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Stabilization for degenerate diffusion with absorption

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Abstract

The purpose of this paper is to study the limit in $L^1(\Omega)$ of solutions of general initial-boundary-value problems of the form $u_t = \Delta w - g(x,u)$ and $u \in \beta(w)$ in a bounded domain Ω with general boundary conditions of the form $\partial_\eta w + \gamma(w) \ni 0$, where β and γ are maximal monotone graphs and $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a nonincreasing continuous function in $r \in \mathbb{R}$. We prove that a solution stabilizes in $L^1(\Omega)$ as $t \to \infty$ to a function $\underline{u} \in L^1(\Omega)$ which satisfies $\underline{u}(x) \in \varphi^{-1}(c) \cap g(x,.)^{-1}(0)$ a.e. $x \in \Omega$, with $c \in \gamma^{-1}(0)$. So, if for instance $\gamma^{-1}(0) = \varphi^{-1}(0) \cap g(x,.)^{-1}(0) = \{0\}$, then a solution stabilizes by converging to 0, in $L^1(\Omega)$, as $t \to \infty$.

MSC: 35K65; 35B40; 47H20

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1. Introduction

Consider the initial-boundary-value problem

$$\begin{cases} u_t - \Delta\varphi(u) + \sigma(x)|u|^{p-1}u = 0 & \text{in } Q := (0,\infty) \times \Omega, \\ \dfrac{\partial}{\partial\eta}\varphi(u) + a\varphi(u) = 0 & \text{on } \Sigma := (0,\infty) \times \Gamma, \\ u(0) = u_0, & \end{cases} \tag{1.1}$$

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where Ω is a bounded domain of $\mathbb{R}^N$ with a smooth boundary Γ, $\partial w/\partial\eta$ is the normal derivative of w, φ is a nondecreasing continuous function such that $\varphi(0)=0$, $p>0$, $\sigma\in BV(\Omega)$, $\sigma\geqslant 0$ and $a\in[0,\infty]$ (the case $a=\infty$ corresponds to the Dirichlet boundary condition). It is known that for any $u_0\in L^\infty(\Omega)$, (1.1) has a unique weak solution u (see for instance [38]). We are interested to the asymptotic behavior of $u(t)$, as $t\to\infty$.

Problems of type (1.1), or some special case of it, arise in many different physical contexts. With respect to stabilization of solutions, the case where φ is increasing (strictly) and continuous is probably the most covered in the literature. For instance, the linear case, i.e. $\varphi(r)=r$ for every $r\in\mathbb{R}$, corresponds to semilinear heat equations (see for instance [8,18,24,28,32–35]). The evolution equations (1.1) with φ increasing and continuous arise in modelling gas flow in porous media [9], and the spread of biological populations (cf. [27,39]); for the stabilization of solutions of this type of problems one can see the works [3,5,21,22,31] (see also [4,13]). Among the results of [3] it is proved that, if φ is increasing (strictly) then a solution of problem (1.1) stabilizes as $t\to\infty$ by converging to a constant function in $L^1(\Omega)$. Our main interest lies in the case where φ is a nondecreasing function for which the evolution equation (1.1) arises in the study of various phenomena with changes of states (see [19,41]). Recently, in [29] we studied the case $\sigma\equiv 0$ with general nonlinearities (φ is any maximal monotone graph); we proved that a solution $u(t)$ stabilizes as $t\to\infty$ by converging to a function $z\in L^1(\Omega)$ such that $z(x)\in\varphi^{-1}(0)$ a.e. $x\in\Omega$; i.e. the limit z remains in the plane region $[\varphi=0]$. Since in general such z is not unique, we also gave a characterization of the true limit for a large class of initial data (see [29]), but the problem of such characterization remains open in general. In this work, we generalize a part of this results to the case $\sigma\not\equiv 0$. In particular, we prove that in the presence of the absorption term $\sigma(x)u$ the solution of (1.1) stabilizes by converging to 0, in $L^1(\Omega)$, as $t\to\infty$, for any $u_0\in L^\infty(\Omega)$.

The main application we have in mind concerns evolution problems of Stefan type. These problems are described by (1.1) with

$$\varphi(r)=\begin{cases}(r-1)^+ & \text{if } r\geqslant 0,\\ r & \text{if } r<0.\end{cases}\tag{1.2}$$

The function u then represents the *enthalpy*, $\varphi(u)$ the *temperature* and $\varphi(u)=0$ the *melting temperature* of the material (see for instance [19,41] and the references therein). The limit of the solution $u(t)$ as $t\to\infty$, is closely connected to a problem that attracted considerable interest; it concerns the nature and the evolution of the so-called "Mushy region" the set which separates the two phases, which is the interior of the set in which $\varphi(u)=0$, i.e. $M=[0<u<1]$. In the classical formulation, M is assumed to be empty. In the case of Dirichlet boundary conditions and starting from a weak formulation, conditions were obtained by Oleinik [40] and Friedman [23] which ensure that indeed $M=\emptyset$. On the other hand it was shown numerically by Atthey [6], and analytically by Meirmanov [37] (see also [25,26,42]) that interior heating may cause M to have nonempty interior. In [12], the authors obtain by means of comparison methods in one dimension a number of qualitative statements about the existence and nonexistence of the set M. In [29], we proved that, if $\sigma\equiv 0$, then the mushy region is nonincreasing and may remains nonempty as $t \to \infty$, and a characterization of the mushy region that will never be reduced by the diffusion was given for a large class of initial data u_0. Actually, in the presence of the absorption term $\sigma(x)|u|^{p-1}u$, we prove that the mushy region disappears completely, as $t \to \infty$, for any initial data u_0.

In fact, we will consider the general evolution equation of the form

$$(\mathrm{P}) \quad \begin{cases} u_t - \Delta w + g(x,u) = 0, \quad u \in \beta(w) & \text{in } Q = (0,\infty) \times \Omega, \\ \dfrac{\partial w}{\partial \eta} = -z, \quad z \in \gamma(w) & \text{on } \Sigma = (0,\infty) \times \partial\Omega, \\ u(0) = u_0 & \text{in } \Omega, \end{cases}$$

where β and γ are maximal monotone graphs in $\mathbb{R}$ (see [14]) such that $\mathscr{D}(\gamma) \cap \mathscr{D}(\beta) \neq \emptyset$ and $g : \Omega \times \mathbb{R} \to \mathbb{R}$ such that $g(x,r)$ is nonincreasing continuous function in r and integrable in x. In particular, γ may be multivalued and this allows the boundary condition to include the Dirichlet (taking $\gamma = \{0\} \times \mathbb{R}$) and the Neumann condition (taking $\gamma = \mathbb{R} \times \{0\}$) as well as many other possibilities. Also, β may be multivalued, so that (P) is a mathematical model of various phenomena with changes of states. On the other hand, β may be a continuous function in $\mathbb{R}$, then (P) is the filtration equation which includes the flow of liquids or gazes through porous media, the heat propagation in plasmas, population dynamics, spread of thin viscous films and others. In [11], the authors treat (P) in the case $g \equiv 0$ in the contest of nonlinear semigroups theory and proved that problem of type (P) has a unique generalized solution. Assuming that $g \not\equiv 0$, we will prove that (P) still has a unique generalized solution u, which is also a solution in a usual weak sense if $u_0 \in L^\infty(\Omega)$. Moreover, $u(t) = S(t)u_0$ where $S(t)$ is a continuous nonlinear semigroup of order preserving contractions in $L^1(\Omega)$. We are interested in the limit of $S(t)u_0$, as $t \to \infty$. In order to prove stabilization result, we need the orbits of the semigroup $S(t)$, i.e. $\{S(t)u_0;\ t \geqslant 0\}$, to be relatively compact in $L^1(\Omega)$. For this aim, we will prove that the resolvent of $S(t)$ are relatively compact from $L^\infty(\Omega)$ into $L^1(\Omega)$, so that using the same arguments of [3] (see also [29,36]) the relative compactness in $L^1(\Omega)$ of the orbits follows. On the other hand, we will use estimates of energy type to describe the limit function $u(t)$, as $t \to \infty$.

The paper is organized as follows. The main results (cf. Theorem 1 and Corollary 1) concerning the stabilization of the solution of (P) is stated and proved in Section 3. In Section 2, we state assumptions on the data that will hold throughout the paper and prove that problem of type (P) is well posed and governed by an order preserving contraction in $L^1(\Omega)$. We also prove energy estimates that are useful for the description of the limit function.

2. Preliminaries

In the sequel, Ω is a bounded domain of $\mathbb{R}^N$ with smooth boundary Γ, φ and γ are maximal monotone graphs in $\mathbb{R}$ such that

$$(\mathrm{H}_1) \qquad \mathscr{D}(\varphi) = \mathbb{R},$$

$$(\mathrm{H}_2) \qquad \text{either } \mathscr{D}(\gamma) = \mathbb{R} \quad \text{or } \mathscr{D}(\gamma) = \{0\},$$

$$(\mathrm{H}_3) \qquad 0 \in \varphi(0) \cap \gamma(0)$$

and $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is such that

$$(\mathrm{H}_4) \qquad \begin{array}{l} \text{for almost all } x \in \Omega,\ r \to g(x,r) \text{ is continuous, nondecreasing} \\ \text{and for every } r \in \mathbb{R},\ x \to g(x,r) \text{ is in } L^1(\Omega) \text{ with } g(.,0) \equiv 0. \end{array}$$

We consider the following evolution problem

$$(\mathrm{E}) \qquad \begin{cases} u_t = \Delta w - g(x,u), \quad w \in \varphi(u) & \text{in } Q = (0,\infty) \times \Omega, \\ \dfrac{\partial w}{\partial \eta} = -z, \quad z \in \gamma(w) & \text{on } \Sigma = (0,\infty) \times \partial\Omega, \\ u(0) = u_0 & \text{in } \Omega \end{cases}$$

with $u_0 \in L^1(\Omega)$. In order to study the problem in the context of nonlinear semigroup theory, we define the operator (possibly multivalued) $A_{g\varphi\gamma}$, in $L^1(\Omega)$ by

$$A_{g\varphi\gamma} = \Big\{ (v,f) \in L^1(\Omega) \times L^1(\Omega);\ g(.,v(.)) \in L^1(\Omega),\ \exists\ w \in W^{1,1}(\Omega),$$
$$\exists\ z \in L^1(\Gamma) \text{ s.t. } w \in \varphi(v) \text{ a.e. in } \Omega,\ z \in \gamma(w) \text{ a.e. on } \Gamma \text{ and}$$
$$\int_\Omega \mathrm{D}w\,\mathrm{D}\xi + \int_\Omega g(.,v)\xi + \int_\Gamma z\xi = \int_\Omega f\xi \text{ for any } \xi \in W^{1,\infty}(\Omega) \Big\}.$$

Proposition 1.

(1) *$A_{g\varphi\gamma}$ is m-accretif in $L^1(\Omega)$, i.e. $A_{g\varphi\gamma}$ has a nonexpansive resolvent $\mathscr{J}_\lambda = (I + \lambda A_{g\varphi\gamma})^{-1}$ everywhere defined in $L^1(\Omega)$, for every $\lambda > 0$.*
(2) *For any $f \in L^p(\Omega)$, with $1 \leqslant p \leqslant \infty$, we have*

$$\|(I + \lambda A_{g\varphi\gamma})^{-1} f\|_{L^p} \leqslant \|f\|_{L^p}.$$

(3) $\overline{\mathscr{D}(A_{g\varphi\gamma})}^{L^1} = L^1(\Omega)$.

Proof.

(1) Thanks to [11], if $g \equiv 0$, we know that the corresponding operator $A_{0\varphi\gamma}$ is m-accretive in $L^1(\Omega)$ and for any $(u,v) \in A_{0\varphi\gamma}$

$$\int_\Omega p(u)v \geqslant 0 \quad \text{for any } p \in \mathbb{P}_0, \tag{2.1}$$

where

$$\mathbb{P}_0 = \{ p \in Lip(\Omega);\ \ p \text{ nondecreasing},\ \ p(0) = 0 \text{ and } supp(p')\text{compact}\}.$$

Now, let B_g be the single-valued operator in $L^1(\Omega)$ defined by $B_g u(x) = g(x, u(x))$ a.e. $x \in \Omega$ with $\mathscr{D}(B_\varphi) = \{u \in L^1(\Omega);\ g(.,u(.)) \in L^1(\Omega)\}$. Since B_g is continuous, accretif in $L^1(\Omega)$ and

$$A_{g\varphi\gamma} = A_{0\varphi\gamma} + B_g, \tag{2.2}$$

then $A_{g\varphi\gamma}$ is accretif in $L^1(\Omega)$. On the other hand, using Corollary 3.1 of [3] and the fact that $A_{0\varphi\gamma}$ satisfies (2.1), we deduce that $A_{g\varphi\gamma}$ is m-accretif in $L^1(\Omega)$.

(2) Since $g(.,0) \equiv 0$, then $A_{g\varphi\gamma}$ also satisfies (2.1), i.e. for any $(u,f) \in A_{g\varphi\gamma}$, we have

$$\int_\Omega p(u) f \geqslant 0 \quad \text{for any } p \in \mathbb{P}_0. \tag{2.3}$$

Indeed, if $(u,f) \in A_{g\varphi\gamma}$ then $(u, f - g(.,u)) \in A_{0\varphi\gamma}$ so that, (2.1) implies that

$$\int_\Omega p(u) f - \int_\Omega p(u) g(.,u) \geqslant 0$$

and (2.3) follows by using the fact that $p(u)g(.,u) \geqslant 0$ a.e. in Ω. Then, Part 2 of the proposition is an immediate consequence of Corollaries 2.1 and 2.2 of [10].

(3) For Part 3 of the proposition, it is enough to prove that

$$L^\infty(\Omega) \subseteq \overline{\mathscr{D}(A_{g\varphi\gamma})}. \tag{2.4}$$

For $u \in L^\infty(\Omega)$, set $u_\varepsilon = (I + \varepsilon A_{g\varphi\gamma})^{-1} u$ for any $\varepsilon > 0$, then $u_\varepsilon = (I + \varepsilon A_{0\varphi\gamma})^{-1}(u - \varepsilon g(.,u_\varepsilon))$. Since $\|u_\varepsilon\|_{L^\infty(\Omega)} \leqslant \|u\|_{L^\infty(\Omega)}$ (cf. Part 2 of the proposition) then $u - \varepsilon g(.,u_\varepsilon) \to u$ in $L^1(\Omega)$, as $\varepsilon \to 0$, so that using Theorem B of [11] and using the fact that $\varepsilon A_{0\varphi\gamma} = A_{0\varphi_\varepsilon\gamma_\varepsilon}$, with $\varphi_\varepsilon := \varepsilon\ \varphi$ and $\gamma_\varepsilon := \varepsilon\ \gamma(./\varepsilon)$, we deduce that $u_\varepsilon \to u$ in $L^1(\Omega)$, as $\varepsilon \to 0$. □

Using the general theory of nonlinear semigroups of evolution equations, $A_{g\varphi\gamma}$ generates a continuous nonlinear semigroup of order preserving contractions $S(t)$, in $L^1(\Omega)$. Moreover, for any $u_0 \in L^1(\Omega)$, $S(t)u_0$ is the unique generalized solution of (E) (cf. Theorem I of [11]). By definition of $S(t)$,

$$S(t)u_0 = L^1 - \lim_{\varepsilon \to 0} u_\varepsilon(t) \tag{2.5}$$

uniformly for $t \in [0,\tau]$, where for $\varepsilon > 0$, u_ε is an ε-approximate solution corresponding to a subdivision $t_0 = 0 < t_1 < \cdots < t_{n-1} < \tau \leqslant t_n$, with $t_i - t_{i-1} = \varepsilon$ and defined by $u_\varepsilon(0) = u_0$, $u_\varepsilon(t) = u_i$ for $t \in]t_{i-1}, t_i]$ where $u_i \in L^1(\Omega)$ satisfies

$$\frac{u_i - u_{i-1}}{\varepsilon} + A_{g\varphi\gamma} u_i \ni 0. \tag{2.6}$$

In other words, the generalized solution u of (E) is given by the exponential formula

$$u(t) = S(t)u_0$$

$$= \mathrm{e}^{-tA_{g\varphi\gamma}} u_0$$
$$= \lim_{n\to\infty} \left(I + \frac{t}{n} A_{g\varphi\gamma}\right)^{-n} u_0. \tag{2.7}$$

Proposition 2. *If $u_0 \in L^\infty(\Omega)$, then the generalized solution u of (E) satisfies*

$$\begin{cases} u \in L^\infty(Q),\ \exists\ w \in L^2_{\text{loc}}([0,\infty), H^1(\Omega)),\ \exists\ z \in L^2_{\text{loc}}([0,\infty), L^2(\Gamma)), \\ w \in \varphi(u) \text{ a.e. in } Q,\ z \in \gamma(w) \text{ a.e. in } \Sigma, \\ \int_0^\tau \int_\Omega \mathrm{D}w\, \mathrm{D}\xi + \int_0^\tau \int_\Gamma \xi z + \int_0^\tau \int_\Omega g(x,u)\xi \\ \quad = \int_0^\tau \int_\Omega \xi_t u + \int_\Omega \xi(0) u_0, \quad \forall\ \xi \in \mathscr{C}^1([0,\tau]\times\bar\Omega),\ \tau > 0 \text{ and } \xi(\tau) \equiv 0. \end{cases} \tag{2.8}$$

Moreover, for any $\tau \geqslant 0$,

$$\|u(\tau)\|_\infty \leqslant \|u_0\|_\infty, \tag{2.9}$$

$$\int_\Omega j(u(\tau)) + \int_0^\tau \int_\Omega |\mathrm{D}w|^2 + \int_0^\tau \int_\Gamma zw + \int_0^\tau \int_\Omega g(.,u)w \leqslant \int_\Omega j(u_0), \tag{2.10}$$

where $j : \mathbb{R} \to [0,\infty]$ is a proper convex s.c.i. function such that $\varphi = \partial j$, and

$$\int_\Omega |u(\tau)| + \int_0^\tau \int_\Omega |g(.,u)| \leqslant \int_\Omega |u_0|. \tag{2.11}$$

Before proving this proposition, we give some consequences of Proposition 1 and results of [11] that will be useful for the sequel. For any $f \in L^1(\Omega)$, there exists a unique (u,w,z) solution of

$$S(f,g,\varphi,\gamma) \quad \begin{cases} v - \Delta w + g(.,v) = f,\ w \in \varphi(v) & \text{in } \Omega, \\ \dfrac{\partial w}{\partial \eta} + z = 0,\ z \in \gamma(w) & \text{on } \Gamma \end{cases}$$

in the sense

$$\begin{cases} v \in L^1(\Omega),\ g(.,v) \in L^1(\Omega),\ w \in W^{1,1}(\Omega),\ z \in L^1(\Gamma), \\ w \in \varphi(v) \text{ a.e. in } \Omega,\ z \in \gamma(w) \text{ a.e. on } \Gamma \text{ and} \\ \int_\Omega \mathrm{D}w\, \mathrm{D}\xi + \int_\Omega g(.,v)\xi + \int_\Gamma z\xi = \int_\Omega (f-v)\xi \\ \text{for any } \xi \in W^{1,\infty}(\Omega). \end{cases} \tag{2.12}$$

In addition, applying Proposition E of [11], for any $f_1,\ f_2 \in L^1(\Omega)$, if (v_i, w_i, z_i) is the solution of $S(f_i, g, \varphi, \gamma)$ for $i = 1, 2$, then

$$\int_\Omega (v_1 - v_2)^+ + \int_\Omega (g(.,v_1) - g(.,v_2))^+ + \int_\Gamma (z_1 - z_2)^+ \leqslant \int_\Omega (f_1 - f_2)^+$$

and

$$\int_\Omega |v_1 - v_2| + \int_\Omega |g(.,v_1) - g(.,v_2)| + \int_\Gamma |z_1 - z_2| \leqslant \int_\Omega |f_1 - f_2|. \tag{2.13}$$

Moreover, if $f \in L^\infty(\Omega)$ then the solution $(v, w, z) \in L^\infty(\Omega) \times H^2(\Omega) \times L^2(\Gamma)$ and one has the following estimates:

$$\|v\|_\infty \leqslant \|f\|_\infty, \tag{2.14}$$

and

$$\|w\|_{H^1(\Omega)} \leqslant C\|f\|_\infty, \tag{2.15}$$

where C is a constant which depends only on Ω and $\|f\|_1$.

Proof of Proposition 2. Using (2.5) and (2.6), let u_ε be the ε-approximate solution with $\varepsilon = \tau/n$, and, for $i = 1, \ldots, n$, let $(w_i, z_i) \in H^2(\Omega) \times L^2(\Gamma)$ such that

$$\begin{cases} u_i + \varepsilon g(.,u_i) - \varepsilon \Delta w_i = u_{i-1},\ \ w_i \in \varphi(u_i) & \text{in } \Omega, \\ \dfrac{\partial w_i}{\partial \eta} + z_i = 0,\ \ z_i \in \gamma(w_i) & \text{on } \Gamma. \end{cases} \tag{2.16}$$

Thanks to (2.14), it follows that $u_i \in L^\infty(\Omega)$ and $\|u_i\|_\infty \leqslant \|u_0\|_\infty$, so that

$$\|u_\varepsilon\|_\infty \leqslant \|u_0\|_\infty \tag{2.17}$$

and, thanks to (2.13), we have

$$\int_\Omega |u_i| + \varepsilon \int_\Omega |g(.,u_i)| + \varepsilon \int_\Gamma |z_i| \leqslant \int_\Omega |u_{i-1}|. \tag{2.18}$$

On the other hand, multiplying (2.16) by w_i and using the fact that

$$\int_\Omega (u_{i-1} - u_i) w_i \leqslant \int_\Omega j(u_{i-1}) - \int_\Omega j(u_i)$$

we deduce that

$$\int_\Omega j(u_i) + \varepsilon \int_\Omega |\mathrm{D}w_i|^2 + \varepsilon \int_\Omega g(.,u_i) w_i + \varepsilon \int_\Gamma z_i w_i \leqslant \int_\Omega j(u_{i-1}). \tag{2.19}$$

Adding (2.18) and (2.19) for $i=1,\dots,n$, we get

$$\int_\Omega |u_\varepsilon| + \int_0^\tau\int_\Omega |g(.,u_\varepsilon)| + \int_0^\tau |z_\varepsilon| \leqslant \int_\Omega |u_0| \tag{2.20}$$

and

$$\int_\Omega j(u_\varepsilon(\tau)) + \int_0^\tau\int_\Omega |\mathrm{D}w_\varepsilon|^2 + \int_0^\tau\int_\Omega g(.,u_\varepsilon)w_\varepsilon + \int_0^\tau\int_\Gamma w_\varepsilon z_\varepsilon \leqslant \int_\Omega j(u_0), \tag{2.21}$$

where $w_\varepsilon:[0,\tau]\to H^1(\Omega)$ and $z_\varepsilon:[0,\tau]\to L^2(\Gamma)$ with $w_\varepsilon(t)=w_i$ and $z_\varepsilon(t)=z_i$, for any $t\in\,]t_{i-1},t_i]$, $i=1,\dots,n$. Thanks to (H_1) and (2.17), w_ε is bounded in $L^\infty((0,\tau)\times\Omega)$, and, thanks to (H_2), z_ε is bounded in $L^\infty((0,\tau)\times\Gamma)$. On the other hand, using the fact that $j\geqslant 0$, $g(.,u_\varepsilon)w_\varepsilon\geqslant 0$ a.e. in $[0,\tau]\times\Omega$ and $z_\varepsilon w_\varepsilon\geqslant 0$, a.e. in $[0,\tau]\times\Gamma$, we deduce from (2.21) that w_ε is bounded in $L^2(0,\tau;H^1(\Omega))$.

Let $w\in L^2(0,\tau;H^1(\Omega))$, $z\in L^2((0,\tau)\times\Gamma)$ and $\varepsilon_k\to 0$, such that $z_{\varepsilon_k}\to z$ weakly in $L^2((0,\tau)\times\Gamma)$, $w_{\varepsilon_k}\to w$ weakly in $L^2(0,\tau;H^1(\Omega))$ and in $L^2((0,\tau)\times\Gamma)$. Since, for any $t>0$, $w_{\varepsilon_k}(t)\in\varphi(u_{\varepsilon_k}(t))$ a.e. in Ω, then by monotonicity argument we deduce that $w(t)\in\varphi(u(t))$ a.e. in Ω. Now, let $\tilde u_\varepsilon$ be the function from $[0,\tau]$ into $L^1(\Omega)$, defined by $\tilde u_\varepsilon(t_i)=u_i$, $\tilde u_\varepsilon$ is linear in $[t_{i-1},t_i]$, then (2.16) implies that

$$\begin{aligned}&\int_0^\tau\int_\Omega \mathrm{D}w_\varepsilon\,\mathrm{D}\xi + \int_0^\tau\int_\Omega g(.,u_\varepsilon)\xi + \int_0^\infty\int_\Gamma \xi z_\varepsilon\\ &\quad= \int_0^\tau\int_\Omega \tilde u_\varepsilon\xi_t + \int_\Omega \xi(0)u_0\end{aligned} \tag{2.22}$$

for any $\xi\in\mathscr{C}^1([0,\tau]\times\bar\Omega)$. Letting $\varepsilon\to 0$ in (2.17), (2.20) and (2.22), we get (2.9), (2.11) and

$$-\int_0^\tau\int_\Omega u\xi_t - \int_\Omega \xi(0)u_0 + \int_0^\tau\int_\Omega \mathrm{D}w\,\mathrm{D}\xi + \int_0^\tau\int_\Omega g(.,u)\xi + \int_0^\tau\int_\Gamma \xi z = 0 \tag{2.23}$$

for any $\xi\in\mathscr{C}^1([0,\tau]\times\bar\Omega)$. It remains to prove that

$$z\in\gamma(w) \quad \text{a.e. in } \Sigma. \tag{2.24}$$

To this aim, let us consider the operator (possibly multivalued) G defined in $L^2(\Sigma)$ by

$$G\eta = \{z\in L^2(\Sigma);\ z\in\gamma(\eta) \quad \text{a.e. in } \Sigma\}.$$

It is clear that G is a maximal monotone graph in $L^2(\Sigma)\times L^2(\Sigma)$ and (2.24) is equivalent to $z\in G(w)$. Since, $z_\varepsilon\in G\ w_\varepsilon$, $z_\varepsilon\to z$ weakly in $L^2(\Sigma)$ and $w_\varepsilon\to w$ weakly in $L^2(\Sigma)$, then, thanks to Proposition 2.5 of [14], it is enough to prove that

$$\liminf_{\varepsilon\to 0}\iint_\Sigma z_\varepsilon\ w_\varepsilon \leqslant \iint_\Sigma zw. \tag{2.25}$$

Firstly, one sees that letting $\varepsilon \to 0$ in (2.21), we have

$$\liminf_{\varepsilon\to 0} \iint_\Sigma w_\varepsilon\ z_\varepsilon \leqslant \int_\Omega j(u_0) - \iint_Q |\mathrm{D}w|^2 - \int_\Omega j(u(\tau)) - \iint_\Omega g(.,u)w.$$

On the other hand, since w satisfies (2.23), then, one proves exactly in the same way of Lemma 4.6 of [16] (see also Lemma 1.5 of [2]), that

$$\int_\Omega j(u(\tau)) + \iint_Q |\mathrm{D}w|^2 + \iint_Q g(x,u)w + \iint_\Sigma zw = \int_\Omega j(u_0),$$

which implies that (2.25) is fulfilled. □

Remark 1.

(1) If $u_0 \in L^\infty(\Omega)$, then Proposition 1 implies that $S(t)u_0$ is also a solution of (E) in the usual weak sense. But, we do not know if weak solutions are unique in general. However, this is true in the case of linear boundary conditions and also in the case where γ and φ are locally Lipschitz continuous functions (see for instance [38]).

(2) If $u_0 \in L^1(\Omega)$, we do not know in which sense $S(t)u_0$ satisfies (E) in general. However, in the case of Dirichlet boundary condition we know that, if either φ or φ^{-1} is a nondecreasing continuous function, then $S(t)u_0$ is the unique solution of (E) in the renormalized solution (cf. [17,30] and the references therein).

(3) In the case $\mathscr{D}(\gamma) = \{0\}$, i.e. Dirichlet boundary condition, assumption (H_1) is not necessary, and all the results of Proposition 1 remains true even if $\mathscr{D}(\varphi) \neq \mathbb{R}$. Indeed, with Dirichlet boundary condition on Γ, the Poincarè inequality gives directly a control of the H^1-norm of $w_\varepsilon(t)$ with the L^2-norm of $\mathrm{D}w_\varepsilon(t)$. With the compactness of u_ε, this is enough to pass to the limit in the equation and the inequalities satisfied by u_ε and w_ε.

3. Stabilization results

Throughout this section, we assume that g satisfies

$$(\mathrm{H}_4') \qquad g(.,r) \in BV(\Omega) \quad \text{uniformly for } r \in [r_1, r_2]$$

for any $-\infty < r_1 < r_2 < +\infty$. We also introduce the set $\mathscr{K}$, defined by

$$\mathscr{K} = \{z \in L^1(\Omega);\ \exists\ c \in \gamma^{-1}(0),\ z(x) \in \varphi^{-1}(c) \cap g(x,.)^{-1}(0) \text{ a.e. } x \in \Omega\}.$$

It is not difficult to see that $\mathscr{K}$ is a nonempty closed subset of $L^1(\Omega)$ and, moreover, $\mathscr{K}$ is contained in the set of stationary solutions of (E), i.e. for any $z \in \mathscr{K}$, $S(t)z=z$, for any $t \geqslant 0$. Indeed, thanks to (H_3) and (H_4), we see that $0 \in \mathscr{K}$, which implies that $\mathscr{K} \neq \emptyset$. On the other hand, for any $z \in \mathscr{K}$, it is not difficult to verify that $(I + \lambda\ A_{g\varphi\gamma})^{-1} z = z$, for any $\lambda > 0$, so that

$$S(t)z = L^1 - \lim_{n\to\infty} \left(I + \frac{t}{n} A_{g\varphi\gamma}\right)^{-n} z = z.$$

Theorem 1. *For any $u_0 \in L^1(\Omega)$, there exists a unique $\underline{u} \in \mathscr{K}$, such that*

$$S(t)u_0 \to \underline{u} \quad \text{in } L^1(\Omega), \text{ as } t \to \infty.$$

In particular we have

Corollary 1. *If, $\gamma^{-1}(0) = \varphi^{-1}(0) \cap g(x,.)^{-1}(0) = \{0\}$, then*

$$S(t)u_0 \to 0 \quad \text{in } L^1(\Omega), \text{ as } t \to \infty.$$

The proof of Theorem 1 will follows as a corollary of a sequence of lemmas that we next present. First, we need the orbits of the semigroup $S(t)$, i.e. $\{S(t)u_0;\ t \geqslant 0\}$, to be relatively compact in $L^1(\Omega)$. Now, it is not possible to obtain this result from the compactness of the semigroup because it is known for the Dirichlet boundary condition case, that if $g \equiv 0$ and $\beta(r) = |r|^{(1/m)-1}r$, then $S(t): L^1(\Omega) \to L^1(\Omega)$ is compact if $m > ((N-2)/N)$ $(N \geqslant 3)$ (see [7]) but for $0 < m \leqslant (N-2)/N$, even the resolvents are not compact (see [15]). For general boundary condition and β an increasing (strictly) continuous function everywhere defined, Mazon and Toledo proved in [36] (see also [3]) that $S(t)u_0$ is relatively compact in $L^1(\Omega)$, for any $u_0 \in L^1(\Omega)$ (one can see also [1,20] for Dirichlet and Neumann boundary conditions, respectively). In [29], we proved that this result remains true if φ and γ are maximal monotone graphs satisfying (H_1)–(H_3). Next (cf. Lemma 2), we will generalize this result to (P), with an absorption g satisfying (H_4').

Lemma 1. *Let $f \in L^\infty(\Omega)$, $\lambda > 0$ and $v = \mathscr{J}_\lambda f$. For any $y \in \mathbb{R}^N$ and $\xi \in \mathscr{C}^2(\Omega)$ supported in $\{x \in \Omega;\ distance(x,\Gamma) < |y|\}$, we have*

$$\int_\Omega \xi(x)|v(x+y) - v(x)|\,dx + \lambda \int_\Omega \xi(x)|g(x+y, v(x+y)) - g(x+y, v(x))|\,dx$$

$$\leqslant C|y|\|\Delta\xi\|_\infty\ \|f\|_\infty + \int_\Omega \xi(x)|f(x+y) - f(x)|\,dx$$

$$+\lambda \int_\Omega \xi(x)|g(x+y, v(x)) - g(x, v(x))|\,dx,$$

where C is a constant depending only on Ω and $\|f\|_1$.

Proof. Let ξ as above and $(w,z) \in H^1(\Omega) \times L^2(\Gamma)$ such that (v,w,z) is the solution of $S(f, \lambda g, \lambda\varphi, \lambda\gamma)$. First, let us prove that

$$\int_\Omega \xi(x)|v(x+y) - v(x)|\,dx + \lambda \int_\Omega \xi(x)|g(x+y, v(x+y)) - g(x+y, v(x))|\,dx$$

$$\leqslant \int_\Omega |\Delta\xi||w(x+y) - w(x)|\,dx + \int_\Omega \xi(x)|f(x+y) - f(x)|dx$$

$$+\lambda \int_\Omega \xi(x)|g(x+y, v(x)) - g(x, v(x))|. \tag{3.1}$$

Setting $V=v(x+y)-v(x)$, $W=w(x+y)-w(x)$, $G=g(x+y,v(x+y))-g(x+y,v(x))$ and $F=f(x+y)-f(x)+\lambda(g(x+y,v(x))-g(x,v(x)))$, we observe that

$$-\lambda\Delta W = F - U - \lambda G \quad \text{in } \mathscr{D}'(\Omega).$$

Applying Lemma F of [11], we get

$$\int_{[w>0]} \{(F-U-\lambda G)\xi + W\Delta\xi\} \geqslant \int_{[W=0]} (F-U-\lambda G)^{-}\xi,$$

so that using the fact that $(F-U-\lambda G)^{-} \geqslant (U+\lambda G)^{+} - F^{+}$, $(U+\lambda G)^{+} = U^{+}+\lambda G^{+}$, $\int_\Omega U^{+} = \int_{[W>0]} U + \int_{[W=0]} U^{+}$ and $\int_\Omega G^{+} = \int_{[W>0]} G + \int_{[W=0]} G^{+}$, we conclude that

$$\begin{aligned}\int \xi U^{+} + \lambda \int \xi G^{+} &\leqslant \int_{[W>0]} \xi F + \int_{[W=0]} \xi F^{+} \\ &\leqslant \int \xi F^{+}. \end{aligned} \tag{3.2}$$

In a similar way, one proves that

$$\int \xi U^{-} + \lambda \int \xi G^{-} \leqslant \int \xi F^{-}. \tag{3.3}$$

Adding (3.2) and (3.3) one gets (3.1). At last, since

$$\int_\Omega |\Delta\xi||W|\,\mathrm{d}x \leqslant |y||\Delta\xi\|_\infty |\Omega|^{1/2}\|\nabla w\|_2,$$

then (3.1) implies the result of the lemma. □

Lemma 2. *Under the assumptions* (H_1), (H_2), (H_3), *and* (H_4'), *for any* $u_0 \in L^\infty(\Omega) \cap \mathscr{D}(A_{g\varphi\gamma})$, $S(t)u_0$ *is relatively compact in* $L^1(\Omega)$.

Proof. First, using Lemma 1, we see that for any $\lambda > 0$ fixed and B a bounded subset of $L^\infty(\Omega)$, $\mathscr{J}_\lambda B$ is a relatively compact subset of $L^1(\Omega)$. Indeed, for any $\{f_n\} \subseteq B$, with an appropriate choice of ξ and using (H_4'), we have

$$\lim_{|y|\to 0} \sup_{t>0} \int_{\Omega'} |\mathscr{J}_\lambda f_n(x+y) - \mathscr{J}_\lambda f_n(x)| = 0$$

for any $\Omega' \subset\subset \Omega$, which implies, with (2.9), that $\{\mathscr{J}_\lambda f_n\}$ is relatively compact in $L^1(\Omega)$. Now, since $u_0 \in L^\infty(\Omega)$, then thanks to (2.9) and the first part of the proof, we deduce that, for any fixed $\lambda > 0$, $\mathscr{J}_\lambda S(t)u_0$ is relatively compact in $L^1(\Omega)$. On the other hand, since $u_0 \in \mathscr{D}(A_{g\varphi\gamma})$, then

$$\|S(t)u_0 - \mathscr{J}_\lambda S(t)u_0\|_1 \leqslant \lambda \inf\{\|v\|_1,\ v \in A_{g\varphi\gamma}u_0\} \tag{3.4}$$

and the relative compactness of $S(t)u_0$ in $L^1(\Omega)$ follows. Indeed, we know that, for any fixed $\lambda > 0$, there exists a subsequence $t_n \to \infty$ such that, $\mathscr{J}_\lambda S(t_n)u_0$ converges in $L^1(\Omega)$, as $t \to \infty$, and, by using (3.4), one proves easily that $S(t_n)u_0$ is a Cauchy sequence in $L^1(\Omega)$. $\square$

Now, for any $u_0 \in L^1(\Omega)$, we define the ω-limit set of (E) by

$$\omega(u_0) = \left\{ \underline{u} \in L^1(\Omega);\ \underline{u} = L^1 - \lim_{t_n \to \infty} S(t_n)u_0 \text{ for some sequence } t_n \right\}.$$

As a corollary of the preceeding lemma, we have

Corollary 2. *For any $u_0 \in L^\infty(\Omega) \cap \mathscr{D}(A_{g\varphi\gamma})$, $\omega(u_0) \neq \emptyset$.*

Lemma 3. *Assuming $u_0 \in L^\infty(\Omega)$, we have*

$$\omega(u_0) \subseteq \{z \in L^1(\Omega);\ g(x, z(x)) = 0 \ a.e.\ x \in \Omega\}.$$

Proof. Let $\underline{u} \in \omega(u_0)$ and let $t_k \to \infty$, such that $u(t_k) \to \underline{u}$ in $L^1(\Omega)$. It is clear that $u(t + t_n) = S(t)u(t_n)$, so that by using the continuity of the semigroup $S(t)$ on $L^1(\Omega)$, we get

$$u(t + t_n) \to S(t)\underline{u} \quad \text{in } L^1(\Omega) \quad \text{as } t_n \to \infty, \tag{3.5}$$

uniformly in $t \in [0, \tau]$, for any $\tau > 0$. On the other hand, using (2.11), we have

$$\int_0^\tau \int_\Omega |g(x, u(t + t_n, x))| \, dx \, dt = \int_{t_n}^{t_n+\tau} \int_\Omega |g(x, u(t, x))| \, dx \, dt \to 0$$

as $t_n \to \infty$; so that (3.5) and (2.9) implies that

$$\int_0^\tau \int_\Omega |g(x, S(t)\underline{u})| \, dx \, dt = 0 \quad \text{for any } \tau > 0$$

and then $g(x, \underline{u}(x)) = 0$ a.e. $x \in \Omega$. This ends up the proof of the lemma. $\square$

Lemma 4. *Assuming $u_0 \in L^\infty(\Omega) \cap \mathscr{D}(A_{g\varphi\gamma})$, we have*

$$\omega(u_0) \cap \mathscr{K} \neq \emptyset.$$

Proof. Using Proposition 2, let us consider $(w, z) \in H^1(\Omega) \times L^2(\Gamma)$, such that (u, w, z) satisfy (2.12) with $u(t) = S(t)u_0$. Thanks to (2.10) and since $j \geqslant 0$ and, for any $t \geqslant 0$, $w(t)z(t) \geqslant 0$ and $g(., u(t))w(t) \geqslant 0$, a.e. in Ω, there exists $t_n \to \infty$, such that

$$\lim_{t_n \to \infty} \left(\int_\Omega |Dw(t_n)|^2 + \int_\Gamma z(t_n)w(t_n) \right) = 0. \tag{3.6}$$

So, using (H_1), (H_2), (3.6) and Poincaré inequality, we deduce that $w(t_n)$ is bounded in $H^1(\Omega)$ and $z(t_n)$ is bounded in $L^\infty(\Gamma)$, as $t_n \to \infty$. Thanks to Lemma 2, let $t_{nk} \to \infty$ such that $u(t_{nk}) \to \underline{u}$ in $L^1(\Omega)$, $z(t_{nk}) \to \underline{z}$ weakly in $L^2(\Gamma)$ and $w(t_{nk}) \to \underline{w}$

weakly in $H^1(\Omega)$ and strongly in $L^2(\Omega)$. As in the proof of Proposition 1, by using standard compactness and monotony arguments, we get $\underline{w}(x) \in \varphi(\underline{u}(x))$ a.e. in $x \in \Omega$ and $\underline{z}(x) \in \gamma(\underline{w}(x))$, a.e. $x \in \Gamma$. Passing to the limit in (3.6), through the subsequence t_{nk}, we get

$$\int_\Omega |D\underline{w}|^2 + \int_\Gamma \underline{z}\,\underline{w} = 0,$$

so that, by using the fact that $\underline{z}\,\underline{w} \geqslant 0$ a.e. in Γ, we have

$$\underline{w} \equiv c \quad \text{in} \quad \Omega \quad \text{and} \quad \underline{z}\ c \equiv 0 \quad \text{on } \Gamma. \tag{3.7}$$

Since $\underline{z}\ c \equiv 0$ on Γ, then $c \in \gamma^{-1}(0)$ and we deduce, by using Lemma 3, that $\underline{u} \in \mathscr{K}$. This ends up the proof of the lemma. □

Proof of Theorem 1. First, we see that the result of the theorem is true under the assumption of Lemma 4. Indeed, assuming that $u_0 \in L^\infty(\Omega) \cap \mathscr{D}(A_{g\varphi\gamma})$, we know (by Lemma 4) that there exists a subsequence $t_n \to \infty$ and $\underline{u} \in \mathscr{K}$, such that $S(t_n)u_0 \to \underline{u}$, in $L^1(\Omega)$. On the other hand, since $S(t)$ is a contraction in $L^1(\Omega)$ and $S(t)\underline{u} = \underline{u}$, for any $t \geqslant t_n$, then

$$\begin{aligned} \|S(t)u_0 - \underline{u}\|_1 &= \|S(t-t_n)S(t_n)u_0 - \underline{u}\|_1 \\ &= \|S(t-t_n)S(t_n)u_0 - S(t-t_n)\underline{u}\|_1 \\ &\leqslant \|S(t_n)u_0 - \underline{u}\|_1, \end{aligned}$$

so that, by letting $t_n \to \infty$, we deduce that $S(t)u_0 \to \underline{u}$, in $L^1(\Omega)$ as $t \to \infty$. Now, if $u_0 \in L^1(\Omega)$, then thanks to Propositions 1–3, we consider a sequence $(u_{0n})_{n\in\mathbb{N}}$ of $L^\infty(\Omega) \cap \mathscr{D}(A_{g\varphi\gamma})$ such that $u_{0n} \to u_0$ in $L^1(\Omega)$. Using the first part of the proof, we know that there exists $\underline{u}_{0n} \in \mathscr{K}$, such that, for any $n \in \mathbb{N}$, $S(t)u_{0n} \to \underline{u}_{0n}$ in $L^1(\Omega)$, as $t \to \infty$. Now, using the contraction property of $S(t)$, it is not difficult to see that $\underline{u}_{0n}$ is a Cauchy sequence in $L^1(\Omega)$, so that if $\underline{u}_0$ is the L^1-limit of $\underline{u}_{0n}$, as $n \to \infty$, then $\underline{u}_0 \in \mathscr{K}$ and by using, again, the contraction of $S(t)$, one sees that $S(t) \to \underline{u}_0$ in $L^1(\Omega)$, as $t \to \infty$. □

Remark 2.

(1) Notice that Corollary 1 corresponds to the case where $\mathscr{K} = \{0\}$. In general, we do not know the true value of the limit of $S(t)u_0$, as $t \to \infty$, among the elements of $\mathscr{K}$. In the case $g \equiv 0$, we gave in [29] a characterization of this limit for a large class of initial data u_0. It would be interesting to generalize this results to the case $g \not\equiv 0$.

(2) In a similar way of Remark 1, one sees that in the case of Dirichlet boundary condition, results of Theorem 1 and Corollary 1 remain true without the assumption (H_1). In this direction, notice that an interesting application of Corollary 1

is the porous medium equation of quasilinear elliptic-parabolic type:

$$\begin{cases} b(v)_t - \Delta v + g(x, b(v)) = 0 & \text{in } Q := (0,\infty) \times \Omega, \\ v = 0 & \text{on } \Sigma = (0,\infty) \times \partial\Omega, \\ u(0) = u_0 & \text{in } \Omega, \end{cases} \tag{3.8}$$

where $b : \mathbb{R} \to \mathbb{R}$ is nondecreasing and continuous such that $\varphi(0) = g(x,0) = 0$ a.e. $x \in \Omega$. This equation appears in the study of one saturated-unsaturated of water through a porous medium, where the gravity force is neglected. Then $u = b(v)$ represents the concentration of the water, v the pressure and $g(x,u)$ the absorption term. The function b is given by experiments and usually is a continuous nondecreasing function such that $\mathrm{Im}(b) \neq \mathbb{R}$, so that $b = \varphi^{-1}$ may be a maximal monotone graph not define in all $\mathbb{R}$.

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