Singular limit of perturbed nonlinear semigroups

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Abstract

In this paper, we consider evolution problems

$$\begin{cases} \dfrac{du_m}{dt} + A_m u_m \ni F_m(.,u_m) \text{ in } (0,T), \\ u_m(0) = u_{0m} \end{cases}$$

where for $m = 1, 2, ...\infty$, A_m are m-accretive operators in a Banach space X, $F_m : (0,T) \times \overline{\mathcal{D}(A_m)} \to X$ are Caratheodory functions satisfying some assumptions, $u_{0m} \in \overline{\mathcal{D}(A_m)}$ and u_m the mild solution of the problems. Assuming that, as $m \to \infty$, $A_m \to A_\infty$ in the sense of resolvent and $F_m \to F_\infty$ in the natural sense, we prove that if $e^{-tA_m}u_{0m} \to e^{-tA_\infty}u_{0\infty}$ for $t > 0$, then $u_m \to u_\infty$ in $\mathcal{C}\left((0,T);X\right)$. And, we apply this result to the limit as $m \to \infty$, of the solution u_m of

$$\begin{cases} u_t = \Delta u^m + g(.,u) \quad \text{on } (0,T) \times \Omega, \\ u^m = 0 \text{ on } (0,T) \times \partial\Omega, \; u(0,.) = f \geq 0 \text{ on } \Omega. \end{cases}$$

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1 Introduction.

Consider the following problem

$$(P_m) \qquad \begin{cases} u_t = \Delta u^m + g(u) & \text{on } Q = (0,T)\times\Omega \\ u^m = 0 & \text{on } \Sigma = (0,T)\times\partial\Omega \\ u(0,x) = f(x) & \text{on } \Omega \end{cases}$$

where Ω is a bounded open set in $\mathbb{R}^N$, $f \in L^\infty(Q)$, $f \geq 0$ and $g : \mathbb{R}_+ \to \mathbb{R}$ is continuous with

$$(1.1) \qquad g(0) \geq 0,\ \frac{dg}{dr} \leq K \quad \text{in } \mathcal{D}'(0,\infty),$$

with $K \in \mathcal{C}(\mathbb{R}_+)$; the time $T > 0$ is such that the solution of the o.d.e

$$M' = g(M), \quad g(0) = \|f\|_\infty$$

is defined on $[0,T)$. Then for every $m \geq 1$, there exists a unique solution of (P_m) in the sense

$$(1.2) \qquad \begin{cases} u \in \mathcal{C}\left([0,T); L^1(\Omega)\right) \cap L^\infty_{loc}\left([0,T)\times\overline{\Omega}\right), u \geq 0,\ u(0,.) = f, \\ u^m \in L^2_{loc}\left([0,T); H^1_0(\Omega)\right),\ u_t = \Delta u^m + g(u) \text{ in } \mathcal{D}'(Q). \end{cases}$$

Let denote this solution by u_m ; one has

$$(1.3) \qquad 0 \leq u_m \leq M \quad \text{a.e. on } Q \quad \text{for every } m \geq 1.$$

In the case $g \equiv 0$, it has been proved in [4] (see also [11], [13]) that

$$u_m(t) \to \underline{f} = f\chi_{[\underline{w}=0]} + \chi_{[\underline{w}>0]} \quad \text{in } L^1(\Omega) \text{ for any } t \in (0,T),$$

where w is the unique solution of the 'mesa problem'

$$\underline{w} \in H^1_0(\Omega),\ \Delta\underline{w} \in L^\infty(\Omega),\ \underline{w} \geq 0,$$
$$0 \leq \Delta\underline{w} + f \leq 1,\ \underline{w}(\Delta\underline{w} + f - 1) = 0 \text{ a.e } \Omega.$$

We extend this result for any function g satisfying the assumptions above, and prove that

$$u_m \to u_\infty \quad \text{in } \mathcal{C}\left((0,T); L^1(\Omega)\right),$$

where u_∞ is the unique solution of

$$(P_\infty) \qquad \begin{cases} u_\infty \in \mathcal{C}\left([0,T);L^1(\Omega)\right) \cap L^\infty_{loc}\left([0,T)\times\overline{\Omega}\right),\ 0 \le u_\infty \le 1, \\ u_\infty(0,.) = \underline{f},\ \exists w_\infty \in L^2_{loc}\left([0,T);H^1_0(\Omega)\right),\ w_\infty \ge 0, \\ w_\infty(u_\infty - 1) = 0 \text{ and } \dfrac{\partial u_\infty}{\partial t} = \Delta w_\infty + g(u_\infty) \text{ in } \mathcal{D}'(Q), \end{cases}$$

with $\underline{f}$ defined above.

Let $\underline{u}$ be the solution of the o.d.e

$$\frac{\partial \underline{u}}{\partial t} = g(\underline{u}) \text{ on } Q, \quad \underline{u}(0,.) = \underline{f} \text{ on } \Omega.$$

Notice that $\underline{u}$ is well defined on Q, $\underline{u} \ge 0$ and that

$$\underline{u} \le 1 \text{ on } Q \quad \Leftrightarrow \quad u_\infty = \underline{u} \text{ on } Q.$$

This is in particular the case if $g(1) \le 0$. In other words, if $g(1) \le 0$ then

$$u_m \to \underline{u} \quad \text{in } \mathcal{C}\left((0,T),L^1(\Omega)\right) \text{ as } m \to \infty.$$

This last convergence has been shown in [14] for $g(u) \equiv -u^p$, by proving again in the perturbed problem all the estimates of the case $g \equiv 0$. Our approach is completely different .

We will obtain the results above in an abstract framework of perturbation of nonlinear problem in a Banach space X. We consider evolutions problems

$$\begin{cases} \dfrac{du_m}{dt} + A_m u_m \ni F_m(.,u_m) \text{ in } (0,T), \\ u_m(0) = u_{0m} \end{cases}$$

for $m = 1,2,...\infty$, where A_m are m-accretive operators in X, $F_m : (0,T) \times \overline{\mathcal{D}(A_m)} \to X$ are Caratheodory functions satisfying assumptions, made precise below (corresponding to (1.1) in the concrete case above) and $u_{0m} \in \overline{\mathcal{D}(A_m)}$. Our main result is : assume that, as $m \to \infty$, $A_m \to A_\infty$ in the sense of resolvent and $F_m \to F_\infty$ in the natural sense (made precise below), if $e^{-tA_m}u_{0m} \to e^{-tA_\infty}u_{0\infty}$ in X for $t > 0$, then $u_m \to u_\infty$ in

$\mathcal{C}\left((0,T);X\right)$.

The asumptions and the main result in the abstract framework are presented in section 2. In section 3, we show how it applies for the concrete problems (P_m) and in section 4, we made present other examples which will be developed in [15].

2 Abstract framework.

Let X be a Banach space with norm $|.|$ and braket $[.,.]$ defined by :

$$[x,y] = \inf_{\lambda>0} \frac{|x+\lambda y| - |x|}{\lambda}.$$

If A is a m-accretive operator in X, i.e $A : X \to \mathcal{P}(X)$ has a nonexpansive resolvent $\mathcal{J}_\lambda^A = (I+\lambda A)^{-1}$ everywhere defined in X for every $\lambda > 0$, then for $u_0 \in \overline{\mathcal{D}(A)}$ (the closure in X of the effective domain $\mathcal{D}(A) = \{x \in X \ ; \ Ax \neq \emptyset\}$) and $f \in L^1_{loc}\left([0,T),X\right)$, the evolution problem

$$\begin{cases} \dfrac{du}{dt} + Au \ni f & \text{on } (0,T) \\ u(0) = u_0 \end{cases}$$

is well posed in the sense of mild solution (or integral solution) (see [3], [5], [12]). If $f \equiv 0$, this solution is given by the exponential formula

$$u(t) = e^{-tA}u_0 := \lim_{n\to\infty} \left(I + \frac{t}{n}A\right)^{-n} u_0.$$

In general one has :

Lemma 1 . *Let A be m-accretive in X, $u_0 \in \overline{\mathcal{D}(A)}$ and $F : (0,T)\times\overline{\mathcal{D}(A)} \to X$ satisfy*

i) *F is Caratheodory , i.e. $t \to F(t,x)$ is measurable for any $x \in \overline{\mathcal{D}(A)}$, and $x \to F(t,x)$ is continuous for a.a. $t \in (0,T)$*

ii) *$[x-\hat{x}, F(t,x) - F(t,\hat{x})] \le k(t)|x-\hat{x}|$, for every x $\hat{x} \in \overline{\mathcal{D}(A)}$, and a.a. $t \in (0,T)$ with $k \in L^1_{loc}([0,T))$*

iii) *$|F(t,x)| \le c(t)$, for any $x \in \overline{\mathcal{D}(A)}$ and a.a. $t \in (0,T)$, with $c \in L^1_{loc}([0,T))$.*

Then there exists a unique $u \in \mathcal{C}([0,T);X)$ *mild solution of*

$$P(A,F,u_0) \qquad \begin{cases} \dfrac{du}{dt} + Au \ni F(.,u) & \text{on } (0,T), \\ u(0) = u_0. \end{cases}$$

Notice that, by the assumptions, $F(.,u) \in L^1_{loc}(0,T,X)$ for any $u \in \mathcal{C}([0,T);X)$. Lemma 1 is well known if $F(t,x) = f(t) + F_0(x)$ (since then $A - F_0 + kI$ is m-accretive) ; we will give a proof of the general case at the end of this section.

We state the main result of this section :

Theorem 1 . *For* $m = 1, 2, ...\infty$, *let* A_m *be m-accretive operators in* X, $u_{0m} \in \overline{\mathcal{D}(A_m)}$, $F_m : (0,T) \times \overline{\mathcal{D}(A_m)} \to X$ *satisfy the assumptions* $i), ii), iii)$ *of Lemma 1 with* k, c *independent of* m *and* $u_m \in \mathcal{C}([0,T);X)$ *be the mild solution of* $P(A_m, F_m, u_{0m})$. *Assume that, as* $m \to \infty$,

$a)$ $(I + A_m)^{-1}x \to (I + A_\infty)^{-1}x$ *in* X *for any* $x \in X$,

$b)$ $F_m(t, x_m) \to F_\infty(t, x_\infty)$ *in* X *for a.a.* $t \in (0,T)$ *and* $(x_m) \in \prod_{m=1,2...\infty} \overline{\mathcal{D}(A_m)}$ *such that* $x_\infty = \lim_{m\to\infty} x_m$,

$c)$ $e^{-tA_m}u_{0m} \to e^{-tA_\infty}u_{0\infty}$ *in* X *for* $t > 0$.

Then $u_m \to u_\infty$ *in* $\mathcal{C}((0,T);X)$ *as* $m \to \infty$.

Proof of Theorem 1. Let first assume, instead of $c)$; that

$$u_{0m} \to u_{0\infty} \quad \text{in } X \text{ as } m \to \infty \ ;$$

as it is well known (see [9]), this assumption (together with $a)$) implies $c)$, more generally (see [3], [5]) : let $f \in L^1_{loc}(0,T,X)$ and, for $m = 1, 2, ...\infty$, v_m be the mild solution of

$$\begin{cases} \dfrac{dv_m}{dt} + A_m v_m \ni f & \text{on } (0,T), \\ v_m(0) = u_{0m} \ ; \end{cases}$$

then $v_m \to v_\infty$ in $\mathcal{C}([0,T);X)$ as $m \to \infty$. We apply this result with $f = F_\infty(., u_\infty)$ such that $v_\infty = u_\infty$. We have (see [3], [5])

$$\begin{aligned} \frac{d}{dt}|u_m - v_m| &\le [u_m - v_m, F_m(.,u_m) - F_\infty(.,u_\infty)] \\ &\le k|u_m - v_m| + |F_m(.,v_m) - F_\infty(.,u_\infty)| \quad \text{in } \mathcal{D}'(0,T) \end{aligned}$$

where we have used the assumption ii). Then

$$|u_m(t) - v_m(t)| \le \int_0^t e^{\int_s^t k(\tau)d\tau} \varepsilon_m(s)ds$$

with $\varepsilon_m = |F_m(.,v_m) - F_\infty(.,u_\infty)|$. Since $v_m \to u_\infty$ in $\mathcal{C}\left([0,T);X\right)$, thanks to b) and iii), one has $\varepsilon_m \to 0$ in $L^1_{loc}([0,T))$, and then $u_m - v_m \to 0$ in $\mathcal{C}\left([0,T);X\right)$. The conclusion $u_m \to u_\infty$ in $\mathcal{C}\left([0,T);X\right)$ follows.

We prove now the result with the general assumption c) ; for $\delta \in (0,T)$, set

$$F_m^\delta(t,x) = \chi_{(\delta,T)} F_m(t,x)$$

and let u_m^δ be the mild solution of $P(A_m, u_{0m}, F_m^\delta)$. Clearly

$$u_m^\delta(t) = e^{-tA_m} u_{0m} \quad \text{for } t \in [0,\delta],$$

and then by assumption c)

$$u_m^\delta(\delta) \to u_\infty^\delta(\delta) \quad \text{as } m \to \infty.$$

Applying the first part of the proof on the interval (δ, T), one has

$$u_m^\delta \to u_\infty^\delta \quad \text{in } \mathcal{C}\left([\delta,T),X\right) \text{ as } m \to \infty.$$

On the other hand, using $ii), iii)$, we have

$$\begin{aligned} \frac{d}{dt}|u_m - u_m^\delta| &\le [u_m - u_m^\delta, F_m(.,u_m) - F_m^\delta(.,u_m^\delta)] \\ &\le c\chi_{[0,\,\delta]} + k|u_m - u_m^\delta|\chi_{(\delta,\,T)} \quad \text{in } \mathcal{D}'(0,T), \end{aligned}$$

such that

$$|u_m(t) - u_m^\delta(t)| \le e^{\int_\delta^t k(\tau)d\tau} \int_0^\delta c(s)ds \quad \text{for } t \in [\delta, T).$$

Then for $0 < \delta \le t_1 < t_2 < T$,

$$\limsup_{m\to\infty,\ t\in[t_1,t_2]} |u_m(t) - u_\infty(t)| \le 2e^{\int_\delta^{t_2} k(\tau)d\tau} \int_0^\delta c(s)ds,$$

such that the conclusion $\quad u_m \to u_\infty \quad$ in $\mathcal{C}\left((0,T);X\right)$ follows.
□

Proof of the Lemma 1. Uniqueness is clear since, if u, $\hat{u}$ are two mild solutions of $P(A,F,u_0)$, one has

$$\begin{aligned} \frac{d}{dt}|u-\hat{u}| &\leq [u-\hat{u}, F(.,u)-F(.,\hat{u})] \\ &\leq k|u-\hat{u}| \quad \text{in } \mathcal{D}'((0,T)), \end{aligned}$$

and then $u=\hat{u}$. To prove existence we may assume $T<\infty$ and $k,c \in L^1(0,T)$.
Let $\mathcal{X}$ be the Banach space $L^1\left(0,T,\rho dt,X\right)$ with the weight $\rho(t)=e^{-\int_0^t(1+k(\tau))d\tau}$ and $\mathcal{A}$ be the operator in $\mathcal{X}$ defined by

$$f\in\mathcal{A}u \qquad \Leftrightarrow \qquad f\in L^1\left(0,T,X\right),\ u\in\mathcal{C}\left([0,T];X\right) \text{ is a mild solution of } \begin{cases} \dfrac{du}{dt}+Au-(k+1)u\ni f & \text{on } (0,T) \\ u(0)=u_0. \end{cases}$$

The operator $\mathcal{A}$ is m-accretive in $\mathcal{X}$; indeed for $f\in\mathcal{A}u$, $\hat{f}\in\mathcal{A}\hat{u}$

$$\rho[u-\hat{u},f-\hat{f}] \geq \rho\frac{d}{dt}|u-\hat{u}|+\rho'|u-\hat{u}| \quad \text{in } \mathcal{D}'(0,T),$$

such that

$$\begin{aligned} [u-\hat{u},f-\hat{f}]_{\mathcal{X}} &= \int_0^T [u-\hat{u},f-\hat{f}]\rho dt \\ &\geq \rho(T)|u(T)-\hat{u}(T)| \geq 0\ ; \end{aligned}$$

on the other hand for $f\in L^1\left(0,T,X\right)$, using Banach fixed point Theorem, one proves the existence of a mild solution of $\dfrac{du}{dt}+Au\ni ku+f$ on $(0,T)$, $u(0)=u_0$, and then $R\left(I+\mathcal{A}\right)=\mathcal{X}$.
Let $\mathcal{B}$ be the map defined on

$$\overline{\mathcal{D}\left(\mathcal{A}\right)}\subset\left\{u\in\mathcal{X}\ ;\ u(t)\in\overline{\mathcal{D}(A)} \text{ a.a. } t\in(0,T)\right\},$$

by $\quad \mathcal{B}u(t)=k(t)u(t)-F(t,u(t))$; using *ii*), $\mathcal{B}$ is accretive and, thanks to *i*) and *iii*), $\mathcal{B}$ is continuous in $\mathcal{X}$. Then $\mathcal{A}+\mathcal{B}$ is m-accretive (see [1], [5]) and

there exists $u \in \overline{\mathcal{D}(\mathcal{A})}$ satisfying $\quad u + \mathcal{A}u + \mathcal{B}u \ni 0$, that is a mild solution of $P(A, F, u_0)$.
□

Remarks. The assumption $iii)$ in Lemma 1 may be relaxed : for instance with the same proof, one has existence of a mild solution to $P(A, F, u_0)$ only with $i), ii)$ and

$$|F(t,x)| \leq c_0|x| + c, \quad c_0 \in \mathbb{R}_+, c \in L^1_{loc}([0,T]).$$

Notice that one could also take multivalued perturbations F_m in Theorem 1 : see [15] for such examples.

3 An application.

Let Ω be a bounded open set in $\mathbb{R}^N$, $f \in L^\infty(\Omega)$ with $f \geq 0$, $T > 0$, $Q = (0,T) \times \Omega$, and $g : Q \times \mathbb{R}_+ \to \mathbb{R}$ satisfy

$i)$ $\quad g(t,x,r)$ is continuous in $r \in \mathbb{R}_+$ for a.a. $(t,x) \in Q$,

$ii)$ $\quad g(.,r) \in L^1_{loc}\left([0,T) \times \overline{\Omega}\right)$ for any $r \in \mathbb{R}_+$,

$iii)$ $\quad \dfrac{\partial g}{\partial r}(t,x,.) \leq K$ in $\mathcal{D}'(0,\infty)$ for a.a. $(t,x) \in Q$ with $K \in \mathcal{C}(\mathbb{R}_+)$,

$iv)$ $\quad g(.,0) \geq 0 \quad$ a.e. on Q,

$v)$ $\quad$ there exists $M \in W^{1,1}_{loc}([0,T))$ such that
$M'(t) \geq g(t,x,M(t)) \quad$ for a.a.. $(t,x) \in Q$ and $M(0) \geq \|f\|_\infty$.

Notice that these assumptions implies

$$g(.,u) \in L^1_{loc}\left([0,T) \times \overline{\Omega}\right) \qquad \text{for any } u \in L^\infty_{loc}\left([0,T) \times \Omega\right)$$

since

$$g(.,R) - \tilde{K}(R).R \leq g(.,r) \leq g(.,0) + \tilde{K}(R).R \quad \text{for } 0 \leq r \leq R,$$

where $\tilde{K}(R) = \max\limits_{[0,R]} K$.
For $m \geq 1$, we consider the problem

$$(P_m) \qquad \begin{cases} u_t = \Delta u^m + g(.,u) & \text{on } Q \\ u = 0 & \text{on } \Sigma = (0,T) \times \partial\Omega \\ u(0,.) = f & \text{on } \Omega. \end{cases}$$

One has :

Lemma 2 . *Under assumptions above, for any $m \geq 1$ there exists a unique solution of (P_m) in the sense*

$$(3.1)\quad \begin{cases} u \in L^\infty_{loc}([0,T) \times \Omega) \cap \mathcal{C}\left([0,T); L^1(\Omega)\right), \; u \geq 0, \; u(0,.) = f(.), \\ u^m \in L^2_{loc}\left((0,T); H^1(\Omega)\right) \; \text{and} \; \dfrac{\partial u}{\partial t} = \Delta u^m + g(.,u) \; \text{in} \; \mathcal{D}'(Q). \end{cases}$$

Moreover $\quad u \leq M \quad$ *a.e. on Q.*

This result follows from the general theory of porous medium problem. We will below relate it to Lemma 1.

As $m \to \infty$, the problem (P_m) formally tends to

$$(P_\infty)\qquad \begin{cases} \dfrac{\partial u}{\partial t} = \Delta w + g(.,u) & \text{on } Q, \\ 0 \leq u \leq 1, \; w \geq 0, \; (u-1)w = 0 & \text{on } Q, \\ w = 0 & \text{on } \Sigma, \\ u(0,.) = \underline{f} & \text{on } \Omega, \end{cases}$$

where $\underline{f} = f\chi_{[\underline{w}=0]} + \chi_{[\underline{w}>0]}$, with $\underline{w}$ the unique solution of the 'mesa problem'

$$\underline{w} \in H^1_0(\Omega), \; \Delta\underline{w} \in L^\infty(\Omega), \; \underline{w} \geq 0,$$
$$0 \leq \Delta\underline{w} + f \leq 1, \; \underline{w}(\Delta\underline{w} + f - 1) = 0 \text{ a.e } \Omega.$$

One has

Theorem 2 . *Under assumptions above, for $m \geq 1$, let u_m be the solution of (P_m), given in Lemma 2. Then,*
1) $\quad u_m \to u_\infty \quad$ *in $\mathcal{C}\left((0,T); L^1(\Omega)\right)$ as $m \to \infty$.*
2) *Assuming $\quad g(.,1) \leq \tilde{g} \quad$ in $\mathcal{D}'(Q)$ with $\tilde{g} \in L^2_{loc}\left([0,T), H^{-1}(\Omega)\right)$, there exists a unique (u,w) solution of (P_∞) in the sense*

$$(3.2)\qquad \begin{cases} u \in \mathcal{C}\left([0,T); L^1(\Omega)\right), \; w \in L^2_{loc}\left((0,T), H^1_0(\Omega)\right), \\ u(0,.) = \underline{f}(.), \; 0 \leq u \leq 1, \; w \geq 0, \; (u-1)w = 0 \\ \text{and} \; \dfrac{\partial u}{\partial t} = \Delta w + g(.,u) \; \text{in} \; \mathcal{D}'(Q), \end{cases}$$

and we have $u_\infty = u$.
3) *Assuming* $g(.,1) \leq 0$ *a.e. on* Q, $u_\infty = \underline{u}$ *where* $\underline{u}$ *is the solution of the o.d.e*

$$\frac{\partial \underline{u}}{\partial t} = g(t,x,\underline{u}) \text{ on } Q, \quad \underline{u}(0) = \underline{f} \text{ on } \Omega.$$

Proof of Theorem 2. To apply the result of Theorem 1, let $X = L^1(\Omega)$ and consider L the (linear) Dirichlet-Laplace operator in $L^1(\Omega)$: $Lu = \Delta u$, with $\mathcal{D}(L) = \{u \in W_0^{1,1}(\Omega)\ ;\ \Delta u \in L^1(\Omega)$ and $\int_\Omega u\Delta v = \int_\Omega v\Delta u$ for any $v \in H_0^1(\Omega) \cap L^\infty(\Omega)$ with $\Delta v \in L^\infty(\Omega)\}$; notice that if Ω has a smooth boundary, then $\mathcal{D}(L) = \{u \in W_0^{1,1}(\Omega)\ ;\ \Delta u \in L^1(\Omega)\}$ (see [10]).
For $m \geq 1$, we define the singlevalued operator A_m in X by

$$A_m u = -\Delta(|u|^{m-1}u), \quad \mathcal{D}(A_m) = \{u \in L^m(\Omega)\ ;\ |u|^{m-1}u \in \mathcal{D}(L)\}.$$

For $m = \infty$, we define the multivalued operator A_∞ in X by

$$A_\infty u = \{-\Delta w\ ;\ w \in \mathcal{D}(L),\ u \in sign(w) \text{ a.e. on } \Omega\}.$$

Thanks to [10], A_m is m-accretive in X for $m \in [1,\infty]$; and, thanks to [6], we have

$$(3.3)\quad (I + A_m)^{-1}u \to (I + A_\infty)^{-1}u \quad \text{in } X \text{ as } m \to \infty \text{ for any } u \in X.$$

At last, thanks to [4], we have

$$(3.4)\qquad e^{-tA_m} f \to e^{-tA_\infty}\underline{f} \quad \text{in } X \text{ for any } t > 0.$$

As in the proof of Lemma 1, we may assume without loss of generality that $T < \infty$ and the function $M(t)$ is bounded on $[0,T)$.
Let $R \geq \max_{[0,T)} M$, and define $F : (0,T) \times X \to X$ by

$$F(t,u) = g(t,.,u^+ \wedge R), \quad \text{for a.a. } t \in (0,T) \text{ and any } u \in L^1(\Omega).$$

Thanks to the assumptions on g, F satisfies the assumptions of Lemma 1, with $k(t) = \tilde{K}(R)$ and $c(t) = \|g(t,.,0)^+\|_1 + \|g(t,.,R)^-\|_1 + |\Omega|R\tilde{K}(R)$. The relation between problems (P_m) and the abstract framework is given by the next Lemma

Lemma 3 . *For $m \geq 1$, the unique mild solution u (see Lemma 1) of*

$$\frac{du}{dt} + A_m u \ni F(.,u) \ \text{on}\ (0,T), \quad u(0) = f \tag{3.5}$$

is caracterised by (3.1) of Lemma 2 (and in particular independent of $R \geq \max_{[0,T)} M$.)

This Lemma 3, together with Lemma 1, proves Lemma 2. Also using Lemma 3 together with (3.3), (3.4) the part 1) of the Theorem 2 follows immediatly from Theorem 1 : actually

$$u_m \to u_\infty \quad \text{in } \mathcal{C}\left((0,T); L^1(\Omega)\right),$$

where u_∞ is the mild solution of

$$\begin{cases} \dfrac{du_\infty}{dt} + A_\infty u_\infty \ni F(.,u_\infty) & \text{on } (0,T), \\ u_\infty(0) = \underline{f}. \end{cases}$$

The part 3) of Theorem 2 is an immediate consequence of the part 2) : if $g(.,1) \leq 0$ a.e. on Q, since $0 \leq \underline{f} \leq 1$, the solution of the o.d.e satisfies $0 \leq \underline{u} \leq 1$ such that $(\underline{u}, 0)$ is the solution of (3.2).

At last since $u_\infty \in \mathcal{C}\left([0,T); L^1(\Omega)\right)$ and $0 \leq u_\infty \leq 1$, the part 2) of Theorem 2 follows clearly from the next Lemma. This will end the proof of the results.

Lemma 4 . *Let $u \in \mathcal{C}\left([0,T); L^1(\Omega)\right)$, $0 \leq u \leq 1$ a.e. on Q and $h \in L^1(Q)$ with $h\chi_{[u=1]} \leq \tilde{g}$ in $\mathcal{D}'(Q)$ where $\tilde{g} \in L^2\left(0,T; H^{-1}(\Omega)\right)$. Then u is a mild solution of*

$$\frac{du}{dt} + A_\infty u \ni h \quad \text{on } (0,T) \tag{3.6}$$

iff

$$\begin{cases} \exists w \in L^2\left(0,T; H_0^1(\Omega)\right), \ w \geq 0, \ w(u-1) = 0 \\ \text{and } \dfrac{\partial u}{\partial t} = \Delta w + h \ \text{in } \mathcal{D}'(Q). \end{cases}$$

□

Proof of Lemma 3. First we show that the mild solution u of (3.5), satisfy $0 \le u \le M$; as a consequence the mild solution is independent of $R \ge \max_{[0,T]} M$. Recall that A_m is T-accretive. We have

$$\begin{aligned} \frac{d}{dt}\int (0-u(t))^+ &\le \int_{[0\ge u(t)]} (0-F(t,u(t)))^+ \\ &\le \int (-g(t,.,0))^+ \;=\; 0, \end{aligned}$$

and then $u \ge 0$ a.e. on Q. On the other hand $v \equiv M \in W^{1,1}\Big(0,T;L^1(\Omega)\Big)$ is a supersolution of $\quad \dfrac{dv}{dt} + A_m v \ni M' \quad$ in the sense of [2] ; then we have

$$\begin{aligned} \frac{d}{dt}\int (u(t)-M(t))^+ &\le \int_{[u(t)\ge M(t)]} (F(t,u(t))-M'(t))^+ \\ &\le \int (g(t,.,M(t))-M'(t))^+ + k(t)\int (u(t)-M(t))^+ \\ &\le k(t)\int (u(t)-M(t))^+ \end{aligned}$$

and the conclusion $u \le M$ follows.

Denote by H the Hilbert space $H^{-1}(\Omega)$ with the scalar product $(.,.)_H = \; < (-\Delta)^{-1}.,. >$, where $< .,. >$ is the duality between $H_0^1(\Omega)$ and $H^{-1}(\Omega)$, and $\phi :\; H \to [0,\infty)$ be the convex l.s.c functionnal defined by

$$\phi(u) = \frac{1}{m+1}\int_\Omega |u|^{m+1} \quad \text{on} \quad \mathcal{D}\left(\phi\right) = L^{m+1}(\Omega).$$

One has (see [8])

$$\begin{aligned} &\partial\phi(u) = -\Delta(|u|^{m-1}u) \text{ on} \\ &\mathcal{D}(\partial\phi) = \Big\{u \in L^{m+1}(\Omega)\ ;\ |u|^{m-1}u \in H_0^1(\Omega)\Big\}\ ; \end{aligned}$$

in particular $\partial\phi \cap (L^1(\Omega)\times L^1(\Omega)) = A_m \cap (H\times H)$. Denote by Y the space $L^1(\Omega) + H^{-1}(\Omega)$ endowed with the norm

$$\|u\|_Y = \inf\Big\{\|u_1\|_{L^1} + \|u_2\|_{H^{-1}} \;\; ; \;\; u_1 \in L^1(\Omega),\ u_2 \in H^{-1}(\Omega)\ ; \; u = u_1 + u_2\Big\}.$$

We have $\overline{A_m}^Y = \overline{\partial\phi}^Y$, and by classical interpolation, this operator denoted by B is m-accretive in Y.

Now let u be solution of (3.1). Since $h = g(.,u) \in L^1(Q)$ and $u^m \in L^2\left(0,T;H_0^1(\Omega)\right)$, we have $u \in W^{1,1}\left(0,T,L^1(\Omega)+H^{-1}(\Omega)\right)$ and $\dfrac{du}{dt}(t) + \partial\phi(u(t)) \ni h(t)$ for a.e. $t \in [0,T)$; then u is mild solution (in Y) of $\dfrac{du}{dt} + Bu \ni h$; since the mild solution (in X) of $\dfrac{du}{dt} + Au \ni h$, $u(0) = f$ is clearly mild solution (in Y) of $\dfrac{du}{dt} + Bu \ni h$ it follows that u is actually mild solution (in X) of $\dfrac{du}{dt} + A_m u \ni h$. We may assume $R \geq \|u\|_\infty$, such that $h = F(.,u)$ and then u is the mild solution of (3.5).

To end up the proof we show that the mild solution u of (3.5) satisfies (3.1). We already know that $u \in L^\infty(Q)$, $u \geq 0$, $h := F(.,u) = g(.,u)$. Set $h_{n,l} = (h \wedge n) \vee (-l)$ and let $u_{n,l}$ be the mild solution of

$$\frac{du_{n,l}}{dt} + A_m u_{n,l} \ni h_{n,l}\ , \quad u_{n,l}(0) = f.$$

We have $u_{n,l} \downarrow u_n$ as $l \uparrow \infty$ and $u_n \uparrow u$ as $n \uparrow \infty$. Since $h_{n,l} \in L^\infty(Q) \subset L^2\left(0,T;H^{-1}(\Omega)\right)$, $u_{n,l}$ is solution (in H) of $\dfrac{du_{n,l}}{dt} + \partial\phi(u_{n,l}) \ni h_{n,l}$, that is

$$\text{(3.7)} \qquad \begin{cases} w_{n,l} := |u_{n,l}|^{m-1}u_{n,l} \in L^2\left(0,T;H_0^1(\Omega)\right) \text{ and} \\ \dfrac{\partial u_{n,l}}{\partial t} = \Delta w_{n,l} + h_{n,l} \quad \text{in } \mathcal{D}'(Q). \end{cases}$$

First, since $(u_{n,l}^+)^m \in L^2\left(0,T;H_0^1(\Omega)\right)$, we have

$$\iint |\nabla(u_{n,l}^+)^m|^2 \leq \frac{1}{m+1}\int_\Omega f^{m+1} + \iint h_{n,l}(u_{n,l}^+)^m\ ;$$

since

$$\iint h_{n,l}(u_{n,l}^+)^m \leq \iint (h^+ \wedge n)(u_{n,l}^+)^m \downarrow \iint (h^+ \wedge n)(u_n^+)^m \text{ as } l \uparrow \infty$$

and

$$\iint (h^+ \wedge n)(u_n^+)^m \leq \iint h^+ u^m \leq \|h\|_1 R^m,$$

we deduce that

$$\limsup_{n\to\infty}\limsup_{l\to\infty}\iint |\nabla(u_{n,l}^{+})^{m}|^{2} < \infty,$$

and then $u^m = u^{+m} \in L^2\left(0,T;H_0^1(\Omega)\right)$.
On the other hand integrating (3.7) in time,

$$-\Delta(\int_0^T |w_{n,l}(s,.)ds) = f - u_{n,l}(T,.) + \int_0^T h_{n,l}(s,.)ds$$

is bounded in $L^1(\Omega)$, and then $\iint w_{n,l}$ is bounded ; by monotone convergence Theorem, it follows that $w_{n,l} \to w_n := |u_n|^{m-1}u_n$ in $L^1(Q)$ as $l \to \infty$ and $w_n \to u^m$ in $L^1(Q)$; passing to the limit in (3.7), we get $\dfrac{\partial u}{\partial t} = \Delta u^m + h$ in $\mathcal{D}'(Q)$. □

Proof of Lemma 4. To proof the 'only if' part, we exactly follow the second part of the proof of Lemma 3, using the l.s.c convex functionnal ϕ on $H = H^{-1}(\Omega)$, defined by

$$\phi(u) = 0 \quad \text{on} \quad \mathcal{D}(\phi) = \{u \in L^\infty(\Omega)\ ;\ |u| \leq 1\}.$$

We have,

$$\partial\phi(u) = \Big\{-\Delta w\ ;\ w \in H_0^1(\Omega),\ u \in \ \mathrm{sign}(w)\Big\}$$

and as in the proof of Lemma 3 : $\partial\phi \cap (L^1(\Omega)\times L^1(\Omega)) = A_\infty \cap (H \times H)$, $B = \overline{A_\infty}^Y = \overline{\partial\phi}^Y$ is m-acretive in Y. For $h \in L^1(Q)$, if $u \in \mathcal{C}\left([0,T);L^1(\Omega)\right)$ satisfies

$$\begin{cases} \exists w \in L^2\left(0,T;H_0^1(\Omega)\right),\ u \in \ \mathrm{sign}(w) \text{ a.e. on } Q \text{ and} \\ \dfrac{\partial u}{\partial t} = \Delta w + h \text{ in } \mathcal{D}'(Q), \end{cases}$$

then u is mild solution in Y of $\dfrac{du}{dt} + Bu \ni h$ and then it is mild solution in X of (3.6).

Conversly let u be mild solution of $\dfrac{du}{dt} + A_\infty u \ni h$ with $0 \leq u \leq 1$ and $h\chi_{[u=1]} \leq \tilde{g} \in L^2\left(0,T;H^{-1}(\Omega)\right)$. As in the proof of Lemmma 3, let

$h_{n,l} = (h \wedge n) \vee (-l)$ and $u_{n,l}$ the corresponding solution ; there exists $w_{n,l} \in L^2\left(0,T;H_0^1(\Omega)\right)$ such that

$$\begin{cases} u_{n,l} \in \text{sign}(w_{n,l}) \text{ a.e. on } Q, \text{ and} \\ \dfrac{\partial u_{n,l}}{\partial t} = \Delta w_{n,l} + h_{n,l} \text{ in } \mathcal{D}'(Q). \end{cases}$$

$w_{n,l}$ is unique and actually

$$w_{n,l} = w - L^2\left(0,T;H_0^1(\Omega)\right) - \lim_{m \to \infty} w_{n,l}^{(m)},$$

where $u_{n,l}^{(m)}$ and $w_{n,l}^{(m)}$ are the solution of (3.7) with $u_{n,l}^{(m)}(0) = \underline{f}$; indeed $u_{n,l}^{(m)} \to u_{n,l}$ in $\mathcal{C}\left([0,T);L^1(\Omega)\right)$ as $m \to \infty$ and $\left\{\dfrac{\partial u_{n,l}^{(m)}}{\partial t} \ ; \ m \geq 1\right\}$ is bounded in $L^2\left(0,T;H^{-1}(\Omega)\right)$ since

$$\|\frac{\partial u_{n,l}^{(m)}}{\partial t}\|^2_{L^2(0,T,H^{-1}(\Omega))} \leq \frac{|\Omega|}{m+1} + \|h_{n,l}\|^2_{L^2(0,T,H^{-1}(\Omega))}\|\frac{\partial u_{n,l}^{(m)}}{\partial t}\|^2_{L^2(0,T,H^{-1}(\Omega))}.$$

It follows that $w_{n,l} \downarrow w_n$ as $l \uparrow \infty$ and $w_n \uparrow w$ as $n \uparrow \infty$; we have $u_n \in \text{sign}(w_n)$ and $u \in \text{sign}(w)$; then in particular $w_n^+ = 0$ on $[u < 1]$ and $w \geq 0$. We have

$$\begin{aligned} \iint |\nabla w_{n,l}^+|^2 &= \iint h_{n,l} w_{n,l}^+ \\ &\leq n \iint w_{n,l}^+. \end{aligned}$$

It follows that $w_{n,l}^+ \in L^2\left(0,T;H_0^1(\Omega)\right)$, $(h \wedge n)w_{n,l}^+ \in L^1(Q)$ and

$$\begin{aligned} \iint |\nabla w_{n,l}^+|^2 &\leq \iint (h \wedge n) w_{n,l}^+ = \iint_{[u=1]} (h \wedge n) w_{n,l}^+ \\ &\leq \iint_{[u=1]} h w_{n,l}^+ \leq < \tilde{g}, w_{n,l}^+ > \\ &\leq C \|\nabla w_{n,l}^+\|_{L^2(0,T,H_0^1(\Omega)} \ ; \end{aligned}$$

then $w \in L^2\left(0,T;H_0^1(\Omega)\right)$. Exactly like in the proof of Lemma 3, we have $\dfrac{\partial u}{\partial t} = \Delta w + h$ in $\mathcal{D}'(Q)$ and this end up the proof. □

4 Remarks.

Similar results may be obtained for other boundary value problems. Let us mention the following cases developped in [15] :

a) Neuman boundary conditions : We assume that Ω has a sufficiently smooth boundary, g and f being as in section 3. For $m \geq 1$ there exists a unique u_m solution of

$$u_t = \Delta u^m + g(.,u) \text{ on } Q, \ \frac{\partial u^m}{\partial n} = 0 \text{ on } \Sigma, \ u(0,.) = f \text{ on } \Omega,$$

in the sense

$$\begin{cases} u \in \mathcal{C}\left([0,T);L^1(\Omega)\right) \cap L^\infty_{loc}([0,T)\times\overline{\Omega}), \ u \geq 0, \ u^m \in L^2_{loc}\left([0,T),H^1(\Omega)\right) \\ \text{and } \iint u\frac{\partial \xi}{\partial t} + \iint g(.,u)\xi = \iint \nabla u^m \nabla \xi + \int f\xi(0,.), \ \forall \xi \in \mathcal{C}^\infty(\overline{Q}) \\ \text{with } supp(\xi) \subset [0,T)\times\overline{\Omega}. \end{cases}$$

In the case $g \equiv 0$, it is shown in [4] that $u_m \to \underline{f}$ in $\mathcal{C}\left((0,T);L^1(\Omega)\right)$ as $m \to \infty$, where

$$\begin{cases} \underline{f} \equiv 1 \quad \text{if } \frac{1}{|\Omega|}\int_\Omega f \geq 1, \\ \underline{f} = f\chi_{[\underline{w}=0]} + \chi_{[\underline{w}>0]} \quad \text{if } \frac{1}{|\Omega|}\int_\Omega f < 1, \end{cases}$$

with $\underline{w}$ the unique solution of the variationnal problem

$$\underline{w} \in H^2(\Omega), \ \underline{w} \geq 0, \ 0 \leq \Delta\underline{w} + f \leq 1,$$
$$\underline{w}(\Delta\underline{w} + f - 1) = 0 \text{ a.e } \Omega \ \text{ and } \frac{\partial \underline{w}}{\partial n} = 0 \text{ on } \Sigma.$$

With the same technics, the corresponding conclusion of Theorem 2 holds :

i) $u_m \to u_\infty$ in $\mathcal{C}\left((0,T);L^1(\Omega)\right)$ as $m \to \infty$.

ii) If $\iint g(.,1)\xi \leq \iint \tilde{g}_0\xi + \sum_{i=1}^{i=N} \tilde{g}_i \frac{\partial \xi}{\partial x_i}$ for any $\xi \in \mathcal{C}^\infty(\overline{Q})$, $\xi \geq 0$ and

$supp(\xi) \subset [0,T) \times \overline{\Omega}$, with $g_0 ... g_n \in L^2_{loc}\left([0,T) \times \overline{\Omega}\right)$, then there exists a unique (u,v) solution of

$$\begin{cases} u \in \mathcal{C}\left([0,T); L^1(\Omega)\right) \cap L^\infty_{loc}([0,T) \times \overline{\Omega}),\ w \in L^2_{loc}\left([0,T), H^1(\Omega)\right), \\ 0 \le u \le 1,\ w \ge 0,\ w(u-1) = 0 \text{ and } \iint u \dfrac{\partial \xi}{\partial t} + \iint g(.,u)\xi = \\ \iint \nabla w \nabla \xi + \int f \xi(0,.),\ \forall \xi \in \mathcal{C}^\infty(\overline{Q}) \text{ with } supp(\xi) \subset [0,T) \times \overline{\Omega}. \end{cases}$$

and we have $\quad u_\infty = u.$

iii) If $g(.,1) \le 0$, then $u_\infty = \underline{u}$ where $\underline{u}$ is the solution of the o.d.e

$$\frac{\partial \underline{u}}{\partial t} = g(t,x,\underline{u}) \text{ on } Q, \quad \underline{u}(0) = \underline{f} \text{ on } \Omega.$$

b) Cauchy problem : Let $\Omega = \mathbb{R}^N$ and $g,\ f$ satisfies the assumptions of section 3 with moreover $g(.,0) \in L^1(Q_\tau)$ for any $\tau \in (0,T)$ where $Q_\tau = [0,\tau) \times \mathbb{R}^N$ and $f \in L^1(\mathbb{R}^N)$. Then for any $m \ge 1$, there exists a unique solution u_m of

$$\begin{cases} u \in \mathcal{C}\left([0,T); L^1(\Omega)\right) \cap L^\infty(Q_\tau) \text{ for any } \tau \in (0,T),\ u \ge 0, \\ u(0,.) = f(.),\ g(.,u) \in L^1(Q_\tau) \text{ for any } \tau \in (0,T), \text{ and} \\ \dfrac{\partial u}{\partial t} = \Delta u^m + g(.,u) \text{ in } \mathcal{D}'(Q). \end{cases}$$

As $m \to \infty$, $u_m \to u_\infty$ in $\mathcal{C}\left((0,T); L^1(\Omega)\right)$, where u_∞ is the unique solution of

$$\begin{cases} u_\infty \in \mathcal{C}\left([0,T); L^1(\Omega)\right),\ 0 \le u_\infty \le 1,\ u_\infty(0,.) = \underline{f},\ g(.,u_\infty) \in L^1(Q_\tau) \\ \text{for any } \tau \in (0,T),\ \exists w_\infty \in L^1(Q_\tau) \text{ for any } \tau \in (0,T) \text{ s.t. } w_\infty \ge 0, \\ w_\infty(u_\infty - 1) = 0 \text{ and } \dfrac{\partial u}{\partial t} = \Delta w + g(.,u) \text{ in } \mathcal{D}'(Q), \end{cases}$$

where $\underline{f} = f\chi_{[\underline{w}=0]} + \chi_{[\underline{w}>0]}$, $\underline{w}$ is the unique solution of the mesa problem

$$\begin{gathered} w \in H^2(\Omega),\ w \ge 0,\ 0 \le \Delta w + f \le 1, \\ \text{and } w(\Delta w + f - 1) = 0 \text{ a.e } \mathbb{R}^N. \end{gathered}$$

The case $g \equiv 0$ is shown in [4] (see also [11]). In the case $g(.,1) \leq 0$, $u_\infty = \underline{u}$ the unique solution of the o.d.e :

$$\frac{\partial \underline{u}}{\partial t} = g(t,x,\underline{u}) \text{ on } Q, \quad \underline{u}(0) = \underline{f} \text{ on } \mathbb{R}^N.$$

a) Nonlinear diffusion : Let Ω, g and f as in **b**) and $1 < p < \infty$. For any $m \geq 1$, there exists a unique solution u_m of

$$u_t = \Delta_p u^m + g(.,u) \text{ on } Q, \ u(0,.) = f \text{ on } \mathbb{R}^N,$$

in the sense

$$\begin{cases} u \in \mathcal{C}\left([0,T); L^1(\Omega)\right) \cap L^\infty(Q_\tau) \text{ for any } \tau \in (0,T), \ u \geq 0, \\ u(0,.) = f(.), \ g(.,u) \in L^1(Q_\tau) \text{ for any } \tau \in (0,T), \\ u^m \in L^p_{loc}\left([0,T); W^{1,p}(\mathbb{R}^N)\right) \text{ and } \dfrac{\partial u}{\partial t} = \Delta_p u^m + g(.,u) \text{ in } \mathcal{D}'(Q). \end{cases}$$

Assuming that f is radial nonincreasing, i.e. $f(x) = \tilde{f}(|x|)$ with $\tilde{f} : \mathbb{R}^N \to \mathbb{R}_+$ nonincreasing, then $u_m \to u_\infty$ in $\mathcal{C}\left((0,T); L^1(\mathbb{R}^N)\right)$ as $m \to \infty$. If moreover $g(.,1) \leq 0$, then $u_\infty = \underline{u}$ the unique solution of the o.d.e.

$$\frac{\partial \underline{u}}{\partial t} = g(t,x,\underline{u}) \text{ a.e. } Q, \quad \underline{u}(0) = f\chi_{[|x|<a]} + \chi_{[|x|\geq a]} \text{ on } \mathbb{R}^N,$$

with a the unique positive number such that

$$\int_0^1 \tilde{f}(ar)dr^N = 1.$$

The case $g \equiv 0$ is shown in [7] (see also [15]).

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