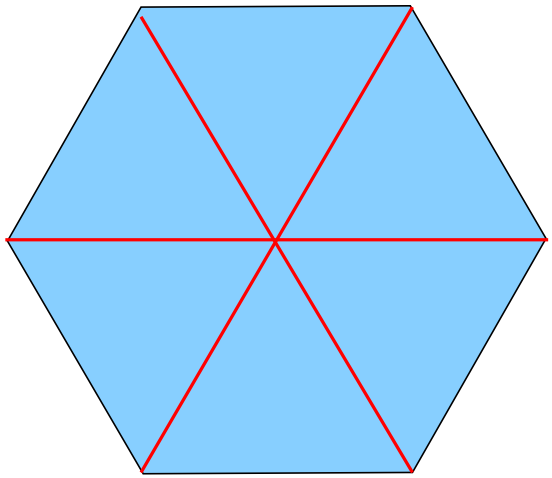

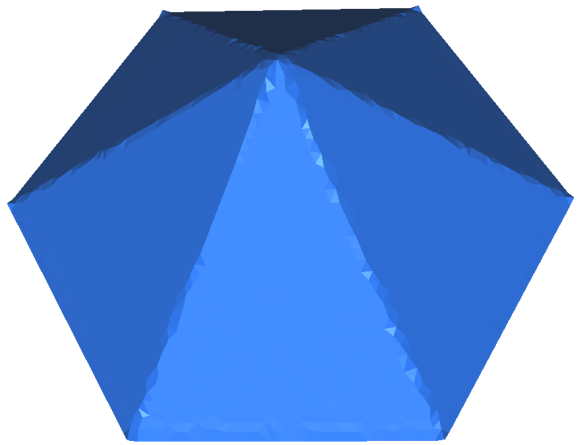

The singular set (in red) for different table geometries

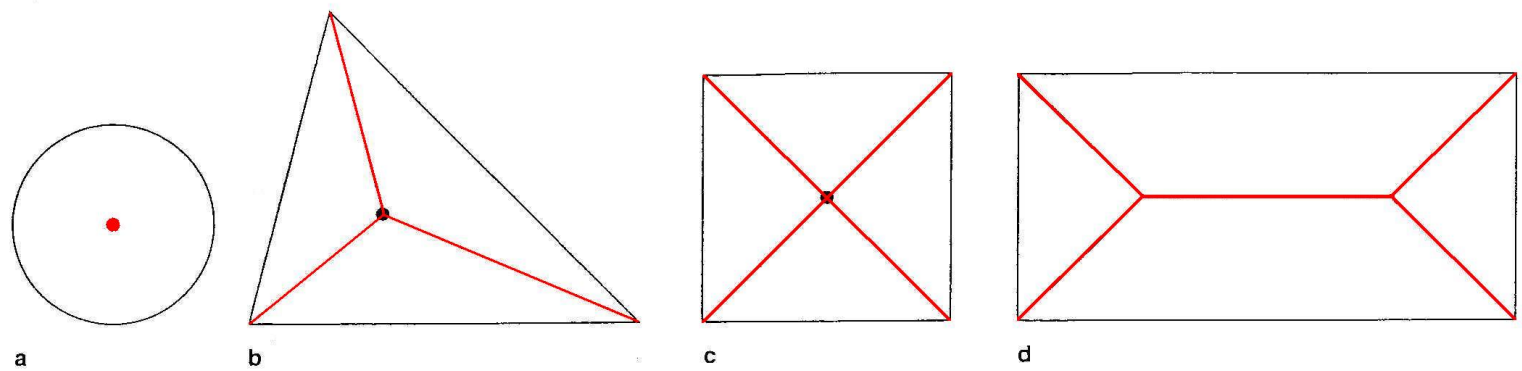

Fig. 2a–d. The singular set for some simple geometries

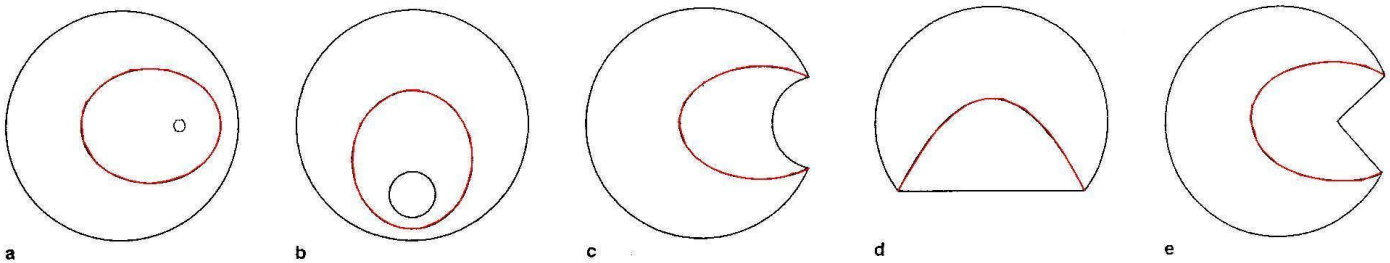

Fig. 3a–e. Singular set for geometries with circular and straight boundaries

Non trivial equilibria for the open table problem

For a given, independent of time, source $f(x) \neq 0$, which is the form of the stationary solutions, if any?
Experiments on real sandpiles say that conical piles grow on the table until they touches the boundary at a point x; at this moment the tip of the pile moves slightly in the opposite direction with respect to x, and all the additional matter flows down towards x leaving the table: the configuration remains essentially stable (except for much longer scales of time).
The mathematical expression for this physically meaningful profile of the standing layer is

$$u_f(x) = \max_{y \in D_f} \{d(y) - |x - y|\}^+,$$

that is the union of maximal cones centered at the points of the support of the source. It is of course the smallest (with respect to f) stationary profile, and depends only on the support of f, not on its values. The corresponding rolling layer v_f can be completely characterized in terms of Ω and f.

Asymptotic behavior and equilibria

- Existence of equilibria when $f = f(x)$:

$$u_t \geq 0 \ , \quad u(.,t) \leq d(.) \quad \Rightarrow \quad u(x,t) \rightarrow \ \overline{u}(x) \ .$$

Asymptotic behavior and equilibria

- Existence of equilibria when $f = f(x)$:

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- Formally, the admissible equilibrium configurations are solutions of the PDE system

$$(E) \begin{cases} -\nabla \cdot (v \nabla u) = f & \text{in } \Omega \\ |\nabla u| = 1 & \text{in } \{v > 0\} \\ |\nabla u| \leq 1\ , \quad u, v \geq 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

Remark

System (E) is not able to determine u in regions where $v = 0$

Proposition (1D explicit form of equilibria)

Let $supp(f) = [x_1, x_2] \subset [0, L]$. Then an equilibrium solution of system (E) is given by the pair (u_f, v_f) defined as follows:

1. If $x_2 < L/2$ then

$$u_f(x) = \begin{cases} x & [0, x_2] \\ 2x_2 - x & [x_2, 2x_2] \\ 0 & [2x_2, L] \end{cases} \qquad v_f(x) = \begin{cases} \int_x^{x_2} f(y)dy & [0, x_2] \\ \\ 0 & [x_2, L] \end{cases}$$

2. If $L/2 < x_1$ then the solution can be obtained by reflection from the previous case.
3. If $x_1 \leq L/2 \leq x_2$ then

$$u_f(x) = d(x) = \begin{cases} x & [0, L/2] \\ \\ L - x & [L/2, L] \end{cases} \qquad v_f(x) = \begin{cases} \int_x^{L/2} f(y)dy & [0, L/2] \\ \\ \int_{L/2}^{x} f(y)dy & [L/2, L] \end{cases}$$

Proof (1)

System (E) reduces in 1D to

$$(u_x v)_x + f = 0, \quad v = |u_x| v, \quad u(0) = u(L) = 0$$

By integration on the first equation:

$$u_x(x)v(x) = -\int_0^x f(y)dy + C$$

where the constant C corresponds to the flux (vu_x) at $x = 0$. From the second relation and $v > 0$ we get

$$v(x) = |u_x(x)|v(x) = \left| C - \int_0^x f(y)dy \right|$$

Three cases:

- $C > \int_0^x f(y)dy \;\Rightarrow\; v > 0\,,\; u_x = 1$
- $C < \int_0^x f(y)dy \;\Rightarrow\; v > 0\,,\; u_x = -1$
- $C = \int_0^x f(y)dy \;\Rightarrow\; v = 0\,,\; u_x$ undetermined

Proof (2)

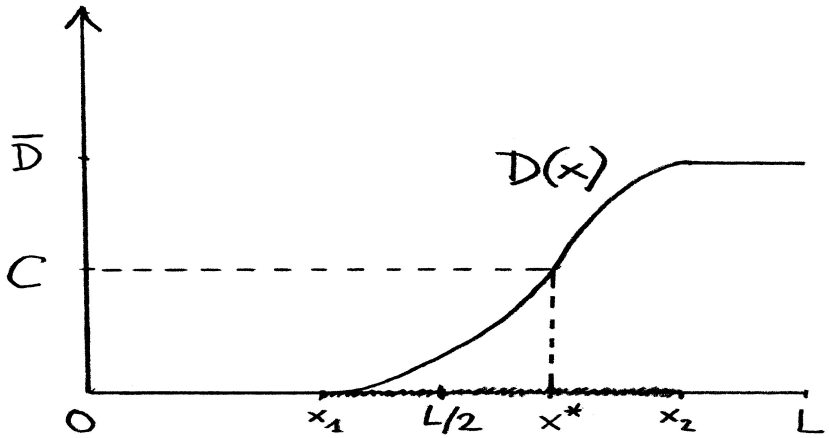
Case (3): $0 < x_1 < L/2 < x_2 < L$. Let

$$D(x) = \int_0^x f(y)dy = \begin{cases} 0 & [0, x_1] \\ \nearrow & [x_1, x_2] \\ \overline{D} & [x_2, L] \end{cases}$$

Then if $C \in (0, \overline{D}) \Rightarrow \exists!\ x^* \in (0, L) : C = D(x^*) \Rightarrow$
$u(0) = 0, u_x = 1$ in $(0, x^*)$, $u_x = -1$ in (x^*, L), $u(L) = 0 \Rightarrow$

$$x^* = L/2\ , \qquad v(x) = \left| \int_0^{L/2} f(y)dy - \int_0^x f(y)dy \right| \ (QED)$$

Case (1): Now $D(L/2) = \overline{D}$ and the only way to satisfy the B.C. is to take $C = \overline{D}$. Then $u_x = 1$ in $(0, x_2)$ and $v = 0,\ u_x$ undetermined in $(x_2, L) \Rightarrow$
$u = u_f$ is an admissible solution (BUT NOT THE UNIQUE ONE !!) $\square$


$\overline{D}$
C
$D(x)$
0
x_1
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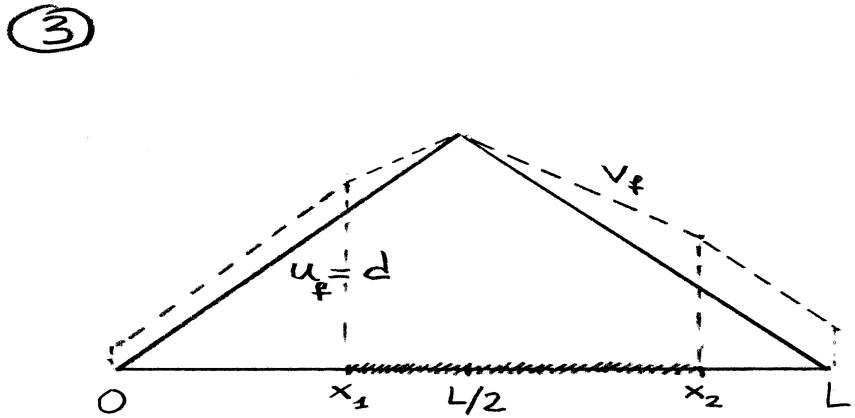
x^*
x_2
L


3
v_f
$u_f = d$
0
x_1
$L/2$
x_2
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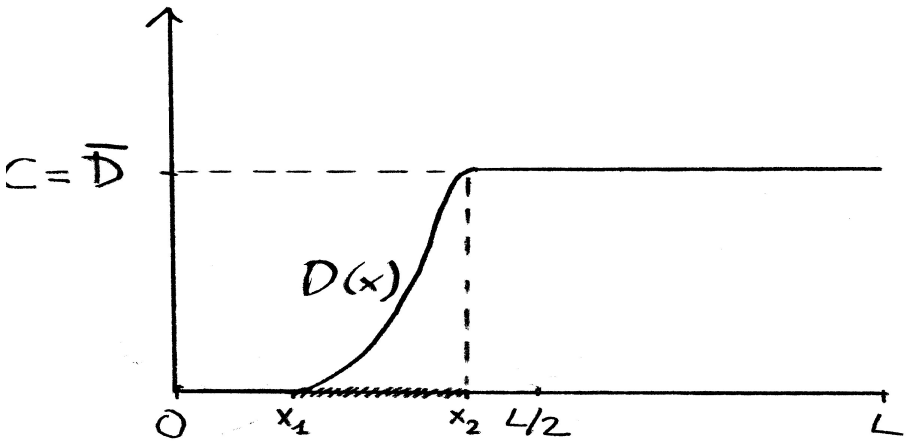
$C = \overline{D}$
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x_1
x_2
$L/2$
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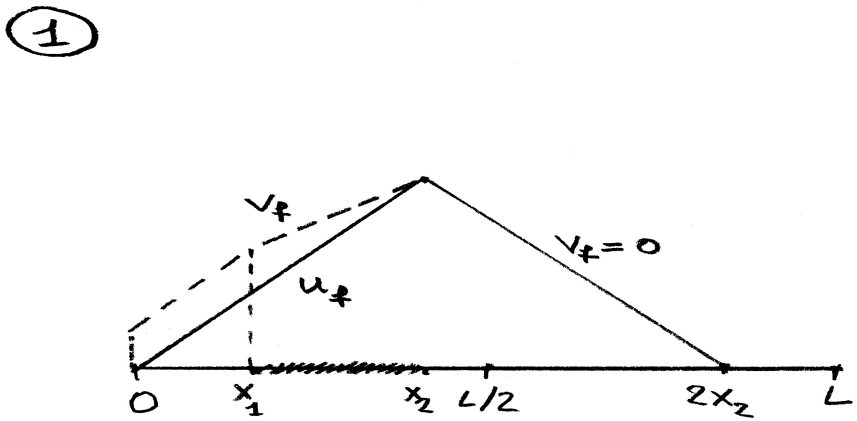
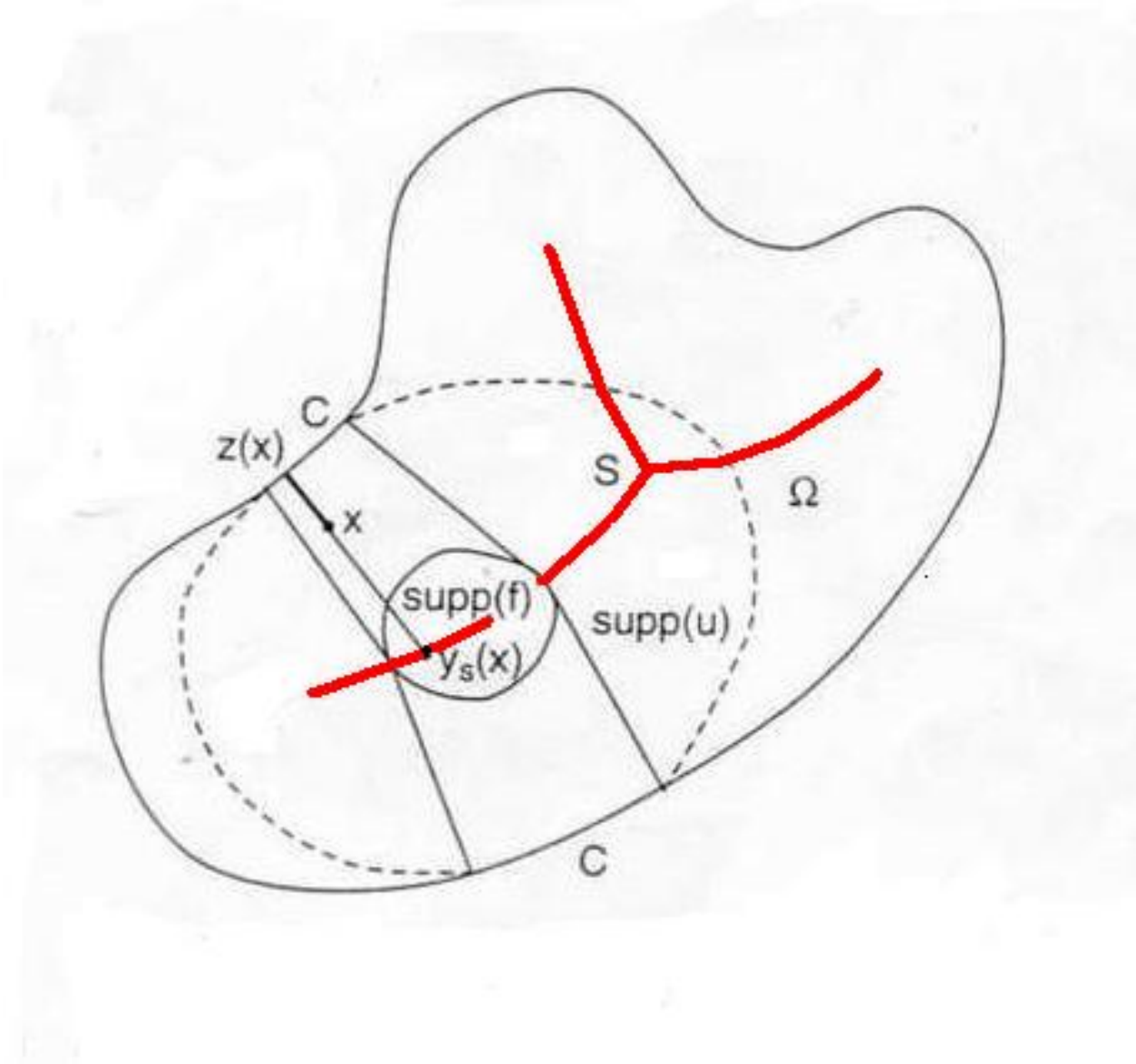
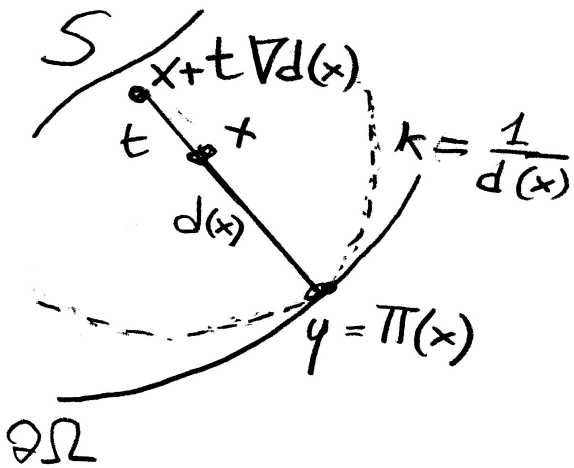

1
v_f
$v_f = 0$
u_f
0
x_1
x_2
$L/2$
$2x_2$
L

Source support, pile support and singular set in 2D

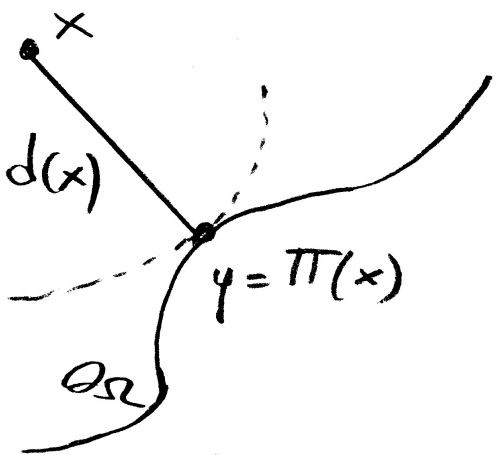


Characterization of equilibria

- $\tau(x) = \min\{t \geq 0 : x + t\nabla d(x) \in \overline{S}\}$ *normal distance* to S
- $k(x)$ *curvature* of $\partial\Omega$ at the boundary projection $\Pi(x)$ of $x \in \overline{\Omega}\backslash\overline{S}$ ($k \geq 0$ if Ω convex).
- $\forall x \in \overline{\Omega},\ y \in \Pi(x) \Rightarrow k(y)d(x) \leq 1$.
- $x \in \overline{\Omega}\backslash\overline{S} \Rightarrow k(x)d(x) < 1,\ \ \Delta d(x) = -\frac{k(x)}{1-d(x)k(x)}$.



$$0 < k(y) \leq \frac{1}{d(x)}$$



$$k(y) < 0 < \frac{1}{d(x)}$$

Characterization of equilibria

[Cannarsa-Cardaliaguet '04], [Cannarsa-Cardaliaguet-Sinestrari '08]

Theorem

Let $\partial\Omega \in C^2$, $f \in C^0(\Omega)$ $[L^1(\Omega)]$; then :

- *Existence*: $(d, v_f) \in W_0^{1,\infty} \times C^0$ $[W_0^{1,\infty} \times L^1]$ *solves (E), with*

$$(*) \quad v_f(x) = \int_0^{\tau(x)} f(x + t\nabla d(x)) \frac{1 - (d(x) + t)k(x)}{1 - d(x)k(x)}\, dt, \ in\ \Omega\backslash\overline{S}$$

$$v_f = 0 \ on\ \overline{S}$$

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$$v_f = 0 \ \ on\ \overline{S}$$

- *Almost uniqueness*: *if* (u, v) *is another solution of* (E), *then*

$$v = v_f \quad in\ \Omega\ , \quad u = d \quad in\ supp(v_f).$$

Asymptotic behavior of the dynamic model

1. $\boxed{\overline{S} \subset D_f}$: (E) **has only the solution** (d, v_f)
   $\Rightarrow$ the system (HK) evolves in finite time towards the maximum equilibrium configuration (same behaviour of the (P) model)

Asymptotic behavior of the dynamic model

1. $\boxed{\overline{S} \subset D_f}$: (E) **has only the solution** (d, v_f)
   $\Rightarrow$ the system (HK) evolves in finite time towards the maximum equilibrium configuration (same behaviour of the (P) model)

2. $\boxed{\overline{S} \not\subset D_f}$: **no uniqueness for** u in (E) and $\boxed{u_f \leq \overline{u} \leq d}$
   $\Rightarrow \overline{u}$ and the active region $\Omega^+ = \{x \in \Omega : \overline{u} > 0\}$ for (HK) can be characterized only numerically: they depend on source support and intensity (for the (P) model $u(x,t) \to u_f(x)$ in finite time)
   $[u_f = \overline{u} = d$ in $\{v_f > 0\}]$

Special equilibria

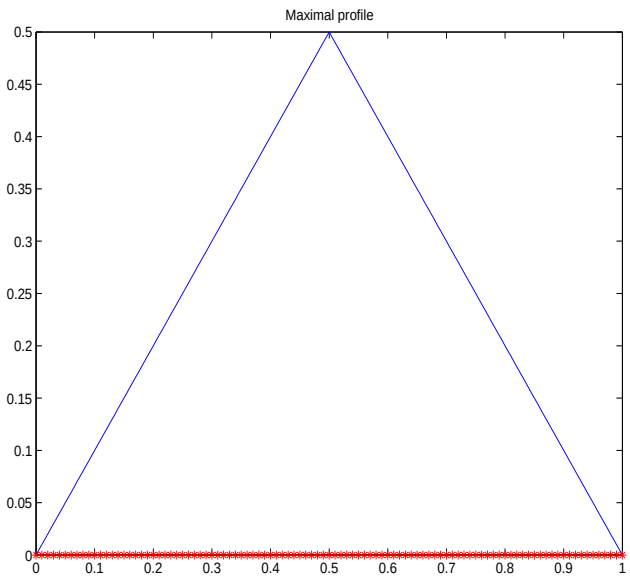
1. Maximal:

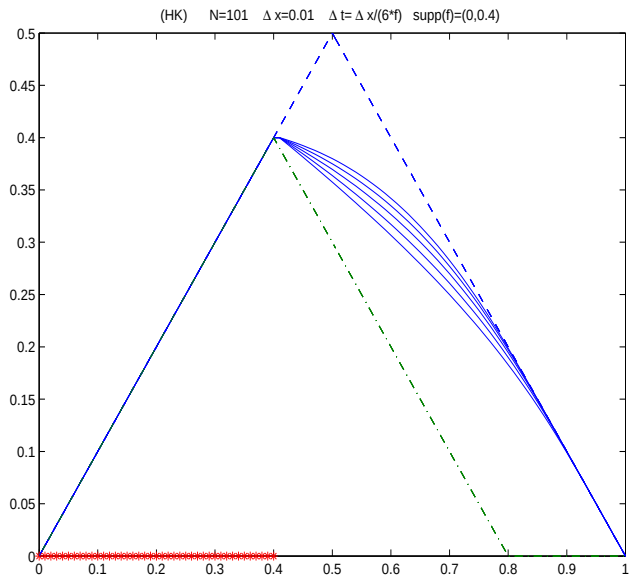
$$u(x)=d(x)$$

2. Minimal w.r. to f:

$$u_f(x) = \max_{y \in D_f} \{d(y) - |x-y|\}^+$$

$$u_f(x) \leq \overline{u}(x) \leq d(x)$$





Numerical approximation and simulations

An explicit f.d. scheme for the two-layer system (HK)

[Falcone - F.V., SIAM J.Sci.Comput., '06]

$$(HK_{1D}) \begin{cases} v_t + [-u_x v]_x = v_t + [F_u(v)]_x = f - (1 - |u_x|)v \\ u_t = (1 - |u_x|)v \\ u(0,t) = u(1,t) = 0\ , \qquad u(x,0) = 0 \quad x \in \Omega = (0,1)\ . \end{cases}$$

$$(FD) \begin{cases} v_i^{n+1} = v_i^n - \frac{\Delta t}{h}(H^n_{i+\frac{1}{2}} - H^n_{i-\frac{1}{2}}) + \Delta t\,[\ f_i - (1 - |Du_i^n|)\ v_i^n] \\ u_i^{n+1} = u_i^n + \Delta t\ \ (1 - |Du_i^n|)\ v_i^{n+1} \\ u_i^0 = v_i^0 = 0 \quad (i = 1, \ldots, N)\ , \quad u_1^n = u_N^n = 0 \quad \forall n. \end{cases}$$

Finite difference formulas

$$\text{At any node } x_i : \quad D^+ u_i = \frac{u_{i+1}-u_i}{h}, \qquad D^- u_i = \frac{u_i - u_{i-1}}{h}$$

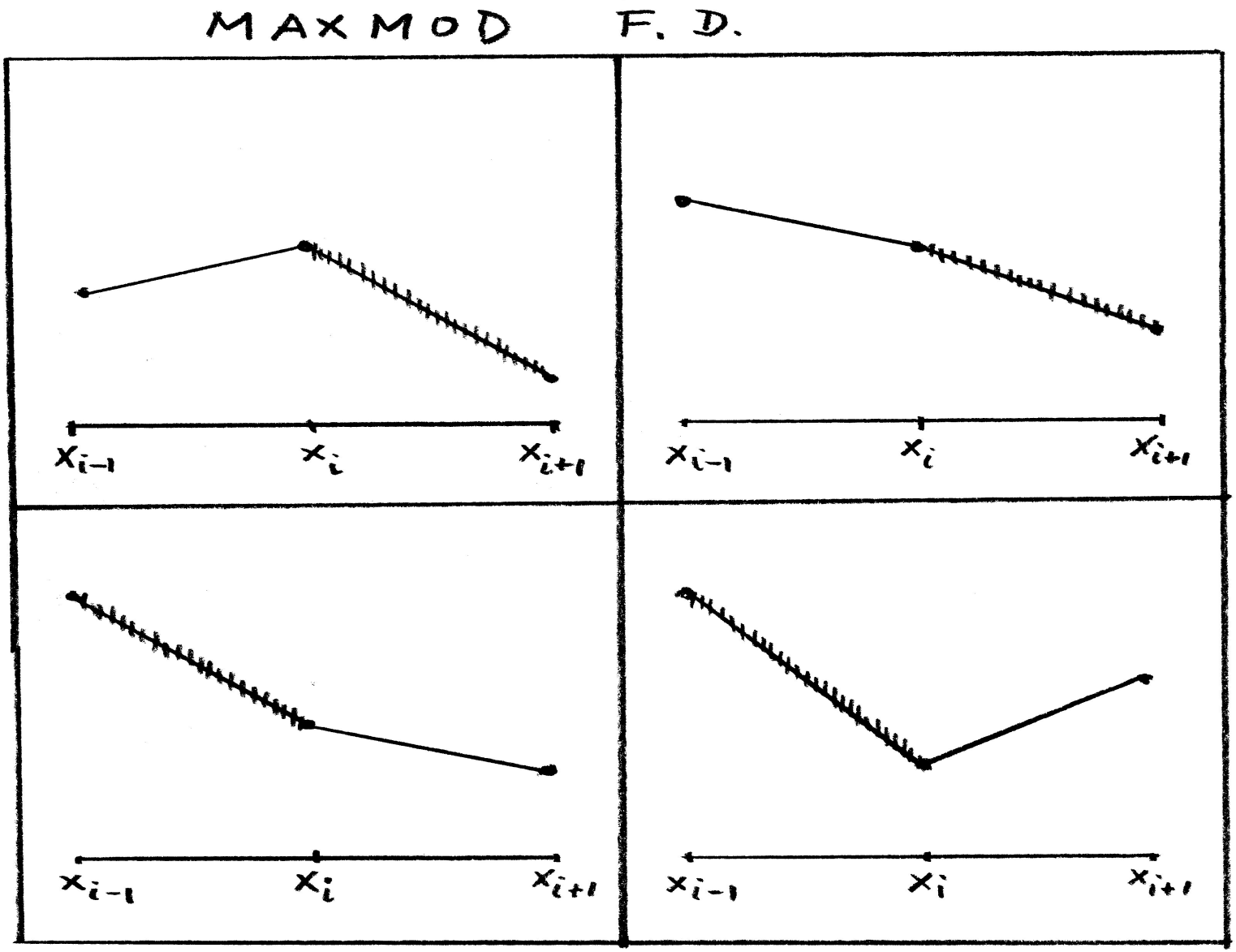
maxmod difference

$$|Du_i| \equiv \max\left(\left|D^+ u_i\right|, \left|D^- u_i\right|\right)$$

upwind numeric flow in (x_i, x_{i+1})

$$H_{i+\frac{1}{2}} \equiv -\frac{u_{i+1} - u_i}{h}\ \mathrm{upw}(v_i, v_{i+1})$$

$$\mathrm{upw}(v_i, v_{i+1}) \equiv \begin{cases} v_i & \text{if } u_i > u_{i+1} \\ v_{i+1} & \text{if } u_i \le u_{i+1} \end{cases}$$


MAXMOD F. D.
x_{i-1}
x_i
x_{i+1}
x_{i-1}
x_i
x_{i+1}
x_{i-1}
x_i
x_{i+1}
x_{i-1}
x_i
x_{i+1}

Properties of the scheme

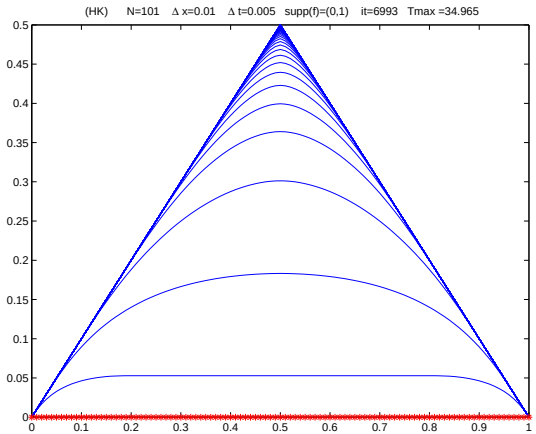
Theorem

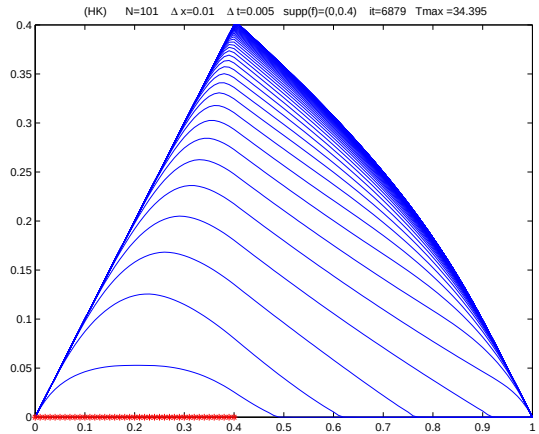
Let $f \geq 0$ *in* Ω, *and* $\frac{\Delta t}{h} \leq \min\left(\frac{1}{2}, \frac{c}{\|f\|_\infty}\right)$ *; then for any* n *:*

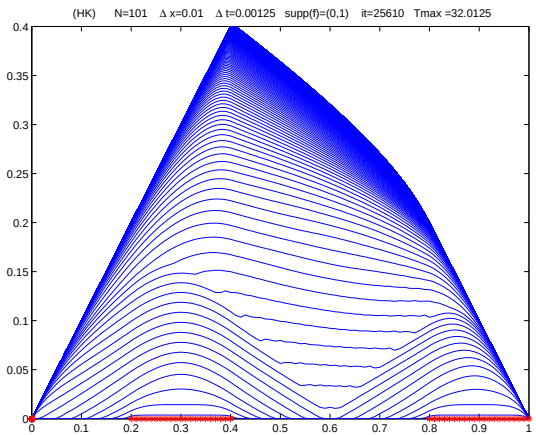
- *(Positivity and monotonicity in* u*)* $\quad 0 \leq u^n \leq u^{n+1}$
- *(Positivity in* v*)* $\quad v^n \geq 0$
- *(Gradient constraint in* u*)* $\quad |Du^n| \leq 1$

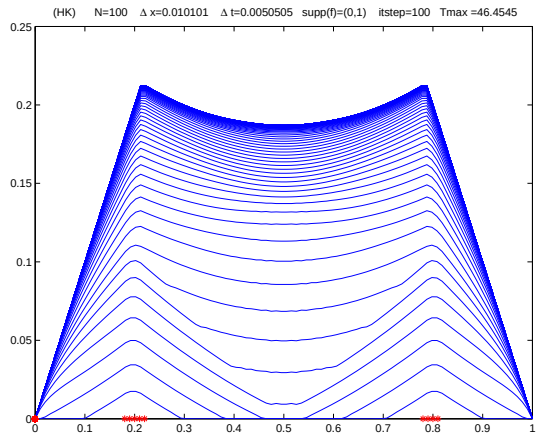
$\Rightarrow$ *Under the previous stability conditions:* $(u^n, v^n) \to (\bar{u}, \bar{v})$, *equilibrium of the discrete system such that* $(1 - |D\bar{u}|)\bar{v} = 0$.

Examples of growing sandpiles for different sources





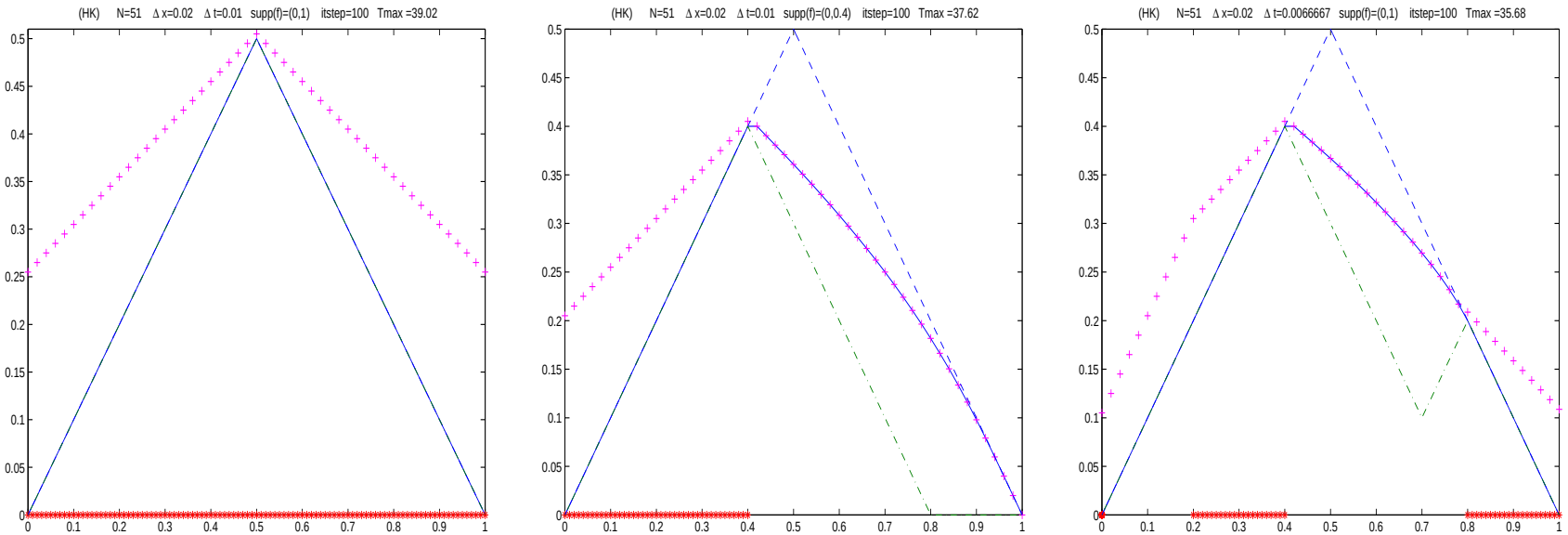




$f \equiv 0.5$, $u_h(\cdot, 100n\Delta t)$; D_f=$[0,1]$, $[0,0.4]$, $[0.2,0.4] \cup [0.8,1]$, $[0.18,0.22] \cup [0.78,0.82]$.

Examples of equilibria for different source supports

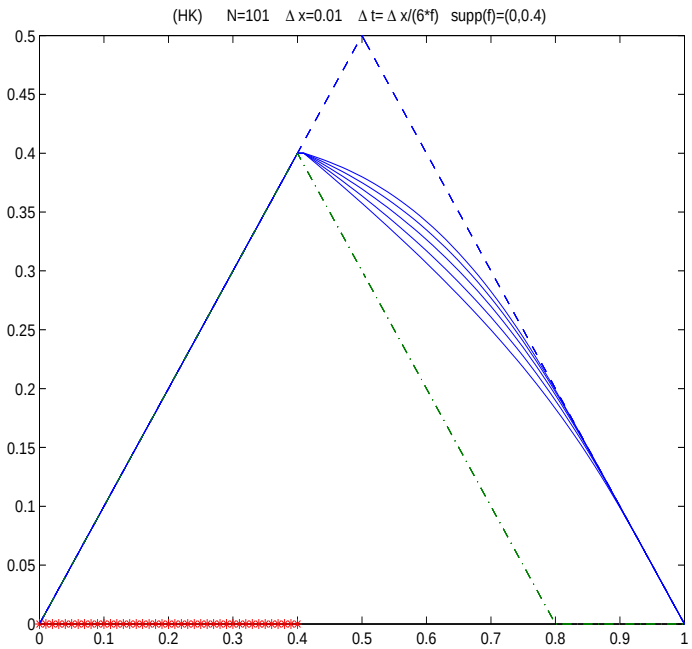
$\overline{u}$ versus u_f

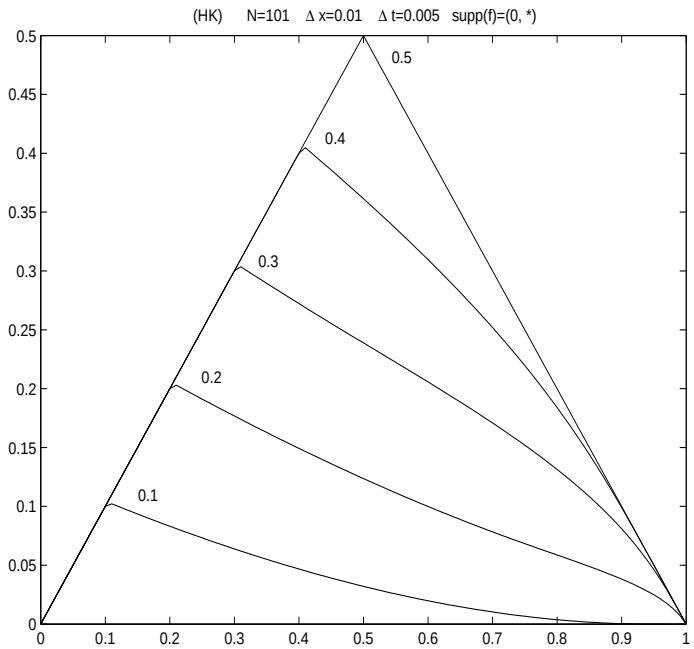


$f \equiv 0.5$, stationary u_h [—] and $(u_h + v_h)$ [++], the distance function d [- -] and the minimal solution u_f [-·-] when D_f is $[0,1]$, $[0,0.4]$ and $[.2,0.4] \cup [0.8,1]$.

Solution dependence on the source intensity and support

$$0 \le f_1 \le f_2 \quad \text{in } D_f \quad \Rightarrow \quad u_f \le \overline{u_1} \le \overline{u_2} \le d \quad \text{in } \Omega$$





u_h [—], d [- -] and u_f [-·-]. Left: $f \equiv 0.5, .., 2.5$; right: $f = 0.5,\ D_f = (0, 0.1), .., (0, 0.5)$

Two-dimensional extension of the scheme (FD)

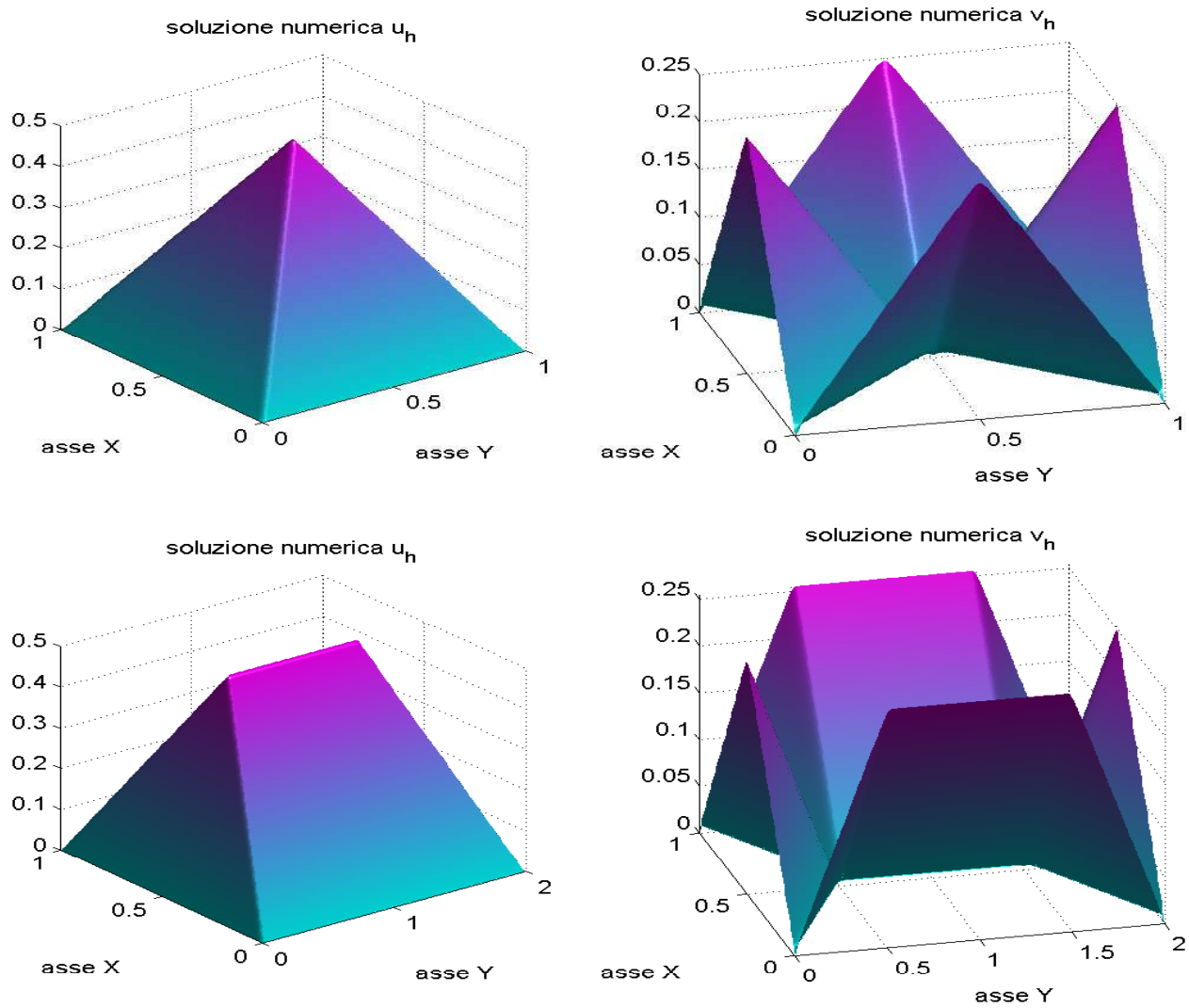
Explicit formulas for the equilibria are not easy to implement, except for simple geometries and source data

The scheme (FD) naturally extends, and pile evolution is not mesh dependent

Stationary solutions have in general subcritical regions and a larger support w.r.t. u_f

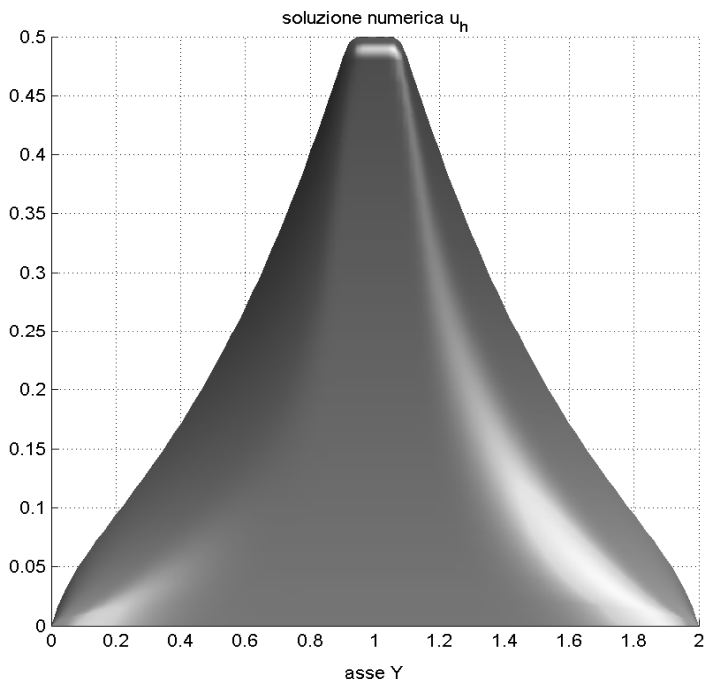
The contact angle of the stationary piles is critical at the boundary, zero inside

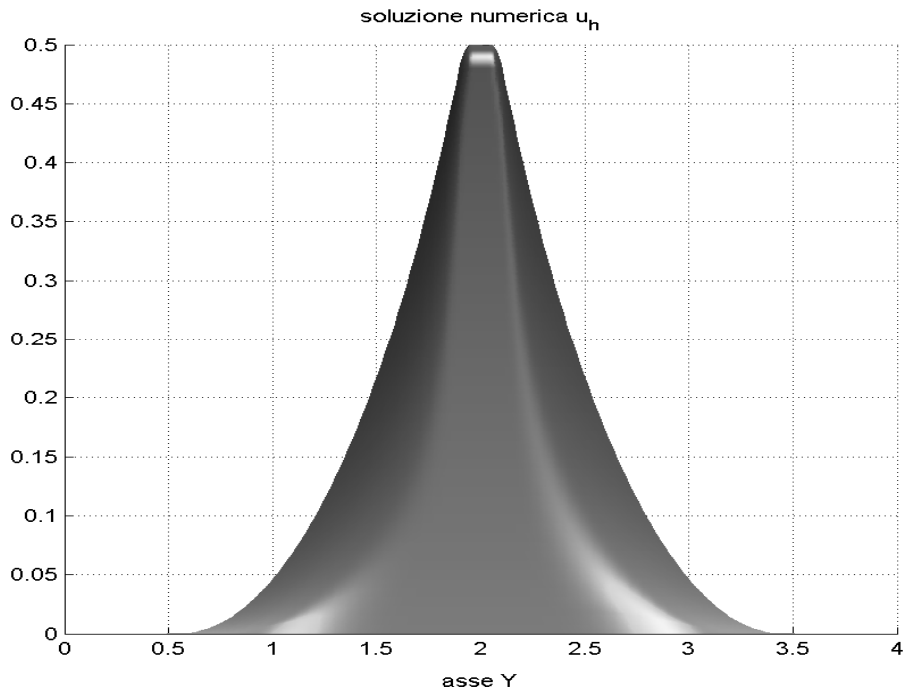
Maximal equilibria (square and rectangular tables)



HK-2D : $D_f = \Omega$, $\Omega = (0,1)^2$ (top), $\Omega = (0,1) \times (0,2)$ (bottom).

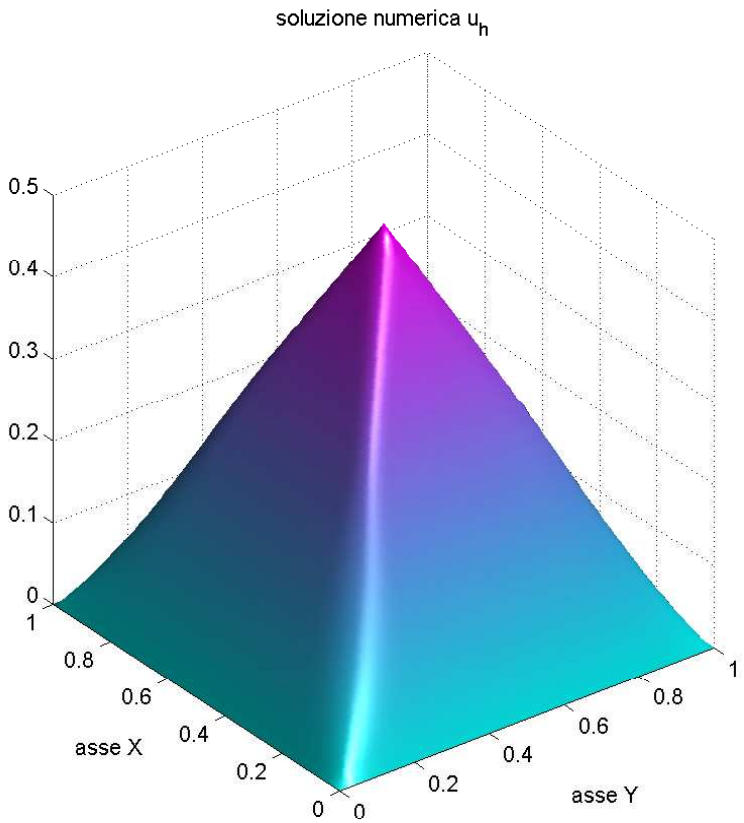
Contact angles for $\overline{u}$: critical on $\partial\Omega$, zero inside Ω

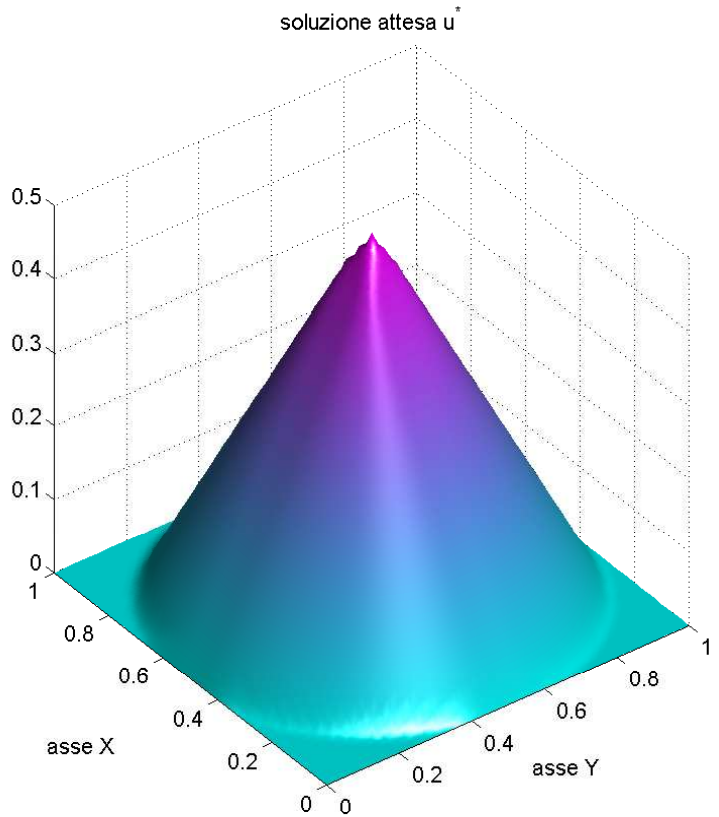




HK-2D : left: $\Omega = (0,1) \times (0,2)$, right: $\Omega = (0,1) \times (0,4)$, D_f small, $N = 51$.

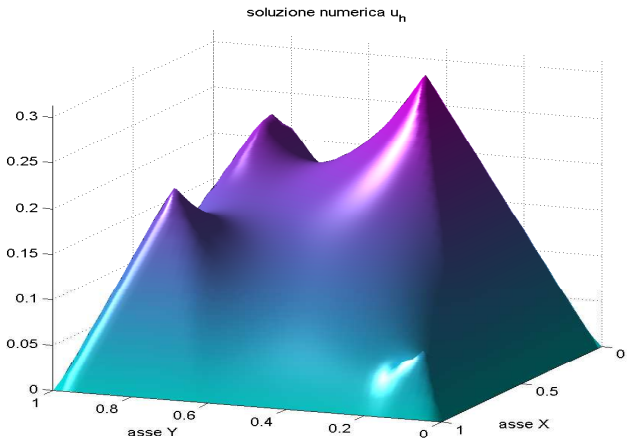
Support of stationary solutions ($\overline{u} > u_f$)

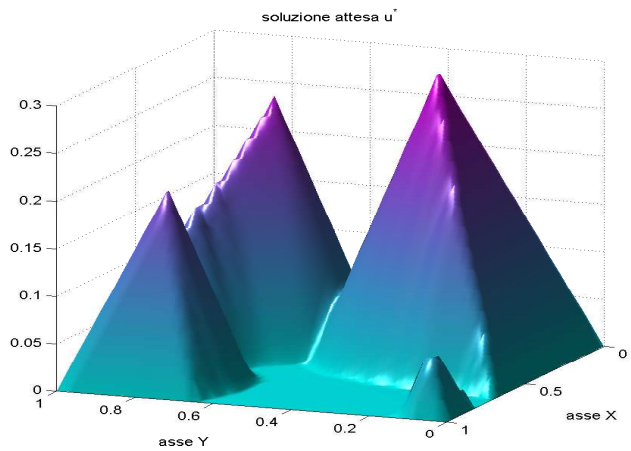


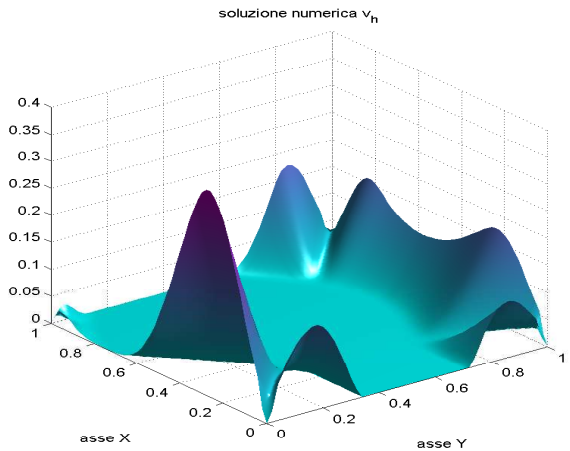


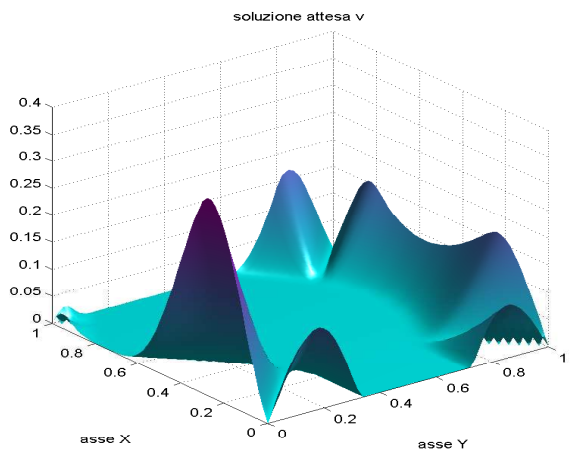
HK-2D : $\Omega = (0,1)^2$, $D_f = [0.45, 0.55]^2$, $N = 51$, u_h versus u_f.

Different standing (same rolling) layers at equilibrium



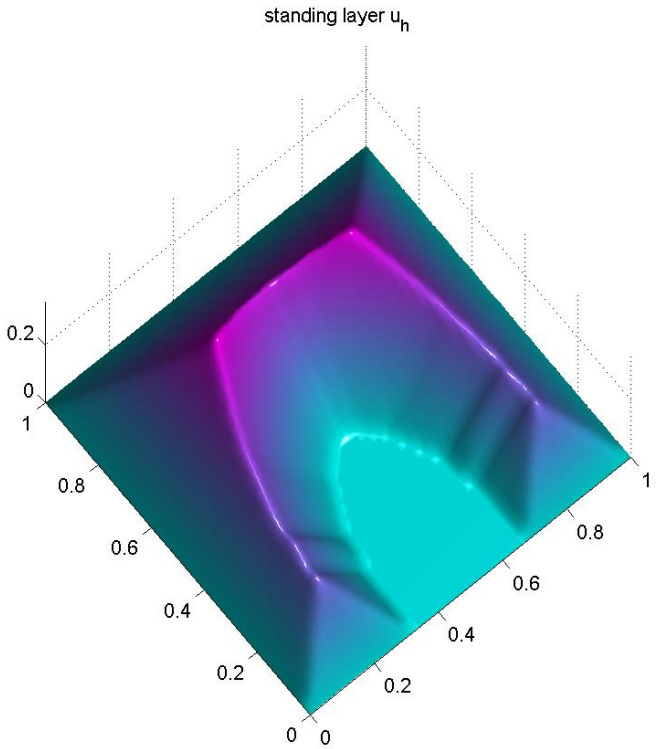


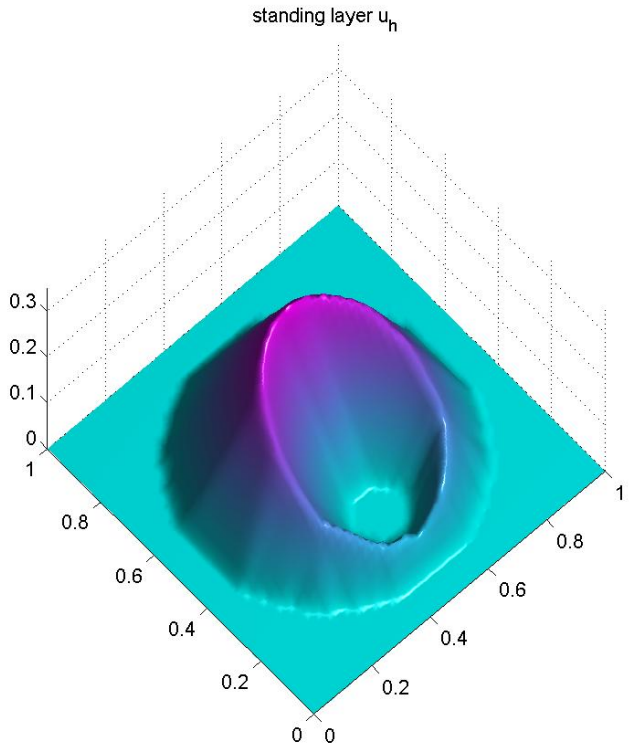




HK-2D : $\Omega = (0,1)^2$, $D_f \subset \Omega$, $N = 51$, u_h versus u_f, v_h versus v_f, when $f(x,y) = \max\{0, \sin(2\pi x) + \cos(2\pi y) - \sin(2\pi xy)\}$.

More general tables: $|\nabla u| \leq 1_{\Omega'}(x),\ \Omega' \subset \Omega$





HK-2D : examples of non convex or non simply connected open tables

An explicit semi-Lagrangian scheme for system (HK)

[Falcone - F.V., Proc. Enumath07, Graz , '08]

$$(HK_{1D}) \begin{cases} u_t = v(1-|u_x|) = v\left(1 + \min\limits_{a\in B(0,1)} a u_x\right) , \\ v_t = u_x v_x + v u_{xx} - v(1-|u_x|) + f , \\ u(0,t) = u(1,t) = 0 , \quad u(x,0) = 0 \quad x \in \Omega = (0,1) . \end{cases}$$

$$(SL) \begin{cases} u_i^{n+1} = u_i^n + \Delta t\ v_i^n \left(1 + \min_{a\in\{-1,0,1\}} \frac{u^n(x_i + a\Delta t) - u_i^n}{\Delta t}\right) \\ v_i^{n+1} = v^n(x_i + \Delta t D u_i^n) - (u_i^{n+1} - u_i^n) + \Delta t(v_i^n D^2 u_i^n + f_i) \\ u_i^0 = v_i^0 = 0 \quad (i = 1, ..., N) , \quad u_1^n = u_N^n = 0 \quad \forall n. \end{cases}$$

Features of the (SL) scheme

At any node x_i :

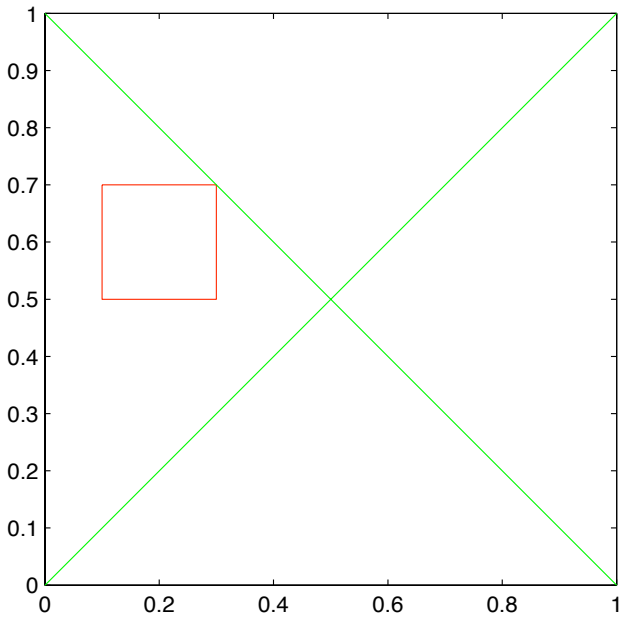
- $|u_x| \simeq -\min_{a\in\{-1,0,1\}} \frac{u(x_i+a\Delta t)-u_i}{\Delta t}$
- u_x replaced by Du_i (*maxmod difference*)
- u_{xx} replaced by D^2u_i (*second order central difference*)
- $u_x v_x$ replaced by the discrete directional derivative of v in the direction of u_x: $\frac{v(x_i+\Delta t Du_i)-v_i}{\Delta t}$

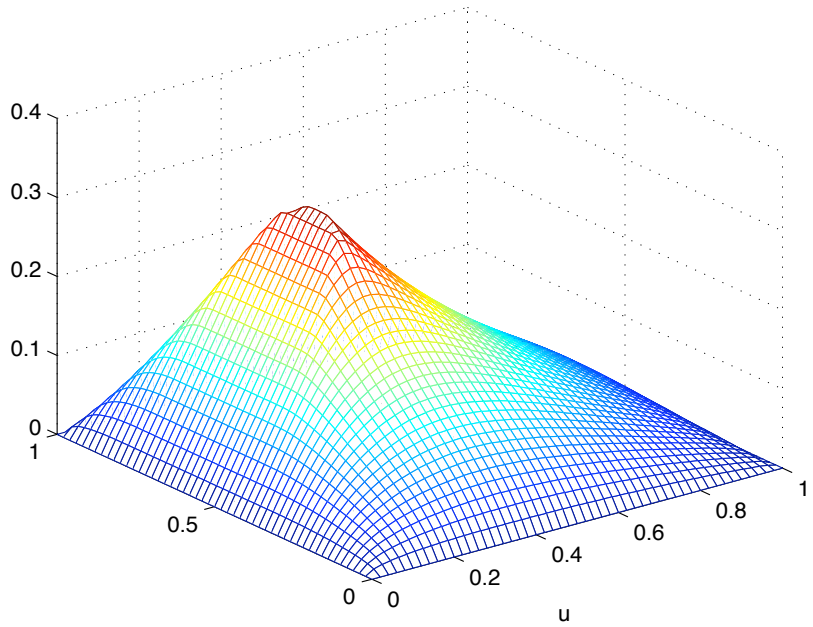
If $\Delta t \neq \Delta x$ and $a \neq 0$, the points $x_i + a\Delta t$, $x_i + \Delta t Du_i^n$ are not nodes of the grid and the values of u^n and v^n at those points have to be computed by interpolation.

The nonlinear term in the second equation is replaced in the scheme by the previously computed difference $(u_i^{n+1} - u_i^n)$.

Same essential properties of the (FD) scheme, more expensive but more accurate in the detection of level lines.

An example of growing sandpile for a small support

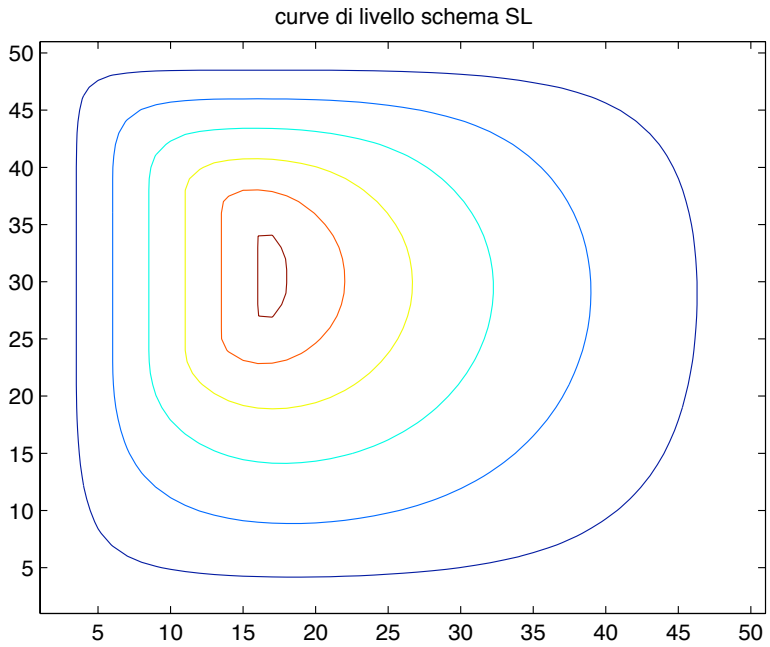


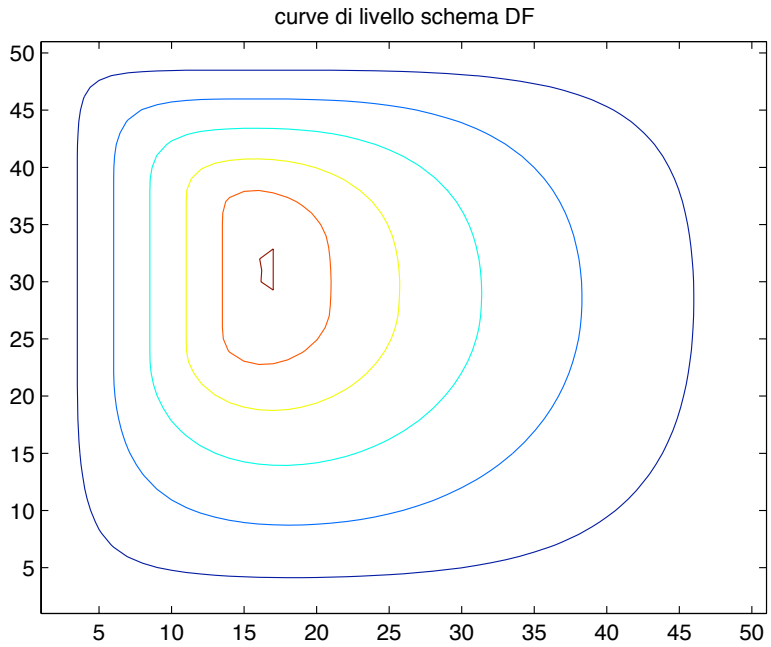


The source support and the computed standing layer ((SL) scheme)

In 2D the minimum term on the unit ball is computed on a discrete set of directions (for example 8 or 16).

Level lines comparison: (SL) versus (FD)





Level curves for u with the (SL) and the (FD) schemes

The scheme (SL) is more expensive but more accurate.

Viscosity solutions

- u is a viscosity subsolution of the eikonal equation if for every smooth ϕ such that $u - \phi$ has a local maximum at $x_0 \in \Omega$ then

$$|\nabla \phi(x_0)| \leq 1 \ ;$$

- u is a viscosity supersolution of the eikonal equation if for every smooth ϕ such that $u - \phi$ has a local minimum at $x_0 \in \Omega$ then

$$|\nabla \phi(x_0)| \geq 1 \ ;$$

- u is a viscosity solution of the eikonal equation if it is both a viscosity subsolution and a viscosity supersolution.

Remark. It can be proved that the viscosity solution is the largest function u such that $\|\nabla u\|_\infty \leq 1$ and $u = 0$ on the boundary.

Definition introduced by [Crandall-Lions, '81] (classical sol. $\Rightarrow$ visco sol.)

back