Localized large reaction for a non linear Reaction-Diffusion system

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Abstract

This paper is concerned with the asymptotic limit of the solution of a nonlinear reaction-diffusion system when the reaction is very large in a localized region.
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1 Introduction and main results

Let $\Omega \subseteq \mathbb{R}^N$ be bounded domain with smooth boundary Γ. In Ω, we consider the Reaction-Diffusion systems of the form

$$(P)\quad \begin{cases} u_{1t} - d_1 \triangle w_1 + \mathcal{R}\, g(w_1 - w_2) = f_1, \quad w_1 = \varphi_1(u_1) & \text{in } Q := \Omega \times (0,T) \\ u_{2t} - d_2 \triangle w_2 - \mathcal{R}\, g(w_1 - w_2) = f_2, \quad w_2 = \varphi_2(u_2) & \text{in } Q := \Omega \times (0,T) \\ \partial_{\vec{n}} w_1 + z_1 = 0, \quad z_1 \in \gamma_1(w_1) & \text{in } \Sigma := \Gamma \times (0,T) \\ \partial_{\vec{n}} w_2 + z_2 = 0, \quad z_2 \in \gamma_2(w_2) & \text{in } \Sigma := \Gamma \times (0,T) \\ u_1(x,0) = u_{01}(x) \quad u_2(x,0) = u_{02}(x) & \text{in } \Omega, \end{cases}$$

where for $i = 1, 2$; d_i is a positive constant, φ_i is strictly increasing continuous function with $\varphi_i(0) = 0$ and γ_i is maximal monotone graph in $\mathbb{R} \times \mathbb{R}$ such that $0 \in \gamma_i(0)$ and

$$(H_1) \qquad \mathcal{D}(\gamma_i) = \mathbb{R} \ \text{ or } \ \mathcal{D}(\gamma_i) = \{0\}.$$

Many particular cases of φ_i correspond to problems that arise in many applications. For instance, $\varphi_i(r) = |r|^{m_i} Sign_0(r)$, with $m_i > 1$ describes a slow nonlinear diffusion ([2], [15]); $m_i = 1$ corresponds to the classical heat equations and $0 < m_i < 1$ is the fast diffusion equation case. On the other hand, γ_i may be multivalued and this allows the

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boundary condition to include the Dirichlet (taking $\gamma_i = \{0\} \times \mathbb{R}$) and the Neumann boundary condition (taking $\gamma_i = \mathbb{R} \times \{0\}$) as well as many other possibilities.

This kind of systems appears in numerous application with nondecreasing continuous function g such that $g^{-1}\{0\} = 0$ and nonnegative reaction rate. It was introduced in [9] to describe some problems intervening in model of heat conduction in a composite material, diffusion phenomena in a heterogeneous medium as fractured porous medium. It arises also from modelling interacting evolution of two mobile species for the reversible chemical reactions (cf. [8], [10], [2], [4], [14]). Roughly speaking, g describes the exchange between the two components w_1 and w_2, and $\mathcal{R}$ is the rate of the exchange. For instance, in modelling a diffusion processes within a medium composed of two components which involves phase change, the term $\mathcal{R}\, g$ is related to the surface area common to the two components and measure the homogeneity of the material (for more details concerning the physical situations of system (P), one can see [14]).

Existence and uniqueness of a solution for this kind of system is more or less well known. For instance, in the case of Dirichlet boundary condition and $\varphi_2 = 0$, existence and uniqueness of a solution was established in [9] by using the H^{-1} theory. Then in [16] and in [13] the authors show that the theory of nonlinear semigroups may be applied to prove the existence of an unique *mild solution* (nonlinear semigroups solution). Among our result in this paper, we extend these results to the case where $\mathcal{R}$ depends on space (see Proposition 3.3) and we prove that *mild solution* is the unique weak solution of (P).

In many situation changes due to reaction are very fast compared with diffusive effects. This corresponds to large value for the rate $\mathcal{R}$, and a natural question then is the passage to the limit in the system by letting $\mathcal{R} \to \infty$. This kind of question is familiar in the context of reaction-diffusion systems (see [14], [8]). For the problem (P) related situations was studied in [16] and [8]. As expected, the two components w_1 and w_2 coincides in the limit and satisfy the nonlinear heat equation.

Our main interest in this paper remains in the study of the case where the reaction is very large in a localized region of Ω. More precisely, assume $\Omega_0 \subseteq \Omega$ and $\Omega_1 = \Omega \setminus \overline{\Omega}_0$,

$$\mathcal{R} = \mathcal{R}_k(x) = a(k)\chi_{\Omega_0}(x) + \mathcal{R}_1(x)\chi_{\Omega_1}(x) \qquad \text{a.e. in } \Omega$$

and $a(k)$ is a strictly positive function such that $a(k) \to \infty$ as $k \to \infty$. Let us denote by (u_1^k, u_2^k) the solution of the system

$$(P^k) \begin{cases} u_{1t} - d_1 \triangle w_1 + \mathcal{R}_k\, g(w_1 - w_2) = f_1, \quad w_1 = \varphi_1(u_1) & \text{in } Q := \Omega \times (0,T) \\ u_{2t} - d_2 \triangle w_2 - \mathcal{R}_k\, g(w_1 - w_2) = f_2, \quad w_2 = \varphi_2(u_2) & \text{in } Q := \Omega \times (0,T) \\ \partial_{\vec{n}} w_1 + z_1 = 0, \quad z_1 \in \gamma_1(w_1) & \text{in } \Sigma := \Gamma \times (0,T) \\ \partial_{\vec{n}} w_2 + z_2 = 0, \quad z_2 \in \gamma_2(w_2) & \text{in } \Sigma := \Gamma \times (0,T) \\ u_1(x,0) = u_{01}(x) \quad u_2(x,0) = u_{02}(x) & \text{in } \Omega, \end{cases}$$

In contrast, of the case where the reaction is large in all Ω, our situation is more difficult. Indeed, by letting $k \to \infty$, the components w_1 and w_2 coincide in Ω_0, where they satisfy some nonlinear heat equation and in Ω_1, (w_1, w_2) satisfies the reaction-diffusion system with $\mathcal{R}_1\, g$ as a reaction term. But, in order to close the problem one needs to add some conditions on the boundary of Ω_0. To handle the problem, we introduce a new notion of solution for the limiting problem which includes the condition on the solution in all Ω. Moreover, notice that the limit is singular. Indeed, since in the limiting problem $\varphi_1(u_1)$ and $\varphi_2(u_2)$ coincide in Ω_0, then this needs to be true also for the initial data. So, in the passage to the limit with non compatible initial data, it appears a boundary layer in the neighborhood of $t = 0$. To overcame this difficulty, we use nonlinear semigroup theory and our recent paper [12]. To summarize our main result, assume that $\mathcal{R}_k : \Omega \to \mathbb{R}^+$ is a measurable function such that $\mathcal{R}_k \in BV(\Omega)$ and we introduce the following set

$$\mathcal{K} = \Big\{(\xi_1,\, \xi_2) \in L^1(\Omega)\times L^1(\Omega) \ \text{ such that } \ \xi_1 = \xi_2 \ \text{ a.e. in } \ \Omega_0\Big\} \quad \text{and } \ \underline{U}_0 := (\underline{u}_{01}, \underline{u}_{02})$$

where

$$\underline{u}_{01}(x) = \begin{cases} u_{01}(x) & \text{in } \ \Omega_1 \\ \tilde{u}_{01}(x) & \text{in } \ \Omega_0 \end{cases} \qquad \underline{u}_{02}(x) = \begin{cases} u_{02}(x) & \text{in } \ \Omega_1 \\ \tilde{u}_{01}(x) & \text{in } \ \Omega_0 \end{cases},$$

$\tilde{u}_{01}(x) := [I + \varphi_2^{-1}\circ\varphi_1]^{-1}(u_{01} + u_{02})$ and $\tilde{u}_{02}(x) := [I + \varphi_1^{-1}\circ\varphi_2]^{-1}(u_{01} + u_{02})$. As usual, we denote by T_l the truncation function defined by $T_l(s) = \min(\max(s, -l),\, l)$, $\forall\, l > 0$. Our main result is

Theorem 1.1 *Let $U_0 = (u_{01},\, u_{02}) \in L^\infty(\Omega)\times L^\infty(\Omega)$, $(f_1,\, f_2) \in L^\infty(Q)\times L^\infty(Q)$. Then (P^k) has a unique weak solution $U^k := (u_1^k,\, u_2^k)$ (in the sense of Proposition 3.3). As $k \longrightarrow \infty$, we have*

$$U^k \longrightarrow U := (u_1,\, u_2) \quad \text{in } \ \mathcal{C}\Big((0,T),\ L^1(\Omega)\times L^1(\Omega)\Big),$$

where $(u_1,\, u_2)$ is the unique solution in the following sense: $U(0) = \underline{U}_0$ and for $i = 1, 2$; $u_i \in \mathcal{C}\Big([0,\, T];\ L^1(\Omega)\Big)\cap L^\infty(Q)$, $\exists\, w_i \in L^2\Big(0,\, T;\ H^1(\Omega)\Big)$ and $\exists\, z_i \in L^2(\Sigma)$ such that $w_i = \varphi_i(u_i)$ a.e. in Q, $z_i \in \gamma_i(w_i)$ a.e. in Σ, and for any $l > 0$, we have

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega\int_0^{u_1} T_l(\varphi_1(r) - \xi_1)dr + d_1\int_\Omega Dw_1 DT_l(w_1 - \xi_1) + \int_\Gamma z_1 T_l(w_1 - \xi_1)\\
&+\frac{d}{dt}\int_\Omega\int_0^{u_2} T_l(\varphi_2(r) - \xi_2)dr + d_2\int_\Omega Dw_2 DT_l(w_2 - \xi_2) + \int_\Gamma z_2 T_l(w_2 - \xi_2)\\
&+\int_{\Omega_1}\mathcal{R}_1(x) g(w_1 - w_2)\Big(T_l(w_1 - \xi_1) - T_l(w_2 - \xi_2)\Big) \le \int_\Omega f_1 T_l(w_1 - \xi_1)\\
&+\int_\Omega f_2 T_l(w_2 - \xi_2) \quad \text{in } \mathcal{D}'(0,T),
\end{aligned}$$

for any $(\xi_1, \xi_2) \in \Big(\mathcal{C}^1(\overline{\Omega})\times\mathcal{C}^1(\overline{\Omega})\Big)\cap\mathcal{K}$.

To use nonlinear semigroup theory and the results of [12], we begin by studying the elliptic problem associated with (P^k). This is the aim of the following section. Section 3 is devoted to the evolution problem; we also prove that the mild solution is the unique weak solution of (P^k) and state the convergence result. In section 4 we use the notion of integral solutions to prove the uniqueness of solutions.

2 Stationary problem

Through the implicit discretization in time arising in the theory of nonlinear semigroup, the study of (P^k) is closely connected to the associate stationary problem which is

$$S^k(f_1,f_2)\quad\begin{cases} v_1 - d_1\,\triangle w_1 + \mathcal{R}_k(x)g(w_1-w_2) = f_1, \quad w_1=\varphi_1(v_1) & \text{in } \Omega \\ v_2 - d_2\,\triangle w_2 - \mathcal{R}_k(x)g(w_1-w_2) = f_2, \quad w_2=\varphi_2(v_2) & \text{in } \Omega \\ \partial_{\vec{n}} w_1 + z_1 = 0, \quad z_1\in\gamma_1(w_1) & \text{in } \Gamma \\ \partial_{\vec{n}} w_2 + z_2 = 0, \quad z_2\in\gamma_2(w_2) & \text{in } \Gamma. \end{cases}$$

Proposition 2.1 *Let $(f_1,\, f_2)\in L^\infty(\Omega)\times L^\infty(\Omega)$, there exists a unique solution $(v_1,\, v_2)\in L^\infty(\Omega)\times L^\infty(\Omega)$ of $S^k(f_1,\, f_2)$ in the sense that for $i=1,2$; there exists $w_i\in H^1(\Omega)$ and $z_i\in L^2(\Gamma)$ such that $w_i=\varphi_i(v_i)$ a.e. in Ω, $z_i\in\gamma_i(w_i)$ a.e. in Γ, and*

$$d_i\int_\Omega Dw_iD\xi_i + (-1)^{i-1}\int_\Omega \mathcal{R}_k(x)g(w_1-w_2)\xi_i + \int_\Gamma z_i\xi_i = \int_\Omega (f_i-v_i)\xi_i \qquad (1)$$

for any $\xi_i\in H^1(\Omega)$. Moreover, we have the following estimates:

1. $||(v_1,\, v_2)||_\infty \le ||(f_1,\, f_2)||_\infty$.
2. $\displaystyle\int_\Omega (|Dw_1|^2+|Dw_2|^2)\le C'$, *where $C'$is a constant independent of k.*

Lemma 2.2 *Given $(f_1,f_2)\in L^\infty(\Omega)\times L^\infty(\Omega)$ and for $i=1,2$; we suppose that φ_i, φ_i^{-1}, γ_i^{-1} and g are Lipschitz continuous functions, then there exist a unique solution of $S^k(f_1,\, f_2)$ in the sense of the proposition (2.1).*

Proof. The proof follows the same manner as in [13]. ∎

Proof of Proposition 2.1. For $i=1,2$, let $\varphi_i^\lambda, \gamma_i^\lambda$ and g^λ be the Yoshida approximations of φ_i, γ_i and g respectively, and we define $\hat{\varphi}_i^\lambda(r)=\varphi_i^\lambda(r)+\lambda r$. It is clear that $\hat{\varphi}_i^\lambda$, $(\hat{\varphi}_i^\lambda)^{-1}$, γ_i^λ and g^λ are Lipschitz continuous functions, so that by Lemma 2.2, there exists a unique weak solution $(v_1^\lambda,\, v_2^\lambda)$ of the problem $S^k(f_1,\, f_2)$, moreover we have

$$||(v_1^\lambda,\, v_2^\lambda)||_\infty \le ||(f_1,\, f_2)||_\infty \qquad (2)$$

and, since $|\varphi_i^\lambda(r)|\le|\varphi_i^0(r)|$, then

$$\begin{aligned} ||\hat{\varphi}_i^\lambda(v_i^\lambda)||_{L^\infty(\Omega)} &\le ||\varphi_i^0(v_i^\lambda)||_{L^\infty(\Omega)} + \lambda||v_i^\lambda||_{L^\infty(\Omega)} \\ &\le \varphi_i^0(||(f_1,\, f_2)||_\infty) + \lambda||(f_1,\, f_2)||_\infty, \end{aligned}$$

and for λ small enough, we have

$$||\hat{\varphi}_i^\lambda(v_i^\lambda)||_{L^\infty(\Omega)} \leq M_1 \tag{3}$$

where M_1 is a constant depending only on Ω and $\varphi_i^0(||(f_1,\ f_2)||_\infty)$. A similar argument shows that

$$||g^\lambda(\hat{\varphi}_1^\lambda(v_1^\lambda) - \hat{\varphi}_2^\lambda(v_2^\lambda))||_{L^\infty(\Omega)} \leq g(||\hat{\varphi}_1^\lambda(v_1^\lambda)||_{L^\infty(\Omega)} + ||\hat{\varphi}_2^\lambda(v_2^\lambda)||_{L^\infty(\Omega)})$$

and

$$||\mathcal{R}_k g^\lambda(\hat{\varphi}_1^\lambda(v_1^\lambda) - \hat{\varphi}_2^\lambda(v_2^\lambda))||_{L^\infty(\Omega)} \quad \leq \quad M_2 \tag{4}$$

where M_2 is a constant depending only on Ω, g, M_1 and $||\mathcal{R}_k||_\infty$. Now, taking $w_i^\lambda = \hat{\varphi}_i^\lambda(v_i^\lambda)$ as test function in the definition of solution of $S^k(f_1,\ f_2)$, we get

$$d_i \int_\Omega |Dw_i^\lambda|^2 + (-1)^{i-1} \int_\Omega \mathcal{R}_k(x) g^\lambda(w_1^\lambda - w_2^\lambda) w_i^\lambda + \int_\Gamma z_i^\lambda w_i^\lambda = \int_\Omega (f_i - v_i^\lambda) w_i^\lambda$$

where $z_i^\lambda = \gamma_i^\lambda(w_i^\lambda)$, so that by using the fact that γ_i^λ and $\hat{\varphi}_i^\lambda$ are nondecreasing, we deduce that

$$\begin{aligned} d_i \int_\Omega |Dw_i^\lambda|^2 \quad &\leq \ (-1)^i \int_\Omega \mathcal{R}_k(x) g^\lambda(w_1^\lambda - w_2^\lambda) w_i^\lambda + \int_\Omega f_i w_i^\lambda \\ &\leq \ ||\mathcal{R}_k g^\lambda(w_1^\lambda - w_2^\lambda)||_{L^\infty(\Omega)} ||w_i^\lambda||_{L^1(\Omega)} + ||f_i||_{L^1(\Omega)} ||w_i^\lambda||_{L^\infty(\Omega)} \\ &\leq M_3 \end{aligned} \tag{5}$$

with M_3 is independent of λ. Therefore $(\hat{\varphi}_1^\lambda(v_1^\lambda),\ \hat{\varphi}_2^\lambda(v_2^\lambda))$ is bounded in $H^1(\Omega) \times H^1(\Omega)$, hence weakly sequentially compact in $L^2(\Omega) \times L^2(\Omega)$. On the other hand, by Lemma 4.4 (see the appendix) we have

$$\begin{aligned} &\int_{\Omega''} \psi(x)(|v_1^\lambda(x+y) - v_1^\lambda(x)| + |v_2^\lambda(x+y) - v_2^\lambda(x)|) \\ &\leq max(d_1,\ d_2) \int_{\Omega'} |\Delta\psi(x)|(|w_1^\lambda(x+y) - w_1^\lambda(x)| + |w_2^\lambda(x+y) - w_2^\lambda(x)|) + \\ &\int_{\Omega'} \psi(x)(|f_1(x+y) - f_1(x)| + |f_2(x+y) - f_2(x)|) \\ &+2 \int_{\Omega'} \psi(x)(|\mathcal{R}_k(x+y) - \mathcal{R}_k(x)||g^\lambda(w_1^\lambda - w_2^\lambda)|) \end{aligned}$$

where Ω' and Ω'' are open subsets of Ω such that $\overline{\Omega'} \subseteq \Omega$, $\overline{\Omega''} \subseteq \Omega'$.

Using (5) and the fact that $\mathcal{R}_k \in BV(\Omega)$, we obtain

$$\limsup_{y\to 0\, \lambda>0} \int_{\Omega''} (|v_1^\lambda(x+y) - v_1^\lambda(x)| + |v_2^\lambda(x+y) - v_2^\lambda(x)|) = 0. \tag{6}$$

Combining (2) and (6), then we deduce that $(v_1^\lambda, v_2^\lambda)$ is precompact in $L^1(\Omega)\times L^1(\Omega)$. Moreover, by (4) we have $g^\lambda\Big(\hat\varphi_1^\lambda(v_1^\lambda)-\hat\varphi_2^\lambda(v_2^\lambda)\Big)$ is weakly sequentially compact in $L^1(\Omega)$. Then, there exists $\lambda_n\to 0$, such that

$$\begin{cases} (v_1^{\lambda_n},\, v_2^{\lambda_n}) \;\to\; (v_1, v_2) \quad \text{strongly in } L^1(\Omega)\times L^1(\Omega)\\[2mm] (\hat\varphi_1^{\lambda_n}(v_1^{\lambda_n}), \hat\varphi_2^{\lambda_n}(v_2^{\lambda_n})) \to (w_1, w_2) \quad \text{strongly in } \ L^2(\Omega)\times L^2(\Omega)\\[2mm] (z_1^{\lambda_n}, z_2^{\lambda_n}) \to (z_1, z_2) \quad \text{weakly in } \ L^2(\Gamma)\times L^2(\Gamma)\\[2mm] g^{\lambda_n}(\hat\varphi_1^{\lambda_n}(v_1^{\lambda_n}) - \hat\varphi_{n2}^{\lambda_n}(v_2^{\lambda_n})) \to h \quad \text{weakly in } \ L^1(\Omega). \end{cases}$$

Then, by using a standard monotonicity argument, we deduce that $h = g(w_1 - w_2)$ and for $i = 1, 2$ $w_i = \varphi_i(v_i)$ and $z_i \in \gamma_i(w_i)$. Finally, passing to the limit in

$$d_i \int_\Omega Dw_i^\lambda D\xi_i + (-1)^{i-1}\int_\Omega \mathcal{R}_k(x) g^\lambda(w_1^\lambda - w_2^\lambda)\xi_i + \int_\Gamma z_i^\lambda \xi_i = \int_\Omega (f_i - v_i^\lambda)\xi_i$$

we obtain (1), and the first part of the proof is complete.
Now, to finish the proof we prove the estimates 1) and 2). Passing to the limit in (2), we obtain 1).
To prove 2), we take w_i^k as test function in (1), then we have

$$d_i \int_\Omega |Dw_i^k|^2 + (-1)^{i-1}\int_\Omega \mathcal{R}_k(x) g(w_1^k - w_2^k) w_i^k + \int_\Gamma z_i^k w_i^k = \int_\Omega (f_i - v_i^k) w_i^k.$$

So we have γ_i and φ_i are monotones, whence it follows that $v_i^k w_i^k \geq 0$ and $z_i^k w_i^k \geq 0$. Summing the equation satisfied for $i = 1$ and the equation satisfied for $i = 2$ we deduce

$$d_1 \int_\Omega |Dw_1^k|^2 + d_2 \int_\Omega |Dw_2^k|^2 + \int_\Omega \mathcal{R}_k(x) g(w_1^k - w_2^k)(w_1^k - w_2^k) \leq \int_\Omega (f_1 w_1^k + f_2 w_2^k)$$

whence it follows that

$$d_1 \int_\Omega |Dw_1^k|^2 + d_2 \int_\Omega |Dw_2^k|^2 \leq ||f_1||_{L^1(\Omega)}\varphi_1^0(||v_1^k||_{L^\infty(\Omega)}) + ||f_2||_{L^1(\Omega)}\varphi_2^0(||v_2^k||_{L^\infty(\Omega)}).$$

This ends up the proof of proposition. ■

Proposition 2.3 *Let $(f_1,\ f_2)\in L^\infty(\Omega)\times L^\infty(\Omega)$ and $(v_1^k,\ v_2^k)$ the solution of $S^k(f_1,\ f_2)$. Then, as $k\longrightarrow +\infty$, we have, for $i = 1, 2$ $v_i^k \longrightarrow v_i$, in $L^1(\Omega)$, $z_i^k \longrightarrow z_i$, in $L^1(\Gamma)$-weak and $w_i^k \longrightarrow w_i$ in $H^1(\Omega)$-weak. The triplet $(v_i, w_i, z_i) \in \Big(L^1(\Omega), H^1(\Omega), L^1(\Gamma)\Big)$ satisfies:*

$w_i = \varphi_i(v_i)$ a.e. Ω, $z_i \in \gamma_i(w_i)$ a.e. Γ, $w_1 = w_2$ a.e. Ω_0, and

$$\begin{aligned}&\int_\Omega v_1(w_1-\xi_1)+\int_\Omega v_2(w_2-\xi_2)+d_1\int_\Omega Dw_1D(w_1-\xi_1)+d_2\int_\Omega Dw_2D(w_2-\xi_2)\\&+\int_{\Omega_1}\mathcal{R}_1(x)g(w_1-w_2)\Big((w_1-\xi_1)-(w_2-\xi_2)\Big)+\int_\Gamma z_1(w_1-\xi_1)\\&+\int_\Gamma z_2(w_2-\xi_2)\le\int_\Omega f_1(w_1-\xi_1)+\int_\Omega f_2(w_2-\xi_2)\end{aligned}\tag{7}$$

for any $\xi := (\xi_1, \xi_2) \in \Big(H^1(\Omega) \times H^1(\Omega)\Big) \cap \mathcal{K}$. Moreover, the couple (v_1, v_2) is unique.

First, we prove the following lemma.

Lemma 2.4 *For any a, b, $, \hat{a}$, $\hat{b} \in \mathbb{R}$ we have*

$$\Big(g(a-b)\ -\ g(\hat{a}-\hat{b})\Big)\Big(T_l(a-\hat{a})-T_l(b-\hat{b})\Big)\ \geq\ 0\qquad\forall\, l>0.$$

Proof. Let $l > 0$ and we set $I_l := \Big(g(a-b)\ -\ g(\hat{a}-\hat{b})\Big)\Big(T_l(a-\hat{a})-T_l(b-\hat{b})\Big)$ then if $a - \hat{a} \geq l$ and $b - \hat{b} \leq -l$ we have $I_l = 2l\Big(g(a-b)\ -\ g(\hat{a}-\hat{b})\Big) \geq 0$.
If $a - \hat{a} \geq l$ and $-l \leq b - \hat{b} \leq l$ then $I_l = \Big(g(a-b)\ -\ g(\hat{a}-\hat{b})\Big)\underbrace{\Big(l-(b-\hat{b})\Big)}_{\geq 0} \geq 0$.

If $a - \hat{a} \leq -l$ and $-l \leq b - \hat{b} \leq l$ then $I_l = \Big(g(a-b)\ -\ g(\hat{a}-\hat{b})\Big)\underbrace{\Big(-l-(b-\hat{b})\Big)}_{\leq 0} \geq 0$.

If $a - \hat{a}$ and $b - \hat{b}$ are on the same area of T_l then $I_l = 0$. The remaining cases result by permuting a with $-\hat{a}$ and b with $-\hat{b}$. ■

Proof of Proposition 2.3. Convergence: Let $(f_1, f_2) \in L^\infty(\Omega)\times L^\infty(\Omega)$ and (v_1^k, v_2^k) the

solution of $S^k(f_1, f_2)$, then for $i = 1, 2$ we have $v_i^k \in L^\infty(\Omega)$, $w_i^k \in H^1(\Omega)$ and $z_i^k \in L^2(\Gamma)$. On the other hand, using Proposition 2.1, we deduce that w_i^k is bounded in $H^1(\Omega)$, v_i^k is bounded in $L^\infty(\Omega)$ and $v_i^k = \varphi_i^{-1}(w_i^k)$ then by choosing a subsequence v_i^k (respectively w_i^k) if necessary, such that as k goes to $+\infty$, we have v_i^k converges strongly to v_i, in $L^1(\Omega)$ and w_i^k converges to w_i, weakly in $H^1(\Omega)$ and strongly in $L^2(\Omega)$. Using (H_1), we have z_i^k is bounded in $L^\infty(\Gamma)$, therefore $z_i^k \rightharpoonup z_i$, in $L^1(\Gamma)$. Now, by the Dominate Convergence Theorem,

$$g(w_1^k - w_2^k) \longrightarrow g(w_1 - w_2),\ \text{ in }\ L^1(\Omega).$$

On the other hand, we multiply the equation (1) by $[a(k)]^{-1}$ and we let $k \to +\infty$, we get $\lim_{k\to\infty}\int_{\Omega_0} g(w_1^k - w_2^k)\xi_i = 0$ for all $\xi_i \in H^1(\Omega)$. Since $g(w_1^k - w_2^k)$ converge to $g(w_1 - w_2)$ in $L^1(\Omega)$, we deduce that $|g(w_1 - w_2)|_{L^2(\Omega_0)} = 0$; hence $g(w_1 - w_2) = 0$ and $w_1 = w_2$ a.e. Ω_0. Now, taking $(w_i^k - \xi_i)$ as test function in (1) and adding both expressions, we obtain

$$
\begin{aligned}
&\int_\Omega v_1^k(w_1^k-\xi_1)+\int_\Omega v_2^k(w_2^k-\xi_2)+d_1\int_\Omega Dw_1^kD(w_1^k-\xi_1)\\
&+d_2\int_\Omega Dw_2^kD(w_2^k-\xi_2)+\int_\Gamma z_1^k(w_1^k-\xi_1)+\int_\Gamma z_2^k(w_2^k-\xi_2)\\
&+\int_\Omega \Theta^k(x)=\int_\Omega f_1(w_1^k-\xi_1)+f_2(w_2^k-\xi_2)
\end{aligned}
\tag{8}
$$

for any $\xi_1,\xi_2\in H^1(\Omega)$, where $\Theta^k(x)=\mathcal{R}_k(x)g(w_1^k-w_2^k)\Big((w_1^k-\xi_1)-(w_2^k-\xi_2)\Big)$. Assume now, that $\xi:=(\xi_1,\,\xi_2)\in\mathcal{K}$ and we decompose the integral of Θ^k in the following way

$$
\int_\Omega \Theta^k(x)=\int_{\Omega_0}\Theta^k(x)+\int_{\Omega_1}\Theta^k(x). \tag{9}
$$

It is not difficult to see that $\displaystyle\int_{\Omega_0}\Theta^k(x)\geq 0$, so that $\displaystyle\int_\Omega\Theta^k(x)\geq\int_{\Omega_1}\Theta^k(x)$ and

$$
\begin{aligned}
&\int_\Omega v_1^k(w_1^k-\xi_1)+\int_\Omega v_2^k(w_2^k-\xi_2)+d_1\int_\Omega Dw_1^kD(w_1^k-\xi_1)+d_2\int_\Omega Dw_2^kD(w_2^k-\xi_2)\\
&+\int_{\Omega_1}\mathcal{R}_1(x)g(w_1^k-w_2^k)\Big((w_1^k-\xi_1)-(w_2^k-\xi_2)\Big)+\int_\Gamma z_1^k(w_1^k-\xi_1)\\
&+\int_\Gamma z_2^k(w_2^k-\xi_2)\leq\int_\Omega f_1(w_1^k-\xi_1)+f_2(w_2^k-\xi_2)
\end{aligned}
\tag{10}
$$

for any $(\xi_1,\xi_2)\in\Big(H^1(\Omega)\times H^1(\Omega)\Big)\cap\mathcal{K}$. Now, letting $k\to\infty$ and using Fatou Lemma, we deduce that

$$
\begin{aligned}
&\int_\Omega v_1(w_1-\xi_1)+\int_\Omega v_2(w_2-\xi_2)+d_1\int_\Omega Dw_1D(w_1-\xi_1)+d_2\int_\Omega Dw_2D(w_2-\xi_2)\\
&+\int_{\Omega_1}\mathcal{R}_1(x)g(w_1-w_2)\Big((w_1-\xi_1)-(w_2-\xi_2)\Big)+\int_\Gamma z_1(w_1-\xi_1)\\
&+\int_\Gamma z_2(w_2-\xi_2)\leq\int_\Omega f_1(w_1-\xi_1)+\int_\Omega f_2(w_2-\xi_2)
\end{aligned}
$$

which end up the proof of existence.

Uniqueness: Let $V:=(v_1,\,v_2)$ and $\tilde V:=(\tilde v_1,\,\tilde v_2)$ be two solutions respectively in the sense of (7). Taking $\xi_i=\tilde w_i-T_l(w_i-\tilde w_i)$ for $i=1,2$; as a test function in the inequalities satisfied by $(v_1,\,v_2)$, we get

$$
\begin{aligned}
&\int_\Omega v_1T_l(w_1-\tilde w_1)+d_1\int_\Omega Dw_1DT_l(w_1-\tilde w_1)+\int_\Gamma z_1T_l(w_1-\tilde w_1)\\
&+\int_\Omega v_2T_l(w_2-\tilde w_2)+d_2\int_\Omega Dw_2DT_l(w_2-\tilde w_2)+\int_\Gamma z_2T_l(w_2-\tilde w_2)\\
&+\int_{\Omega_1}\mathcal{R}_1(x)g(w_1-w_2)(T_l(w_1-\tilde w_1)-T_l(w_2-\tilde w_2))\\
&\leq\int_\Omega f_1T_l(w_1-\tilde w_1)+f_2T_l(w_2-\tilde w_2)\qquad\forall l>0.
\end{aligned}
\tag{11}
$$

Now, taking $\xi_i = w_i - T_l(\tilde{w}_i - w_i)$ for $i = 1, 2$; as test function in the inequalities satisfied by $(\tilde{v}_1,\ \tilde{v}_2)$

$$\begin{aligned}
&\int_\Omega \tilde{v}_1 T_l(\tilde{w}_1 - w_1) + d_1 \int_\Omega D\tilde{w}_1 DT_l(\tilde{w}_1 - w_1) + \int_\Gamma \tilde{z}_1 T_l(\tilde{w}_1 - w_1) \\
&+ \int_\Omega \tilde{v}_2 T_l(\tilde{w}_2 - w_2) + d_2 \int_\Omega D\tilde{w}_2 DT_l(\tilde{w}_2 - w_2) + \int_\Gamma \tilde{z}_2 T_l(\tilde{w}_2 - w_2) \\
&+ \int_{\Omega_1} \mathcal{R}_1(x) g(\tilde{w}_1 - \tilde{w}_2)\Big(T_l(\tilde{w}_1 - w_1) - T_l(\tilde{w}_2 - w_2)\Big) \\
&\leq \int_\Omega \tilde{f}_1 T_l(\tilde{w}_1 - w_1) + \tilde{f}_2 T_l(\tilde{w}_2 - w_2) \qquad \forall l > 0.
\end{aligned} \tag{12}$$

Adding the inequalities (11) and (12), we get

$$\begin{aligned}
&\int_\Omega (v_1 - \tilde{v}_1) T_l(w_1 - \tilde{w}_1) + \int_\Omega (v_2 - \tilde{v}_2) T_l(w_2 - \tilde{w}_2) \\
&+ d_1 \int_\Omega D(w_1 - \tilde{w}_1) DT_l(w_1 - \tilde{w}_1) + d_2 \int_\Omega D(w_2 - \tilde{w}_2) DT_l(w_2 - \tilde{w}_2) \\
&+ \int_\Gamma (z_1 - \tilde{z}_1) T_l(w_1 - \tilde{w}_1) + \int_\Gamma (z_2 - \tilde{z}_2) T_l(w_2 - \tilde{w}_2) \\
&+ \int_{\Omega_1} \tilde{\Theta}_l(x) \leq (f_1 - \tilde{f}_1) T_l(w_1 - \tilde{w}_1) + (f_2 - \tilde{f}_2) T_l(w_2 - \tilde{w}_2) \qquad \forall l > 0,
\end{aligned} \tag{13}$$

where

$$\tilde{\Theta}_l(x) = \mathcal{R}_1(x)\Big(g(w_1 - w_2) - g(\tilde{w}_1 - \tilde{w}_2)\Big)\Big(T_l(w_1 - \tilde{w}_1) - T_l(w_2 - \tilde{w}_2)\Big).$$

Since γ_1, γ_2 are monotone then $(z_i - \tilde{z}_i) T_l(w_i - \tilde{w}_i) \geq 0$ for $i = 1, 2$; and by lemma 2.4, we get $\tilde{\Theta}_l(x) \geq 0$ a.e. $x \in \Omega_1$. We multiply (13) by $\dfrac{1}{l}$ and letting $l \to 0$, we obtain

$$||v_1 - \tilde{v}_1||_{L^1(\Omega)} + ||v_2 - \tilde{v}_2||_{L^1(\Omega)} \leq ||f_1 - \tilde{f}_1||_{L^1(\Omega)} + ||f_2 - \tilde{f}_2||_{L^1(\Omega)},$$

and the proof of the Proposition concludes. ∎

3 The evolution problem

In order to rewrite (P^k) in an abstract form, let $X = L^1(\Omega) \times L^1(\Omega)$ a Banach space endowed with the natural norm $|(u_1, u_2)|_X = ||u_1||_{L^1(\Omega)} + ||u_2||_{L^1(\Omega)}$, for any $(u_1,\ u_2) \in X$. By considering $U = (u_1,\ u_2)$ as map from $[0, T]$ into X, the abstract Cauchy problem associated to (P^k) is

$$CP^k(U_0, f_1, f_2) \begin{cases} U_t + \mathcal{A}^k U \ni (f_1, f_2) & \text{in } (0, T) \\ \\ U(0) := U_0 = (u_{01},\ u_{02}), \end{cases}$$

where $\mathcal{A}^k$ is the operator in X defined by:

$$F \in \mathcal{A}^k U \Leftrightarrow F := (f_1,\ f_2) \in X,\ U := (u_1,\ u_2) \in X,\ f_1 \in \mathcal{A}^k_{g,1} u_1 \ \text{ and } \ f_2 \in \mathcal{A}^k_{g,2} u_2;$$

where for $i = 1, 2$; the operators $\mathcal{A}^k_{g,i}$ are defined by

$$f_i \in \mathcal{A}^k_{g,i} u_i \Leftrightarrow \begin{cases} u_i,\ f_i \in L^1(\Omega),\ \exists w_i \in W^{1,1}(\Omega),\ \exists z_i \in L^1(\Gamma) \\ w_i = \varphi_i(u_i) \ \text{ a.e. } \ \Omega,\ z_i \in \gamma_i(w_i) \ \text{ a.e.}\,\Gamma, \ \text{ and} \\ d_i \displaystyle\int_\Omega Dw_i D\xi_i + (-1)^{i-1} \int_\Omega \mathcal{R}_k(x) g(w_1 - w_2)\xi_i + \int_\Gamma z_i \xi_i = \int_\Omega f_i \xi_i \\ \text{ for any } \ \xi_i \in W^{1,\infty}(\Omega). \end{cases}$$

We equipped X with the usual partial ordering $(f_1,\ f_2) \le (g_1,\ g_2)$ if and only if $f_1 \le g_1$ and $f_2 \le g_2$ a.e. in Ω.

Remark 3.1 *Thanks to [13] the operator $\mathcal{A}^k$ is T-accretive in X, moreover the relation between the resolvents of the operators $\mathcal{A}^k$, $\mathcal{A}_{0,1}$ and $\mathcal{A}_{0,2}$ is: $(v_1, v_2) = (I + \varepsilon\mathcal{A}^k)^{-1}(f_1,\ f_2)$ if and only if for $i = 1, 2$; $v_i = (I + \varepsilon\mathcal{A}_{0,i})^{-1}(f_i - (-1)^{i-1}\varepsilon\mathcal{R}_k(.)g(w_1 - w_2))$.*

Remark 3.2 *The operator $\overline{\mathcal{A}^k}$ is m-T-accretive in X. Indeed, from the Remark 3.1, we have $\mathcal{A}^k$ is T-accretive in X then $\overline{\mathcal{A}^k}$ is T-accretive, and by the Proposition 2.1 we have $L^\infty(\Omega) \times L^\infty(\Omega) \subseteq \mathcal{R}(I + \lambda\mathcal{A}^k)$ so by density we obtain $\mathcal{R}(I + \lambda\overline{\mathcal{A}^k}) = X$ for all $\lambda > 0$.*

3.1 Existence of solution for (P^k)

Using the general theory of nonlinear semigroups and thanks to remark 3.2, for any $f_1, f_2 \in L^1(Q)$ and $u_{01},\ u_{02} \in L^1(\Omega)$ the problem $CP^k(U_0, f_1, f_2)$ has a unique *mild solution.* Let us denote it by $(u_1,\ u_2)$ and show that is also a weak solution of (P^k), thus obtaining.

Proposition 3.3 *Let $(u_{01},\ u_{02}) \in L^\infty(\Omega) \times L^\infty(\Omega)$ and $(f_1,\ f_2) \in L^\infty(Q) \times L^\infty(Q)$, then $(u_1,\ u_2)$ is a weak solution in the sense: for $i = 1,\ 2$; $u_i \in \mathcal{C}\Big([0,\ T];\ \ L^1(\Omega)\Big) \cap L^\infty(Q)$, there exists $w_i \in L^2\Big(0,\ T;\ \ H^1(\Omega)\Big)$ and $z_i \in L^2(\Sigma)$ such that $w_i = \varphi_i(u_i)$ a.e. in Q, $z_i \in \gamma_i(w_i)$ a.e. in Σ, and*

$$\begin{aligned} d_1 \int\!\!\int_Q Dw_1 D\xi_1 + \int\!\!\int_Q \mathcal{R}_k(x) g(w_1 - w_2)\xi_1 + \int\!\!\int_\Sigma z_1\xi_1 &= \int\!\!\int_Q u_1 \xi_{1t} \\ &\quad + \int_\Omega u_{01}\xi_1(0) + \int_Q f_1\xi_1 \\ d_2 \int\!\!\int_Q Dw_2 D\xi_2 - \int\!\!\int_Q \mathcal{R}_k(x) g(w_1 - w_2)\xi_2 + \int\!\!\int_\Sigma z_2\xi_2 &= \int\!\!\int_Q u_2 \xi_{2t} \\ &\quad + \int_\Omega u_{02}\xi_2(0) + \int_Q f_2\xi_2, \end{aligned}$$

for any $\xi_1,\, \xi_2 \in \mathcal{C}^1\Big([0,\,T]\times\overline{\Omega}\Big)$ *such that* $\xi_1(.,T) = \xi_2(.,T) \equiv 0$. *Moreover, for any* $t \geq 0$, *we have*

1. $||(u_1(t),\, u_2(t))||_\infty \leq ||(u_{01},\, u_{02})||_\infty + T||(f_1,\, f_2)||_\infty$
2. $\displaystyle\int_0^t \int_\Omega (|Dw_1^k|^2 + |Dw_2^k|^2) \leq C$

where $||(u_1,\, u_2)||_\infty = \sup\Big(||u_1||_{L^\infty(Q)},\, ||u_2||_{L^\infty(Q)}\Big)$ *and* C *is a constant independent of* k.

Proof. Let $(u_1^\epsilon,\, u_2^\epsilon)$ be the ϵ- approximate solution, and for $\epsilon = t/n$, $i = 1, ..., n$, we have

$$\left\{\begin{array}{ll} u_1^i - \epsilon d_1 \triangle w_1^i + \epsilon \mathcal{R}_k(x) g(w_1^i - w_2^i) = u_1^{i-1} + \epsilon f_1^i \quad w_1^i = \varphi_1(u_1^i) & \text{in } \Omega \\ u_2^i - \epsilon d_2 \triangle w_2^i - \epsilon \mathcal{R}_k(x) g(w_1^i - w_2^i) = u_2^{i-1} + \epsilon f_2^i \quad w_2^i = \varphi_2(u_2^i) & \text{in } \Omega \\ \partial_{\vec{n}} w_1^i + z_1^i = 0 \quad z_1^i \in \gamma_1(w_1^i) & \text{in } \Gamma \\ \partial_{\vec{n}} w_2^i + z_2^i = 0, \quad z_2^i \in \gamma_2(w_2^i) & \text{in } \Gamma. \end{array}\right. \tag{14}$$

For $q = 1, 2$, the function u_q^ϵ is given by $u_q^\epsilon(t) = u_q^i$, for $t \in]t_{i-1}, t_i]$, $u_q^\epsilon(0) = u_{0q}$ and $f_q^1, ..., f_q^n \in L^\infty(\Omega)$ with $\displaystyle\sum_{i=1}^n \int_{t_{i-1}}^{t_i} ||f_q(t) - f_q^i||_{L^1(\Omega)} \leq \varepsilon$. Then, by Proposition 2.1, we have $\quad ||(u_1^i,\, u_2^i)||_\infty \leq ||(u_{01},\, u_{02})||_\infty + \epsilon \displaystyle\sum_{j=1}^N ||(f_1^j,\, f_2^j)||_\infty$, hence

$$||(u_1^\epsilon,\, u_2^\epsilon)||_\infty \leq ||(u_{01},\, u_{02})||_\infty + \int_0^T ||(f_1,\, f_2)||_\infty. \tag{15}$$

On the other hand, we know that $w_q^\epsilon = \varphi_q(u_q^\epsilon)$ then

$$||w_q^\epsilon||_{L^\infty(\Omega)} \leq \varphi_q(||(u_1^\epsilon,\, u_2^\epsilon)||_\infty) := M' \tag{16}$$

where M' is a constant independent of ϵ. Now, we take w_1^i (respectively w_2^i) as test function in the first (respectively the second) equation of (14) and we sum the equations obtained, we find

$$\begin{aligned} \int_\Omega (u_1^i - u_1^{i-1}) w_1^i + \int_\Omega (u_2^i - u_2^{i-1}) w_2^i + \epsilon d_1 \int_\Omega |Dw_1^i|^2 + \epsilon d_2 \int_\Omega |Dw_2^i|^2 + \epsilon \int_\Gamma z_1^i w_1^i \\ + \epsilon \int_\Gamma z_2^i w_2^i + \epsilon \int_\Omega \mathcal{R}_k(x) g(w_1^i - w_2^i)(w_1^i - w_2^i) = \epsilon \int_\Omega f_1^i w_1^i + \epsilon \int_\Omega f_2^i w_2^i. \end{aligned} \tag{17}$$

Let $j_q : \mathbb{R} \to [0, \infty]$ be a convex l.s.c function such that $j_q(r) = \displaystyle\int_0^r \varphi_q(s)ds$, then we have

$$\int_\Omega (u_q^{i-1} - u_q^i)w_q^i \le \int_\Omega (j_q(u_q^{i-1}) - j_q(u_q^i)),$$

and, by using the monotonicity of φ_q, γ_q and g in (17), we see that

$$\begin{aligned} \int_\Omega j_1(u_1^i) + \int_\Omega j_2(u_2^i) + \epsilon d_1 \int_\Omega |Dw_1^i|^2 + \epsilon d_2 \int_\Omega |Dw_2^i|^2 \le \int_\Omega j_1(u_1^{i-1}) \\ + \int_\Omega j_2(u_2^{i-1}) + \epsilon \int_\Omega f_1^i w_1^i + \epsilon \int_\Omega f_2^i w_2^i. \end{aligned} \tag{18}$$

Summing (18) for $i = 1$ to n then

$$\begin{aligned} \int_\Omega j_1(u_1^\epsilon) + \int_\Omega j_2(u_2^\epsilon) + d_1 \int_0^t \int_\Omega |Dw_1^\epsilon|^2 + d_2 \int_0^t \int_\Omega |Dw_2^\epsilon|^2 \le \int_\Omega j_1(u_{01}) \\ + \int_\Omega j_2(u_{02}) + \int_0^t \int_\Omega f_1 w_1^\epsilon + \int_0^t \int_\Omega f_2 w_2^\epsilon \end{aligned} \tag{19}$$

and since $j_1, j_2 \ge 0$, it is easy to see that

$$min(d_1, d_2) \int_0^t \int_\Omega |Dw_1^\epsilon|^2 + |Dw_2^\epsilon|^2 \le \int_\Omega (j_1(u_{01}) + j_2(u_{02})) + ||w_1^\epsilon||_\infty ||f_1||_\infty + ||w_2^\epsilon||_\infty ||f_2||_\infty. \tag{20}$$

It follows from (15) and (20) that w_q^ϵ is bounded in $L^2\Big(0,\ T;\ H^1(\Omega)\Big)$, then there exists $w_q \in L^2\Big(0,\ T;\ H^1(\Omega)\Big)$ such that $w_q^\epsilon \to w_q$ weakly in $L^2\Big(0,\ T;\ H^1(\Omega)\Big)$. On the other hand, since $u_q^\epsilon \to u_q$ in $\mathcal{C}\Big([0,\ T];\ L^1(\Omega)\Big)$, w_q^ϵ is bounded in $L^\infty(Q)$ and φ_q is continuous, then by the Dominate Convergence Theorem, we have $w_q^\epsilon \to w_q$ in $L^1(Q)$. By using (H_1) we have $z_q^\epsilon \in \gamma_q(w_q^\epsilon)$ is bounded in $L^\infty(\Sigma)$, so that there exists $z_q \in L^\infty(\Sigma)$ such that $z_q^\epsilon \to z_q$ weakly in $L^2(\Sigma)$. Using the monotonicity argument (cf. Lemma G [6]) we deduce that $w_q = \varphi_q(u_q),\ z_q \in \gamma_q(w_q)$ a.e. in Q. Now, Let $\tilde{u}_q^\epsilon$ be the function defined from $[0, T]$ to $L^1(\Omega)$, by $\tilde{u}_q^\epsilon(t_i) = u_q^i$, and $\tilde{u}_q^\epsilon$ defined by (14) then

$$\begin{aligned} d_1 \int\int_Q Dw_1^\epsilon D\xi_1 + \int\int_Q \mathcal{R}_k(x)g(w_1^\epsilon - w_2^\epsilon)\xi_1 + \int\int_\Sigma z_1^\epsilon \xi_1 = \int\int_Q \tilde{u}_1^\epsilon \xi_{1t} \\ + \int_\Omega u_{01}\xi_1(0) + \int\int_\Omega f_1^\epsilon \xi_1 \\ d_2 \int\int_Q Dw_2^\epsilon D\xi_2 - \int\int_Q \mathcal{R}_k(x)g(w_1^\epsilon - w_2^\epsilon)\xi_2 + \int\int_\Sigma z_2^\epsilon \xi_2 = \int\int_Q \tilde{u}_2^\epsilon \xi_{2t} \\ + \int_\Omega u_{02}\xi_2(0) + \int\int_\Omega f_1^\epsilon \xi_1 \end{aligned}$$

for any $\xi_1,\, \xi_2 \in \mathcal{C}^1((0,T]\times\overline{\Omega})$ such that $\xi_1(.,T)=\xi_2(.,T)\equiv 0$. Taking a limit as ϵ goes to 0 in (15), (20) and the preceding equalities, we obtain respectively 1) , 2) and

$$\begin{aligned} d_1 \int\int_Q Dw_1 D\xi_1 + \int\int_Q \mathcal{R}_k(x) g(w_1-w_2)\xi_1 + \int\int_\Sigma z_1\xi_1 &= \int\int_Q u_1\xi_{1t} \\ &+ \int_\Omega u_{01}\xi_1(0) + \int\int_\Omega f_1\xi_1 \\ d_2 \int\int_Q Dw_2 D\xi_2 - \int\int_Q \mathcal{R}_k(x) g(w_1-w_2)\xi_2 + \int\int_\Sigma z_2\xi_2 &= \int\int_Q u_2\xi_{2t} \\ &+ \int_\Omega u_{02}\xi_2(0) + \int\int_\Omega f_1\xi_1 \end{aligned}$$

for any $\xi_1,\, \xi_2 \in \mathcal{C}^1((0,T]\times\overline{\Omega})$ such that $\xi_1(.,T)=\xi_2(.,T)\equiv 0$. And the proof finished. ∎

3.2 The limit problem

As an immediate consequence of proposition 2.3, we have the following

Corollary 3.4 *As $k \longrightarrow \infty$, the operator $\mathcal{A}^k$ converges in X, in the sense of resolvent, to the T-accretive operator $\mathcal{A}_\infty$ defined, by*

$$(f_1,f_2)\in\mathcal{A}_\infty(v_1,v_2) \Leftrightarrow \begin{cases} \textit{For } i=1,\,2;\ v_i\in L^1(\Omega),\ w_i:=\varphi(v_i)\in H^1(\Omega), \\ \exists z_i\in L^1(\Gamma),\, z_i\in\gamma(w_i) \textit{ a.e. in } \Gamma, \textit{ and} \\ d_1\int_\Omega Dw_1D(w_1-\xi_1) + d_2\int_\Omega Dw_2D(w_2-\xi_2) \\ +\int_\Gamma z_1(w_1-\xi_1)+\int_\Gamma z_2(w_2-\xi_2)+ \\ \int_{\Omega_1}\mathcal{R}_1(x)g(w_1-w_2)\Big((w_1-\xi_1)-(w_2-\xi_2)\Big) \\ \leq \int_\Omega f_1(w_1-\xi_1)+\int_\Omega f_2(w_2-\xi_2) \\ \textit{for any } \xi=(\xi_1,\xi_2)\in\Big(H^1(\Omega)\times H^1(\Omega)\Big)\cap\mathcal{K}. \end{cases} \tag{21}$$

In particular, we deduce that $\mathcal{R}(I+\varepsilon\mathcal{A}_\infty)\supseteq L^\infty(\Omega)\times L^\infty(\Omega)$. This implies that $\overline{\mathcal{A}_\infty}$ is m-T- accretive and $\mathcal{A}_\infty$ generates a nonlinear semigroup of order preserving contractions in X. Moreover we have

Proposition 3.5 $\overline{\mathcal{D}(\mathcal{A}_\infty)} = \{(u_1,u_2)\in X,\ \textit{ such that } \varphi_1(u_1)=\varphi_2(u_2),\ \textit{a.e. in } \Omega_0\} := Y$

In order to prove this proposition, let us prove the following

Lemma 3.6 *Let* $(f_1,\ f_2) \in L^\infty(\Omega) \times L^\infty(\Omega)$ *and* $(v_1,\ v_2)$ *be the solution of the problem* $S^\infty(f_1,\ f_2)$ *in the sense of Proposition 2.3. Then* $(v_1,\ v_2)$ *is a weak solution of*

$$\begin{cases} v_1 - d_1 \triangle w_1 + \mathcal{R}_1(x) g(w_1 - w_2) = f_1 & in \ \ \mathcal{D}'(\Omega_1) \\ v_2 - d_2 \triangle w_2 - \mathcal{R}_1(x) g(w_1 - w_2) = f_2 & in \ \ \mathcal{D}'(\Omega_1) \\ (v_1 + v_2) - (d_1 + d_2) \triangle w = f_1 + f_2, \quad w_1 = w_2 := w & in \ \ \mathcal{D}'(\Omega_0) \end{cases}$$

Proof. Thanks to Proposition 2.3, there exists (v_1, v_2) solution of the problem $S^\infty(f_1,\ f_2)$ in the sense of (7). It is clear that, for any $\xi \in H_0^1(\Omega_1) \cap H_0^1(\Omega)$ the couple $(w_1 \pm \xi,\ w_2)$ is an admissible test function in (7), so

$$\int_{\Omega_1} v_1 \xi + d_1 \int_{\Omega_1} Dw_1 D\xi + \int_{\Omega_1} \mathcal{R}_1(x) g(w_1 - w_2)\xi = \int_{\Omega_1} f_1 \xi. \tag{22}$$

Similarly, we get

$$\int_{\Omega_1} v_2 \xi + d_2 \int_{\Omega_1} Dw_2 D\xi - \int_{\Omega_1} \mathcal{R}_1(x) g(w_1 - w_2)\xi = \int_{\Omega_1} f_2 \xi. \tag{23}$$

Now, let $\xi \in H_0^1(\Omega)$ be such that $\xi = w_1 = w_2 := w$ a.e. Ω_0. Taking $\xi_1 = \xi_2 = \xi$ a.e. Ω in (7) and using the fact that $\mathcal{R}_1(x) g(w_1 - w_2)(w_1 - w_2) \geq 0$ a.e. Ω_1, we obtain

$$\begin{gathered} \int_{\Omega_1} v_1(w_1 - \xi) + d_1 \int_{\Omega_1} Dw_1 D(w_1 - \xi) + \int_{\Omega_1} v_2(w_2 - \xi) + d_2 \int_{\Omega_1} Dw_2 D(w_2 - \xi) \\ \int_{\Omega_0} v_1(w - \xi) + d_1 \int_{\Omega_0} DwD(w - \xi) + \int_{\Omega_0} v_2(w - \xi) + d_2 \int_{\Omega_0} DwD(w - \xi) \\ \leq \int_{\Omega_1} f_1(w_1 - \xi) + \int_{\Omega_1} f_2(w_2 - \xi) + \int_{\Omega_0} f_1(w - \xi) + \int_{\Omega_0} f_2(w - \xi) \end{gathered}$$

Using (22) and (23), we get

$$\int_{\Omega_0} (v_1 + v_2)(w - \xi) + (d_1 \ + d_2) \int_{\Omega_0} DwD(w - \xi) \leq \int_{\Omega_0} (f_1 + f_2)(w - \xi).$$

Taking $\xi = w \pm \eta$ with $\eta \in H_0^1(\Omega_0)$, we deduce that

$$\int_{\Omega_0} (v_1 + v_2)\eta + (d_1 \ + d_2) \int_{\Omega_0} DwD\eta = \int_{\Omega_0} (f_1 + f_2)\eta.$$

and the proof is finished . ∎

Proof of Proposition 3.5. By density and the definition of $\mathcal{A}_\infty$ we have $\overline{\mathcal{D}(\mathcal{A}_\infty)} \subseteq Y$. To prove that $Y \subseteq \overline{\mathcal{D}(\mathcal{A}_\infty)}$, it is enough to prove that $Y \cap \Big(L^\infty(\Omega) \times L^\infty(\Omega) \Big) \subseteq \overline{\mathcal{D}(\mathcal{A}_\infty)}$. So let

$u := (u_1, u_2) \in Y \cap \Big(L^\infty(\Omega)\times L^\infty(\Omega)\Big)$ and consider $u^\varepsilon := (u_1^\varepsilon, u_2^\varepsilon) = (I+\varepsilon\mathcal{A}_\infty)^{-1}(u_1,u_2)$. Since $(u_1^\varepsilon, u_2^\varepsilon) \in \mathcal{D}(\mathcal{A}_\infty)$, it is enough to prove that for $i = 1,2$; $u_i^\varepsilon \to u_i$ in $L^1(\Omega)$. Taking $\xi_1 = \xi_2 = 0$ as a test function in the definition of the operator $\mathcal{A}_\infty$, we get

$$\begin{aligned} d_1 \int_\Omega |Dw_1^\varepsilon|^2 + d_2 \int_\Omega |Dw_2^\varepsilon|^2 \quad &+\int_\Gamma (z_1^\varepsilon w_1^\varepsilon + z_2^\varepsilon w_2^\varepsilon) + \int_{\Omega_1} \mathcal{R}_1(x) g(w_1^\varepsilon - w_2^\varepsilon)(w_1^\varepsilon - w_2^\varepsilon) \\ &\leq \frac{1}{\varepsilon}\Big(\int_\Omega (u_1 - u_1^\varepsilon)w_1^\varepsilon + \int_\Omega (u_2-u_2^\varepsilon)w_2^\varepsilon\Big) \end{aligned}$$

and by the Proposition 2.3, we have

$$||(u_1^\varepsilon, u_2^\varepsilon)||_\infty \leq ||(u_1,u_2)||_\infty \quad \text{ and } ||(w_1^\varepsilon, w_2^\varepsilon)||_\infty \leq C||(u_1,u_2)||_\infty.$$

Using the monotonicity of φ_i, γ_i and g, we get

$$\begin{aligned} d_1 \int_\Omega |Dw_1^\varepsilon|^2 + d_2 \int_\Omega |Dw_2^\varepsilon|^2 \quad &\leq \frac{1}{\varepsilon}\Big(\int_\Omega (u_1 - u_1^\varepsilon)w_1^\varepsilon + \int_\Omega (u_2-u_2^\varepsilon)w_2^\varepsilon\Big) \\ &\leq \frac{1}{\varepsilon}\Big(\int_\Omega u_1 w_1^\varepsilon + \int_\Omega u_2 w_2^\varepsilon\Big) \\ &\leq \frac{1}{\varepsilon} C', \end{aligned}$$

where C' is a constant depending on Ω, $||(u_1,u_2)||_\infty$, φ_1 and φ_2. Hence, for $i = 1,2$, and ε small enough, we have $\varepsilon D w_i^\varepsilon$ is bounded in $L^2(\Omega)$, then as ε goes to 0, $\varepsilon w_i^\varepsilon \longrightarrow 0$, in $H^1(\Omega)$-weak, and by (H_2) we get z_i^ε is bounded in $L^\infty(\Gamma)$ then $z_i^\varepsilon \to z_i$, in $L^2(\partial\Omega)$-weak. On the other hand, using Lemma 3.6, we have $(u_1^\varepsilon,\ u_2^\varepsilon)$ satisfies

$$\begin{cases} u_1^\varepsilon - \varepsilon\, d_1 \triangle w_1^\varepsilon + \varepsilon\, \mathcal{R}_1(x) g(w_1^\varepsilon - w_2^\varepsilon) = u_1 & \text{in } \mathcal{D}'(\Omega_1) \\ u_2^\varepsilon - \varepsilon d_2 \triangle w_2^\varepsilon - \varepsilon\, \mathcal{R}_1(x) g(w_1^\varepsilon - w_2^\varepsilon) = u_2 & \text{in } \mathcal{D}'(\Omega_1) \\ (u_1^\varepsilon + u_2^\varepsilon) - \varepsilon\,(d_1 + d_2)\triangle w^\varepsilon = u_1 + u_2, \quad w_1^\varepsilon = w_2^\varepsilon := w^\varepsilon & \text{in } \mathcal{D}'(\Omega_0). \end{cases} \tag{24}$$

Thanks to lemma 4.4, we have

$$\begin{aligned} &\int_{\Omega"_i} \psi(x)(|u_1^\varepsilon(x+y) - u_1^\varepsilon(x)| + |u_2^\varepsilon(x+y) - u_2^\varepsilon(x)|) \\ &\leq \varepsilon max(d_1,d_2) \int_{\Omega'_i} |\Delta\psi(x)|(|w_1^\varepsilon(x+y) - w_1^\varepsilon(x)| + |w_2^\varepsilon(x+y) - w_2^\varepsilon(x)|) + \\ &\int_{\Omega'_i} \psi(x)(|u_1(x+y) - u_1(x)| + |u_2(x+y) - u_2(x)|) \\ &+2i\varepsilon \int_{\Omega'_i} \psi(x)(|\mathcal{R}_1(x+y) - \mathcal{R}_1(x)|\, |g(w_1^\varepsilon - w_2^\varepsilon)|) \end{aligned}$$

where for $i = 0,1$; Ω'_i and $\Omega"_i$ open sets of Ω_i and $\overline{\Omega'_i} \subseteq \Omega_i$, $\overline{\Omega"_i} \subseteq \Omega'_i$.
Now, using the fact that $\mathcal{R}_1 \in BV(\Omega_1)$, we obtain

$$\lim_{y\to 0}\sup_{\varepsilon>0} \int_{\Omega"_i} (|u_1^\varepsilon(x+y) - u_1^\varepsilon(x)| + |u_2^\varepsilon(x+y) - u_2^\varepsilon(x)|) = 0,$$

hence $(u_1^\varepsilon,\ u_2^\varepsilon)$ is precompact in $L^1(\Omega_i)\times L^1(\Omega_i)$, and consequently we have $(u_1^\varepsilon, u_2^\varepsilon)$ is weakly sequential compact in $L^1(\Omega)\times L^1(\Omega)$. Then, there exist $\varepsilon_n\to 0$ such that for $i=1,2$; $u_i^{\varepsilon_n}\to\tilde u_1$ strongly in $L^1(\Omega)$, $\varepsilon_n Dw_i^{\varepsilon_n}\to 0$ weakly in $L^2(\Omega)$ and $\varepsilon_n g(w_1^{\varepsilon_n}-w_2^{\varepsilon_n})\to 0$ weakly in $L^1(\Omega)$. Passing to limit as $\varepsilon_n\to 0$ in (24) we get $\tilde u_1=u_1$, $\tilde u_2=u_2$ in Ω_1, $\tilde u_1+\tilde u_2=u_1+u_2$ and $\varphi_1(\tilde u_1):=\varphi_2(\tilde u_2)$ in Ω_0, so that $\tilde u_1=u_1$ and $\tilde u_2=u_2$ in Ω. ■

To apply Theorem 2.2 (cf. [12]), we consider the operator $\widetilde{\mathcal{A}}^k$ defined by $\widetilde{\mathcal{A}}^k=[a(k)]^{-1}\mathcal{A}^k$ and we consider the associated elliptic problem of $\widetilde{\mathcal{A}}^k$

$$S_r^k(f_1,f_2)\begin{cases} v_1-d_1\triangle w_1+\mathcal{R}_k^{'}(x)g(w_1-w_2)=f_1, \quad w_1=\varphi_1^k(v_1) & \text{in } \Omega\\ v_2-d_2\triangle w_2-\mathcal{R}_k^{'}(x)g(w_1-w_2)=f_2, \quad w_2=\varphi_2^k(v_2) & \text{in } \Omega\\ \partial_{\vec n}w_1+z_1=0, \quad z_1\in\gamma_1^k(w_1) & \text{in } \Gamma\\ \partial_{\vec n}w_2+z_2=0, \quad z_2\in\gamma_2^k(w_2) & \text{in } \Gamma\end{cases}$$

where $\varphi_i^k=[a(k)]^{-1}\varphi_i$, $\gamma_i^k=[a(k)]^{-1}\gamma_i$ and $\mathcal{R}_k^{'}$ is the positive function defined by

$$\mathcal{R}_k^{'}(x)=\chi_{\Omega_0}(x)+[a(k)]^{-1}\mathcal{R}_1(x)\chi_{\Omega_1}(x) \quad \text{a.e. } \Omega.$$

We have,

Lemma 3.7 *Let $(f_1,f_2)\in L^\infty(\Omega)\times L^\infty(\Omega)$ and $(\overline{v}_1^k,\overline{v}_2^k)$ be the solution of $S_r^k(f_1,f_2)$ in the sense of Proposition 2.1. Then, as k goes to $+\infty$, for $i=1,2$; we have $\overline{v}_i^k\longrightarrow\overline{v}_i$, in $L^1(\Omega)$ and the couple $(\overline{v}_1,\overline{v}_2)$ satisfies :*

$$\begin{cases}\overline{v}_1+g(\varphi_1(\overline{v}_1)-\varphi_2(\overline{v}_2))\chi_{\Omega_0}=f_1, & a.e.\ in\ \Omega\\ \overline{v}_2-g(\varphi_1(\overline{v}_1)-\varphi_2(\overline{v}_2))\chi_{\Omega_0}=f_2, & a.e.\ in\ \Omega.\end{cases}$$

Moreover, $(\overline{v}_1,\overline{v}_2)$ is unique.

Proof. Thanks to Proposition 2.1, there exist $(\overline{v}_1^k,\overline{v}_2^k)$ a unique solution of $S\mathcal{R}^k(f_1,f_2)$ in the sense of (1). For $i=1,2$; we have $\overline{v}_i^k\in L^\infty(\Omega)$, $\overline{w}_i^k\in H^1(\Omega)$, and $\overline{z}_i^k\in L^2(\Gamma)$ with $\overline{w}_i^k=\varphi_i(\overline{v}_i^k)$ and $\overline{z}_i^k\in\gamma_i(\overline{w}_i^k)$. Using a similar argument of the Proposition 2.3, there exists a subsequence that we denote again by $\overline{v}_i^k$, such that as k goes to ∞, $\overline{v}_i^k\to\overline{v}_i$ in $L^1(\Omega)$, $\overline{z}_i^k\rightharpoonup 0$ in $L^1(\Gamma)$ and $D\overline{w}_i^k\rightharpoonup 0$ in $L^2(\Omega)$. By the Dominate Convergence Theorem, we deduce that $\mathcal{R}_k^{'}(x)g(\overline{w}_1^k-\overline{w}_2^k)\longrightarrow g(\overline{w}_1-\overline{w}_2)\chi_{\Omega_0}$, in $L^1(\Omega)$. Passing to the limit in the weak formulation, for $i=1,2$; we obtain

$$\int_\Omega\overline{v}_i\xi_i+(-1)^{i-1}\int_{\Omega_0}g(\overline{w}_1-\overline{w}_2)\xi_i=\int_\Omega f_i\xi_i.$$

■

As consequence of this Lemma we have the following corollary

Corollary 3.8 *As* $k \longrightarrow \infty$, *the operator* $\widetilde{\mathcal{A}}^k$ *converge in* X *to the* T*-accretive operator* $\widetilde{\mathcal{A}}_\infty$ *defined by*

$$(f_1, f_2) \in \widetilde{\mathcal{A}}_\infty(v_1, v_2) \Leftrightarrow \begin{cases} v_1,\ v_2,\ f_1,\ f_2 \in L^1(\Omega) \\ f_1 = f_2 = 0 \quad \text{a.e. in } \Omega_1 \text{ and} \\ f_1 = -f_2 = g(\varphi_1(v_1) - \varphi_2(v_2)) \quad \text{a.e. in } \Omega_0. \end{cases}$$

Proposition 3.9 *Let* $(u_{01}, u_{02}) \in L^\infty(\Omega) \times L^\infty(\Omega)$ *then a mild solution of the problem*

$$\begin{cases} U_t + \widetilde{\mathcal{A}}_\infty\ U \ni (0,\ 0) \quad \text{in } (0,T) \\ \\ U(0) = U_0 \end{cases} \tag{25}$$

is a solution of the ordinary differential systems :

$$ED(g, u_{01}, u_{02}) \begin{cases} c_{1t} + g(\varphi_1(c_1) - \varphi_2(c_2))\chi_{\Omega_0} = 0, & \text{in } (0,T) \\ \\ c_{2t} - g(\varphi_1(c_1) - \varphi_2(c_2))\chi_{\Omega_0} = 0, & \text{in } (0,T) \\ \\ c_1(0) = u_{01}, \quad c_2(0) = u_{02} \end{cases}$$

in the following sense: for $i = 1, 2$; $c_i \in W^{1,1}\Big(0, T;\ L^1(\Omega)\Big)$ *and the system is satisfy for a.e* $(t, x) \in Q$ *with* $c_i(0) = u_{0i}$. *Moreover*

$$\lim_{t\to\infty} (c_1(t), c_2(t)) = (\underline{u}_{01}, \underline{u}_{02}) \quad \text{in } L^1(\Omega) \times L^1(\Omega)$$

and

$$\underline{u}_{01}(x) = \begin{cases} u_{01}(x) & \text{in } \Omega_1 \\ \tilde{u}_{01}(x) & \text{in } \Omega_0 \end{cases} \qquad \underline{u}_{02}(x) = \begin{cases} u_{02}(x) & \text{in } \Omega_1 \\ \tilde{u}_{01}(x) & \text{in } \Omega_0 \end{cases}$$

where, $\tilde{u}_{01}(x) := [I + \varphi_2^{-1} \circ \varphi_1]^{-1}(u_{01} + u_{02})$ *and* $\tilde{u}_{02}(x) := [I + \varphi_1^{-1} \circ \varphi_2]^{-1}(u_{01} + u_{02})$.

Proof. Recall that by corollary 3.8 the operator $\widetilde{\mathcal{A}}^k$ converge to $\widetilde{\mathcal{A}}_\infty$, then a mild solution U^k of the problem

$$\begin{cases} U_t + \widetilde{\mathcal{A}}^k\, U \ni 0 \quad \text{in } (0,T) \\ \\ U(0) = U_0, \end{cases}$$

converges to U (mild solution of (25)). Using, Proposition 3.3, for $i = 1, 2$; $u_i^k \in L^\infty(Q)$, there exists $w_i^k \in L^2\Big(0, T;\ H^1(\Omega)\Big)$, $z_i^k \in L^\infty(\Sigma)$ such that $w_i^k = \varphi_i(u_i^k)$ a.e. in Q, $z_i^k \in \gamma_i(w_i^k)$ a.e. in Σ, and

$$
\begin{aligned}
&[a(k)]^{-1} d_i \int_0^\tau \int_\Omega D w_i^k D\xi + (-1)^{i-1} \int_0^\tau \int_\Omega \mathcal{R}_k'(.) g(w_1^k - w_2^k)\xi \\
&\quad + [a(k)]^{-1} \int_0^\tau \int_\Gamma z_i^k \xi = \int_0^\tau \int_\Omega u_i^k \xi_t + \int_\Omega u_0 \xi(0).
\end{aligned} \tag{26}
$$

Moreover, u_i^k, w_i^k and $g(w_1^k - w_2^k)$ are bounded in $L^\infty(Q)$, z_i^k is bounded in $L^\infty(\Sigma)$ and w_i^k is bounded in $L^2\Big(0,T; H^1(\Omega)\Big)$, then we have $[a(k)]^{-1} D w_i^k \to 0$ weakly in $L^2\Big(0,T; H^1(\Omega)\Big)$ and $[a(k)]^{-1} z_i^k \to 0$ weakly in $L^2(\Sigma)$. Passing to limit in (26), we deduce that

$$
(-1)^{i-1} \int_0^\tau \int_{\Omega_0} g(\varphi_1(c_1) - \varphi_2(c_2))\, \xi = \int_0^\tau \int_\Omega c_i\, \xi_t + \int_\Omega u_{0i}\, \xi(0).
$$

Since, $g(\varphi_1(c_1) - \varphi_2(c_2)) \in L^1\Big((0,T) \times \Omega_0\Big)$ $c_i \in W^{1,1}\Big(0,T;\ L^1(\Omega)\Big)$ and, for a.e. $x \in \Omega$,

$$
\begin{cases}
c_{1t} + g(\varphi_1(c_1) - \varphi_2(c_2))\chi_{\Omega_0} = 0 & \text{in } (0,T) \\
c_{2t} - g(\varphi_1(c_1) - \varphi_2(c_2))\chi_{\Omega_0} = 0 & \text{in } (0,T) \\
c_1(0) = u_{01} \quad c_2(0) = u_{02}.
\end{cases}
$$

and this finishes the first part of the proof. Thanks to [11] there exist a measurable couple of functions $(c_{1\infty}, c_{2\infty})$ such that as $t \to \infty$, we have $c_1(t) \longrightarrow c_{1\infty}$, $c_2(t) \longrightarrow c_{2\infty}$ a.e. Ω, and $g(\varphi_1(c_{1\infty}) - \varphi_2(c_{2\infty})) = 0$ a.e. Ω_0 which implies that $\varphi_1(c_{1\infty}) = \varphi_2(c_{2\infty})$ a.e. Ω_0 . On the other hand, adding both equations, we see that $c_{1t} + c_{2t} = 0$ a.e. Ω_0, therefore $c_{1\infty} + c_{2\infty} = u_{01} + u_{01}$ a.e. Ω_0. Then we conclude that $c_{1\infty} = [I + \varphi_2^{-1} \circ \varphi_1]^{-1}(u_{01} + u_{02})$ and $c_{2\infty} = [I + \varphi_1^{-1} \circ \varphi_2]^{-1}(u_{01} + u_{02})$ a.e. $x \in \Omega$. Thus the Lemma follows. ■

Lemma 3.10 *We suppose that* $U_0 = (u_{01}, u_{02}) \in L^\infty(\Omega) \times L^\infty(\Omega)$, $(f_1, f_2) \in L^\infty(\Omega) \times L^\infty(\Omega)$ *and* $U^k = (u_1^k, u_2^k)$ *a mild solution of problem* $CP^k(U_0, f_1, f_2)$*, Then, as* $k \to \infty$*, we have*

$$
U^k \longrightarrow U := (u_1, u_2) \quad \textit{in } \mathcal{C}\Big((0,T); X\Big)
$$

where U *is a mild solution of problem*

$$
CP_\infty(\underline{U}_0, f_1, f_2) \begin{cases}
U_t + \mathcal{A}_\infty U \ni (f_1, f_2) & \textit{in } (0,T) \\
U(0) = \underline{U}_0 := (\underline{u}_{01}, \underline{u}_{02}).
\end{cases}
$$

and $\underline{u}_{01}$*,* $\underline{u}_{02}$ *are given by the the preceding proposition.*

Proof. Let U^k be a mild solution of problem $CP^k(U_0, f_1, f_2)$, thanks to Corollary 3.4 (respectively Corollary 3.8) the operator $\mathcal{A}^k$ (respectively $\widetilde{\mathcal{A}}^k$) converges to $\mathcal{A}_\infty$ (respectively $\widetilde{\mathcal{A}}_\infty$), using proposition 3.9 we have $\lim_{t\to\infty} e^{-t\widetilde{\mathcal{A}}_\infty} = (\underline{u}_{01}, \underline{u}_{02})$ and we see that

$(\underline{u}_{01}, \underline{u}_{02}) \in \overline{\mathcal{D}(\mathcal{A}_\infty)}$. Then applying Theorem 2.2 (cf. [12]), the result of the lemma follows. ■

Now we show that this mild solution is also a weak solution of limiting problem, thus obtaining

Proposition 3.11 *Let (u_1, u_2) be the mild solution of the problem $CP_\infty(\underline{U}_0, f_1, f_2)$ given by lemma 3.10. Then (u_1, u_2) is a weak solution in the sense of Theorem 1.1.*

Firstly, we need the following Lemma.

Lemma 3.12 *Suppose that $U := (u_1, u_2)$ is a solution of problem (P^k) in the sense of Theorem 3.3. Then, for $i = 1, 2$; we have*

$$\begin{aligned} &\frac{d}{dt}\int_\Omega \int_0^{u_i} T_l(\varphi_i(s) - \xi_i)ds + d_i \int_\Omega Dw_i DT_l(w_i - \xi_i) + \int_\Gamma z_i T_l(w_i - \xi_i) \\ &+(-1)^{i-1}\int_\Omega \mathcal{R}_k(x) g(w_1 - w_2) T_l(w_i - \xi_i) = \int_\Omega f_i T_l(w_i - \xi_i) \quad in\ \mathcal{D}'(0,T) \end{aligned} \tag{27}$$

for any $\xi_i \in H^1(\Omega)$.

Proof. This is an immediate consequence of the Chain rule Lemma (see for instance [1], [3]). ■

Proof of proposition 3.11. Let (u_1^k, u_2^k) a solution of problem (P^k) in the sense of Theorem 3.3 and for $i = 1, 2$ we set $w_i^k = \varphi_i(v_i^k)$ then we have

$$||(w_1^k(t), w_2^k(t))||_\infty \leq \varphi_1(||(u_{01}, u_{02})||_\infty) + \varphi_2(||(u_{01}, u_{02})||_\infty)$$

and

$$\int\int_Q |Dw_1^k|^2 + \int\int_Q |Dw_2^k|^2 \ \leq\ C \quad \text{a.e. } \ t \in (0,T),$$

where C is a constant independent of k. So, for $i = 1, 2$ w_i^k is bounded in $L^2\Big(0, T;\, H^1(\Omega)\Big)$ and there exists $w_i \in L^2\Big(0, T;\, H^1(\Omega)\Big)$ and a subsequence that we denote again by w_i^k such that $w_i^k \longrightarrow w_i$ in $L^2\Big(0, T;\, H^1(\Omega)\Big)$-weak. On the other hand, (u_1^k, u_2^k) is a mild solution, then thanks to proposition 3.10 we have $u_i^k \longrightarrow u_i$, in $\mathcal{C}\Big([0,T],\, L^1(\Omega)\Big)$ and (u_1, u_2) is a mild solution of $CP_\infty(U_0,\ f_1,\ f_2)$. Since $(u_1, u_2) \in \overline{\mathcal{D}(\mathcal{A}_\infty)}$, then

$$w_1 = w_2 \quad \text{ a.e. } \ \Omega_0 \times (0,T).$$

Thanks to lemma 3.12

$$\begin{aligned}&\frac{d}{dt}\int_\Omega\int_0^{u_1^k} T_l(\varphi_1(s)-\xi_1)ds+d_1\int_\Omega Dw_1^kDT_l(w_1^k-\xi_1)+\int_\Gamma z_1^kT_l(w_1^k-\xi_1)\\&+\frac{d}{dt}\int_\Omega\int_0^{u_2^k} T_l(\varphi_2(s)-\xi_2)ds+d_2\int_\Omega Dw_2^kDT_l(w_2^k-\xi_2)+\int_\Gamma z_2^kT_l(w_2^k-\xi_2)\\&+\int_\Omega \Theta^k(x,t)\le\int_\Omega f_1T_l(w_1^k-\xi_1)+\int_\Omega f_2T_l(w_2^k-\xi_2)\ \text{ in } \mathcal{D}'(0,T),\end{aligned}$$

for any $\xi_1,\xi_2\in\mathcal{C}^1(\overline{\Omega})$, where $\Theta^k(x,t)=\mathcal{R}_k(x)g(w_1^k-w_2^k)\Big(T_l(w_1^k-\xi_1)-T_l(w_2^k-\xi_2)\Big)$. Taking $(\xi_1,\xi_2)\in\mathcal{K}$, we decompose the integral $\Theta^k(x,t)$ in the same way as in (28) as follow

$$\int_\Omega\Theta^k(x,\ t)=\int_{\Omega_0}\Theta^k(x,\ t)+\int_{\Omega_1}\Theta^k(x,\ t)\quad\text{for all } t\in(0,T)\tag{28}$$

To treat the first term, we consider the following sets

$$E_1:=\{x\in\Omega_0,\ \text{ such that }\ |w_1^k(t)-\xi|\le l\ \text{ and }\ |w_2^k(t)-\xi|\le l\quad\text{for all } t\in(0,T)\},$$

$$E_2:=\{x\in\Omega_0,\ \text{ such that }\ |w_1^k(t)-\xi|\le l\ \text{ and }\ |w_2^k(t)-\xi|\ge l\quad\text{for all } t\in(0,T)\}$$

and

$$E_3:=\{x\in\Omega_0,\ \text{ such that }\ |w_1^k(t)-\xi|\ge l\ \text{ and }\ |w_2^k(t)-\xi|\le l\quad\text{for all } t\in(0,T)\}.$$

Then

$$\begin{aligned}\int_{\Omega_0}\Theta^k(x,\ t)\ &=\int_{E_1}\underbrace{\mathcal{R}_k(x)g(w_1^k-w_2^k)(w_1^k-w_2^k)}_{\ge 0}+\int_{E_2}\mathcal{R}_k(x)g(w_1^k-w_2^k)(w_1^k-\xi-l)\\&\quad+\int_{E_3}\mathcal{R}_k(x)g(w_1^k-w_2^k)(l-w_2^k+\xi)\quad\text{for all } t\in(0,T)\end{aligned}$$

For a.e. $x\in E_2$ we have :

$$w_1^k-\xi-l\le 0\ \text{ then } w_1^k\le w_2^k\ \text{ and } \mathcal{R}_k(.)g(w_1^k-w_2^k)(w_1^k-\xi-l)\ge 0.$$

For a.e. $x\in E_3$ we have :

$$l-w_2^k+\xi\ge 0\ \text{ and } w_1^k\ge w_2^k\ \text{ then } \mathcal{R}_k(.)g(w_1^k-w_2^k)(l-w_2^k+\xi)\ge 0.$$

Consequently,

$$\int_\Omega\Theta^k(x,t)\ge\int_{\Omega_1}\Theta^k(x,t)\quad\text{ for all } t\in(0,T)$$

so that

$$
\begin{aligned}
&\frac{d}{dt}\int_\Omega\int_0^{u_1^k} T_l(\varphi_1(s)-\xi_1)ds + d_1\int_\Omega Dw_1^k DT_l(w_1^k-\xi_1) + \int_\Gamma z_1^k T_l(w_1^k-\xi_1)\\
&+\frac{d}{dt}\int_\Omega\int_0^{u_2^k} T_l(\varphi_2(s)-\xi_2)ds + d_2\int_\Omega Dw_2^k DT_l(w_2^k-\xi_2) + \int_\Gamma z_2^k T_l(w_2^k-\xi_2)\\
&+\int_{\Omega_1}\mathcal{R}_1(x)g(w_1^k-w_2^k)\Big(T_l(w_1^k-\xi_1)-T_l(w_2^k-\xi_2)\Big)\\
&\le \int_\Omega f_1T_l(w_1^k-\xi_1)+\int_\Omega f_2T_l(w_2^k-\xi_2) \text{ in } \mathcal{D}'(0,T).
\end{aligned}
\tag{29}
$$

On the other hand, since

$$
\int_\Omega Dw_i^k DT_l(w_i^k-\xi_i) = \int_\Omega |DT_l(w_i^k-\xi_i)|^2 + \int_\Omega D\xi_i DT_l(w_i^k-\xi_i) \quad \forall\, t\in(0,T),
$$

then, for $t\in(0,T)$

$$
\liminf_{k\to\infty}\int_\Omega Dw_i^k DT_l(w_i^k-\xi_i) \ge \int_\Omega Dw_i DT_l(w_i-\xi_i)
$$

and passing to the limit in (29) the result of the proposition follows. ∎

4 Uniqueness of weak solution

To prove uniqueness we use the concept of integral solution (cf. [5], [7]). Since $\mathcal{A}^k$ (respectively $\mathcal{A}_\infty$) is accretive in X, it is well known that mild solutions and integral solutions of problem (P^k) (respectively limiting problem) coincide.

Definition 4.1 *A function $U := (u_1, u_2) \in \mathcal{C}([0,T], X)$ is an integral solution of $CP^k(U_0, f_1, f_2)$ (respectively $CP_\infty(\underline{U}_0,\ f_1,\ f_2)$) if for every $(\hat f_1,\ \hat f_2) \in \mathcal{A}^k(\hat u_1,\ \hat u_2)$ (respectively $(\hat f_1,\ \hat f_2) \in \mathcal{A}_\infty(\hat u_1,\ \hat u_2)$), we have*

$$
\frac{d}{dt}\int_\Omega |u_1(t)-\hat u_1| + \frac{d}{dt}\int_\Omega |u_2(t)-\hat u_2| \le \int_{[u_1(t)=\hat u_1]} |f_1(t)-\hat f_1| + \int_{[u_2(t)=\hat u_2]} |f_2(t)-\hat f_2|
$$

$$
+\int_\Omega (f_1(t)-\hat f_1) Sign_0(u_1(t)-\hat u_1) + \int_\Omega (f_2(t)-\hat f_2) Sign_0(u_2(t)-\hat u_2) \quad in\ \mathcal{D}'(0,T),
$$

and $U(0) = U_0$ (respectively $U(0) = \underline{U}_0$).

Remark 4.2 *Let $(u_1,\ u_2)$ be a weak solution of the problem (P^k). Then it is also a solution in the sense of Theorem 1.1.*

Proposition 4.3 *Let $(u_1,\ u_2)$ a solution of problem (P^k) (respectively limiting problem) in the sense of Theorem 1.1. Then $(u_1,\ u_2)$ is an integral solution of problem $CP^k(U_0,\ f_1,\ f_2)$ (respectively $CP_\infty(\underline{U}_0,\ f_1,\ f_2)$).*

Proof. The proof for (P^k) and the limiting problem are similar. To avoid to duplicate similar argument we prove this proposition only for limiting problem. For $i = 1,2$, let $w_i \in L^2\Big(0,T;\ H^1(\Omega)\Big)$, $z_i \in L^2(\Sigma)$ such that $w_i = \varphi_i(u_i)$ a.e. Q and $z_i \in \gamma_i(w_i)$ a.e. Σ, given by Theorem 1.1, and $\hat{w}_i \in H^1(\Omega)$, $\hat{z}_i \in L^2(\Gamma)$ such that $\hat{w}_i = \varphi_i(\hat{u}_i)$ a.e. Ω and $\hat{z}_i \in \gamma_i(\hat{w}_i)$ a.e. Γ in (3.4). Since $(\hat{w}_1, \hat{w}_2) \in \mathcal{K}$ then one can take $(\hat{w}_1, \hat{w}_2)$ as test function in the definition of the solution of Theorem 1.1, so

$$\begin{aligned}&\frac{d}{dt}\int_\Omega\int_0^{u_1} T_l(\varphi_1(r)-\hat{w}_1)dr + d_1\int_\Omega Dw_1DT_l(w_1-\hat{w}_1) + \int_\Gamma z_1T_l(w_1-\hat{w}_1)\\ &+\frac{d}{dt}\int_\Omega\int_0^{u_2} T_l(\varphi_2(r)-\hat{w}_2)dr + d_2\int_\Omega Dw_2DT_l(w_2-\hat{w}_2) + \int_\Gamma z_2T_l(w_2-\hat{w}_2)\\ &+\int_{\Omega_1}\mathcal{R}_1(x)g(w_1-w_2)(T_l(w_1-\hat{w}_1)-T_l(w_2-\hat{w}_2)) \le \int_\Omega f_1T_l(w_1-\xi_1)\\ &+\int_\Omega f_2T_l(w_2-\xi_2)\ \text{ in } \mathcal{D}'(0,T),\end{aligned} \tag{30}$$

On the other hand, using definition of the operator $\mathcal{A}_\infty$ hence

$$\begin{aligned}d_1\int_\Omega D\hat{w}_1DT_l(\hat{w}_1-\xi_1) + d_2\int_\Omega D\hat{w}_2DT_l(\hat{w}_2-\xi_2) + \int_\Gamma \hat{z}_1T_l(\hat{w}_1-\xi_1) + \int_\Gamma \hat{z}_2T_l(\hat{w}_2-\xi_2)+\\ \int_{\Omega_1}\mathcal{R}_1(x)g(\hat{w}_1-\hat{w}_2)\Big(T_l(\hat{w}_1-\xi_1)-T_l(\hat{w}_2-\xi_2)\Big) \le \int_\Omega \hat{f}_1T_l(\hat{w}_1-\xi_1) + \int_\Omega \hat{f}_2T_l(\hat{w}_2-\xi_2)\end{aligned}$$

for any $\xi = (\xi_1,\xi_2) \in \Big(H^1(\Omega)\times H^1(\Omega)\Big)\cap\mathcal{K}$. For a.e. $(x,t)\in(0,T)\times\Omega_0$ we have $w_1(t) = w_2(t)$, then (w_1,w_2) is an admissible test function in the definition of the operator $\mathcal{A}_\infty$, hence

$$\begin{aligned}&d_1\int_\Omega D\hat{w}_1DT_l(\hat{w}_1-w_1) + \int_\Gamma \hat{z}_1T_l(\hat{w}_1-w_1) + d_2\int_\Omega D\hat{w}_2DT_l(\hat{w}_2-w_2)\\ &+\int_{\Omega_1}\mathcal{R}_1(x)g(\hat{w}_1-\hat{w}_2)(T_l(\hat{w}_1-w_1)-T_l(\hat{w}_2-w_2)) + \int_\Gamma \hat{z}_2T_l(\hat{w}_2-w_2)\\ &\qquad\le \int_\Omega \hat{f}_1T_l(\hat{w}_1-w_1) + \int_\Omega \hat{f}_2T_l(\hat{w}_2-w_2)\quad \text{for }\ t\in(0,T).\end{aligned} \tag{31}$$

Adding (30) and (31), we find

$$\begin{aligned}&\frac{d}{dt}\int_\Omega\int_0^{u_1} T_l(\varphi_1(r)-\hat{w}_1)dr + \frac{d}{dt}\int_\Omega\int_0^{u_2} T_l(\varphi_2(r)-\hat{w}_1)dr + \int_{\Omega_1}\hat{\Theta}(x,t)dx\\ &+\int_\Gamma (z_1-\hat{z}_1)T_l(w_1-\hat{w}_1) + \int_\Gamma (z_2-\hat{z}_2)T_l(w_2-\hat{w}_1)\\ &+d_1\int_\Omega D(w_1-\hat{w}_1)DT_l(w_1-\hat{w}_1) + d_2\int_\Omega D(w_2-\hat{w}_2)DT_l(w_2-\hat{w}_2)\\ &\le \int_\Omega (f_1-\hat{f}_1)T_l(\hat{w}_1-w_1) + \int_\Omega (f_2-\hat{f}_2)T_l(\hat{w}_2-w_2)\quad \text{ in } \mathcal{D}'(0,T),\end{aligned} \tag{32}$$

where $\hat{\Theta}(x,t) = \mathcal{R}_1(x)(g(w_1 - w_2) - g(\hat{w}_1 - \hat{w}_2))(T_l(w_1 - \hat{w}_1) - T_l(w_2 - \hat{w}_2))$.

Using the fact that $Sign_0(\varphi_i(r) - \varphi_i(\hat{u}_i)) = Sign_0(r - \hat{u}_i)$, then as l tends to 0 we obtain $\displaystyle\int_0^{u_i(x,t)} \frac{1}{l} T_l(\varphi_i(r) - \hat{w}_i(x))dr$ converge to $|u_i(x,t)) - \hat{u}_i(x)|$ a.e. $(t,x) \in Q$. By Lebesgue dominate convergence Theorem, it follows that, the term $\displaystyle\frac{d}{dt}\int_0^{u_i(x,t)} \frac{1}{l} T_l(\varphi_i(r) - \hat{w}_i(x))dr$ converge to $\displaystyle\frac{d}{dt}\int_\Omega |u_i(x,t)) - \hat{u}_i(x)|$ and $\displaystyle\int_\Omega \frac{1}{l}(f_i - \hat{f}_i)T_l(\hat{w}_i - w_i)$ converge to $\displaystyle\int_\Omega (f_i - \hat{f}_i) Sign_0(\hat{w}_i - w_i)$.
We multiply (32) by $\frac{1}{l}$, using lemma 2.4 and passing to limit as l goes to 0, we obtain

$$\frac{d}{dt}\int_\Omega |u_1(t) - \hat{u}_1| + \frac{d}{dt}\int_\Omega |u_2(t) - \hat{u}_2| \le -\int_\Omega (f_1 - \hat{f}_1) Sign_0(u_1(t) - \hat{u}_1)$$
$$-\int_\Omega (f_2 - \hat{f}_2) Sign_0(u_2(t) - \hat{u}_2) + \int_{[u_1(t)=\hat{u}_1]} |f_1 - \hat{f}_1| + \int_{[u_2(t)=\hat{u}_2]} |f_2 - \hat{f}_2| \quad \text{in } \mathcal{D}'(0,T).$$

This finishes the proof of the Proposition. ■

Appendix

Lemma 4.4 *Let $F := (f_1, f_2) \in L^\infty(\Omega) \times L^\infty(\Omega)$. If $V := (v_1, v_2)$ is the solution of the problem $S^k(f_1, f_2)$, then for $y \in \mathbb{R}^N$, $\psi \ge 0$ and $\psi \in \mathcal{C}^2(\Omega)$ of support $\Omega' := \{x \in \Omega;\ distance(x, \Gamma) < |y|\}$, then we have*

$$\begin{aligned}
&\int_{\Omega'} \psi(x)(|v_1(x+y) - v_1(x)| + |v_2(x+y) - v_2(x)|)\\
&\le max(d_1, d_2)\int_{\Omega'} |\Delta\psi(x)|(|w_1(x+y) - w_1(x)| + |w_2(x+y) - w_2(x)|)\\
&+\int_{\Omega'} \psi(x)(|f_1(x+y) - f_1(x)| + |f_2(x+y) - f_2(x)|)\\
&+2\int_{\Omega'} \psi(x)(|\mathcal{R}_k(x+y) - \mathcal{R}_k(x)||g(w_1 - w_2)|)
\end{aligned} \tag{33}$$

Proof. Using a similar techniques in [16]. Let (v_1, v_2) be the solution of the problem $S^k(f_1, f_2)$ in the sense of Proposition 2.1, we have

$$\begin{aligned}
&d_1\int_{\Omega'} D(w_1(x+y) - w_1(x))D\xi dx + \int_{\Omega'} (v_1(x+y) - v_1(x))\xi dx\\
&+\int_{\Omega'} [\mathcal{R}_k(x+y)g(w_1(x+y) - w_2(x+y)) - \mathcal{R}_k(x)g(w_1(x) - w_2(x))]\xi dx\\
&= \int_{\Omega'} (f_1(x+y) - f_1(x))\xi dx
\end{aligned} \tag{34}$$

for all $\xi \in H^1(\Omega')$ such that support$(\xi) \subset\subset \Omega'$. Let $\psi \in \mathcal{C}^2(\Omega')$, $\psi \geq 0$ taking $\xi = \psi\mathcal{H}_\varepsilon(w_1(x+y) - w_1(x))$ as function test in (34) and let I_1^ε the left term of the equation (34), then

$$\begin{aligned} I_1^\varepsilon : &= \int_{\Omega'} DW[D\psi\mathcal{H}_\varepsilon(W) + \psi\mathcal{H'}_\varepsilon(W)D\mathcal{H}_\varepsilon(W)] \\ &\geq \int_{\Omega'} DWD\psi\mathcal{H}_\varepsilon(W) \\ &= -\int_{\Omega'} W\Delta\psi\mathcal{H}_\varepsilon(W) - \int_{\Omega'} WD\psi\mathcal{H'}_\varepsilon(W)D\mathcal{H}_\varepsilon(W), \end{aligned}$$

where $W = w_1(x+y) - w_1(x)$ and $\mathcal{H}_\varepsilon(s) = \inf(s^+/\varepsilon, 1)$ for $\varepsilon > 0$ and $s \in \mathbb{R}$. On the other hand, by the definition of $\mathcal{H}_\varepsilon$ that, as $\varepsilon \to 0$ we have

$$W\mathcal{H'}_\varepsilon(W) \to 0 \quad \text{and} \quad |W\mathcal{H'}_\varepsilon(W)| \leq 1 \quad \text{a.e. } \Omega'.$$

Consequently

$$\lim_{\varepsilon\to 0} \int_{\Omega'} WD\psi\mathcal{H'}_\varepsilon(W)D\mathcal{H}_\varepsilon(W) = 0.$$

By letting $\varepsilon \to 0$ in (I_1^ε), we get

$$\lim_{\varepsilon\to 0} I_1^\varepsilon = \lim_{\varepsilon\to 0} \int_{\Omega'} W\Delta\psi\mathcal{H}_\varepsilon(W) \geq -\int_{\Omega'} |W||\Delta\psi|. \tag{35}$$

Therefore, using the fact that $\varphi_i, \varphi_i^{-1}$, are continuous strictly increasing we obtain

$$\lim_{\varepsilon\to 0} \int_{\Omega'} (v_1(x+y) - v_1(x))\psi\mathcal{H}_\varepsilon(W)dx \geq \int_{\Omega'} |v_1(x+y) - v_1(x)|\psi. \tag{36}$$

Passing to limit in (34) as ε goes to 0 and using (35) and (36), we deduce that

$$\begin{aligned} &\int_{\Omega'} \psi(x)|v_1(x+y) - v_1(x)| + \int_{\Omega'} t(x)\psi(x)Sign_0(w_1(x+y) - w_1(x)) \\ &\leq d_1 \int_{\Omega'} |\Delta\psi(x)|(|w_1(x+y) - w_1(x)|) + \int_{\Omega'} \psi(x)(|f_1(x+y) - f_1(x)|) \end{aligned} \tag{37}$$

where $t(x) = \mathcal{R}_k(x+y)g(w_1(x+y) - w_2(x+y)) - \mathcal{R}_k(x)g(w_1(x) - w_2(x))$. A similar argument shows

$$\begin{aligned} &\int_{\Omega'} \psi(x)|v_2(x+y) - v_2(x)| - \int_{\Omega'} t(x)\psi(x)Sign_0(w_2(x+y) - w_2(x)) \\ &\leq d_2 \int_{\Omega'} |\Delta\psi(x)|(|w_2(x+y) - w_2(x)|) + \int_{\Omega'} \psi(x)(|f_2(x+y) - f_2(x)|). \end{aligned} \tag{38}$$

On the other hand, we have

$$
\begin{aligned}
J: \; &= t(x)\Big(Sign_0(w_1(x+y)-w_1(x)) - Sign_0(w_2(x+y)-w_2(x))\Big) \\
&\geq \Big(\mathcal{R}_k(x+y)-\mathcal{R}_k(x)\Big)g(w_1(x)-w_2(x))\Big(Sign_0(w_1(x+y)-w_1(x)) \\
&-Sign_0(w_2(x+y)-w_2(x))\Big) \\
&\geq -2\Big(|(\mathcal{R}_k(x+y)-\mathcal{R}_k(x))g(w_1-w_2)|\Big).
\end{aligned}
\tag{39}
$$

Adding (37) and (38) and using (39), the result of lemma follows. ∎

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