ON A DUAL FORMULATION FOR THE GROWING SANDPILE PROBLEM WITH MIXED BOUNDARIES CONDITIONS

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ABSTRACT. We study a Prigozhin model type for a growing sandpile with mixed boundaries conditions. Using semi-group theory, we prove the existence and uniqueness of the solution for the model. For numerical analysis we use duality approach.

1. INTRODUCTION

In this paper we are interested in the theorical and numerical study of the Prigozhin model for growing sandpile in the case of mixed boundary condition, that corresponds to the following PDE:

$$(P)\begin{cases} u_t - \nabla.\left(m\nabla u\right) = f & \text{in} \quad [0,T]\times\Omega \\ |\nabla u| \le 1, \quad m \ge 0 & m\left(|\nabla u|-1\right)=0 \ \text{ in } \ [0,T]\times\Omega \\ u = 0 & \text{on} \quad [0,T]\times\Gamma_D \\ m\frac{\partial u}{\partial \nu} = g & \text{on} \quad [0,T]\times\Gamma_N \\ u\left(0\right) = u_0 & \end{cases}$$

where $\Omega \subset \mathbb{R}^D$ is a bounded open domain with boundary Γ, in this model Γ is splitting into two parts Γ_D and Γ_N, in the boundary Γ_D we are applying a homogeneous Dirichlet condition on u, in other word we are assume that at this boundary the sand cand fall down. The boundary Γ_N correspond to nonhomogenous Neumann boundary condition, this situation correspond to the case where in this part the sand is locked by a wall. Solution u is the height of the surface, f represente the source and $m = m(x,t)$ is an unknown scalar function.

Let's recall that the open table case i.e when one imposes Dirichlet condition on the whole boundary Γ was studied by many authors by duality arguments (see [3,17,18,28]). In this situation, for theorical and numerical analysis of the problem in [3] the authors are used a dual problem in the space of vector valued Radon measure. Another approach based on semi-group theory was introduced by Dumont and Igbida [17,18]. More precisely, in [17] the following problem was investiguated

$$\begin{cases} u_t - \nabla.\left(m\nabla u\right) = f \quad \text{in} \quad [0,T]\times\Omega \\ |\nabla u| \le 1, m \ge 0, \quad m\left(|\nabla u|-1\right)=0 \quad \text{in} \quad [0,T]\times\Omega \\ u = 0 \quad \text{on} \quad [0;T]\times\partial\Omega \\ u\left(0\right) = u_0, \end{cases} \tag{1.1}$$

in this paper authors are used the semi-group theory to prove the existence and uniqueness of the solution of Euler implicite discretization problem in time associated with (1.1), the that permitted them to prove the existence and uniqueness of variational solution of the problem (1.1). For numerical analysis they begin by transform the Euler implicite

discretization problem onto projections problems on convex sets, then they use an Gauss Seidel algorithm type to compute the solution of dual problem associated to the every projection problem.
Our main goal here is to extend the approach developped in [17] to a larger boundary condition which includes Dirichlet boundary condition. Firstly, we use the nonlinear semi-group theory to get the existence and uniqueness of variational solution of problem (P). Next we show that when $t \to \infty$ this solution converges to some function $\tilde{u}$ where $\tilde{u}$ is solution of a stationary problem associated with (P). For numerical analysis we focalise our attention on the Euler implicit discretization in time problem associated with (P), we show how one can compute the solution of discretization via duality argument. The paper is organized as follows : the next section is devoted to the theorical analysis of (P). In section 3, we expose our method for the computation of the solution of the Euler implicite discretization problem associated with (P). At last, in section 4, we give some results of numerical simulations for (P).

2. Theorical Study of problem (P)

For $\epsilon > 0$, we say that $(t_i, f_i, g_i)_{i=1,\ldots,n}$ is ε-discretization for the problem (P), if $t_0 = 0 < t_1 < \ldots < t_{n-1} < T = t_n$ with $t_i - t_{i-1} \leq \epsilon$, $f_1, \cdots, f_n \in L^2(\Omega)$, $g_1, \ldots, g_n \in L^2(\Gamma_N)$, such that

$$\sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \|f(t) - f_i\|_{L^2(\Omega)} \leq \epsilon$$

and

$$\sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \|g(t) - g_i\|_{L^2(\Gamma_N)} \leq \varepsilon.$$

For any $\varepsilon > 0$, we say that u_ε is an ε-approximate solution of (P), if there exists $(t_i, f_i, g_i)_{i=1,\cdots,n}$ an ε-discretization for the problem (P) such that

$$u_\varepsilon(t) = \begin{cases} u_0 & \text{for} \quad t \in]0, t_1] \\ u_i & \text{for} \quad t \in]t_{i-1}, t_i], i = 1, \ldots, n \end{cases} \tag{2.1}$$

and u_i solves the Euler implicit time discretization of (P)

$$\begin{cases} |\nabla u_i| \leq 1, \ \exists m_i \geq 0, m_i(|\nabla u_i| - 1) = 0 \quad \text{in} \quad \Omega \\ u_i - \varepsilon \nabla.(m_i \nabla u_i) = \varepsilon f_i + u_{i-1} \quad \text{in} \quad \Omega \\ u_i = 0 \quad \text{on} \quad \Gamma_D \\ m_i \frac{\partial u_i}{\partial \nu} = g_i \quad \text{on} \quad \Gamma_N. \end{cases} \tag{2.2}$$

The problem (2.2) is particular case of the following problem

$$(S_2) \begin{cases} v - \nabla.(m \nabla v) = f & \text{in} \quad \Omega \\ |\nabla v| \leq 1, \ m \geq 0, & m(|\nabla v| - 1) = 0 \ \text{in} \ \Omega \\ v = 0 & \text{on} \quad \Gamma_D \\ m \dfrac{\partial v}{\partial \nu} = g & \text{on} \quad \Gamma_N, \end{cases}$$

where $f \in L^2(\Omega)$ and $g \in L^2(\Gamma_N)$. It clear that the study of (S_2) give somes ideas for solving the problem (2.2). In the following, we prove that the solution v of (S_2) is

also solution of some minimization problem. For numerical analysis of the minimization problem associated with of (S_2), we use duality arguments. This method was already used in many papers (see [2, 17, 18, 28]).
Let's recall that the stationnary problem associated with (P) is given by

$$(S_1)\begin{cases} -\nabla.\left(m\nabla\tilde{u}\right)=\tilde{f} & \text{in} \quad \Omega \\ |\nabla\tilde{u}|\leq 1, \quad m\geq 0, & m\left(|\nabla u|-1\right)=0 \text{ in } \Omega \\ \tilde{u}=0 & \text{on} \quad \Gamma_D \\ m\dfrac{\partial\tilde{u}}{\partial\nu}=g & \text{on} \quad \Gamma_N, \end{cases}$$

The problem like (S_1) appears in the study of the optimal mass transport problem of the Monge-Kantorovich type (cf [1, 20, 21].
As in [17], we introduce the following set

$$K=\left\{z\in W^{1,\infty}\left(\Omega\right)\cap H_D^1\left(\Omega\right);\ |\nabla z\left(x\right)|\leq 1\,\text{a.e.}\,x\in\Omega\right\}$$

with

$$H_D^1\left(\Omega\right)=\left\{z\in H^1\left(\Omega\right);\ z=0 \text{ on } \Gamma_D\right\}.$$

We define $\Pi_K$the indicator function of K defined by

$$\Pi_K\left(z\right)=\begin{cases} 0 & \text{if}\, z\in K \\ +\infty & \text{otherwise,}\end{cases}$$

and the function $F_g : L^2\left(\Omega\right)\to\mathbb{R}$ by

$$F_g\left(z\right)=\begin{cases} -\displaystyle\int_{\Gamma_N} gzds & \text{if}\, z\in K \\ +\infty & \text{otherwise,}\end{cases}$$

We denote by ∂F_g the sub-differential of F_g in $L^2\left(\Omega\right)$ defined by $f\in\partial F_g\left(v\right)$ if only if $F_g\left(z\right)\geq F_g\left(v\right)+\left(f,z-v\right)$ for any $z\in L^2\left(\Omega\right)$, which is equivalent to

$$\int_\Omega f\left(z-v\right)dx+\int_{\Gamma_N} g\left(z-v\right)ds\leq 0.$$

With a view to define a notion of variational for problems (S_1), (S_2), and (P) we do the proof of the following results

Lemma 2.1. *If $v\in K$ is a solution of (S_2) in the sense that*

$$\int_\Omega vzdx+\int_\Omega m\nabla v.\nabla zdx=\int_\Omega fzdx+\int_{\Gamma_N} gzds \quad \forall z\in H_D^1\left(\Omega\right), \tag{2.3}$$

then v is also solution of the following optimization problem

$$\max_{z\in K}\left\{\int_\Omega\left(f-v\right)zdx+\int_{\Gamma_N} gzds\right\}. \tag{2.4}$$

Proof Taking $z\in K$ as test function, we get

$$\int_\Omega\left(f-v\right)zdx+\int_{\Gamma_N} gzds=\int_\Omega m\nabla v.\nabla zdx. \tag{2.5}$$

Thus using the fact that $|\nabla z| \leq 1$, we get

$$\int_{\Omega} (f - v)\, z dx + \int_{\Gamma_N} gzds \leq \int_{\Omega} m\, |\nabla v|\, dx \tag{2.6}$$

and taking $z = v$ in (2.5) we obtain

$$-\int_{\Omega} (f - v)\, v dx - \int_{\Gamma_2} gzds = \int_{\Omega} -m\, |\nabla v|^2\, dx. \tag{2.7}$$

So adding (2.6) and (2.7) and using the fact that $m\,(|\nabla v| - 1) = 0$ a.e. in Ω we obtain

$$\int_{\Omega} (f - v)\, z + \int_{\Gamma_N} gzds - \left(\int_{\Omega} (f - v)\, v + \int_{\Gamma_N} gvds \right) dx \leq 0 \tag{2.8}$$

□

Lemma 2.2. (1) *∂F_g is maximal monotone graph in $L^2(\Omega)$.*
(2) *v is solution of* (2.4) *if and only if $v + \partial F_g(v) \ni f$.*

Proof 1. It is not difficult to see ∂F_g is monotone, to end the proof of this part it suffices to show that F_g is closed. Let $G(F_g) = \{(z, F_g(z))\ ;\ z \in K\}$ the graph of F_g and considering $(z_n, F_g(z_n))$ a subsequences in $G(F_g)$ such that

$$z_n \to z \text{ in } L^2(\Omega) \text{ and } F_g(z_n) \to Z \text{ in } \mathbb{R}.$$

Since the restriction of F_g on K is linear and continuous application, we deduce that

$$\lim F_g(z_n) = F_g(z) = -\int_{\Gamma_N} gzd\sigma.$$

Hence $(z, Z) = (z, F_g z) \in G(F_g)$ ie F_g is closed.
2. If v is solution of

$$v + \partial F_g(v) \ni f$$

then

$$F_g(z) \geq F_g(v) + (f - v, z - v) \text{ for any } z \in K$$

which implies that

$$(f - v, v) - F_g(v) \geq (f - v, z) - F_g(z) \text{ for any } z \in K$$

ie

$$\int_{\Omega} (f - v)\, v dx + \int_{\Gamma_N} gvds \geq \int_{\Omega} (f - v)\, z dx + \int_{\Gamma_N} gzd\sigma \text{ for any } z \in K.$$

Now suppose that v is solution of (2.4) and let $z \in L^2(\Omega)$.
If $z \in K$ we have

$$\int_{\Omega} (f - v)\, v dx + \int_{\Gamma_N} gvds \geq \int_{\Omega} (f - v)\, z dx + \int_{\Gamma_N} gzds$$

it follows that

$$-\int_{\Gamma_N} gvds \geq -\int_{\Gamma_N} gzds + \int_{\Omega} (f - v)\,(z - v)\, dx$$

thus we have

$$F_g(z) \geq F_g(v) + (f - v, z - v).$$

Otherwise we have

$$F_g(z) = +\infty \geq F_g(v) + (f - v, z - v)$$

□

Definition 2.3. For a given $f, \tilde{f} \in L^2(\Omega)$, $g \in L^2(\Gamma_N)$ we say that v is a variational solution of (S_1) (resp.(S_2)) if $v \in K$ and $\int_\Omega \tilde{f}(z-v)\,dx + \int_{\Gamma_N} g(z-v)\,d\sigma \leq 0$ (resp. $\int_\Omega (f-v)(z-v)\,dx + \int_{\Gamma_N} g(z-v)\,d\sigma \leq 0$) for any $z \in K$.

Definition 2.4. For a given $f \in L^2_{loc}(0,T;L^2(\Omega))$, $g \in L^2_{loc}(\Gamma_N)$ and $u_0 \in K$, we say that u (resp. u_ε) is a variational solution (resp. ε-approximate solution) of (P) if $u \in W^{1,1}\left(0,T;L^2(\Omega)\right)$, $u(0) = u_0$ and for any $t \in (0,T)$, $u(t) \in K$ and $\int_\Omega (f - u_t(t))(z - u(t))\,dx + \int_{\Gamma_N} g(z - u(t))\,d\sigma \leq 0$ for any $z \in K$ (resp. u_ε is given by (2.1) and u_i is a variational solution of (2.2)).

By using the nonlinear semi-group theory in Hilbert space for evolution problems governed by a sub-differential operator ($cf.$[9]), we have the following result.

Theorem 2.5. *Let $u_0 \in K$, $T > 0$ and $f \in L^2_{loc}(0,T;L^2(\Omega))$, $g \in L^2_{loc}(\Gamma_N)$. Then,*

1. *for any $\varepsilon > 0$ and any ε-discretization of (P), there exists a unique ε-approximate variational solution of (P).*
2. *There exist $u \in \mathcal{C}\left([0,T);L^2(\Omega)\right)$ such that $u(0) = u_0$, and as $\varepsilon \to 0$,*

$$u_\varepsilon \to u \ \text{ in } \mathcal{C}\left([0,T);L^2(\Omega)\right).$$

3. *The u function given by 2. is a unique variationnal solution of (P). Moreover if for $i = 1, 2$ u_i is a solution corresponding for f_i, then*

$$\frac{d}{dt}\int_\Omega (u_1 - u_2)^+ \leq \int_\Omega (f_1 - f_2)^+ \ \text{ in } \mathcal{D}'(0,T).$$

In particular, if $f \geq 0$, then $u \geq 0$ a.e. in Ω

By using Theorem 3.11 of [9], we have the following

Theorem 2.6. *Let $f \in L^2_{loc}(0,\infty;L^2(\Omega))$, $u_0 \in K$ and u be the variational solution of (P). If there exists $f_\infty \in L^2(\Omega)$ such that $f - f_\infty \in L^2_{loc}(0,\infty;L^2(\Omega))$ then there exists $u_\infty \in K$ such that $u_\infty \in K$ is a variational solution of (S_1), and, as $t \to \infty$, $u(t) \to u_\infty$ in $L^2(\Omega)$.*

3. Dual formulation and numerical approximation of the maximization problem

In this section we present and treat the dual problem associated with (S_2). At first we have the following result.

Lemma 3.1. *Let $v \in K$, then v is solution of* (2.4) *if and only if v is solution of the following minimization problem :*

$$\min_{z\in K}\left\{\frac{1}{2}\int_{\Omega}|z-f|^2\,dx-\int_{\Gamma_N} gzds\right\}. \tag{3.1}$$

Proof Let $v \in K$ be a solution of (2.4). Then we have

$$\int_{\Omega}(f-v)vdx+\int_{\Gamma_N} gvd\sigma \geq \int_{\Omega}(f-v)zdx+\int_{\Gamma_N} gzds \text{ for all } z\in K.$$

Hence

$$\int_{\Omega}(f-v)(z-v)dx+\int_{\Gamma_N} g(z-v)ds \leq 0 \text{ for all } z\in K. \tag{3.2}$$

Since

$$\|v-f\|^2_{L^2(\Omega)}-\|z-f\|^2_{L^2(\Omega)}=2\,(f-v,z-v)-\|v-z\|^2_{L^2(\Omega)}$$

which implies that

$$\frac{1}{2}\int_{\Omega}|v-f|^2\,dx-\frac{1}{2}\int_{\Omega}|z-f|^2\,dx=\int_{\Omega}(f-v)\,(z-v)\,dx-\frac{1}{2}\int_{\Omega}|v-z|^2\,dx.$$

Thus

$$\begin{aligned}\frac{1}{2}\int_{\Omega}|v-f|^2\,dx-\frac{1}{2}\int_{\Omega}|z-f|^2\,dx+\int_{\Gamma_N} g\,(z-v) &= \int_{\Omega}(f-v)\,(z-v)\,dx+\int_{\Gamma_N} g\,(z-v)\,ds\\ &\quad -\frac{1}{2}\int_{\Omega}|v-z|^2\,dx.\end{aligned}$$

Consequently by using (3.2) we deduce that

$$\frac{1}{2}\int_{\Omega}|v-f|^2\,dx-\frac{1}{2}\int_{\Omega}|z-f|^2\,dx+\int_{\Gamma_N} g\,(z-v)\,ds\leq 0$$

ie

$$\frac{1}{2}\int_{\Omega}|v-f|^2\,dx-\int_{\Gamma_N} gvds\leq\frac{1}{2}\int_{\Omega}|z-f|^2\,dx-\int_{\Gamma_N} gzd\sigma \text{ for all } z\in K.$$

Now suppose that $v\in K$ is solution of minimization problem (3.1). Let $z_0\in K$, $t\in[0,1]$ and setting $z=(1-t)\,v+tz_0$. Since K is convex then $z\in K$,
which implies that

$$\begin{aligned}\frac{1}{2}\int_{\Omega}|v-f|^2\,dx-\int_{\Gamma_N} gvd\sigma &\leq \frac{1}{2}\int_{\Omega}|f-((1-t)\,v+tz_0)|^2\,dx-\int_{\Gamma_N} g\,((1-t)\,v+tz_0)\,ds\\ &\leq \frac{1}{2}\int_{\Omega}|(f-v)-t\,(z_0-v)|^2\,dx-\int_{\Gamma_N} gvd\sigma-t\int_{\Gamma_N} g\,(z_0-v)\,ds\\ &\leq \frac{1}{2}\,\|(f-v)-t\,(z_0-v)\|^2_{L^2(\Omega)}-\int_{\Gamma_N} gvd\sigma-t\int_{\Gamma_N} g\,(z_0-v)\,ds.\end{aligned}$$

ie

$$\frac{1}{2}\int_{\Omega}|v-f|^{2}dx-\int_{\Gamma_{N}}gvds \leq \frac{1}{2}\int_{\Omega}|v-f|^{2}dx-t\left(f-v,z_{0}-v\right)+\frac{t^{2}}{2}\int_{\Omega}|z_{0}-v|^{2}dx$$
$$-\int_{\Gamma_{N}}gvd\sigma-t\int_{\Gamma_{N}}g\left(z_{0}-v\right)ds.$$

Therefore we have

$$\left(f-v,z_{0}-v\right)+\int_{\Gamma_{N}}g\left(z_{0}-v\right)ds\leq\frac{t}{2}\int_{\Omega}|z_{0}-v|^{2}dx. \tag{3.3}$$

Hence by letting $t\to 0$ in (3.3) we obtain

$$\int_{\Omega}\left(f-v\right)\left(z_{0}-v\right)dx+\int_{\Gamma_{N}}g\left(z_{0}-v\right)ds\leq 0 \text{ for all } z_{0}\in K$$

which implies that

$$\int_{\Omega}\left(f-v\right)z_{0}dx+\int_{\Gamma_{N}}gz_{0}ds\leq\int_{\Omega}\left(f-v\right)vdx+\int_{\Gamma_{N}}gvds \text{ for all } z_{0}\in K. \tag{3.4}$$

□

The minimization problem (3.1) is equivalent to

$$\min_{z\in H^{1}(\Omega)}\left\{\frac{1}{2}\int_{\Omega}|z-f|^{2}dx-\int_{\Gamma_{N}}gzds+\Pi_{K}\left(z\right)\right\} \tag{3.5}$$

and since $\mathcal{C}^{\infty}\left(\bar{\Omega}\right)$ is dense in $H^{1}\left(\Omega\right)$, then the problem (3.5) is equivalent to

$$\min\left\{F(z)+H(\Lambda z);z\in\mathcal{C}^{\infty}\left(\bar{\Omega}\right)\right\}, \tag{3.6}$$

where $\Lambda z:=\nabla z$ is linear operator from $\mathcal{C}^{\infty}(\bar{\Omega})$ to $\mathcal{C}\left(\bar{\Omega}\right)^{N}$ and $F:\mathcal{C}^{\infty}(\bar{\Omega})\to\mathbb{R}^{+}$ and $H=:\mathcal{C}(\Omega)^{N}\to\bar{\mathbb{R}}$ are convex functions defined by

$$F(z)=\frac{1}{2}\int_{\Omega}|z-f|^{2}dx-\int_{\Gamma_{N}}gzds$$

and

$$H(\sigma)=\begin{cases}0 & \text{if} \quad |\sigma(x)|\leq 1 \quad \forall x\in\Omega,\\ +\infty & \text{otherwise.}\end{cases}$$

then, the dual problem associated with (3.6) (see [19])is given by

$$\min\left\{-F^{*}\left(\Lambda^{*}\sigma\right)+H^{*}\left(-\sigma\right);\sigma\in\left(\mathcal{C}\left(\bar{\Omega}\right)^{N}\right)^{*}\right\}. \tag{3.7}$$

In this work, we consider a simple case where we approximate the dual problem (3.7) by the following optimazation problem

$$\sup\left\{-\mathcal{G}\left(\sigma\right);\sigma\in H_{\mathrm{div},g}\left(\Omega\right)\right\} \tag{3.8}$$

where

$$\mathcal{G}\left(\sigma\right)=\frac{1}{2}\int_{\Omega}\left(\mathrm{div}\left(\sigma\right)\right)^{2}dx+\int_{\Omega}f\mathrm{div}\left(\sigma\right)dx+\int_{\Omega}|\sigma|\,dx \tag{3.9}$$

and

$$H_{\text{div},g}(\Omega) = \left\{ \sigma \in H_{\text{div}}(\Omega)\,;\; \int_\Omega -\text{div}(\sigma)\,\xi dx = \int_\Omega \sigma.\nabla\xi dx - \int_{\Gamma_N} g\xi ds \;\forall \xi \in H^1_D(\Omega) \right\}.$$

Recall that

$$H_{\text{div}}(\Omega) = \left\{ w \in \left(L^2(\Omega)\right)^N;\; \text{div}(w) \in L^2(\Omega) \right\}.$$

In the rest of paper, we establish the connection between the dual problem (3.8) and primal problem (3.1). We start by prove the following result

Lemma 3.2. *For any* $f \in L^2(\Omega)$, $g \in L^2(\Gamma_N)$, $w \in H_{div,g}(\Omega)$ *and* $z \in K$, *we have*

$$-G(w) \le J(z),$$

where $J(z) = \frac{1}{2}\int_\Omega |z - f|^2 dx - \int_{\Gamma_N} gvds.$

Proof Let $w \in H_{\text{div},g}(\Omega)$ and $z \in K$ be fixed, since

$$\frac{1}{2}\left(div(w) - (z - f)\right)^2 \ge 0$$

we have

$$-\frac{1}{2}\int_\Omega (div(w))^2 dx - \int_\Omega div(w) f dx + \int_\Omega div(w) z dx \le \frac{1}{2}\int_\Omega (z - f)^2 dx$$

which implies that

$$-\frac{1}{2}\int_\Omega (div(w))^2 dx - \int_\Omega div(w) f dx \le \frac{1}{2}\int_\Omega (z - f)^2 dx - \int_\Omega div(w) z dx.$$

Now we use the fact that $w \in H_{\text{div},g}(\Omega)$ to obtain

$$-\frac{1}{2}\int_\Omega (div(w))^2 dx - \int_\Omega div(w) f dx \le \frac{1}{2}\int_\Omega (z - f)^2 dx + \int_\Omega w.\nabla z dx - \int_{\Gamma_N} gz ds.$$

Therefore

$$-G(w) \le \frac{1}{2}\int_\Omega (z - f)^2 dx - \int_{\Gamma_N} gz ds + \int_\Omega w.\nabla z dx - \int_\Omega |w| dx.$$

Notice that since $z \in K$ we have

$$\int_\Omega w.\nabla z dx - \int_\Omega |w| dx \le 0,$$

consequently

$$-G(w) \le \frac{1}{2}\int_\Omega (z - f)^2 dx - \int_{\Gamma_N} gz ds$$

□

For numerical treatment we need to work in the subspace of $H_{\text{div}}(\Omega)$, then introduce the following result

Lemma 3.3. *We have:*

$$\sup\left\{-\mathcal{G}(\sigma):\ \sigma\in H_{div,g}(\Omega)\right\}=\sup\left\{-\mathcal{G}(\phi):\ \phi\in\mathcal{H}\right\}$$

where $\mathcal{G}(\phi)=G(\phi+\nabla Z)$ *with* Z *the variational solution of Laplace equation*

$$\begin{cases}-\Delta Z &= 0 \quad in\ \Omega\\ Z &= 0 \quad on\ \Gamma_D\\ \nabla Z.\eta &= g \quad on\ \Gamma_N.\end{cases}\tag{3.10}$$

and

$$\mathcal{H}=\left\{\phi\in H_{div}(\Omega):\ \int_\Omega -div(\phi)\,\xi=\int_\Omega \phi.\nabla\xi ds\right\}$$

Proof For all $\sigma\in H_{\mathrm{div},g}(\Omega)$ we have : $\sigma=(\sigma-\nabla Z)+\nabla Z$ where Z is the solution of the problem (3.10). The terms $(\sigma-\nabla Z)$ belongs to $H_{\mathrm{div}}(\Omega)$ and

$$\begin{aligned}\int_\Omega -\mathrm{div}(\sigma-\nabla Z)\,\xi &= \int_\Omega -\mathrm{div}(\sigma)\,\xi+\int_\Omega \mathrm{div}(\nabla Z)\,\xi dx\\ &= \int_\Omega \sigma.\nabla\xi dx-\int_{\Gamma_N} g\xi ds-\int_\Omega \nabla Z\nabla\xi dx+\int_{\Gamma_N} g\xi ds\\ &= \int_\Omega (\sigma-\nabla Z).\nabla\xi dx\ \forall\xi\in H_D^1(\Omega),\end{aligned}$$

hence $\sigma\in\{\phi+\nabla Z:\ \phi\in\mathcal{H}\}$, then $H_{\mathrm{div},g}(\Omega)\subset\{\phi+\nabla Z:\ \phi\in\mathcal{H}\}$. Also for any $\phi+\nabla Z$ with $\phi\in\mathcal{H}$, we have $\phi+\nabla Z\in H_{\mathrm{div}}(\Omega)$ and

$$\begin{aligned}\int_\Omega -\mathrm{div}(\phi+\nabla Z)\,\xi &= \int_\Omega -\mathrm{div}(\phi)\,\xi+\int_\Omega -\mathrm{div}(\nabla Z)\,\xi dx\\ &= \int_\Omega \phi.\nabla\xi dx+\int_\Omega \nabla Z.\nabla\xi dx-\int_{\Gamma_N} g\xi ds\\ &= \int_\Omega (\phi+\nabla Z).\nabla\xi dx-\int_{\Gamma_N} g\xi ds\ \forall\xi\in H_D^1(\Omega).\end{aligned}$$

which implies that $\{\phi+\nabla Z:\ \phi\in\mathcal{H}\}\subset H_{\mathrm{div},g}(\Omega)$, therefore $H_{\mathrm{div},g}(\Omega)=\{\phi+\nabla Z:\ \phi\in\mathcal{H}\}$. Then, we have

$$\begin{aligned}\sup\left\{-G(\sigma):\ \sigma\in H_{\mathrm{div},g}(\Omega)\right\} &= \sup\left\{-G(\phi+\nabla Z):\ \phi\in\mathcal{H}\right\}\\ &= \sup\left\{-\mathcal{G}(\phi):\ \phi\in\mathcal{H}\right\}\end{aligned}$$

□

Remark 3.4. Thanks to the Lemma 3.3, it's possible to deal the dual problem in the subspace $\mathcal{H}$ of $H_{\mathrm{div}}(\Omega)$. This approach is important for numerical simulation since it permits to use the algorithm developed in [17] for the computation of $\sup\{-G(\phi+\nabla Z):\ \phi\in\mathcal{H}\}$.

Theorem 3.5. *Let* $f\in L^2(\Omega)$, $g\in L^2(\Gamma_N)$ *and* v *the solution of* (3.1). *Then, there exists a sequence* $(w_\varepsilon)_{\varepsilon>0}$ *in* $H_{div,g}(\Omega)$ *such that, as* $\varepsilon\to 0$,

$$\int_\Omega |w_\varepsilon|\,dx\to\int_\Omega v(f-v)dx+\int_{\Gamma_N} gvds,\tag{3.11}$$

$$div(w_\varepsilon) \to v - f \quad in \quad L^2(\Omega) \tag{3.12}$$

and

$$\begin{aligned} \lim_{\varepsilon\to 0} G(w_\varepsilon) &= \inf_{w\in H_{div,g}(\Omega)} G(w) \\ &= -\min_{z\in K} J(z) \\ &= -\left[\frac{1}{2}\int_\Omega |f-v|^2\,dx - \int_{\Gamma_N} gvds\right]. \end{aligned} \tag{3.13}$$

To prove this result, let us consider the following elliptic equation

$$(S_\varepsilon)\begin{cases} v_\varepsilon - \nabla.w_\varepsilon &= f \quad \text{in} \quad \Omega \\ w_\varepsilon &= \phi_\varepsilon(\nabla v_\varepsilon) \quad \text{in} \quad \Omega \\ v_\varepsilon &= 0 \quad \text{on} \quad \Gamma_D \\ w_\varepsilon.\eta &= g \quad \text{on} \quad \Gamma_N, \end{cases}$$

where, for any $\varepsilon > 0$, $\phi_\varepsilon : \mathbb{R}^N \to \mathbb{R}^N$ is given by

$$\phi_\varepsilon(r) = \frac{1}{\varepsilon}(|r|-1)^+\frac{r}{|r|} \quad \text{for all} \quad r \in \mathbb{R}^N,$$

and ϕ_ε satisfies the following properties :
(i) for any $r_1, r_2 \in \mathbb{R}^N$, $(\phi_\varepsilon(r_1) - \phi_\varepsilon(r_2)).(r_1 - r_2) \geq 0$.
(ii) there exist $\varepsilon_0 > 0$ and $A > 1$ such that $\phi_\varepsilon(r).r \geq |r|^2$ for any $|r| \geq A$ and $\varepsilon < \varepsilon_0$
(iii) for any $\varepsilon > 0$ and $r \in \mathbb{R}^N$, $|\phi_\varepsilon(r)| \leq \phi_\varepsilon(r).r$.

Lemma 3.6. *There exists a unique weak solution v_ε for problem $(S_\varepsilon)_{0<\varepsilon<\varepsilon_0}$ in the sense that $v_\varepsilon \in H^1_D(\Omega)$, $w_\varepsilon = \phi_\varepsilon(\nabla v_\varepsilon) \in \left(L^2(\Omega)\right)^D$ and $\forall z \in H^1_D(\Omega)$*

$$\int_\Omega v_\varepsilon z dx + \int_\Omega \phi_\varepsilon(\nabla v_\varepsilon).\nabla z dx = \int_\Omega f z dx + \int_{\Gamma_N} gzds. \tag{3.14}$$

Moreover $(v_\varepsilon)_{0<\varepsilon<\varepsilon_0}$ is bounded in $H^1_D(\Omega)$ and for any Borel set $B \subseteq \Omega$ we have

$$\liminf_{\varepsilon\to 0}\int_B |\nabla v_\varepsilon|\,dx \leq |B|. \tag{3.15}$$

Proof i) We define the operator $A_\varepsilon : H^1_D(\Omega) \to \left(H^1_D(\Omega)\right)'$ by

$$\langle A_\varepsilon v, z\rangle = \int_\Omega vzdx + \int_\Omega \phi_\varepsilon(\nabla v_\varepsilon).\nabla z dx.$$

The operator A_ε is monotone,coercive, hemicontinous and bounded. In fact, the propertie (i) implies that A_ε is monotone.

For any $v, z \in H_D^1(\Omega)$, we have

$$\begin{aligned} |\langle A_\varepsilon v, z\rangle| &\leq \int_\Omega |v|\,|z|\,dx + \frac{1}{\varepsilon}\int_{[|\nabla v|>1]} \left|(|\nabla v|-1)^+\right| |\nabla z|\,dx \\ &\leq \|v\|_{L^2(\Omega)}\|z\|_{L^2(\Omega)} + \frac{1}{\varepsilon}\int_{[|\nabla v|>1]} |\nabla v|\,|\nabla z|\,dx \\ &\leq \|v\|_{H^1(\Omega)}\|z\|_{H^1(\Omega)} + \frac{1}{\varepsilon}\|\nabla v\|_{L^2(\Omega)}\|\nabla z\|_{L^2(\Omega)} \\ &\leq \|v\|_{H^1(\Omega)}\|z\|_{H^1(\Omega)} + \frac{1}{\varepsilon}\|v\|_{H^1(\Omega)}\|z\|_{H^1(\Omega)} \\ &\leq \left(1+\frac{1}{\varepsilon}\right)\|v\|_{H^1(\Omega)}\|z\|_{H^1(\Omega)} \end{aligned}$$

which implies that

$$\|A_\varepsilon(v)\|_{\left(H_D^1(\Omega)\right)'} \leq \left(1+\frac{1}{\varepsilon}\right)\|v\|_{H^1(\Omega)}.$$

Since $\varepsilon, \varepsilon_0 \in \mathbb{R}$ there exist $n \in \mathbb{N}^*$ such that $\varepsilon_0 \leq n\varepsilon$.
Consequently

$$\|A_\varepsilon(v)\|_{\left(H_D^1(\Omega)\right)'} \leq \left(1+\frac{n}{\varepsilon_0}\right)\|v\|_{H^1(\Omega)}.$$

Let B be a bounded set of $H_D^1(\Omega)$, there exist a constant $M > 0$ such that

$$\|A_\varepsilon(v)\|_{\left(H_D^1(\Omega)\right)'} \leq \left(1+\frac{n}{\varepsilon_0}\right)M, \forall\, v \in B.$$

Hence A_ε is a bounded.
Moreover, using the propertie (ii) of ϕ_ε we obtain

$$\begin{aligned} \langle A_\varepsilon(v), v\rangle &= \int_\Omega v^2 dx + \int_\Omega \Phi_\varepsilon(\nabla v).\nabla v dx \\ &\geq \int_\Omega v^2 dx + \int_\Omega |\nabla v|^2 dx \\ &\geq \|v\|_{H^1(\Omega)}^2 \end{aligned}$$

Then

$$\frac{\langle A_\varepsilon(v), v\rangle}{\|v\|_{H^1(\Omega)}} \geq \|v\|_{H^1(\Omega)}$$

Letting $\|v\|_{H^1(\Omega)} \to +\infty$ we deduce that A_ε is coercive.
To end the first part of the proof of the we consider the map $F : \mathbb{R} \to \mathbb{R}$ defined by

$$\begin{aligned} F(\lambda) &= \langle A_\varepsilon(u+\lambda v), w\rangle \\ &= \int_\Omega (u+\lambda v)\, w dx + \int_\Omega \phi_\varepsilon(\nabla u + \lambda \nabla v).\nabla w dx \\ &= \lambda \int_\Omega v w dx + \int_\Omega u w dx + \int_\Omega \phi_\varepsilon(\nabla u + \lambda\nabla v).\nabla w dx \end{aligned}$$

with u,v,w in $H_D^1(\Omega)$.
Since u,v,w in $H_D^1(\Omega)$ the function

$$\lambda \mapsto \lambda \int_\Omega vwdx + \int_\Omega uwdx \text{ is continuous.}$$

The functions $x \mapsto \phi_\varepsilon(\nabla u + \lambda \nabla v).\nabla w$, $\lambda \mapsto \phi_\varepsilon(\nabla u + \lambda \nabla v).\nabla w$ are respectively mesurable and continuous a.e.in Ω
Let $(\lambda_n)_n$ be such that $\lambda_n \to \lambda$, so there exist a constant $c>0$ such that $|\lambda_n| \le c$.
Wich implies that

$$|\phi_\varepsilon(\nabla u + \lambda_n \nabla v).\nabla w| \le \frac{1}{\varepsilon}(|\nabla u| + c|\nabla v| - 1)^+ |\nabla w| \in L^1(\Omega).$$

Letting $n \to +\infty$ and using the fact that the function $\lambda_n \mapsto \phi_\varepsilon(\nabla u + \lambda_n \nabla v).\nabla w$ is continuous we deduce that

$$|\phi_\varepsilon(\nabla u(x) + \lambda \nabla v(x)).\nabla w(x)| \le \frac{1}{\varepsilon}(|\nabla u(x)| + c|\nabla v(x)| - 1)^+ |\nabla w(x)| \le \text{ for all } x \text{ in } \Omega.$$

Therefore, using Lebesgue continuity theorem we can say that the function

$$\lambda \mapsto \int_\Omega \phi_\varepsilon(\nabla u + \lambda \nabla v).\nabla w dx \text{ is continuous,}$$

Thus F is continuous, hence the operator A_ε is hemi-continuous.
Let $G : H_D^1(\Omega) \to \mathbb{R}$ defined by

$$G(z) = \int_\Omega fzdx + \int_{\Gamma_N} gzds,$$

since the linear form G belongs to $\left(H_D^1(\Omega)\right)'$, then thanks to [25] there exists $v_\varepsilon \in H_D^1(\Omega)$ such that

$$\langle Av_\varepsilon, z\rangle = \langle F, z\rangle \text{ for all } z \in H_D^1(\Omega)$$

ie

$$\int_\Omega v_\varepsilon zdx + \int_\Omega \phi_\varepsilon(\nabla v_\varepsilon).\nabla zdx = \int_\Omega fzdx + \int_{\Gamma_N} gzds.$$

For the uniqueness, let us suppose that $(S_\varepsilon)_{0<\varepsilon\le\varepsilon_0}$ admits two solutions u and v, then we subtract the two equations obtained by replacing respectively v_ε by u and v in (3.14) to get

$$\int_\Omega (u-v)zdx + \int_\Omega (\phi_\varepsilon(\nabla u) - \phi_\varepsilon(\nabla u))\nabla zdx = 0 \quad \forall z \in H_D^1(\Omega). \tag{3.16}$$

We take $u-v$ as test function in (3.16) to obtain

$$\int_\Omega (u-v)^2 dx + \int_\Omega (\phi_\varepsilon(\nabla u) - \phi_\varepsilon(\nabla u)).(\nabla u - \nabla v)dx = 0 \tag{3.17}$$

thus using the property (i) of ϕ_ε we deduce that $u=v$.
Notice that $|w_\varepsilon| = |\phi_\varepsilon(\nabla v_\varepsilon)| \le |\nabla v_\varepsilon|$, which implies that $w_\varepsilon \in \left(L^2(\Omega)\right)^D$
Taking v_ε as test function in (3.14), we get

$$\int_\Omega v_\varepsilon^2 dx + \frac{1}{\varepsilon}\int_\Omega (|\nabla v_\varepsilon| - 1)^+ |\nabla v_\varepsilon| dx = \int_\Omega fv_\varepsilon dx + \int_{\Gamma_N} gv_\varepsilon ds.$$

So,

$$\begin{aligned}\int_\Omega v_\varepsilon^2 dx &\leq \int_\Omega f v_\varepsilon dx + \int_{\Gamma_N} g v_\varepsilon ds\\ &\leq \|f\|_{L^2(\Omega)} \|v_\varepsilon\|_{L^2(\Omega)} + \|g\|_{L^2(\Gamma_N)} \|v_\varepsilon\|_{L^2(\Gamma_N)}\\ &\leq \|f\|_{L^2(\Omega)} \|v_\varepsilon\|_{H^1(\Omega)} + \|g\|_{L^2(\Gamma_N)} \|v_\varepsilon\|_{L^2(\Gamma_N)}\end{aligned}$$

Recall the following trace inequality

$$\int_\Gamma |v_\varepsilon|^2 ds \leq C_1 \left(\int_\Omega |v_\varepsilon|^2 dx + \int_\Omega |\nabla v_\varepsilon|^2 dx \right)$$

where C_1 is a constant, so using this inequality we get

$$\|v_\varepsilon\|_{L^2(\Omega)} \leq \left(\|f\|_{L^2(\Omega)} + C_1 \|g\|_{L^2(\Gamma_N)} \right) \|v_\varepsilon\|_{H^1(\Omega)} \tag{3.18}$$

and also we have

$$\frac{1}{\varepsilon} \int_\Omega (|\nabla v_\varepsilon| - 1)^+ |\nabla v_\varepsilon| \leq \left(\|f\|_{L^2(\Omega)} + C_1 \|g\|_{L^2(\Gamma_N)} \right) \|v_\varepsilon\|_{H^1(\Omega)} \tag{3.19}$$

Combining (3.19) and propertie (ii) of ϕ_ε, for any $0 < \varepsilon < \varepsilon_0$ we get

$$\begin{aligned}\int_\Omega |\nabla v_\varepsilon|^2 dx &\leq \int_{[|\nabla v_\varepsilon| \leq A]} |\nabla v_\varepsilon|^2 dx + \int_{[|\nabla v_\varepsilon| > A]} |\nabla v_\varepsilon|^2 dx\\ &\leq \int_{[|\nabla v_\varepsilon| \leq A]} |\nabla v_\varepsilon|^2 dx + \frac{1}{\varepsilon} \int_\Omega (|\nabla v_\varepsilon| - 1)^+ |\nabla v_\varepsilon| dx\\ &\leq \int_{[|\nabla v_\varepsilon| \leq A]} |A|^2 dx + \left(\|f\|_{L^2(\Omega)} + C_1 \|g\|_{L^2(\Gamma_N)} \right) \|v_\varepsilon\|_{H^1(\Omega)}\\ &\leq |A|^2 |\Omega| + \left(\|f\|_{L^2(\Omega)} + C_1 \|g\|_{L^2(\Gamma_N)} \right) \|v_\varepsilon\|_{H^1(\Omega)}. \end{aligned} \tag{3.20}$$

Thus, adding (3.18) and (3.20) obtain

$$\|v_\varepsilon\|^2_{H^1(\Omega)} \leq |A|^2 |\Omega| + 2 \left(\|f\|_{L^2(\Omega)} + C_1 \|g\|_{L^2(\Gamma_N)} \right) \|v_\varepsilon\|_{H^1(\Omega)}. \tag{3.21}$$

By applying Young inequality we get

$$\|v_\varepsilon\|^2_{H^1(\Omega)} \leq |A|^2 |\Omega| + \frac{1}{2} \left[2 \left(\|f\|_{L^2(\Omega)} + C_1 \|g\|_{L^2(\Gamma_N)} \right) \right]^2 + \frac{1}{2} \|v_\varepsilon\|^2_{H^1(\Omega)}$$

which implies that

$$\|v_\varepsilon\|^2_{H^1(\Omega)} \leq 2 |A|^2 |\Omega| + 4 \left(\|f\|_{L^2(\Omega)} + C_1 \|g\|_{L^2(\Gamma_N)} \right)^2. \tag{3.22}$$

Thus v_ε is bounded in $H^1(\Omega)$.
Now, let $B \subseteq \Omega$ be fixed Borel set. We have,

$$\begin{aligned}
\|\nabla v_\varepsilon\|_{L^1(B)} &\le \left\|(|\nabla v_\varepsilon|-1)^+ + 1\right\|_{L^1(B)} \\
&\le \left\|(|\nabla v_\varepsilon|-1)^+\right\|_{L^1(B)} + \|1\|_{L^1(B)} \\
&\le \int_B \left|(|\nabla v_\varepsilon|-1)^+\right| dx + |B| \\
&\le \int_B (|\nabla v_\varepsilon|-1)^+ |\nabla v_\varepsilon| \, dx + |B| \\
&\le \varepsilon \left(\|f\|_{L^2(\Omega)} + C_1 \|g\|_{L^2(\Gamma_N)}\right) \|v_\varepsilon\|_{H^1(\Omega)} + |B|.
\end{aligned}$$

Letting $\varepsilon \to 0$, and using the fact that v_ε is bounded in $H^1(\Omega)$, we obtain

$$\liminf_{\varepsilon \to 0} \int_\Omega |\nabla v_\varepsilon| \, dx \le |B|.$$

Proof of the Theorem 3.4 From the Lemma 3.6, we know that the sequence (v_ε) is bounded in $H^1_D(\Omega)$, so we can extract a subsequence (still denoted by (v_ε)) such that

$$v_\varepsilon \to \tilde{v} \text{ in } H^1_D(\Omega) - \text{weak and in } L^2(\Omega),$$

hence by the equation $(S_\varepsilon)_{0<\varepsilon<\varepsilon}$ we deduce that

$$div\,(w_\varepsilon) \to \tilde{v} - f \text{ in } L^2(\Omega).$$

Now, we show that $\tilde{v} \in K$, for this we introduce we introduce the following the set $A_\delta = [|\nabla \tilde{v}| \ge 1+\delta]$, with $\delta > 0$. Since as $\varepsilon \to 0$, $\nabla v_\varepsilon \to \nabla \tilde{v}$ in $\left(L^1(\Omega)\right)^N$-weak then

$$\begin{aligned}
(1+\delta)|A_\delta| &\le \int_{A_\delta} |\nabla \tilde{v}| \, dx \\
&\le \liminf_{\varepsilon \to 0} \int_{A_\delta} |\nabla v_\varepsilon| \, dx.
\end{aligned}$$

Taking $B = A_\delta$ in Lemma 3.6, we obtain

$$(1+\delta)|A_\delta| \le |A_\delta|,$$

since δ is nonnegative arbitrary constant, it follows that $|A_\delta| = 0$. Therefore $|\nabla \tilde{v}| \le 1$ a.e. in Ω and $\tilde{v} \in K$. Let us observe that $\tilde{v}$ is also solution of the minimisation problem (3.1).

For any $z \in K$ we have

$$\begin{aligned}
\int_{\Omega} (f - \tilde{v})(z - \tilde{v})\, dx + \int_{\Gamma_N} g(z - \tilde{v})\, ds &= \lim_{\varepsilon \to 0} \left(\int_{\Omega} (f - v_{\varepsilon})(z - \tilde{v})\, dx + \int_{\Gamma_N} g(z - \tilde{v})\, ds \right) \\
&= \lim_{\varepsilon \to 0} \left(\int_{\Omega} -\nabla.\phi_{\varepsilon}(\nabla v_{\varepsilon})(z - \tilde{v})\, dx + \int_{\Gamma_N} g(z - \tilde{v})\, ds \right) \\
&= \lim_{\varepsilon \to 0} \int_{\Omega} \phi_{\varepsilon}(\nabla v_{\varepsilon}).\nabla(z - \tilde{v})\, dx - \int_{\Gamma_N} \phi_{\varepsilon}(\nabla v_{\varepsilon}).\eta(z - \tilde{v})\, ds \\
&\quad + \int_{\Gamma_N} g(z - \tilde{v})\, ds \\
&= \lim_{\varepsilon \to 0} \int_{\Omega} \phi_{\varepsilon}(\nabla v_{\varepsilon}).\nabla(z - \tilde{v})\, dx. \qquad (3.23)
\end{aligned}$$

Since $z \in K$ we have $\phi_{\varepsilon}(\nabla z) = 0$, hence

$$\lim_{\varepsilon \to 0} \int_{\Omega} \phi_{\varepsilon}(\nabla v_{\varepsilon}).\nabla(z - \tilde{v})\, dx = \lim_{\varepsilon \to 0} \int_{\Omega} (\phi_{\varepsilon}(\nabla v_{\varepsilon}) - \phi_{\varepsilon}(\nabla z)).(\nabla z - \nabla \tilde{v}) \leq 0. \qquad (3.24)$$

Thus, (3.23) becomes

$$\int_{\Omega} (f - \tilde{v})(z - \tilde{v})\, dx + \int_{\Gamma_N} g(z - \tilde{v})\, ds \leq 0 \text{ for all } z \in K. \qquad (3.25)$$

Consequently, $\tilde{v}$ is solution of the maximization problem (2.3), thus from the Lemma 3.1, we conclude that $\tilde{v}$ is also solution of (3.1).
Let us shows that w_{ε} satisfies (3.8), by the property (iii) of ϕ_{ε} we have

$$\begin{aligned}
\limsup_{\varepsilon \to 0} \int_{\Omega} |w_{\varepsilon}|\, dx &\leq \limsup_{\varepsilon \to 0} \int_{\Omega} \phi(\nabla v_{\varepsilon}).\nabla v_{\varepsilon} dx \\
&\leq \limsup_{\varepsilon \to 0} \left(\int_{\Gamma_N} \phi(\nabla v_{\varepsilon}).\eta v_{\varepsilon} dx - \int_{\Omega} \nabla.\phi(\nabla v_{\varepsilon}) v_{\varepsilon} dx \right) \\
&\leq \limsup_{\varepsilon \to 0} \left(\int_{\Omega} (f - v_{\varepsilon}) v_{\varepsilon} dx + \int_{\Gamma_N} g v_{\varepsilon} ds \right) \\
&\leq \int_{\Omega} (f - \tilde{v}) \tilde{v} dx + \int_{\Gamma_N} g \tilde{v} ds \qquad (3.26)
\end{aligned}$$

and since v is solution of the maximization problem (2.3), (3.26) become

$$\limsup_{\varepsilon \to 0} \int_{\Omega} |w_{\varepsilon}|\, dx \leq \int_{\Omega} (f - v) v dx + \int_{\Gamma_N} g v ds. \qquad (3.27)$$

We use the fact that $\tilde{v}$ is also solution the probleme have (2.3) to get

$$\begin{aligned} \int_\Omega (f-v)\,vdx + \int_{\Gamma_N} gvd\sigma &\le \int_\Omega (f-\tilde{v})\,\tilde{v}dx + \int_{\Gamma_N} g\tilde{v}ds \qquad (3.28)\\ &= \lim_{\varepsilon\to 0}\int_\Omega (f-v_\varepsilon)\,\tilde{v}dx + \int_{\Gamma_N} g\tilde{v}ds\\ &= \lim_{\varepsilon\to 0}\int_\Omega -\nabla.w_\varepsilon\tilde{v}dx + \int_{\Gamma_N} g\tilde{v}ds\\ &= \lim_{\varepsilon\to 0}\int_\Omega w_\varepsilon.\nabla\tilde{v}dx\\ &\le \lim_{\varepsilon\to 0}\int_\Omega |w_\varepsilon|\,dx \qquad (3.29)\end{aligned}$$

Thus (3.26) and (3.28) implies that

$$\lim_{\varepsilon\to 0}\int_\Omega |w_\varepsilon|\,dx = \int_\Omega (f-v)\,vdx + \int_{\Gamma_N} gvds. \qquad (3.30)$$

Too proof (3.1) , we first use (3.11) , (3.12) and (3.30) to get

$$\begin{aligned} \lim_{\varepsilon\to 0}(-G(w_\varepsilon)) &= \lim_{\varepsilon\to 0}\left(-\frac{1}{2}\int_\Omega \operatorname{div}(w_\varepsilon)^2\,dx - \int_\Omega \operatorname{div}(w_\varepsilon)\,fdx - \int_\Omega |w_\varepsilon|\right)\\ &= -\frac{1}{2}\int_\Omega (\tilde{v}-f)^2\,dx - \int_\Omega (\tilde{v}-f)\,fdx - \int_\Omega (f-\tilde{v})\,vdx - \int_{\Gamma_N} g\tilde{v}ds\\ &= -\frac{1}{2}\int_\Omega (\tilde{v}-f)^2\,dx + \int_\Omega (\tilde{v}-f)^2\,dx - \int_{\Gamma_N} g\tilde{v}ds\\ &= \frac{1}{2}\int_\Omega (\tilde{v}-f)^2\,dx - \int_{\Gamma_N} g\tilde{v}ds \end{aligned}$$

i.e

$$\lim_{\varepsilon\to 0}(-G(w_\varepsilon)) = J(\tilde{v}) = J(v)\,. \qquad (3.31)$$

Thus by using the Lemma 3.1, we have

$$\sup_{w\in H_{\operatorname{div},g}(\Omega)}(-G(w)) \le J(v) = \lim_{\varepsilon\to 0}(-G(w_\varepsilon))\,, \qquad (3.32)$$

moreover, we know that

$$J(v) = \lim_{\varepsilon\to 0}(-G(w_\varepsilon)) \le \sup_{w\in H}(-G(w))\,. \qquad (3.33)$$

Which implies that

$$\lim_{\varepsilon\to 0} -G(w_\varepsilon) = \sup_{w\in H_{\operatorname{div},g}(\Omega)}(-G(w)) = -\min_{z\in K} J(z)\,. \qquad (3.34)$$

□

Remark 3.7. Thanks to Theorem3.5, the solution v of the minimization problem (3.1) is characterized by

$$\begin{cases} v - \operatorname{div}(w) &= f \quad \text{in} \quad \left(H_D^1(\Omega)\right)' \\ |w|(\Omega) &= \int_\Omega v(f-v) + \int_{\Gamma_N} gvds \end{cases} \tag{3.35}$$

where $w \in (M_b(\Omega))^D$ the weak limit of w_ε, in $(M_b(\Omega))^D$.

From the Lemma3.3, the optimization problem

$$\inf\{G(\phi + \nabla Z) : \ \phi \in \mathcal{H}\} \tag{3.36}$$

can be consider as dual problem associated with the minimization problem (3.1). To approximate numericaly the solution of(3.36) , we use the finite element method, at first we make the following assumption:

- Domain Ω is bounded, open, polyhedral subset of $\mathbb{R}^2$.
- T_h will be regular partionning (quadrangulation) of $\overline{\Omega}$ by n disjoint open simplices τ of diameter no greater than a given real h, with $\overline{\Omega} = \cup_{\tau \in T_h} \bar{\tau}$.

For numericaly discretization of the flux ϕ we need the lowest-order Raviart-Thomas finite elements (cf) spaces $RT_0(T_h)$ defined by

$$\begin{aligned} RT_0(T_h) &= \Big\{ q_h \in \left(L^2(\Omega)\right)^2 : q_\tau^h = a_\tau + b_\tau x, a \in \mathbb{R}^2, b \in \mathbb{R}, \forall \tau \in T_h, \\ &\quad \text{and } \left(q_\tau^h - q_{\tau'}^h\right).\nu_{\partial\tau} = 0 \text{ on } \partial\tau \cap \partial\tau'. \Big\} \end{aligned}$$

where $\nu_{\partial\tau}$ represente the outward unit normal to $\partial\tau$, the boundary of τ.
Let V_h be a finite dimensional subspace of $RT_0(T_h) \cap \mathcal{H}$ with a dimension equal to $N = N(h)$. We denote by r_h the intrpolation operator onto the V_h given in Theorem 6.1 of [29], then thanks to [29] we have for all $w \in H_{\text{div}}(\Omega)$

$$r_h(w) \to w \,\text{in} \left(L^2(\Omega)\right)^2 \text{ and } \quad \operatorname{div}(r_h(w)) \to \operatorname{div}(w) \text{ in} \in L^2(\Omega) \tag{3.37}$$

as $h \to 0$.

Theorem 3.8. *Let $f \in L^2(\Omega)$, $g \in L^2(\Gamma_N)$, v a solution of minimization problem* 3.1 *and w_h a solution of optimizing problem*

$$\inf\{\mathcal{G}(q_h); q_h \in V_h\}. \tag{3.38}$$

Then, as $h \to 0$,

$$div(w_h) \to v - f \quad in\, L^2(\Omega) \tag{3.39}$$

and

$$-\mathcal{G}(w_h) \to \min_{z \in K} J(z) = \frac{1}{2}\int_\Omega |v-f|^2\, dx - \int_{\Gamma_N} gzds \tag{3.40}$$

Proof Let w_ε be the solution of elliptic problème (S_ε). Then by (3.37) $r_h(w_\varepsilon - \nabla Z)$ belongs to V_h which implies that $\mathcal{G}(w_h) \le \mathcal{G}(r_h(w_\varepsilon - \nabla Z))$. Then, we have

$$\begin{aligned} G(w_h + \nabla Z) &\le G(r_h(w_\varepsilon - \nabla Z) + \nabla Z) \\ &\le \lim_{\varepsilon\to 0}\left(\lim_{h\to 0} G(r_h(w_\varepsilon - \nabla Z) + \nabla Z)\right) \\ &\le \lim_{\varepsilon \to 0} G(w_\varepsilon) \end{aligned} \tag{3.41}$$

ie

$$\frac{1}{2}\int_{\Omega}(\mathrm{div}\,(w_h+\nabla Z))^2\,dx+\int_{\Omega}f\mathrm{div}\,(w_h+\nabla Z)\,dx+\int_{\Omega}|(w_h+\nabla Z)|\,dx\leq\lim_{\varepsilon\to0}G\left(w_\varepsilon\right). \tag{3.42}$$

Now we use (3.42) and the fact that Z is the solution of the problem (S_ε) to get

$$\begin{aligned}
&\frac{1}{2}\int_{\Omega}(\mathrm{div}\,(w_h)-v+f)^2\,dx\\
&\leq\lim_{\varepsilon\to0}G\left(w_\varepsilon\right)+\frac{1}{2}\int_{\Omega}(v-f)^2\,dx-\int_{\Omega}\mathrm{div}\,(w_h)\,v-\int_{\Omega}|(w_h+\nabla Z)|\,dx\\
&\leq\lim_{\varepsilon\to0}G\left(w_\varepsilon\right)+\frac{1}{2}\int_{\Omega}(v-f)^2\,dx+\int_{\Omega}w_h\nabla v dx-\int_{\Omega}|(w_h+\nabla Z)|\,dx\\
&\leq\lim_{\varepsilon\to0}G\left(w_\varepsilon\right)+\frac{1}{2}\int_{\Omega}(v-f)^2\,dx+\int_{\Omega}w_h\nabla v dx-\int_{\Omega}|(w_h+\nabla Z)|\,dx+\int_{\Omega}\nabla Z\nabla v dx-\int_{\Gamma_N}gvds\\
&\leq\lim_{\varepsilon\to0}G\left(w_\varepsilon\right)+\frac{1}{2}\int_{\Omega}(v-f)^2\,dx-\int_{\Gamma_N}gvds+\int_{\Omega}(w_h+\nabla Z)\,\nabla v dx-\int_{\Omega}|(w_h+\nabla Z)|\,dx.
\end{aligned}$$

Since $v\in K$, we have

$$\int_{\Omega}(w_h+\nabla Z)\,\nabla v dx-\int_{\Omega}|(w_h+\nabla Z)|\,dx\leq0$$

and from Theorem , we have

$$\lim_{\varepsilon\to0}G\left(w_\varepsilon\right)+\frac{1}{2}\int_{\Omega}(v-f)^2\,dx-\int_{\Gamma_N}gvds\leq0.$$

Hence, it follows that

$$\limsup_{h\to0}\frac{1}{2}\int_{\Omega}(\mathrm{div}\,(w_h)-v+f)^2\,dx\leq0. \tag{3.43}$$

Which implies that

$$\mathrm{div}\,(w_h)\to f-v\quad\text{in}\quad L^2\left(\Omega\right) \tag{3.44}$$

From (3.41) and Theorem3.5, we have

$$\mathcal{G}\left(w_h\right)\leq\lim_{\varepsilon\to0}G\left(w_\varepsilon\right)=-J\left(v\right). \tag{3.45}$$

Since $H_{\mathrm{div},g}\left(\Omega\right)=\{\phi+\nabla Z:\ \phi\in\mathcal{H}\}$ (see Lemma3.3), then we use Theorem3.5 to obtain

$$\begin{aligned}
-J\left(v\right)&=\mathrm{Inf}\left\{G\left(w\right):\ w\in H_{\mathrm{div},g}\left(\Omega\right)\right\}\\
&=\mathrm{Inf}\left\{G\left(\phi+\nabla Z\right):\ \phi\in\mathcal{H}\right\}\\
&\leq G\left(w_h+\nabla Z\right)=\mathcal{G}\left(w_h\right)
\end{aligned} \tag{3.46}$$

Consequently, combining to (3.45) and (3.46) we deduduce that

$$\lim_{h\to0}\mathcal{G}\left(w_h\right)=-J\left(v\right). \tag{3.47}$$

□

4. Numerical simulations

We start by giving the mains lines of our approach for numericaly computation of problem (P). For the discretization of the domaine $\Omega = [-1,1] \times [-1,1]$, we take $N \in \mathbb{N}$ and the step of discretization is $h = \dfrac{2}{N+1}$. Let $\partial\Omega^x = \{x_i = x_0 + ih : \ 0 \leq i \leq N+1\}$ be a grid on x-axis and let Let $\partial\Omega^y = \{y_i = y_0 + jh : \ 0 \leq j \leq N+1\}$ be a grid on y-axis. We work with simplices $\tau_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$ $0 \leq i, j \leq N+1$ having uniforme size h^2.

Using this dicretization, we define a basis functions for the space V_h. Let

$$l_i(x) = \begin{cases} \dfrac{x - x_{i-1}}{h} & \text{for} \quad x_{i-1} < x \leq x_i \\ \dfrac{x_i - x}{h} & \text{for} \quad x_i < x \leq x_{i+1} \\ 0 & \text{otherwise} \end{cases} \qquad c_i(x) = \begin{cases} 1 & \text{for} \quad x_{i-1} < x \leq x_i \\ 0 & \text{otherwise.} \end{cases}$$

The flux space V_h is defined as follows: $V_h = V_h^x \times V_h^y$, where $V_h^x = \text{span}\{l_x(x)\, c_j(y)\}$ contains functions that are piecewise linear and continuous on V_h^x and piecewise constant on V_h^y and $V_h^y = \text{span}\{l_j(y)\, c_i(x)\}$ contains functions that are piecewise linear and continuous on V_h^y and piecewise constant on V_h^x. Then, in V_h the functionnal $\mathcal{G}$ take the following form:

$$\mathcal{G}_h(w_h) = \frac{1}{2}\|\text{div}(w_h)\|^2_{L^2(\Omega)} + (f, \text{div}(w_h)) + h^2 \sum_{i,j} |w_h + \nabla Z| \left(P_{\tau_{ij}}\right)$$

where $P_{\tau_{ij}}$ is one of the vertices of τ_{ij}. Therefore, to compute the solution of Euler discretization problem, we must minimize at each time ndt, where $n \in \mathbb{N}$ and dt the time step the non-differential

$$\begin{aligned} \mathcal{G}_h : \ & \mathbb{R}^2 \to \mathbb{R} \\ & w_h \mapsto \mathcal{G}_h(w_h) \ := \tfrac{1}{2}(Aw_h, w_h) + (dtf_h^n + u_{n-1}, \text{div}(w_h)) + h^2 \textstyle\sum_{i,j} |w_h + \nabla Z| \left(P_{\tau_{ij}}\right). \end{aligned}$$

where A is an $2N^2 \times 2N^2$ positive semi-definite matrix. For the minimization of this functionnal, we use Gauss Seidel type algorithm :

- Initiate the algorithm with a vector $q_0 \in \mathbb{R}^{2N^2}$ and, for $k = 0$ until convergence, chose a canonical direction e_j in $\mathbb{R}^{2N^2}$ and find ρ_{jk} minimizing $\varphi_{j,k} : \mathbb{R} \to \mathbb{R}$ defined by $\varphi_{j,k}(\rho) = \mathcal{G}_h(q_k + \rho e_j)$
- Take $q_{k+1} = q_k + \rho_{jk}\omega e_j$, where ω is an over-relaxation parameter.
- When $\varphi_{j,k}$ is differentiable, a Newton algorithm is used to find ρ_{jk}. Otherwise,ρ_{jk} can be computed directly (because is this case ϕ_{jk} is the sum of a polynomial of degree two and an absolute value).
- This algorithm is performed until $\|q_{k+1} - q_k\|_{\mathcal{L}^2\left(\mathbb{R}^{N^2}\right)} \leq \varepsilon$ for a given convergence criterion $\varepsilon > 0$. Afterwards, take $w_h = q_k$.

Then, knowing a minimizer w_h of (3.38), solution u_n of Euler implicit time discretization of (P) is computed using extremality (3.35) relation in a weak sense with piecewise finite elements $P0$.

In all tests $w = 0.5, N = 50$ and the convergence criterion equal to $\varepsilon = 10^{-6}$. In the second test we have fixed the step of time dt to 0.0002 and in the others cases $dt = 0.001$.

The first and the second test have been achieved without the source f; ie f equal zeros. In the first test $\Gamma_N = \{-1 < y < 1, x = -1\}$, we have consider nonhomogenous Neumann boundary condition $g = 10^2$, which simulated the presence of a wall on the boundary.

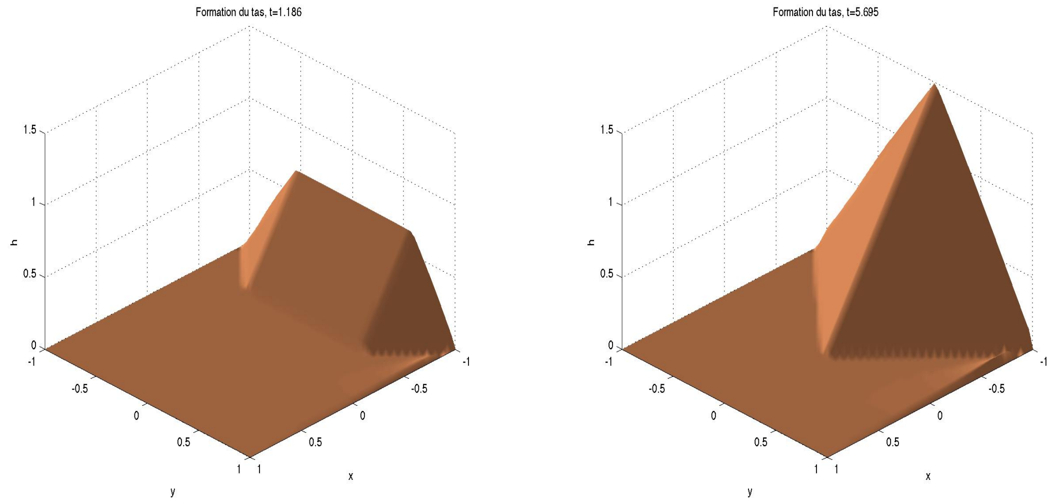

FIGURE 1. Sandpile surface and at $t = 1.186$ and $t = 5.695$ for $f \equiv 0$ and $g \equiv= 10^2$.

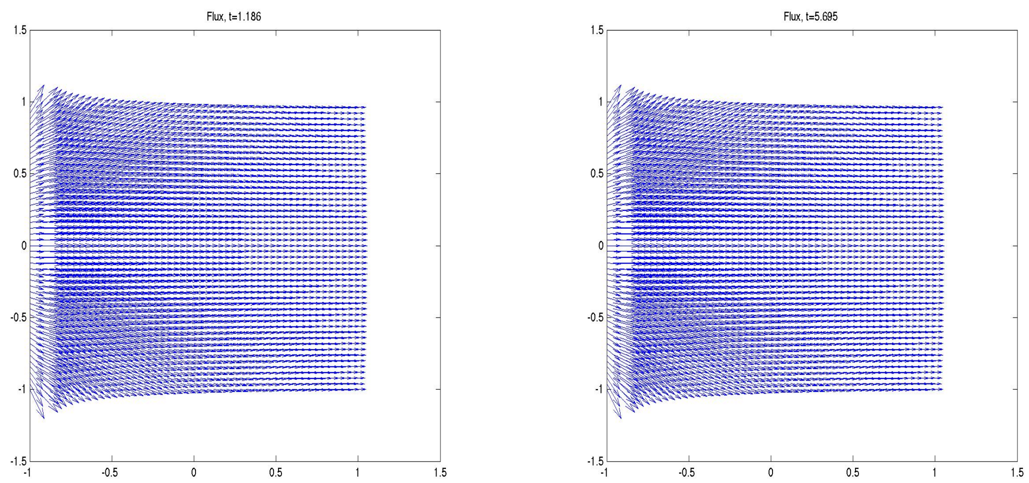

FIGURE 2. sand flux at time $t = 1.186$ and $t = 5.695$ for $f \equiv 0$ and $g \equiv= 10^2$

Figure 1 shows the intermediate profile of sandpile at $t = 1.186$ and the final profile at $t = 5.695$, one notices that in spite of the absence of the source f, the height of the sandpile increases, what proves that the function g acts like a source. Figure 2 shows the flux on the sandpile surface, we can observe that it's quasi-stationary that is due to the fact the flux is equal to $w_h + \nabla Z$. We can also remark that the dynamic of the sand is concentred to the neighborhood of Γ_N.

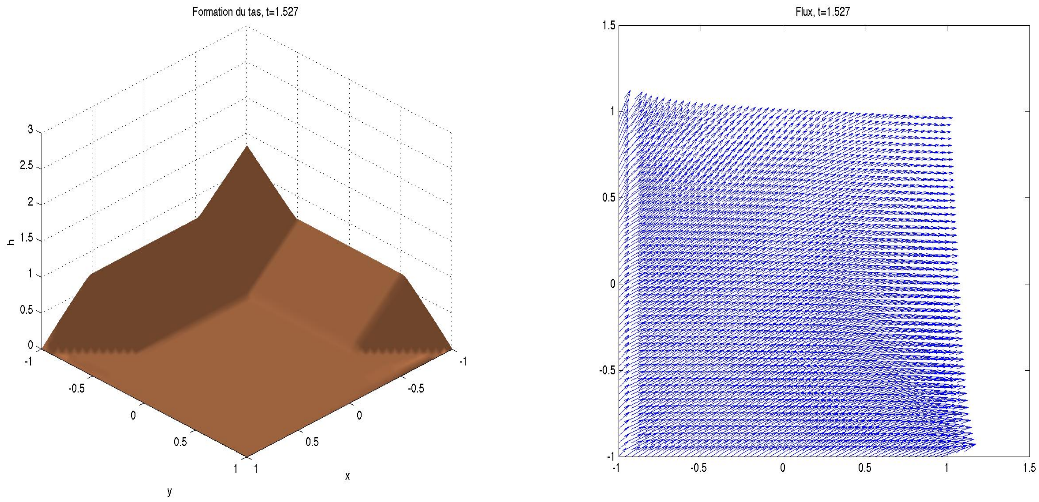

FIGURE 3. Sandpile surface and flux at $t = 1.527$ for $f \equiv 0$ and $g \equiv = 10^2$,

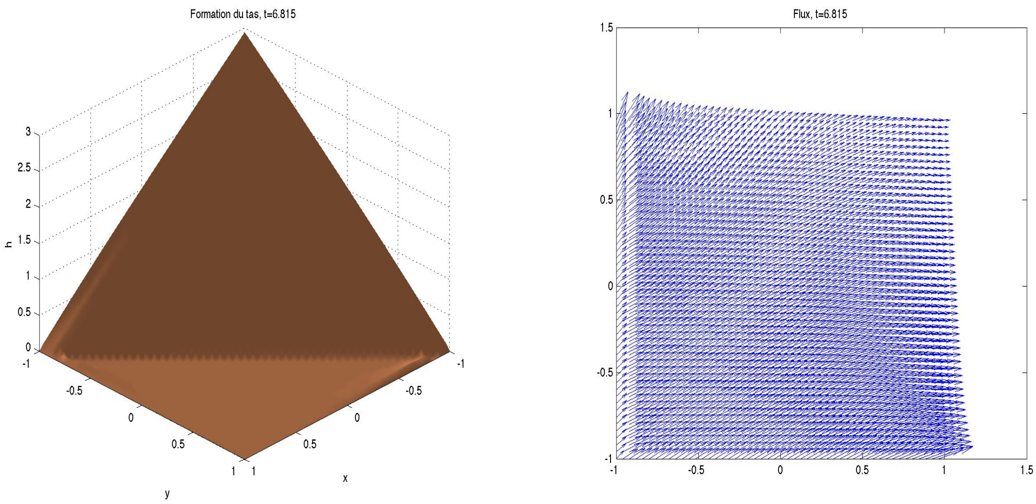

FIGURE 4. sandpile surface and flux at $t = 6.815$ for $f \equiv 0$ and $g = 10^2$,

In the second test $\Gamma_N = \{-1 < y < 1, x = -1\} \cup \{-1 < x < 1, y = -1\}$. Figure (3) and Figure (4) show the intermediate and final profile of the sand.

In the the following test $\Gamma_N = \{-1 < y < 1, x = -1\}$ and we have consider homogenous Neumann boundary condition $g = 0$, which simulated the presence of one wall on the boundary.

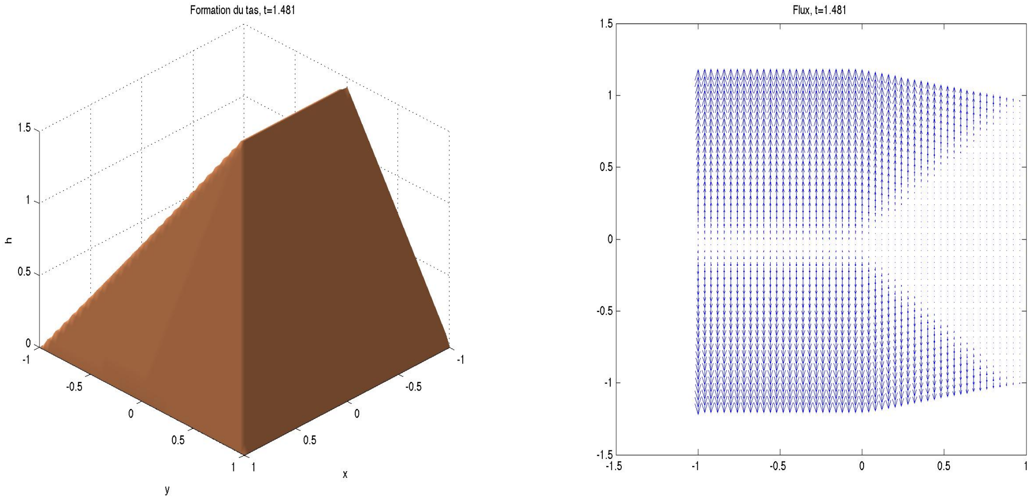

FIGURE 5. Sandpile surface and flux at $t = 1.481$ for $f \equiv 1$ and $g \equiv 0$

Figure 5 show the configuration of the sandpile when it becomes stationary as well as flux to this time. One can observe that the flux is directed unique in two senses and vanishes on the diagonal of square Ω.

In the fourth test we have applying the homegenous Neumann boundary condition $m\dfrac{\partial u}{\partial \nu} = 0$ on the domaine $\Gamma_N = \{-1 < y < 1, x = -1, 1\}$. In others words we simulate the presence of two opposites walls on the boundary.

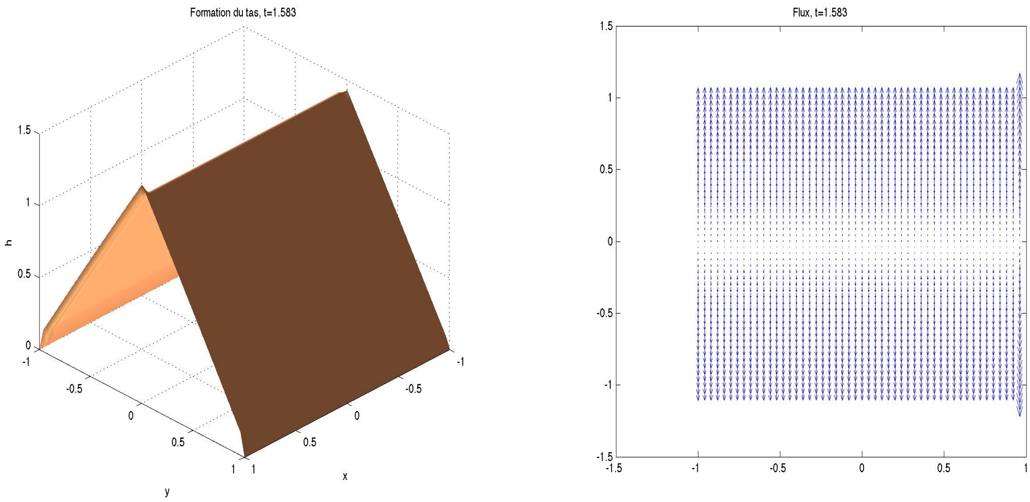

FIGURE 6. Sandpile surface and flux at $t = 1.583$ for $f \equiv 1$ and $g \equiv 0$

The last test simulate the presence of one wall on the boundary in the case where one makes act at the same time the source and g. In this example $\Gamma_N = \{-1 < y < 1, x = -1\}$

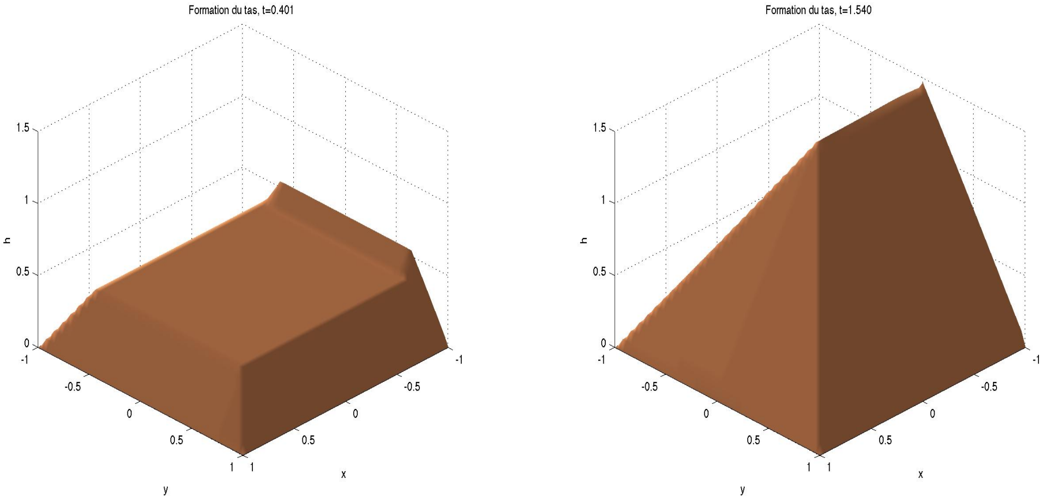

FIGURE 7. Sandpile surface at $t = 0.401$ and $t = 1.540$ for $f \equiv 1$ and $g \equiv 20$

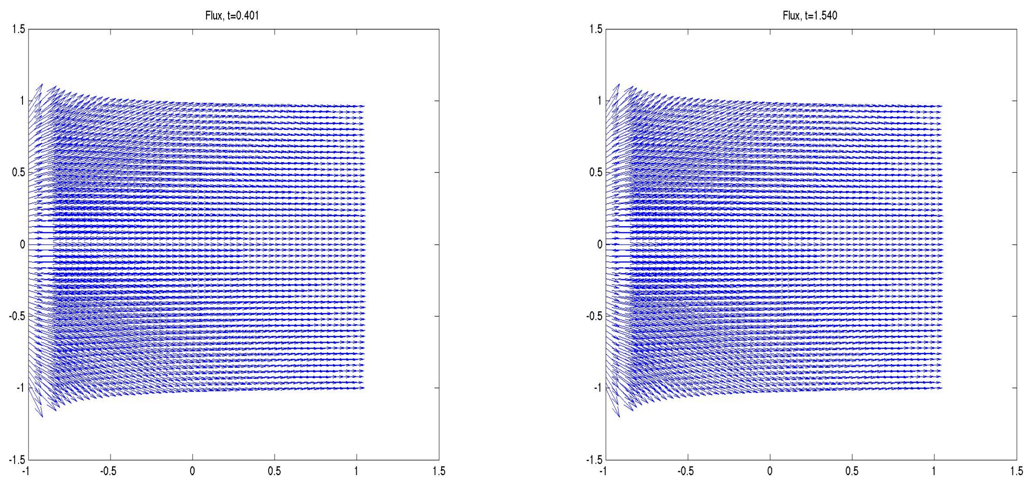

FIGURE 8. Sand flux at $t = 0.401$ and $t = 1.504$ for $f \equiv 0.5$ and $g \equiv 5$

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