Sur l'équation de Monge-Kantorovich

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Monge-Kantorovich evolution equation

Let $\Omega \subset \mathbb{R}^N$ be a bounded regular domain, $N \geqslant 1$, $T > 0$, $Q = \Omega \times (0, T)$ and $\Sigma = \partial\Omega \times (0, T)$

$$(EMK) \qquad \begin{cases} \dfrac{\partial h}{\partial t}(x,t) - \nabla \cdot m(x,t)\nabla h(x,t) = \mu & \text{in } Q \\ m(x,t) \geqslant 0, \ |\nabla h(x,t)| \leq 1 & \text{in } Q \\ m\,(1 - |\nabla h(x,t)|) = 0 & \text{in } Q \\ u = 0 & \text{on } \Sigma \\ u(0) = u_0 & \text{in } \Omega \end{cases}$$

Main interest :

1. Existence and uniqueness of a solution : μ a Radon measure
2. Numerical analysis : representation of $t \in [0, T) \longrightarrow (u(t,x), m(t,x)\,\nabla u(t,x))$.
3. Large time behavior, as $t \longrightarrow \infty$.

Motivation

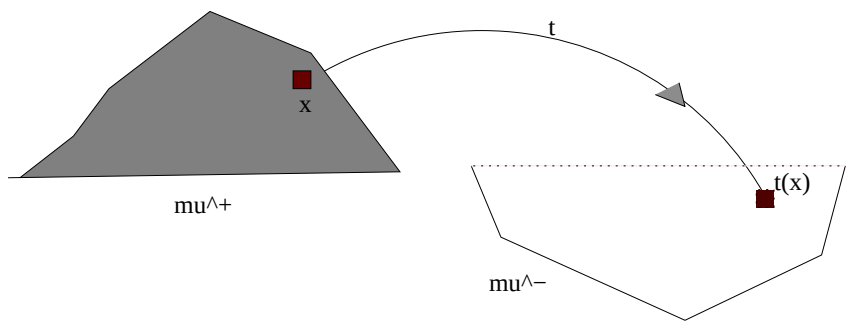
- Optimal mass transportation (Monge-Kantorovich problem) :
  - L. C. Evans and W. Gangbo, Differential equations methods for the Monge-Kantorovich mass transfer problem. Mem. Am. Math. Soc., 137 (1999), no. 653.
- Sandpile model :
  - L. Prigozhin, Variational model of sandpile growth. Euro. J. Appl. Math. , 7 (1996), 225-236.
  - G. Aronson, L. C. Evans and Y. Wu, Fast/Slow diffusion and growing sandpiles. J. Differential Equations, 131 :304-335, 1996.
- Cellular Automaton :
  - L. C. Evans and F. Rezakhanlou. A stochastic model for sandpiles and its continum limit. Comm. Math. Phys., 197 (1998), no. 2, 325-345.
- Mass optimization :
  - G. Bouchitté and G. Buttazzo, Characterization of optimal shapes and masses through Monge-Kantorovich equation. J. Eur. Math. Soc. 3 (2001), no. 2, 139-168.

Motivation I : optimal mass transportation

Monge optimal mass transfer [Monge-1780] : Given two measures μ^+ and μ^- in $\mathbb{R}^N$, such that $\mu^+(\mathbb{R}^N) = \mu^-(\mathbb{R}^N)$ and $\text{spt}(\mu^+) \neq \text{spt}(\mu^-)$

$$Min_{t\in\mathcal{A}} \quad \int_{\mathbb{R}^N} |x - t(x)|\, d\mu^+(x)$$

$$\mathcal{A} = \left\{ \begin{array}{l} t \ : \ \text{spt}(\mu^+) \longrightarrow \text{spt}(\mu^-) \ ; \ \mu^+_{\#t} = \mu^- \\ \text{i.e. } \mu^-(B) = \mu^+(t^{-1}(B)) \end{array} \right\}$$



Monge-Kantorovich problem [Kantorovich-1940] :

- **Realaxed variant :** optimal plan transport

$$\overline{\mu} \in \Pi(\mu^+, \mu^-) := \left\{ \gamma \in \mathcal{M}_b(\boldsymbol{R}^N \times \boldsymbol{R}^N),\ proj_x(\gamma) = \mu^+,\ proj_y(\gamma) = \mu^- \right\}$$

$$\mathcal{J}(\overline{\mu}) = \min_{\gamma\in\Pi(\mu^+,\mu^-)} \mathcal{J}(\gamma) \qquad \mathcal{J}(\gamma) = \iint_{R^N\times R^N} |x - y|\, d\gamma(x, y)$$

  Is $\overline{\mu}$ supported in a graph ? i.e. $\exists$? $t \in \mathcal{A}$, $\overline{\mu}(E) = \mu^+(\{x \in \boldsymbol{R}^N \ ; \ (x, t(x)) \in E\})$

- **Dual variational principle :** Shipper (principe du convoyeur) $\overline{\mu}$ is optimal if and only if

$$\mathcal{J}(\overline{\mu}) = \max_{\xi\in Lip_1(\mathbb{R}^N)} \left\{ \int_{R^N} \xi\, d\mu^+ + \int_{R^N} \xi\, d\mu^- \right\} = \int_{R^N} \overline{u}\, d\mu^+ + \int_{R^N} \overline{u}\, d\mu^- \ .$$

- *Memoire sur la théorie de Déblais et des remblais*, Gaspard Monge, Histoire de l'académie des Sciences de Paris, 1781.
- *On the transfer of massesm*, L.V. Kantorovich, Dokl. Acad. Nauk. SSSR 37, 227-229 (1942).

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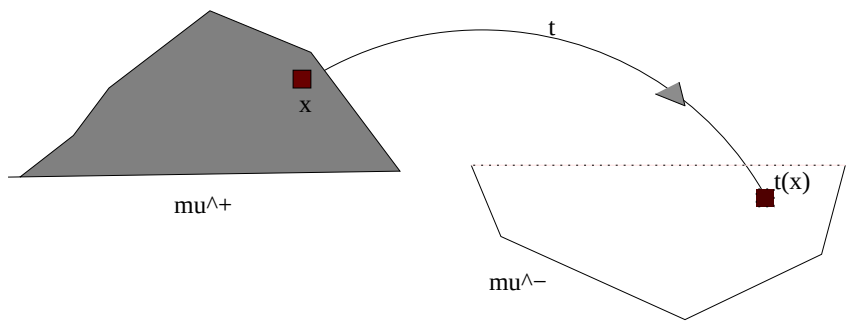

Evans and Gangbo [1999] : **How to fashion an optimal transport t ?**

Monge-Kantorovich equation (Stationary equation of (EMK))

$$(MK) \quad \left\{ \begin{array}{l} -\nabla \cdot (m\, \nabla u) = \mu^+ - \mu^- \\ m \geqslant 0, \ |\nabla u| \leq 1, \ m(|\nabla u| - 1) = 0 \end{array} \right\} \Longrightarrow m\, \nabla u \text{ : the flux transportation}$$

- m gives the transport density
- $-\nabla u$ gives the direction of the optimal transportation

Remark : If (u, m) solves (MK) then u is the Kantorovich potential ; i.e.

$$\max_{\xi\in Lip_1(\mathbb{R}^N)} \int_{R^N} \xi\, d\mu \ = \ \int_{R^N} u\, d\mu$$

Motivation II : growing sandpile model

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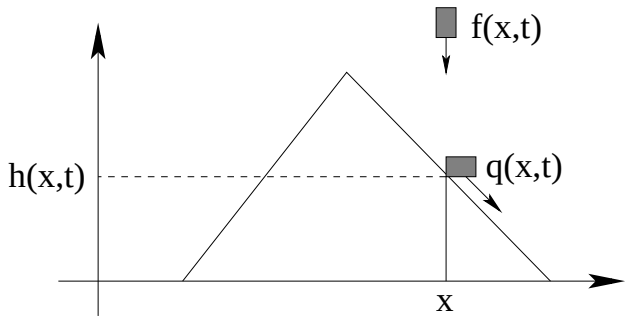


- Conservation of mass :
  - The flow of the granular material is confined in a thin boundary layer moving down the slopes of a growing pile
  - The density of the material is constant

$$\frac{\partial h}{\partial t}(x,t) = -\nabla \cdot q(x,t) + \mu(x,t)$$

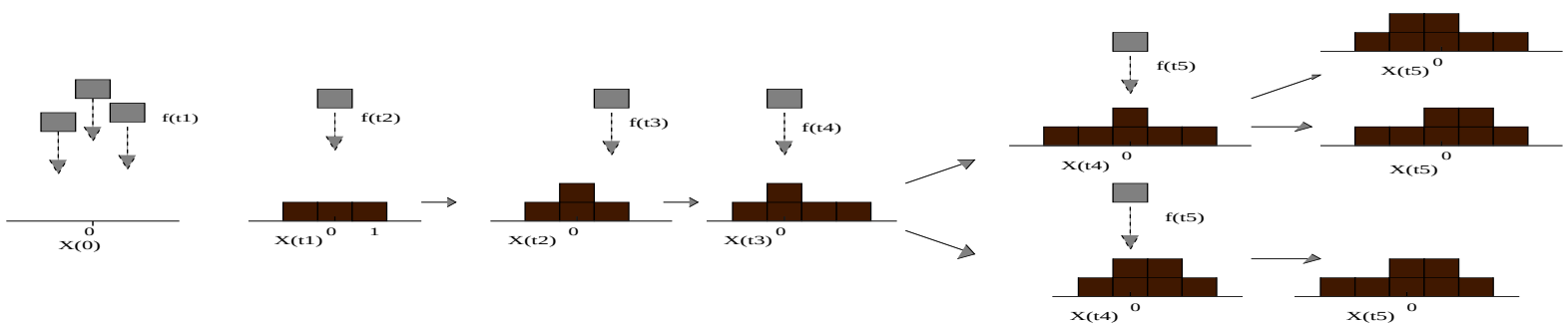
- Surface flow directed by the steepest descent :

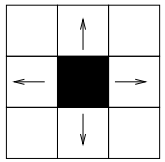
$$\exists\ m = m(x,t) \geqslant 0 \text{ such that } q(x,t) = -m(x,t)\,\nabla h(x,t)$$

- No pouring over the parts of the pile surface inclined less than α :

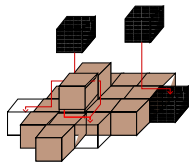
$$|\nabla h(x,t)| < \gamma \Longrightarrow m(x,t) = 0$$

Motivation III : Sandpile Cellular Automaton

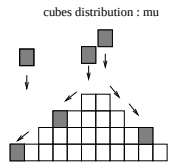




$$i \sim j \Longleftrightarrow |i-j| \leq 1$$



3D



2D

Adding a new cube on an existing pile, we have the following possibilities

- the new cube remains in place
- the new cube moves by falling in one of the forth directions (up, down, left or right).
- there are several downhill "staircases" along which the cube can move, and the cube will randomly select among the allowable downhill paths.

New cubes will move (or not) in order to get a stable configuartion, which means that the heights of any two adjacent columns of cubes can differ by at most one :

$$|\eta(i) - \eta(j)| \leq 1 \text{ if } i \sim j,$$

where $\eta(i)$ denotes the height of the columns situated at the position $i \in \mathbb{Z}^2$.

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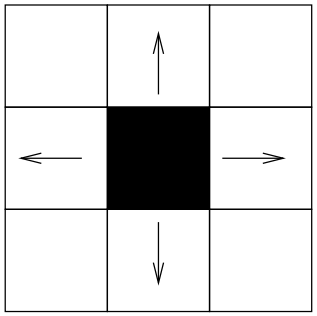
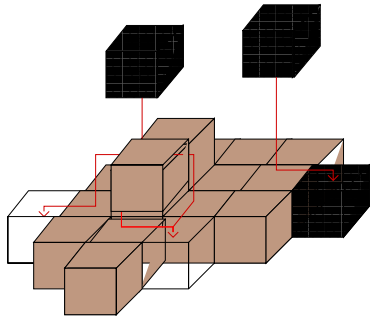
Motivation III : Sandpile Cellular Automaton



$i \sim j \Longleftrightarrow |i-j| \leq 1$



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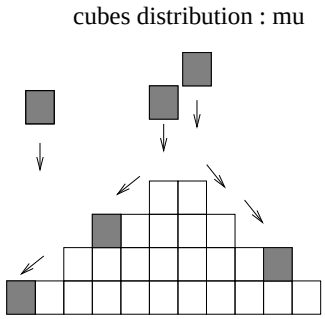


2D

- Let $p(i,j,\eta)$: probability that a cube placed at the position i will end up at j
- So, for any $i,\ j\ \in \mathbb{Z}2$ we have

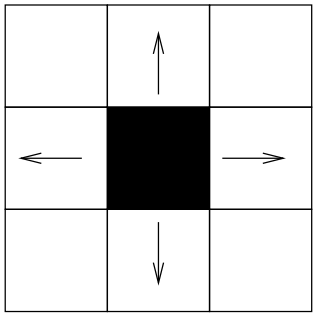
$$0 \leq p(i,j,\xi) \leq 1 \quad \text{and} \quad \sum_{j\in\mathbb{Z}^2} p(i,j,\xi) = 1.$$

- Adding cubes with a rate $\hat{f}$, we have

$$\boldsymbol{E}[\eta(t_{k+1},j) \mid \eta(t_k)] = \eta(t_k,j) + c(j,\eta,t_k), \qquad \text{for any } j \in \mathbb{Z}^2.$$

- $c(j,\eta,\tau)$: (highly nonlocal factor) records the rate, at time τ, new cubes come to rest at j
- $c(j,\eta,t) = \sum_{i\in\mathbb{Z}^2} p(i,j,\eta(t))\,\hat{f}(i,t), \qquad \text{for any } (j,t) \in \mathbb{Z}^2 \times [0,\infty).$

Motivation III : Sandpile Cellular Automaton



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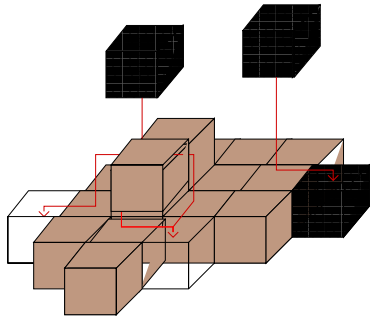

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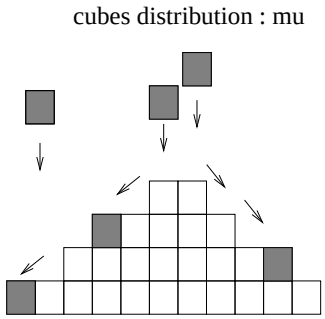


2D

Stochastic Equation (integral form) :

$$\left\{ \begin{array}{l} \forall\ F\ :\ S \times (0,\infty) \to \mathbb{R} \text{ Lipchitz continuous in } t \text{ and } F(\eta(.,0),0) = 0 \\ \\ \boldsymbol{E}\Big[F(\eta(.,t),t) - \displaystyle\int_0^t \left(\frac{\partial F}{\partial s} + \mathcal{L}_s\, F\right)(\eta(.,s))\Big] = 0, \end{array} \right.$$

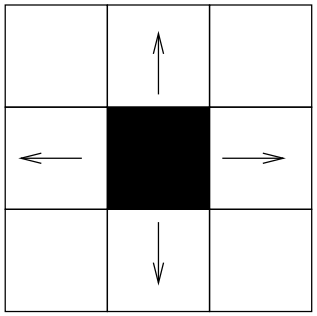
where

$$(\mathcal{L}_t\, F)(\xi) := \sum_{j \in \mathbb{Z}^N} c(j,\xi,t)\Big(F(T_j(\xi)) - F(\xi)\Big).$$

with

$$T_j\ :\ \xi \in S \to T_j(\xi) \in S \quad \text{with} \quad T_j(\xi) = \left\{ \begin{array}{ll} \xi(i)+1 & \text{if } i \sim j \\ \xi(i) & \text{otherwise} \end{array} \right.$$

Motivation III : Sandpile Cellular Automaton



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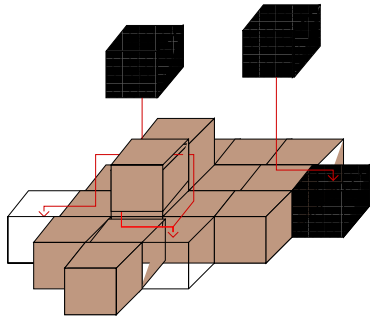

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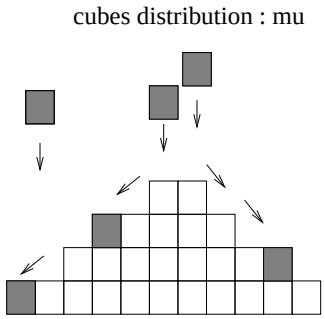

2D

- Assume that $f \in BV(0,T;L^2(\mathbb{R}^2)$
- $\hat{f}(t,i) = \dfrac{1}{N} f\left(\dfrac{t}{N}, \dfrac{i}{N}\right)$
- $(\eta(t), t \geqslant 0)$ the associated Markov processus
- u the solution de (EMK)

Theorem

As $N \longrightarrow \infty$, *we have*

$$\boldsymbol{E}\left[\sup_{x\in\mathbb{R}^2}\left|u(t,x) - \frac{1}{N}\,\eta(t,[N\,x])\right|\right] \longrightarrow 0.$$

- *A stochastic model for growing sandpiles and its continuum limit*, L. C. Evans, F. Rezakhanlou, Comm. Math. Phys., 197 (1998), no2, 325-345.
- *Back on Stochastic Model for Sandpile*, N. Igbida (soumis).

Boundary condition : Dirichlet

- If $\mu^+(\boldsymbol{R}^N) = \mu^-(\boldsymbol{R}^N)$ and μ^+, μ^- are supported in bounded domain : by taking Ω large enough we can assume that $u = 0$ on $\partial\Omega$;

- If $\mu^+(\boldsymbol{R}^N) \neq \mu^-(\boldsymbol{R}^N)$ and μ^+, μ^- are supported in bounded domain, then one needs :
  - There exists Γ such that $\mathcal{H}^{N-1}(\Gamma) = 0$ and replace the euclidean $c(x, y) = |x - y|$ by the semi-geodesic distance taking into account the boundary ; i.e.

$$c_\Gamma(x, y) = \min\Big(|x - y|, \mathit{dist}(x, \Gamma) + \mathit{dist}(y, \Gamma)\Big).$$

  - **Dirichlet Monge-Kantorovich problem** (Bouchité, Buttazzo, De Pascale ...)

$$\inf\Big\{\iint c_\Gamma(x, y)\, d\nu(x, y)\ ;\ \nu \in \mathcal{M}(\Omega \times \Omega) \text{ s.t. } \pi^1_\#\nu - \pi^2_\#\nu = \mu_1 - \mu_2\Big\}$$

Monge-Kantorovich evolution equation with Dirichlet boundary condition

$$(EMK)\qquad \begin{cases} \dfrac{\partial h}{\partial t} - \nabla \cdot (m\, \nabla h) = \mu & \text{in } Q \\ m \geqslant 0,\ |\nabla h| \leq 1 & \text{in } Q \\ m\,(1 - |\nabla h|) = 0 & \text{in } Q \\ u = 0 & \text{on } \Sigma \\ u(0) = u_0 & \text{in } \Omega \end{cases}$$

Main interest :

1. Existence and uniqueness of a solution : μ a Radon measure
2. Numerical analysis : representation of $t \in [0, T) \longrightarrow (u(t,x), m(t,x)\, \nabla u(t,x))$..
3. Large time behavior, as $t \longrightarrow \infty$.

Références de l'exposé

http ://www.mathinfo.u-picardie.fr/igbida/


S. Dumont and N. Igbida,
Back on a Dual Formulation for the growing Sandpile Problem, *European Journal of Applied Mathematics,* vol. 20, pp. 169-185, 2008.


N. Igbida,
Evolution Monge-Kantorovich Equation,*en revision dans Calculus of Variations and Partial Differential Equations*.


N. Igbida,
On Monge-Kantorovich Equation, *sous presse dans Nonlinear Analysis TMA*.

Main Difficulties

If m is regular then

$$\begin{cases} \partial_t u - \nabla \cdot (m \nabla u) = \mu \\ |\nabla u| \leq 1, \ m \geqslant 0, \ m\,(|\nabla u| - 1) = 0 \\ u = 0 \quad \text{on } \partial\Omega \end{cases} \quad \Rightarrow \quad \begin{cases} u_t + \partial I_K(u) \ni \mu, \\ K = \left\{ z \in H_0^1(\Omega) \ ; \ |z| \leq 1 \text{ in } \Omega \right\} \end{cases}$$

i.e. $\forall\ t \in [0, T),\ u(t) \in K$ and $\displaystyle\int_\Omega u(t) \Big(\mu(t) - \partial_t u(t)\Big) = \max_{\xi \in K} \int \Big(\mu(t) - \partial_t u(t)\Big)\, \xi.$

Nonlinear semi-group theory $\Rightarrow$ existence and uniqueness of "variational" solution u

Main Difficulties

- The main information are in the flux $\Phi = m \nabla u$:

Monge-Kantorovich	$\longrightarrow$	flux transport
Sandpile	$\longrightarrow$	numerical analysis

- In general, m is not regular : m is a measure.
- How to define the solution for (MK) and (EMK) : m is a measure and $\nabla u \in L^\infty(\Omega)$.
- How to do numerical analysis for (MK) and (EMK) : regularisation technics are not stable.

Notion of solution : how to define $m\,\nabla u$ (m a Radon measure)

Stationary problem : Theory of tangentiel gradient of u with respect to Radon measure (cf. Bouchitté, Buttazzo, De Pascale, Seppecher ...) :

$$(MK) \qquad \begin{cases} -\nabla\cdot(m\,\nabla_m u) = \mu \quad \text{in } \mathcal{D}'(\Omega) \\ |\nabla_m u| = 1 \quad m- \text{ a.e. in } \Omega. \end{cases} \qquad \text{(Monge-Kantorovich equation)}$$

In particular :

- Let $\nu \in \mathcal{M}_b(\Omega)$ and $1 < p < \infty$

$$w = \nabla_\nu u \ \in L^p_\nu(\Omega) \Leftrightarrow \exists\ u_h \in \mathcal{C}^1(\Omega),\ (u_h, \nabla u_h) \to (u, w) \quad \text{in } L^p_\nu(\Omega)^{N+1}.$$

- For any $F \in L^{p'}_\nu(\Omega)^N$, such that $-\nabla\cdot(F) = \mu \ \in L^p(\Omega)$, we have

$$\int_\Omega \nabla_\nu u \cdot F \ d\nu = -\int_\Omega u\,\mu$$

for any $u \in \mathcal{C}_0(\Omega)$ such that u is Lipchitz.

Existence, uniqueness of $m\,\nabla_m u$, regularity, related transport for Monge problem... : Ambrosio, Bouchitté, Buttazzo, De Pascale, Evans, Seppecher ....

Notion of solution : how to define $m\,\nabla u$ (m a Radon measure)

Evolution problem :

- Theory ? tangentiel derivative for time dependent functions : $t \in (0,T) \longrightarrow \nabla_\nu u(t)$.
- For $\mu \in L^2(Q)$: existence, uniqueness and numerical analysis by dual formulation
  - Prigozhin - 1996 : $m \in \left(L^\infty(Q)\right)^*$?

$$\begin{cases} \displaystyle\int\int_Q (u_t - \mu)\,\xi + \left\langle m, \nabla u \cdot \nabla \xi \right\rangle = 0 \\ \\ \left\langle m, \xi \right\rangle \geqslant 0, \quad \left\langle m, |\nabla u|^2 - 1 >= 0 \right. \end{cases}$$

  - Prigozhin and Barrett - 2006 : $m\,\nabla u =: \Phi \ \in \mathcal{M}_b(Q)^N$, $\nabla \cdot \Phi \in L^2(Q)$.
    - Minimizing problem related to Φ and $\displaystyle\int_0^t \Phi$ in $\mathcal{M}_b(Q)^N$ gives Φ and $u(t) = u_0 + \displaystyle\int_0^t f(s)\,ds + \nabla \cdot \int_0^t \Phi$.
    - Equivalent formulation

$$\begin{cases} u_t - \nabla \cdot \Phi = \mu \quad \text{in } \mathcal{D}'(Q) \\ |\Phi|(Q) = \displaystyle\int\int_Q u\,\nabla \cdot \Phi \end{cases}$$

- **Our aim :** To treat existence and uniqueness for general data (measures), large time behavior and numerical analysis.

Notion of solution : equivalent formulations

Remark

Assume that $\Phi : \Omega \to \mathbb{R}^N$ and $u : \Omega \to \mathbb{R}$ are regular and $|\nabla u| \leq 1$ in Ω. Then

$$\Updownarrow \quad \begin{cases} \bullet & m\,(|\nabla u| - 1) = 0 \ \text{ and } \ \Phi = m\,\nabla u \quad \text{in} \quad \Omega \\ \bullet & m = |\Phi| = \Phi \cdot \nabla u \quad \text{in} \quad \Omega \\ \bullet & m = |\Phi| \ \text{ and } \ \displaystyle\int_\Omega |\Phi| \leq \int_\Omega \Phi \cdot \nabla u \end{cases}$$

Theorem (Ig,2007)

Let μ be a Radon measure and $u \in K$. Then,

$$\int_\Omega u\,d\mu = \max_{\xi \in K} \int_\Omega \xi\,d\mu \quad \Leftrightarrow \quad \exists \Phi \in \mathcal{M}_b(\Omega)^N \quad \begin{cases} -\nabla \cdot (\Phi) = \mu \quad \text{in } \mathcal{D}'(\Omega) \\ |\Phi|(\Omega) \leq \displaystyle\int_\Omega u\,d\mu \end{cases} \quad \textbf{(weak solution)}$$

$$\Longleftrightarrow \quad \begin{cases} -\nabla \cdot (\Phi) = \mu \quad \text{in } \mathcal{D}'(\Omega) \\ \Phi = |\Phi|\,\nabla_{|\Phi|} u \end{cases} \quad \Longleftrightarrow \quad \int_\Omega u\,d\mu = |\Phi|(\Omega) = \min\{|\nu|(\Omega)\ ;\ -\nabla \cdot \nu = \mu\}.$$

Remark

- If (u, Φ) is a weak solution and $\Phi \in L^1(\Omega)^N$, then $\Phi = |\Phi|\,\nabla u$ a.e. in Ω.
- The expression $|\Phi|(\Omega) = \displaystyle\int_\Omega u\,d\mu = \left\langle \nabla \cdot \Phi, u \right\rangle$ is a weak formulation of $|\Phi| = \Phi \cdot \nabla u$.

Notion of solution : Evolution Monge-Kantorovich equation

- $L^q(0,T;w^*\text{-}\mathcal{M}_b(\Omega)) = \Big\{\psi \ : \ (0,T) \to \mathcal{M}_b(\Omega),\ \text{weak}^*\text{measurable} \ ; \ \int_0^T |\psi(t)|^q(\Omega) < \infty\Big\}.$
- $BV(0,T;w^*\text{-}\mathcal{M}_b(\Omega)) = \Big\{\mu \in L^1(0,T;w^*\text{-}\mathcal{M}_b(\Omega)) \ ; \ \limsup_{h\to 0} \frac{1}{h}\int_0^{T-h} |\mu(\tau+h)-\mu(\tau)|(\Omega)\, d\tau < \infty\Big\}.$
- In Q the condition $|\Phi| = \Phi\cdot\nabla u$ mays be represented by one of the following formulation :
  - $|\Phi(t)|(\Omega) \le \Big\langle \nabla\cdot\Phi(t), u(t)\Big\rangle = \Big\langle \mu(t) - \partial_t u(t), u(t)\Big\rangle$ for $t\in(0,T)$
  - $|\Phi|(Q) \le \int_0^T \Big\langle \nabla\cdot\Phi, u\Big\rangle = \int_0^T \Big\langle \mu(t) - \partial_t u(t), u(t)\Big\rangle$

Proposition (Ig,2007)

If $\Phi \in L^1(0,T;w^*\text{-}\mathcal{M}_b(\Omega))^N$ *and* $\nabla\cdot\Phi \ \in L^1(0,T;w^*\text{-}\mathcal{M}_b(\Omega))$, $u\in\mathcal{C}([0,T);L^2(\Omega))\cap K_T$, $u(0)=u_0$, $\partial_t u - \nabla\cdot\Phi = \mu$ *in* $\mathcal{D}'(Q)$, *then*

- $|\Phi(.)|(\Omega) \le -\frac{1}{2}\frac{d}{dt}\int_\Omega u^2(.) + \int_\Omega u(.)\, d\mu(.)$ *in* $\mathcal{D}'(0,T)$
- $|\Phi|(Q) \le \int\int_Q u\, d\mu - \frac{1}{2}\int_\Omega u(T)^2 + \frac{1}{2}\int_\Omega u_0^2$
- *for a.e.* $t\in(0,T)$, $\Phi(t) = \nabla_{|\Phi(t)|}u(t)$, $|\Phi(t)|-$*a.e. in* Ω

Notion of solution : Evolution Monge-Kantorovich equation

Theorem (Ig,2007)

For any $u_0 \in K$ and $\mu \in BV(0,T,w^*\text{-}\mathcal{M}_b(\Omega))$, the problem (EMK) has a weak solution (u,Φ) : $u \in \mathcal{C}([0,T);L^2(\Omega)) \cap K_T$, $\partial_t u \in L^\infty(0,T;w^*\text{-}\mathcal{M}_b(\Omega))$, $\Phi \in L^\infty(0,T;w^*\text{-}\mathcal{M}_b(\Omega)^n)$, $u(0)=u_0$, $\partial_t u - \nabla\cdot\Phi = \mu$ *in* $\mathcal{D}'(Q)$, *and* (u,Φ) *satisfies*

$$\Updownarrow$$

- $|\Phi(.)|(\Omega) \le -\frac{1}{2}\frac{d}{dt}\int_\Omega u^2(.) + \int_\Omega u(.)\,d\mu(.) \quad$ *in* $\mathcal{D}'(0,T)$
- $|\Phi|(Q) \le \int\int_Q u\,\sigma\,d\mu - \frac{1}{2}\int_\Omega u(T)^2 + \frac{1}{2}\int_\Omega u_0^2$
- *for a.e.* $t \in (0,T)$, $\Phi(t) = \nabla_{|\Phi(t)|}u(t)$, $|\Phi(t)| - a.e.$ *in* Ω

Moreover, u is the unique variational solution.

Theorem (Ig,2007)

For any $\mu \in L^1(0,T;w^*\text{-}\mathcal{M}_b(\Omega))$ and $u_0 \in K$, there exists (u,Φ) a weak solution of (EMK) i.e. $u \in \mathcal{C}([0,T);L^2(\Omega)) \cap K_T$, $u(0)=u_0$, $\Phi \in \mathcal{M}_b(Q)^n$, $\partial_t u - \nabla\cdot\Phi = \mu$ *in* $\mathcal{D}'(Q)$ *and*

$$\sup_{\eta\in\mathcal{C}_0(\Omega)^N,\ \|\eta\|_\infty\le 1}\int\int_Q \sigma\,\frac{\Phi}{|\Phi|}\cdot\eta\,d|\Phi| \le \frac{1}{2}\int_0^T\int_\Omega u^2\,\sigma_t + \int_0^T\int_\Omega u\,\sigma\,d\mu$$

for any $\sigma \in \mathcal{D}(0,T)$. Moreover, u is the unique variational solution.

Numerical approximation

Assume that $\mu = f \quad \in L^2(0, T; L^2(\Omega))$.

Time discretisation : Euler implicit schema.

- $0 < t_0 < t_1 < ... < t_n = T$ whith $t_i - t_{i-1} = \varepsilon \quad \forall i = 1, ..., n$
- $f_1, ..., f_n \in L^2(\Omega))$ such that

$$\sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \|f(t) - f_i\|_{L^2(\Omega)} \leq \varepsilon.$$

- consider the approximation of $u(t)$ by :

$$u_\varepsilon(t) = \begin{cases} u_0 & \text{if } t \in]0, t_1] \\ u_i & \text{if } t \in]t_{i-1}, t_i] \end{cases} \qquad i = 1, ..., n$$

where u_i is given by :

$$u_i + \partial I_K(u_i) = \varepsilon f_i + u_{i-1}$$

for $i \geqslant 1$ whith $u_{i=0} = u_0$.

- u_ε is the $\varepsilon-$approximate solution of (P_μ).

Numerical approximation : convergence of the $\varepsilon-$approximate solution

Nonlinear semigroup theory (in $L^2(\Omega)$)

$\Downarrow \quad \Downarrow \quad \Downarrow \quad \Downarrow \quad \Downarrow$

Theorem

Let $u_0 \in L^2(\Omega)$ be such that $|\nabla u_0| \leq 1$ and $f \in BV(0,T;L^2(\Omega))$. Then,

1. *For any $\varepsilon > 0$ and any $\varepsilon-$discretisation, there exists a unique $\varepsilon-$approximation u_ε of (P_f).*
2. *There exists a unique $u \in \mathcal{C}([0,T);L^2(\Omega))$ such that $u(0) = u_0$, and*

$$\|u - u_\varepsilon\|_{C([0,T);L^2(\Omega))} \leq C(\varepsilon) \quad \to 0 \text{ as } \varepsilon \to 0.$$

3. *The function u given by 2. is the unique solution of (P_μ).*

Numerical approximation : a generic problem

The generic problem is

$$v + \partial I_K(v) = g$$

$$\Downarrow\Uparrow \qquad \Downarrow\Uparrow \qquad \Downarrow\Uparrow \qquad \Downarrow\Uparrow$$

$$v = P_K g$$

where

- $K = \left\{ z \in W^{1,\infty}(\Omega) \cap W_0^{1,2}(\Omega) \ ; \ |\nabla u| \leq 1 \right\}$.
- P_K the projection onto the convex K, with respect to the $L^2(\Omega)$ norm :

$$v = \mathbf{P}_K(u) \Leftrightarrow v \in K, \ \int_\Omega (v-u)(v-z) \geqslant 0 \quad \text{for any } z \in K.$$

Question :

For a given $g \in L^2$, how to compute $v = P_K g$?

Numerical approximation : Dual and Primal formulation

Standard duality argument (Ekeland-Temam) : if

$$\mathcal{P} := \min\Big\{F(z) + H(\Lambda z)\ ;\ z \in \mathcal{C}_0^1(\Omega)\Big\},$$

where

$$\begin{array}{rccc} \Lambda & : & \mathcal{C}_0(\Omega) & \longrightarrow & \mathcal{C}(\Omega)^N \\ & & z & \longrightarrow & \Lambda z := \nabla z \end{array} \qquad \begin{array}{rccc} F & : & \mathcal{C}_0(\Omega) & \longrightarrow & \mathbb{R}^+ \\ & & z & \longrightarrow & F(z) = \dfrac{1}{2}\displaystyle\int_\Omega |z-g|^2 \end{array}$$

$$\begin{array}{rccc} H & : & \mathcal{C}(\Omega)^N & \longrightarrow & \overline{\mathbb{R}} \\ & & \sigma & \longrightarrow & H(\sigma) = \left\{ \begin{array}{ll} 0 & \text{si } |\sigma(x)| \le 1\ \forall x \in \Omega \\ +\infty & \text{sinon}, \end{array} \right. \end{array}$$

then

$$\mathcal{D} := \sup_{p \in Y'} -\Big(F^*(\Lambda^* p) + H^*(-p)\Big) \le \mathcal{P}.$$

If,

- $\mathcal{P} < \infty$
- There exists $u_0 \in V$ such that $F(u_0) < \infty$, $H(\Lambda u_0) < \infty$ and H is continuous at Λu_0,

then

$$\mathcal{P} = \mathcal{D}.$$

and

$$\Lambda^* p \in \partial F(u) \quad \text{et} \quad -p \in \partial H(\Lambda u).$$

Numerical approximation : Dual and Primal formulation

We have

$$\Lambda^* \ : \ \Big(\mathcal{C}(\Omega)^N\Big)' \ \to \ \Big(\mathcal{C}_0(\Omega)\Big)' \qquad \Lambda^*\sigma := \nabla\cdot\sigma$$

$$F^* \ : \ \Big(\mathcal{C}_0(\Omega)\Big)' \ \to \ \mathbb{R}^+ \qquad F(z) = \frac{1}{2}\int_\Omega |z-g|^2 + \frac{1}{2}\int_\Omega g^2$$

$$H^* \ : \ \Big(\mathcal{C}(\Omega)^N\Big)' \ \to \ \overline{\mathbb{R}} \qquad H(\sigma) = \int_\Omega |\sigma|.$$

This implies that $G(w) = \frac{1}{2}\int_\Omega (\ \mathrm{div}(w))^2 + \int_\Omega g\ \ \mathrm{div}(w) + \int_\Omega |w|$ and

$$\mathcal{D} = \max\Big\{ -G(\sigma)\ ;\ \sigma\in(\mathcal{C}(\Omega)^N)^*\Big\}.$$

So, one expects

$$-G(\Phi) = \max\Big\{ -G(\sigma)\ ;\ \sigma\in(\mathcal{C}(\Omega)^N)^*\Big\} = \min\Big\{F(z)+H(\Lambda z)\ ;\ z\in\mathcal{C}_0^1(\Omega)\Big\} = F(u).$$

and

$$-\nabla\cdot\Phi = g-u \quad \text{ and } \quad \Phi\in\partial I\!\!I_{B(0,1)}(\nabla u).$$

- the natural space for the study of $\mathcal{P}$ by duality argument is the set of vector valued Radon measures such that the divergence is a L^2.
- This kind of argument was used by Prigozhin and Barrett.
- Our aim is to simplify the analysis by using simply the space $H_{div}(\Omega)$;

Numerical approximation : Dual and Primal formulation

- $J(z) = \frac{1}{2}\int_\Omega |z-g|^2$.
- $H_{div}(\Omega) = \{\sigma \in (L^2(\Omega))^N \ ; \ \mathrm{div}(\sigma) \in L^2(\Omega)\}$
- $G(\sigma) = \frac{1}{2}\int_\Omega (\mathrm{div}(\sigma))^2 + \int_\Omega g\,\mathrm{div}(\sigma) + d\int_\Omega |\sigma|$.

Theorem (Dumont-Ig,2007)

Let $g \in L^2(\Omega)$ and $v = \boldsymbol{P}_k(g)$. Then, there exists a sequence $(w_\epsilon)_{\epsilon>0}$ in $H_{div}(\Omega)$, such that, as $\epsilon \longrightarrow 0$,

$$\begin{aligned} \lim_{\epsilon\longrightarrow\infty}\Big(-G(w_\epsilon)\Big) &= \sup_{w\in H_{div}(\Omega)}\Big(-G(w)\Big) \\ &= \min_{z\in L^2(\Omega)} J(z) \end{aligned}$$

and

$$div(w_\epsilon) \longrightarrow v-g \qquad \text{in } L^2(\Omega).$$

Numerical approximation : main idea of the proof

- $sup_{\sigma \in H_{div}(\Omega)} \Big(- G(\sigma) \Big) \leq \min_{z \in K} J(z)$.
- We consider the following elliptic equation

$$(S_\varepsilon) \qquad \left\{ \begin{array}{l|l} v_\varepsilon - \nabla \cdot w_\varepsilon = g & \\ & \text{in } \Omega \\ w_\varepsilon = \phi_\epsilon(\nabla v_\varepsilon) & \\ & \\ u = 0 & \text{on } \partial\Omega. \end{array} \right.$$

where, for any $\varepsilon > 0$, $\phi_\varepsilon : \mathbb{R}^N \to \mathbb{R}^N$ is given by

$$\phi_\varepsilon(r) = \frac{1}{\varepsilon}(|r| - 1)^+ \frac{r}{|r|}, \qquad \text{for any } r \in \mathbb{R}^N.$$

$$\Downarrow \qquad \Downarrow \qquad \Downarrow \qquad \Downarrow \qquad \Downarrow$$

For any $g \in L^2(\Omega)$, (S_ε) has a unique solution v_ε, in the sense that $v_\varepsilon \in H_0^1(\Omega)$, $w_\varepsilon := \phi_\varepsilon(\nabla v_\varepsilon) \in L^2(\Omega)^N$ and $v_\varepsilon - \nabla \cdot w_\varepsilon = g$ in $\mathcal{D}'(\Omega)$.

$$\Downarrow \qquad \text{letting } \varepsilon \to 0 \qquad \Downarrow$$

The proof of the theorem

Numerical approximation : approximation of $\inf_{\sigma \in H_{div}(\Omega)} G(\sigma)$

We consider

- $V : H_{div}(\Omega) \supset V_h :$ the space of finite elment of Raviard-Thomas
- $r_h : H_{div}(\Omega) \longrightarrow V_h$ the projection operator of Raviard-Thomas : for any $w \in H_{div}(\Omega)$, as $h \longrightarrow 0$, we have

$$(r_h(w), \mathrm{div}(r_h(w)) \longrightarrow (w, \mathrm{div}(w)) \quad \text{in } \left(L^2(\Omega)\right)^{N+1}$$

Theorem (Dumont-Ig,2007)

Let q_h be the solution of

$$G(q_h) = \inf\{G(\sigma_h),\ \sigma_h \in V_h\}$$

and $q \in M_b(\Omega)^N$ such that $div(q) \in L^2(\Omega)$ and q is a solution of the solution of

$$G(q) = \inf\{G(\sigma),\ \sigma \in H_{div}(\Omega)\},$$

then

$$\|div(q) - div(q_h)\|_{L^2(\Omega)} \longrightarrow 0 \quad \text{as } h \longrightarrow 0.$$

Cas d'une source centrale

Ecoulement d'un tas contre un mur

Ecoulement d'un tas sur une table

Cas d'une source qui tourne

Effondrement d'un tas

Related works and works in progress :

- Nonlocale sandpile problem : compound Poisson process
- Moving sand dunes : Barchanes
- Sandpile problem on non flat regions (on obstacles) : lacs and rivers
- Collapsing sandpile problem.
- Hysterisis : angle of stability is different than the angle of avalanches.
- Application to optimal mass transportation : how to use numerical results concerning sandpile to treat Monge problem ?
  Example : $d(., \partial\Omega)$