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Renormalized solution for Stefan type problems: existence and uniqueness

Noureddine Igbida, Karima Sbihi, and Petra Wittbold

Abstract. We consider a class of nonlinear degenerate problems of Stefan type:

$$u_t - \Delta w - \nabla F(u, w) = g(\cdot, u),\ \ w \in \beta(u)$$

where β is a maximal monotone graph in $\mathbb{R}^2$, with homogenous Dirichlet conditions and initial conditions. Under rather general assumptions on F and g, we prove existence and uniqueness of renormalized solutions.

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1. Introduction

Let Ω be a bounded open domain of $\mathbb{R}^N$ with smooth boundary, $T > 0$. For given function g, and $u_0 \in L^1(\Omega)$ we consider the evolution problem

$$\begin{cases} u_t \ - \Delta w - \nabla F(u, w) = g(t, x, u), \quad w \in \beta(u) & \text{in} \quad Q := (0, T) \times \Omega \\ w = 0 & \text{on} \quad \Sigma := (0, T) \times \partial\Omega \\ u(0, .) = u_0(.) & \text{in} \quad \Omega, \end{cases} \tag{Eu_0,g}$$

under the assumptions:

$$\beta \text{ is a maximal monotone graph such that } 0 \in \beta(0), \tag{H_1}$$

$$F(r_1, r_2) = F_1(r_2) + r_1\, F_2(r_2) \ \text{ for any } r_1, r_2 \in \mathbb{R}$$
$$\text{with } F_i \in C(\mathbb{R}; \mathbb{R}) \text{ and } F_2(0) = 0, \tag{H_2}$$

$$\begin{pmatrix} i)\ g(t, x, r) \text{ is continuous in } r \text{ and measurable in } (t, x) \\ ii)\ \dfrac{\partial g}{\partial r}(t, x, r) \leq C \text{ in } \mathcal{D}'(\mathbb{R}),\ C \in \mathbb{R}^+ \\ iii)\ \ |g(t, x, r)| \leq C_1(t, x)|r| + C_2(t, x) \end{pmatrix} \tag{H_3}$$

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with $C_1,\ C_2\ \in L^1(Q)$, There is an extensive literature on this type of problems, since it serves as a mathematical model for a large class of physical problems (see [1,20] and the references therein). A large field of applications corresponds to the case of maximal montone graph β (not continuous) such that β^{-1} is continuous. for which there exists a large number of references. In particular, $E(u_0, g)$ models in this case free boundary problems involving a solid-liquid phase change of Stefan type for which there exists a large number of references. Among them, let us mention the earlier works [1,17,19]. A complete bibliography may be found in [30]. The structure condition (H_2) includes in particular the Stefan problem with a temperature dependent convective term (see for instance [31,32]).

The problem of establishing uniqueness of solutions of $E(u_0, g)$ seems to be complicated in general. The equation in $E(u_0, g)$ has a hyperbolic character in the set where $w = 0$, and we say that $E(u_0, g)$ is of parabolic-hyperbolic type; in general, uniqueness of a weak solution as well as uniqueness of renormalized solution do not hold. In [15], Carrillo proves that problems of type $E(u_0, g)$ are well posed using the concept of "entropy solutions", which are weak solutions that satisfy some additional conditions called entropy conditions. However, under the additional structure condition (H_2), it is well known by now (see [4,15,21,22]) that Problem $E(u_0, g)$ is expected to admit at most one weak solution which, by definition, is a function $u \in L^1(Q)$ such that $w \in L^2(0, T; H_0^1(\Omega))$ and satisfies the equation in $\mathcal{D}'(Q)$. As to the existence of a weak solution, this requires additional assumptions on the data u_0 and g, for instance $u_0 \in L^\infty(\Omega)$ and $g \in L^\infty(Q)$. In this paper, we consider the case where all the right hand side data belongs to L^1. This means that all the sources should have finite energy, which is a physically reasonable requirement.

In order to solve $E(u_0, g)$ for general L^1-data one needs a more general notion of solution. The framework of renormalized solution, which was originally introduced in [18] for study the Boltzmann equation, has proved to be a powerful approach to study a large of class of problems, see, among others, [3,6,11–14,25,27,29].

In the case where β^{-1} is a nondecreasing continuous function, problem $E(u_0, g)$ is a particular case of the so-called elliptic-parabolic problem, and has been studied extensively in the literature (see [1,3,10,23,26], and the references therein). For instance, if F is continuous, it is proved among the results of [16], that, for any $u_0 \in L^1(\Omega)$ and $g \in L^1(Q)$, $E(u_0, g)$ has at most one renormalized solution. Existence of this type of solution has been shown in [3] (see also [13] where the case of a strictly increasing regular function β is treated). The case where β^{-1} is a nondecreasing multivalued function has been studied in [28,29] where the authors established existence and uniqueness of renormalized solutions.

In this paper, we are interested in the case where β is a maximal monotone graph in $\mathbb{R}^2$ with $0 \in \beta(0)$, where the convection term satisfies the structure condition (H_2) and where the data g satisfies Assumption (H_3). We prove that, for any $u_0 \in L^1(\Omega)$, the problem $E(u_0, g)$ is well posed in the renormalized sense. We first consider the case where g is an integrable function f, then we deduce existence of renormalized solution for any g satisfying Assumption (H_3) by using the results of [9].

The proof of existence of renormalized solution consists of two steps: in a first step, for bounded data, we study the non-degenerate problem: (E_k) $u_t - \Delta w - \nabla F(u, w) = f,\ w \in \beta_k(u)$ on Q (+ homogenous Dirichlet boundary conditions and initial conditions), and then we pass to the limit with k. Here β_k is an approximation of the graph β. Existence of weak solutions of this non-degenerate problem is ensured by the work of [15], thanks to the nonlinear semigroups theory (see [7,8]). In order to pass to the limit with k, we need L^∞-estimates and strong convergence in L^1 of the sequence $(w_k)_k$ (see the proof of Proposition 4.2), which are not easy to obtain. To overcome this difficulty we add to Problem (E_k) a monotone function $\psi_{m,n}(w)$. Recall that this type of arguments was already used in [2,3] for elliptic-parabolic problem, and in [5] for parabolic problem of absorption type. Due to the strongly monotone perturbation term, one can prove an L^1-estimate and, in particular, the strong compactness of the sequence of solutions w and also its strong convergence in L^1 to a measurable function. This allows to pass to the limit with k in Problem (E_k) with a fixed perturbation $\psi_{m,n}$.

In the second step, using a bi-monotone approximation $u^0_{m,n}, f_{m,n}$ of the data u_0, f, in the same way of [3], we obtain a monotone sequence of weak solutions $u_{m,n}$ of Problem $E(u^0_{m,n}, f_{m,n}, \psi_{m,n})$. For the convergence of $w_{m,n},\ w_{m,n} \in \beta(u_{m,n})$ (see the proof of Theorem 5.1) we use the monotonicity with respect to m and n, and for the identification of the limit equation essential tool is the regularization method of Landes (see [24]).

The main difficulty when dealing with hyperbolic-parabolic problem is the uniqueness. In [15] the uniqueness of weak solutions was established under the additional assumption that $\beta^{-1}(0) = 0$. In [21], the authors assumed that F_i is Lipschitz continuous, and in [22], it is assumed that F_i is continuous and satisfies $\|F(u, w)\| \leq C\|w\|^2$. Recently, [4] have proved uniqueness of weak solutions under only the structure condition (H_2). In this paper, the uniqueness of renormalized solution is proved by using the result of [4], and the proof goes essentially as follows: we prove that, if u is a renormalized solution of Problem $E(u_0, g)$, then u is a weak solution of some degenerate parabolic problem (see the proof of Proposition 3.1), then, by using the comparison result of [4] of weak solutions, we deduce a comparison result of renormalized solutions, and also uniqueness.

Let us briefly summarize the contents of the paper: In Sect. 2 we fix the notations, give the concept of renormalized solution of Problem $E(u_0, g)$, and state the existence and uniqueness result for renormalized solution of Problem $E(u_0, g)$. In Sect. 3 we prove uniqueness of renormalized solutions by using the results of [4]. Section 4 is devoted to the study of a perturbed problem obtained by adding a monotone term. Existence of weak solution is proved for L^∞-data. In Sect. 5 we give the proof of existence of a renormalized solution for Problem $E(u_0, g)$. It was shown that a weak solution of the perturbed problem $E(u_0, f, \psi_{m,n})$ converges to a renormalized solution. Finally, in Sect. 6 we deduce the corresponding results for the associated stationary problem.

2. Preliminaries and main result

In this section, after some notations, we introduce the concept of renormalized solution for Problem $E(u_0, g)$ and state the existence and uniqueness result for this type of solutions.

We denote by $|A|$ the Lebesgue measure of a set $A \subset \mathbb{R}^N$ and by χ_A the characteristic function of A. For $k \geq 0$, we denote by T_k the truncation function at the level k, defined by

$$T_k(u) = \begin{cases} k\mathrm{sign}_0(u) & \text{if} \quad |u| > k \\ u & \text{if} \quad |u| \leq k, \end{cases} \tag{2.1}$$

where $\mathrm{sign}_0(\cdot)$ denotes the single-valued function defined by $\mathrm{sign}_0(r) = -1$ if $r < 0$, $\mathrm{sign}_0(r) = 1$ if $r > 0$, $\mathrm{sign}_0(r) = 0$ if $r = 0$. We denote by $\mathrm{sign}_0^+(\cdot)$ and $\mathrm{sign}_0^-(\cdot)$ the functions defined by $\mathrm{sign}_0^+(r) = 1$ if $r > 0, = 0$ otherwise, and $\mathrm{sign}_0^-(r) = -1$ if $r < 0, = 0$ otherwise.

For $n \in \mathbb{N}$ we denote

$$h_n(r) = \inf((n + 1 - |r|)^+, 1) \quad \text{and} \quad H_n(r) = \int_0^r h_n(s)\, ds.$$

Throughout the paper, for the sake of simplicity, for u a function of (t, x) and for k a real number, we denote, for example, $\{|u| \leq k\}$ the set $\{(t, x) \in Q; |u(t, x)| \leq k\}$. We also write $\int_Q u$ for $\int_Q u(t, x)dtdx$, *etc...* In the sequel C denotes a constant that may change from line to line.
For a maximal monotone graph β in $\mathbb{R} \times \mathbb{R}$; its main section β_0 is defined by

$$\beta_0(r) = \begin{cases} \inf \beta(r) & \text{if } r > 0 \\ 0 & \text{if } r = 0 \\ \sup \beta(r) & \text{if } r < 0, \end{cases}$$

with the usual convention $\inf \emptyset = +\infty$ and $\sup \emptyset = -\infty$.
An essential tool to prove existence of weak (renormalized) solutions is the following energy estimate similar to the one set of [1].

Let $j, \varphi : \mathbb{R} \to \mathbb{R}$ be a continuous, nondecreasing functions such that $j(0) = \varphi(0) = 0$. For any continuous and monotone function h we define the function

$$B_h(s) = \begin{cases} \displaystyle\int_0^s h(\varphi \circ (j^{-1})_0(r))dr & \text{for } s \in \overline{(h \circ \varphi) \circ j^{-1}} \\ +\infty & \text{otherwise.} \end{cases} \tag{2.2}$$

Lemma 2.1. [15, Lemma 4] *Let $j, \varphi : \mathbb{R} \to \mathbb{R}$ be a continuous and nondecreasing function with $j(0) = \varphi(0) = 0$. Let v be a measurable function such that $j(v) \in L^1(Q), j(v)_t \in L^2(0, T; H^{-1}(\Omega))$ and $j(v)(0) = j(v_0)$, where $v_0 : \Omega \to \mathbb{R}$ is measurable with $j(v_0) \in L^1(\Omega)$. Then*

$$B_h(j(v)) \in L^\infty(0, T; L^1(\Omega))$$

and for a.e. $t \in [0, T]$

$$\int_\Omega B_h(j(v(t))\xi(t) - \int_\Omega B_h(j(v_0))\xi(0) = \int_0^t \langle j(v)_t, h(\varphi(v))\xi\rangle + \int_0^t \int_\Omega B_h(j(v))\xi_t$$

for any $\xi \in C([0,T]\times\overline{\Omega})$ *such that* $h(\varphi(v)) \in L^2(0,T;H_0^1(\Omega))$*, where* $\langle\cdot,\cdot\rangle$ *represents the duality product between* $H^{-1}(\Omega)$ *and* $H_0^1(\Omega)$*.*

We now give the concept of renormalized solution for Problem $E(u_0,g)$.

Definition 2.1. Given $u_0 \in L^1(\Omega)$ and $g : Q\times\mathbb{R}\to\mathbb{R}$ satisfying (H_3), a renormalized solution of $E(u_0,g)$ is a function u such that

(i) $u \in L^1(Q)$

(ii) there exists a measurable function w such that $w \in \beta(u)$ a.e. on Q and

$$T_k w \in L^2\left(0,T;H_0^1(\Omega)\right) \quad \text{for any } k>0$$

(iii) for all $\xi \in \mathcal{D}([0,T)\times\Omega)$ and $h \in \mathcal{C}_c^1(\mathbb{R})$

$$-\int_Q \xi_t \int_{u_0}^{u} h(\beta_0(r))dr + \int_Q (\nabla w + F(u,w))\cdot\nabla(h(w)\xi) = \int_Q g(\cdot,u)\,h(w)\,\xi \tag{2.3}$$

and moreover

$$\int_{Q\cap\{n\le|w|\le n+1\}} |\nabla w|^2 \to 0 \quad \text{as } n\to\infty. \tag{2.4}$$

Remark 2.1. Note that all integrals are well-defined. Indeed, the first one is defined as $|\int_{u_0}^{u} h\circ\beta_0(r)dr| \le \|h\|_\infty |u-u_0|$ and $u \in L^1(Q)$, $u_0 \in L^1(\Omega)$. The second integral must be understood as

$$\int_{\{|w|<k\}} (\nabla T_k w + F(u,T_k w))\cdot\nabla(h(T_k(w))\xi)$$

for $k>0$ such that Supp $h \subset [-k,k]$. Indeed, if Supp $h\subset[-k,k]$, then $h(w)=h(T_k w)$ and $h(w)=0$ a.e. on $\{|w|\ge k\}$; since $T_k w \in L^2(0,T;H_0^1(\Omega))$ it is the same for $h(w)\xi$, and $\nabla(h(w)\xi)=0$ a.e. on $\{|w|\ge k\}$. Similarly the integral (2.4) has to be understood as

$$\int_{Q\cap\{n\le|w|\le n+1\}} |\nabla T_{n+1} w|^2.$$

The main theorem of this paper is

Theorem 2.1. *For any* $u_0 \in L^1(\Omega)$ *and* $g : Q\times\mathbb{R}\to\mathbb{R}$ *satisfying* (H_3)*, there exists a unique renormalized solution* u *of* $E(u_0,g)$*. Moreover,* $u \in C([0,T];L^1(\Omega))$*,* $u(0)=u_0$*, and if* $u_{0i}\in L^1(\Omega)$*,* $g_i : Q\times\mathbb{R}\to\mathbb{R}$ *satisfies* (H_3) *and* u_i *is a renormalized solution of* $E(u_{0i},g_i)$*, for* $i=1,2$*, then, for all* $0\le t\le T$

$$\int_\Omega (u_1(t)-u_2(t))^+ \le \int_\Omega (u_{01}-u_{02})^+ + \int_0^t\int_\Omega \eta\left(g_1(.,u_1)-g_2(.,u_2)\right) \tag{2.5}$$

for some $\eta\in \mathrm{sign}^+(u_1-u_2)$*, where* sign^+ *denotes the usual non-negative sign graph:*

$$\mathrm{sign}^+(s) = \begin{cases} 1 & if\ s>0 \\ [0,1] & if\ s=0 \\ 0 & if\ s<0. \end{cases}$$

3. Uniqueness of renormalized solutions

The proof of Theorem 2.1 will follow as a consequence of Proposition 3.1 below. In fact, we will focus our attention on the problem

$$(E')(v_0, f) \begin{cases} \partial_t j(v) - \Delta\varphi(v) - \nabla F(j(v), \varphi(v)) = f & \text{in } Q \\ \varphi(v) = 0 & \text{on } \Sigma \\ j(v)(0, \cdot) = v_0 & \text{in } \Omega, \end{cases} \tag{3.1}$$

where $f \in L^1(Q)$, $j, \varphi : \mathbb{R} \to \mathbb{R}$ are nondecreasing continuous functions such that $j(0) = \varphi(0) = 0$, and v_0 is measurable function such that $u_0 = j(v_0)$ a.e. on Ω. Indeed, by taking $\varphi = (I + \beta^{-1})^{-1}$, $j = (I + \beta)^{-1}$ and $v := u + w$, one sees that $E(u_0, f)$ and $E'(u_0, f)$ are equivalent.

Remark that

$$D((j + \varphi))^{-1} = \mathbb{R}. \tag{3.2}$$

Next, let us recall the definition of renormalized solution of $E'(v_0, f)$.

Definition 3.1. Given $u_0 \in L^1(\Omega)$ and $f \in L^1(Q)$, a renormalized solution of $E'(v_0, f)$ is a measurable function v such that u is a renormalized solution of $E(u_0, f)$, where $u = j(v)$ and $w = \varphi(v)$.

The main tool we use for the proof of uniqueness of renormalized solution is the following proposition, for which the proof is given at the end of this section.

Proposition 3.1. *For any $f_1, f_2 \in L^1(Q)$ and $u_{01}, u_{02} \in L^1(\Omega)$, if v_i is a renormalized solution of $E'(v_{0i}, f_i)$ for $i = 1, 2$, then, for a.e. $0 < t < T$*

$$\begin{aligned} \int_\Omega \left(\int_{v_2(t)}^{v_1(t)} h_n(\varphi(r)) dj(r) \right)^+ &\le \int_\Omega \left(\int_{v_{02}}^{v_{01}} h_n(\varphi(r)) dj(r) \right)^+ \\ &+ \int_0^t \int_\Omega \eta \left(f_1 h_n(\varphi(v_1)) - f_2 h_n(\varphi(v_2)) \right) \\ &+ \int_0^t \int_\Omega \eta \left((\nabla\varphi(v_1) + F(j(v_1), \varphi(v_1))) \cdot \nabla h_n(\varphi(v_1)) - (\nabla\varphi(v_2) \right. \\ &\left. + F(j(v_2), \varphi(v_2))) \cdot \nabla h_n(\varphi(v_2)) \right), \end{aligned} \tag{3.3}$$

with $\eta \in \mathrm{Sign}^+(v_1 - v_2)$ a.e. in Q.

Corollary 3.1. *For any $u_0 \in L^1(\Omega)$ and g satisfying Assumption (H_3) there exists at most one renormalized solution u of $E(u_0, g)$. Moreover, if $u_{0i} \in L^1(\Omega)$, $g_i : Q \times \mathbb{R} \to \mathbb{R}$ satisfies (H_3) and u_i is the renormalized solution of $E(u_{0i}, g_i)$, for $i = 1, 2$, then (2.5) is fulfilled.*

Proof of Theorem 2.1: **Uniqueness part.**
First, notice that uniqueness of a renormalized solution follows from (2.5). Indeed, if $u_{01} = u_{02}$ and $g_1 = g_2$, then (2.5) and Assumption (H_3) imply that

$$\begin{aligned} \int_\Omega (u_1(t) - u_2(t))^+ &\le \int_0^t \int_{\{u_1 \ge u_2\}} (g(\cdot, u_1) - g(\cdot, u_2)) \\ &\le C \int_0^t \int_\Omega (u_1 - u_2)^+, \end{aligned}$$

which, by Gronwall's Lemma, implies that $u_1 \le u_2$. In the same way, one can prove that $u_2 \le u_1$.

Now, let us prove (2.5). It is clear that if u is a renormalized solution of Problem $E(u_0, g)$ with g satisfying Assumption (H_3), then u is a renormalized solution of Problem $E(u_0, f)$ with $f(t,x) = g(t,x,u(t,x))$ a.e. $(t,x) \in Q$. So, it is enough to prove that if $u_{0i} \in L^1(\Omega)$, $f_i \in L^1(Q)$ and u_i is a renormalized solution of $E(u_{0i}, f_i)$ for $i = 1,2$, then, for a.e. $0 < t < T$

$$\int_\Omega (u_1(t) - u_2(t))^+ \le \int_\Omega (u_{01} - u_{02})^+ + \int_0^t \int_\Omega (f_1 - f_2)^+,$$

or, equivalently, for a.e. $0 < t < T$

$$\int_\Omega (j(v_1)(t) - j(v_2)(t))^+ \le \int_\Omega (j(v_{01}) - j(v_{02}))^+ + \int_0^t \int_\Omega (f_1 - f_2)^+, \quad (3.4)$$

with $u = j(v)$ and $w = \varphi(v)$, where v is a renormalized solution of $E'(v_0, f)$.

To prove the above inequality, we pass to the limit in (3.3) as $n \to \infty$. So, it is clear that

$$\int_\Omega \left(\int_{v_2(t)}^{v_1(t)} h_n(\varphi(s)) dj(s) \right)^+ \to \int_\Omega (j(v_1)(t) - j(v_2)(t))^+ ,$$

$$\int_\Omega \left(\int_{v_{02}}^{v_{01}} h_n(\varphi(s) dj(s) \right)^+ \to \int_\Omega (j(v_{01}) - j(v_{02}))^+$$

and

$$\int_0^t \int_\Omega \eta \left(f_1 h_n(\varphi(v_1)) - f_2 h_n(\varphi(v_2)) \right) \to \int_0^t \int_\Omega \eta (f_1 - f_2)^+.$$

The term

$$\int_0^t \int_\Omega \eta \nabla \varphi(v_1) \cdot \nabla h_n(\varphi(v_1)) - \int_0^t \int_\Omega \eta \nabla \varphi(v_2) \cdot \nabla h_n(\varphi(v_2))$$

converges to 0 as $n \to \infty$ since $\varphi(v)$ satisfies (2.4). Next, let us prove that

$$\int_0^t \int_\Omega \eta \left(F(j(v_1), \varphi(v_1)) \cdot \nabla h_n(\varphi(v_1)) - F(j(v_2), \varphi(v_2)) \cdot \nabla h_n(\varphi(v_2)) \right) = 0. \quad (3.5)$$

Define the set

$$E = \{ r \in \mathbb{R};\ \varphi_0^{-1} \text{ is discontinuous at } r \}.$$

Since φ_0^{-1} is a monotone function, E is a countable subset of $\mathbb{R}^N$; hence we have

$$\nabla \varphi(v) = 0 \ \text{ a.e. on } \{ (t,x) \in Q;\ \varphi(v(t,x)) \in E \}. \quad (3.6)$$

From Assumption (H_2), the term $F(j(v_1), \varphi(v_1)) \cdot \nabla h_n(\varphi(v_1))$ can be decomposed as

$$F_1(\varphi(v_1)) \cdot \nabla h_n(\varphi(v_1)) + j(v_1) F_2(\varphi(v_1)) \cdot \nabla h_n(\varphi(v_1)) =: I_1 + I_2.$$

We have

$$I_1 = \int_Q \text{div} \int_0^{\varphi(v_1)} h_n'(r)F_1(r)dr = 0$$

and

$$\begin{aligned} I_2 &= \int_{E\cap\{n<|\varphi(v_1)|<n+1\}} h_n'(\varphi(v_1))j(v_1)F_2(\varphi(v_1)) \cdot \nabla\varphi(v_1) \\ &\quad + \int_{\overline{E}\cap\{n<|\varphi(v_1)|<n+1\}} h_n'(\varphi(v_1))j(v_1)F_2(\varphi(v_1)) \cdot \nabla\varphi(v_1) \\ &=: I_2^1 + I_2^2, \end{aligned}$$

where $\overline{E}$ stands of the complementary of E in Q.
From (3.6), $I_2^1 = 0$. Since φ^{-1} is a continuous function on the set $\overline{E}$, then

$$I_2^2 = \int_{\overline{E}\cap\{n<|\varphi(v_1)|<n+1\}} \text{div} \int_0^{\varphi(v_1)} h_n'(r)j \circ \varphi^{-1}(r)F_2(r)dr = 0. \tag{3.7}$$

Arguing as above to prove that $\int_Q \eta F(j(v_2), \varphi(v_2)) \cdot \nabla h_n(\varphi(v_2)) = 0$.

Finally, collecting all limits, (3.4) follows. □

Proof of Proposition 3.1. For any $n \in \mathbb{N}$, let $b_n(r) = \int_0^r h_n(\varphi(s))dj(s)$. Remark that, if $|\varphi(v)| \geq n+1$, then $b_n(v) = 0$, and if $|\varphi(v)| < n+1$, then the following structure condition holds: if $v < z$

$$b_n(v) = b_n(z) \Rightarrow j(v) = j(z).$$

By [10], this condition is equivalent to the existence of a continuous function $\widetilde{j}$ such that

$$j(v) = \widetilde{j}(b_n(v)).$$

Let $u_0 \in L^1(\Omega), f \in L^1(Q)$ and v a renormalized solution of $E'(v_0, f)$ with v_0 a measurable function such that $u_0 = j(v_0)$. Hence v satisfies for all $\xi \in \mathcal{D}((0,T) \times \Omega)$

$$-\int_Q \xi_t b_n(v) + \int_Q (\nabla\varphi(v) + F(j(v), \varphi(v))) \cdot \nabla(h_n(\varphi(v))\xi) = \int_Q f h_n(v)\xi. \tag{3.8}$$

Now, let us consider the second integral in (3.8), which can be written as

$$\begin{aligned} &\int_Q (\nabla\varphi(v) + F(j(v), \varphi(v))) \cdot \nabla\xi h_n(\varphi(v)) \\ &\qquad + \int_Q (\nabla\varphi(v) + F(j(v), \varphi(v))) \cdot \nabla h_n(\varphi(v))\xi =: K_1 + K_2. \end{aligned}$$

From Assumption (H_2) the term K_1 can be decomposed into three terms $(K_1^1 + K_1^2 + K_1^3)$

$$\int_Q \nabla\varphi(v)\cdot\nabla\xi h_n(\varphi(v)) + \int_Q F_1(\varphi(v))\cdot\nabla\xi h_n(\varphi(v)) + \int_Q j(v)F_2(\varphi(v))\cdot\nabla\xi h_n(\varphi(v))$$

Note that

$$K_1^1 = \int_Q \nabla H_n(\varphi(v))\cdot\nabla\xi$$

and

$$K_1^2 = \int_Q \widetilde{F}_1(H_n(\varphi(v)))\cdot\nabla\xi,$$

where $\tilde{F}_1$ is a continuous function defined by

$$\widetilde{F}_1(r) = h_n\circ H_n^{-1}(r)F_1(H_n^{-1}(r)).$$

Since $j(v) = \widetilde{j}(b_n(v))$ on the set where $|\varphi(v)| < n+1$, then

$$K_1^3 = \int_Q \widetilde{j}(b_n(v))h_n(\varphi(v))F_2(\varphi(v))\cdot\nabla\xi = \int_Q \widetilde{j}(b_n(v))\widetilde{F}_2(H_n(\varphi(v))\cdot\nabla\xi,$$

where $\tilde{F}_2$ is a continuous function defined by,

$$\widetilde{F}_2(r) = h_n\circ H_n^{-1}(r)F_2(H_n^{-1}(r)).$$

Finally, the term $K_1^2 + K_1^3$ is equal to

$$\int_Q \widetilde{F}(b_n(v), H_n(\varphi(v)))\cdot\nabla\xi,$$

where

$$\widetilde{F}(r_1,r_2) = \widetilde{F}_1(r_2) + r_1\widetilde{F}_2(r_2) = h_n\circ H_n^{-1}(r_2)F(\widetilde{j}(r_1), H_n^{-1}(r_2)).$$

Taking account these decompositions, Eq. (3.8) is rewritten as

$$-\int_Q \xi_t b_n(v) + \int_Q \Big(\nabla H_n(\varphi(v)) + \tilde{F}(b_n(v), H_n(\varphi(v)))\Big)\cdot\nabla\xi = \int_Q f h_n(\varphi(v))\xi - \int_Q \left(\nabla\varphi(v) + F(j(v),\varphi(v))\cdot\nabla h_n(\varphi(v))\right)\xi,$$

which means that v is a weak solution of

$$\begin{cases} \dfrac{\partial}{\partial t}b_n(v) - \nabla\cdot\Big(\nabla H_n(\varphi(v)) + \tilde{F}(b_n(v), H_n(\varphi(v)))\Big) \\ = f h_n(\varphi(v)) - \nabla\varphi(v) - F(j(v),\varphi(v))\cdot\nabla h_n(\varphi(v)) & \text{in } Q \\ H_n(\varphi(v)) = 0 & \text{on } \Sigma \\ b_n(v(0)) = b_n(v_0) & \text{in } \Omega. \end{cases} \qquad (E_n')$$

□

Next, let $u_i, i = 1, 2$ be a renormalized solution of $E(u_{0i}, f_i)$, then $v_i, i = 1, 2$ is a renormalized solution of $E'(v_{0i}, f_i)$. By the preceding computation, $v_i, i = 1, 2$ is also a weak solution of Problem $E'_n(v_{0i}, f_i)$, and thanks to [4, Theorem 1], the result of Proposition 3.1 follows.

4. Existence of weak solutions

To prove existence of renormalized solutions of Problem $E(u_0, f)$, we will proceed by approximation. We need first to prove, for bounded data $f \in L^\infty(Q)$ and $u_0 \in L^\infty(\Omega)$, existence of a weak solution of the parabolic problem with additional strongly monotone perturbation $\psi_{m,n}$, where $\psi_{m,n}(r) = \frac{1}{m}\tan(r)^+ - \frac{1}{n}\tan(r)^-, m, n \in \mathbb{N}$:

$$E'(u_0, f, \psi_{m,n}) \begin{cases} \partial_t j(v) - \Delta\varphi(v) - \nabla F(j(v), \varphi(v)) + \psi_{m,n}(v) = f & \text{in } Q \\ \varphi(v) = 0 & \text{on } \Sigma \\ j(v)(0, \cdot) = v_0 & \text{in } \Omega. \end{cases} \tag{4.9}$$

This is done via approximation by a sequence of non-degenerate parabolic problems

$$E'_k(v_0, f, \psi_{m,n}) \begin{cases} \partial_t j_k(v) - \Delta\varphi_k(v) - \nabla F(j_k(v), \varphi_k(v)) + \psi_{m,n}(v) = f & \text{in } Q \\ \varphi_k(v) = 0 & \text{on } \Sigma \\ j_k(v)(0, \cdot) = v_0 & \text{in } \Omega, \end{cases} \tag{4.10}$$

where $j_k(r) = j(r) + kr,\ \varphi_k(r) = \varphi(r) + kr$ (then $j_k^{-1}, \varphi_k^{-1} \in C_0(\mathbb{R})$).

For these non-degenerate problems we obtain existence of weak solutions with appropriate estimates and monotonicity properties, which allow us to pass to the limit.

So, let us define the operator $A_{m,n}$, in $L^1(\Omega)$, by

$$A_{m,n}(z) = -\Delta\varphi(z) - \nabla F(j(z), \varphi(z)) + \psi_{m,n}(\varphi(z)) \ \text{ in } \mathcal{D}'(\Omega)$$

and

$$\mathcal{D}(A_{m,n}) = \{z \in L^\infty(\Omega);\ \varphi(z) \in H_0^1(\Omega),\ A_{m,n}(z) \in L^1(\Omega)\}.$$

Thanks to the results of [15], we know that $A_{m,n}$ is T-accretive in $L^1(\Omega)$, and $\overline{A_{m,n}}$, the closure of $A_{m,n}$ in L^1, is m-accretive in $L^1(\Omega)$, and moreover $\overline{D(A_{m,n})} = L^1(\Omega)$.

Moreover, if $(j_k)_k, (\varphi_k)_k$ are continuous and nondecreasing functions with $j_k(0) = \varphi_k(0) = 0$ such that $j_k \to j$ and $\varphi_k \to \varphi$ uniformly, then $A_{m,n} \subseteq \liminf_{k\to 0} A^k_{m,n}$, where the operator $A^k_{m,n}$ is defined as $A_{m,n}$, by replacing j and φ by j_k and φ_k respectively.

According to these results, by nonlinear semigroups theory, for any $k, m, n \in \mathbb{N}$, for all $v_0 \in L^1(\Omega)$, for all $f \in L^1(Q)$, there exits a unique mild solution $v^k_{m,n} \in C([0, T]; L^1(\Omega))$ of the abstract Cauchy problem in $L^1(\Omega)$

$$\frac{dv}{dt} + A^k_{m,n} v \ni f,\ v(0) = v_0. \tag{4.11}$$

Moreover, for any $v_0^k \in L^1(\Omega)$ with $v_0^k \to v_0$ in $L^1(\Omega)$ and for all $f \in L^1(Q)$, the mild solution $v_{m,n}^k$ of (4.11) with initial data v_0^k converges in $C([0,T];L^1(\Omega))$ as $k \to 0$ to the mild solution $v_{m,n}$ of the Cauchy problem

$$\frac{dv}{dt} + A_{m,n}v \ni f, \ v(0) = v_0. \tag{4.12}$$

For bounded data, we can prove that the mild solution of the Cauchy problem (4.12) is a "weak solution".

Definition 4.1. A weak solution of $E(u_0, f)$ is a couple of functions (u, w) such that $u \in L^\infty(Q), \ w \in L^2(0,T;H_0^1(\Omega)), \ w \in \beta(u), \ F(u,w) \in (L^2(Q))^N$, and

$$\int_Q [(\nabla w + F(u,w)) \cdot \nabla \xi - u\xi_t] = \int_Q f\xi - \int_\Omega \xi(0)u_0$$

for all $\xi \in \mathcal{D}((-\infty,T) \times \Omega)$.

Next, let us recall the definition of weak solution of $E'(v_0, f)$.

Definition 4.2. Given $u_0 \in L^\infty(\Omega)$ and $f \in L^\infty(Q)$, a weak solution of $E'(v_0, f)$ is a measurable function v such that the couple (u, w) is a weak solution of $E(u_0, f)$, where $u = j(v)$ and $w = \varphi(v)$.

It is proved by Carrillo [15] the following result:

Proposition 4.1. [15] *Let $m, n, k \in \mathbb{N}$, for $f \in L^\infty(Q)$ and $u_0 \in L^\infty(\Omega)$ let $v_{m,n}^k$ be the mild solution of (4.11). Then $v_{m,n}^k$ is a weak solution of $E'_k(v_0, f, \psi_{m,n})$.*

Proposition 4.2. *Given $u_0 \in L^\infty(\Omega)$ and $f \in L^\infty(Q)$ there exists a weak solution of Problem $E'(v_0, f, \psi_{m,n})$.*

Proof of Proposition 4.2. From now on and until Sect. 5, we omit the index m, n to lighten the notations. □

Recall that v_k is the mild solution of $\frac{dv_k}{dt} + A_{m,n}^k v_k \ni f, v_k(0) = v_0^k$, thus, by [15]

$$\|j_k(v_k)\|_\infty \leq C(f, v_0, m, n).$$

Let $B_k(s) = \int_0^s \varphi_k \circ j_k^{-1}(r)dr$. By taking $\xi = \varphi_k(v_k)$ as a test function in the weak formulation of the solution v_k, and by using Lemma 2.1, yields

$$\begin{aligned} \int_\Omega B_k(v_k) &+ \int_Q (\nabla\varphi_k(v_k) + F(j_k(v_k), \varphi_k(v_k))) \cdot \nabla\varphi_k(v_k) \\ &+ \int_Q \psi_{m,n}(\varphi_k(v_k))\varphi_k(v_k) \\ &= \int_Q f\varphi_k(v_k) + \int_\Omega B_k(v_k^0). \end{aligned} \tag{4.13}$$

By monotonicity of $\psi_{m,n}$, $\int_Q \psi_{m,n}(\varphi_k(v_k))\varphi_k(v_k) \geq 0$. The term in the second integral on the right hand side $F(j_k(v_k), \varphi_k(v_k))) \cdot \nabla\varphi_k(v_k)$; from Assumption (H_2) we have

$$F(j_k(v_k),\varphi_k(v_k))\cdot\nabla\varphi_k(v_k) = F_1(\varphi_k(v_k))\cdot\nabla\varphi_k(v_k) \\ +j_k(v_k)F_2(\varphi_k(v_k))\cdot\nabla\varphi_k(v_k),$$

whence

$$\int_Q F_1(\varphi_k(v_k))\cdot\nabla\varphi_k(v_k) = \int_Q \operatorname{div}\int_0^{\varphi_k(v_k)} F_1(r)dr = 0;$$

and, since φ_k^{-1} is a continuous function almost everywhere in Ω, we get

$$\int_Q j_k(v_k)F_2(\varphi_k(v_k))\cdot\nabla\varphi_k(v_k) = \int_Q j_k\circ\varphi_k^{-1}(\varphi_k(v_k))F_2(\varphi_k(v_k))\cdot\nabla\varphi_k(v_k) \\ = \int_Q \operatorname{div}\int_0^{\varphi_k(v_k)} j_k\circ\varphi_k^{-1}(r)F_2(r)dr = 0.$$

Then, we get from (4.13) that $\varphi_k(v_k)$ is bounded in $L^2(0,T;H_0^1(\Omega))$, hence, there exists a subsequence, still denoted by k, such that

$$\varphi_k(v_k)\rightharpoonup w \ \text{ weakly in } L^2(0,T;H_0^1(\Omega)).$$

One can prove exactly as [3, 10] that $\varphi_k(v_k)$ is uniformly bounded in $L^\infty(Q)$.

It remains to prove the strong convergence of $\varphi_k(v_k)$ in $L^1(Q)$.

The proof is based on Kruzhkov's method of doubling of variables. Let $t,s\in[0,T], k,l\in\mathbb{N}$, and consider the weak solution $v_k(t,x)$ as a function of (t,x) and $v_l(s,x)$ as a function of (s,x). Choose in each weak formulation the test function $\phi=\frac{1}{h}\int_t^{t+h}\eta_\delta(\varphi_k(v_k)-\varphi_l(v_l)+\delta\zeta)\xi$, where $\xi\in C_c^\infty([0,T)^2\times\overline{\Omega}),\xi\geq 0,\zeta\in C_c^\infty(\Omega),0\leq\zeta\leq 1$ and $\eta_\delta(r)=\frac{T_\delta(r)}{\delta}$, and integrate in t. Using Lemma 2.1 in each inequality, taking their difference, passing to the limit with $h\to 0$ exactly as in [28, Proposition 4.2.2] (see also [12, Proposition 4.2]) yields

$$\begin{aligned}
&-\int_0^T\int_Q \xi_t[|j_k(v_k)-j_l(v_l)|-|v_k^0-j_l(v_l)|] \\
&-\int_0^T\int_Q \xi_s[|j_k(v_k)-j_l(v_l)|-|v_l^0-j_k(v_k)|] \\
&+\int_0^T\int_Q (\chi_{\{\varphi_k(v_k)>\varphi_l(v_l)\}}-\chi_{\{\varphi_k(v_k)<\varphi_l(v_l)\}})(\nabla(\varphi_k(v_k)-\varphi_l(v_l)))\cdot\nabla\xi \\
&+\int_0^T\int_Q (\chi_{\{\varphi_k(v_k)>\varphi_l(v_l)\}}-\chi_{\{\varphi_k(v_k)<\varphi_l(v_l)\}})(F(j_k(v_k),\varphi_k(v_k)) \\
&-F(j_l(v_k),\varphi_l(v_l)))\cdot\nabla\xi \\
&+\int_0^T\int_Q (\psi_{m,n}(\varphi_k(v_k))-\psi_{m,n}(\varphi_l(v_l)))(\chi_{\{\varphi_k(v_k)>\varphi_l(v_l)\}} \\
&-\chi_{\{\varphi_k(v_k)<\varphi_l(v_l)\}})\xi \\
\leq&\int_0^T\int_Q (f(t,x)-f(s,x))(\chi_{\{\varphi_k(v_k)>\varphi_l(v_l)\}}-\chi_{\{\varphi_k(v_k)<\varphi_l(v_l)\}} \\
&+\chi_{\{\varphi_k(v_k)=\varphi_l(v_l)\}})\xi
\end{aligned} \tag{4.14}$$

The proof of the above inequality is given in [28,29]. The original proof can be found in [12]. We omit here the details in order to avoid the unnecessary duplication of arguments.

Take $\xi = \phi(t)\rho_p(t-s)$, with $\phi \in C_c^\infty([0,T)), \phi \geq 0$ and $(\rho_p)_p$ be a classical sequence of mollifiers in $\mathbb{R}$ with $\mathrm{Supp}(\rho_p) \subset [-\frac{2}{p}, 0]$. Pass to the limit with $p \to 0$, yields

$$\lim_{p\to 0}\lim_{k,l\to 0}\int_0^T\int_Q \phi\rho_p|\psi_{m,n}(\varphi_k(v_k(t,x))) - \psi_{m,n}(\varphi_l(v_l(s,x)))| \leq 0.$$

In the particular case $k = l$, the preceding arguments lead to the estimate

$$\lim_{p\to 0}\lim_{l\to 0}\int_0^T\int_Q |\psi_{m,n}(\varphi_l(v_l(t,x))) - \psi_{m,n}(\varphi_l(v_l(s,x)))|\phi\rho_p \leq 0.$$

By choosing ϕ such that $\phi = 1$ on $[\tau, \theta]$, where $0 < \tau < \theta < T$, we get

$$\begin{aligned}
&\lim_{k,l\to 0}\int_\tau^\theta\int_\Omega |\psi_{m,n}(\varphi_k(v_k(t,x))) - \psi_{m,n}(\varphi(v_l(t,x)))| \\
&\quad \leq \lim_{p\to\infty}\lim_{k,l\to 0}\int_0^T\int_Q |\psi_{m,n}(\varphi_k(v_k(t,x))) - \psi_{m,n}(\varphi_l(v_l(t,x)))|\phi\rho_p \\
&\quad \leq \lim_{p\to\infty}\lim_{k,l\to 0}\left(\int_0^T\int_Q |\psi_{m,n}(\varphi_k(v_k(t,x))) - \psi_{m,n}(\varphi_l(v_l(s,x)))|\phi\rho_p\right. \\
&\qquad \left. + \int_0^T\int_Q |\psi_{m,n}(\varphi_l(v_l(s,x))) - \psi_{m,n}(\varphi_l(v_l(t,x)))|\phi\rho_p\right) \\
&\quad \leq 0.
\end{aligned}$$

As $\psi_{m,n}$ is strictly nondecreasing, it follows that

$$\lim_{k,l\to 0}\int_\tau^\theta\int_\Omega |\varphi_k(v_k) - \varphi_l(v_l)| = 0 \quad \forall 0 < \tau < \theta < T.$$

Since $(\varphi_k(v_k))_k$ is bounded in $L^\infty(Q)$ and $\varphi_k(v_k) \rightharpoonup w$ weakly in $L^2(0,T;H_0^1(\Omega))$, we conclude that

$$\varphi_k(v_k) \to w \text{ strongly in } L^1(Q), \text{ and a.e. on } Q.$$

By nonlinear semigroup theory, $j_k(v_k) \to u$ in $L^\infty(0,T;L^1(\Omega))$, we deduce existence of subsequence of k, still denoted by k, such that

$$j_k(v_k) \to u \text{ a.e. on } Q.$$

The task now is to prove that

$$u = j(v) \text{ and } w = \varphi(v).$$

Since $\varphi = \varphi_k - kI, j = j_k - kI$, and $\varphi_k(v_k), j_k(v_k)$ are uniformly bounded in $L^\infty(Q)$, then, almost everywhere on Q, we have

$$|j(v_k) + \varphi(v_k)| \leq |j_k(v_k) + \varphi_k(v_k)| \leq C.$$

From (3.2) we deduce the existence of a constant C such that

$$\|v_k\|_{L^\infty(Q)} \leq C.$$

Moreover, v_k converges to v *-weakly in $L^\infty(Q)$, still converges in $L^2(Q)$, and kv_k converges to 0 in $L^\infty(Q)$.

We deduce also that $\varphi(v_k) = \varphi_k(v_k) - kv_k$ still converges to w in $L^2(Q)$, whence we deduce that $w = \varphi(v)$. Also we have $u = j(v)$.

Therefore, since $j_k(v_k), \varphi_k(v_k)$ are uniformly bounded in $L^\infty(Q)$, we have $F(j_k(v_k), \varphi_k(v_k))$ is uniformly bounded in $(L^\infty(Q))^N$ since F_1, F_2 are continuous. Hence, from Lebesgue Theorem, we deduce that

$$F(j_k(v_k), \varphi_k(v_k)) \to F(j(v), \varphi(v)) \ \text{ in } L^1(Q).$$

Now, let $\xi \in C_c^\infty([0,T) \times \Omega)$, then

$$\int_Q -j_k(v_k)\xi_t + \int_Q (\nabla\varphi_k(v_k) + F(j_k(v_k), \varphi_k(v_k))) \cdot \nabla\xi + \int_Q \psi_{m,n}(\varphi_k(v_k))\xi$$
$$= \int_Q f\xi - \int_\Omega v_k^0 \xi(0),$$

and by letting $k \to 0$ we get

$$\int_Q -j(v)\xi_t + \int_Q (\nabla\varphi(v) + F(j(v), \varphi(v))) \cdot \nabla\xi + \int_Q \psi_{m,n}(\varphi(v))\xi$$
$$= \int_Q f\xi - \int_\Omega v_0 \xi(0).$$

Hence v is a weak solution of $E'(v_0, f, \psi_{m,n})$. Consequently u is a weak solution of $E(u_0, f, \psi_{m,n})$ with $u = j(v)$ and $w = \varphi(v)$.

5. Existence of renormalized solutions

The main result of this section is

Theorem 5.1. *For all $u_0 \in L^1(\Omega)$ and $f \in L^1(Q)$ Problem $E(u_0, f)$ admits a renormalized solution.*

Proof. Following a standard approach, we obtain the existence of a solution as limit of approximating problems. To this purpose let $u^0_{m,n} = \sup\{\inf\{m, u_0\}, -n\} \in L^\infty(\Omega)$, and $f_{m,n} = \sup\{\inf\{m, f\}, -n\} \in L^\infty(Q)$ be a bi-monotone approximation of u_0 and f in L^1. Then, by Proposition 4.2, there exists a weak solution $u_{m,n}$ of Problem $E(u^0_{m,n}, f_{m,n}, \psi_{m,n})$, i.e.

$$u_{m,n_t} - \Delta w_{m,n} - \nabla F(u_{m,n}, w_{m,n}) + \psi_{m,n}(w_{m,n}) = f_{m,n},$$
$$w_{m,n} \in \beta(u_{m,n}) \ \text{ in } \mathcal{D}'(Q),$$

which is equivalent to

$$j(v_{m,n})_t - \Delta\varphi(v_{m,n}) - \nabla F(j(v_{m,n}), \varphi(v_{m,n})) + \psi_{m,n}(\varphi(v_{m,n}))$$
$$= f_{m,n} \ \text{ in } \mathcal{D}'(Q) \tag{5.15}$$

with $u_{m,n} = j(v_{m,n})$ and $w_{m,n} = \varphi(u_{m,n})$.
We are going to prove that the limit a.e. of $u_{m,n}$, respectively of $j(v_{m,n})$, is a renormalized solution of $E(u_0, f)$, respectively of $E'(v_0, f)$.

By choosing in (5.15) the test function $T_k(\varphi(v_{m,n}))$ and using Lemma 2.1, yields

$$\int_Q |\nabla T_k(\varphi(u_{m,n}))|^2 + \int_{\{|\varphi(u_{m,n})|\leq k\}} F(j(v_{m,n}),\varphi(v_{m,n}))\cdot\nabla\varphi(v_{m,n})$$
$$+\int_Q \psi_{m,n}(\varphi(v_{m,n}))T_k(\varphi(u_{m,n}))$$
$$\leq k\left(\int_Q |f_{m,n}| + \int_\Omega |j(v^0_{m,n})|\right). \tag{5.16}$$

As in (3.5), it follows that $\int_{\{|\varphi(u_{m,n}|\leq k\}} F(j(v_{m,n}),\varphi(v_{m,n}))\cdot\nabla\varphi(v_{m,n}) = 0$. By monotonicity of the function $\psi_{m,n}$ we deduce from inequality (5.16)

$$\int_Q |\nabla T_k\varphi(u_{m,n})|^2 \leq kC,$$

where C is a constant independent of m,n. Thus $T_k\varphi(u_{m,n})$ is bounded in $L^2(0,T;H^1_0(\Omega))$. Hence, up to a subsequence,

$$T_k\varphi(u_{m,n}) \rightharpoonup g \text{ weakly in } L^2(0,T;H^1_0(\Omega)) \text{ as } m,n\to\infty.$$

Now let us prove, up to a subsequence, the strong convergence of the sequence $(\varphi(v_{m,n}))_{m,n}$. For this we will use the following comparison result □

Lemma 5.1. *Let $v_0,\tilde v_0 \in L^\infty(\Omega), f,\tilde f \in L^\infty(Q), \psi,\tilde\psi : \mathbb{R}\to\mathbb{R}$ continuous, strictly increasing functions with $\psi(0) = \tilde\psi(0) = 0$, and let $v,\tilde v$ be weak solutions of $E'(v_0,f,\psi), E'(\tilde v_0,\tilde f,\tilde\psi)$ respectively. Then*

$$\int_\Omega (j(v)(t)-j(\tilde v))^+ + \int_Q \left(\psi(\varphi(v)) - \tilde\psi(\varphi(\tilde v))\right)^+ \leq \int_Q (f-\tilde f)^+ - \int_\Omega (v_0-\tilde v_0)^+.$$

Proof. The proof is adapted exactly from the proof of inequality (4.14). It suffices to take in the equations corresponding to the weak solutions v and $\tilde v$ the test functions $\frac{1}{h}\int_t^{t+h}\eta^+_\delta(\varphi(v)-\varphi(\tilde v)+\delta\zeta)$, where $\eta^+_\delta(r) = \frac{T^+_\delta(r)}{\delta}$.
From Lemma 5.1, we obtain, for $v_{m,n}$ weak solution of $E'(v^0_{m,n}, f_{m,n},\psi_{m,n})$,

$$\int_Q (\psi_{m,n}(\varphi(v_{m,n})) - \psi_{m+1,n}(\varphi(v_{m+1,n})))^+ \leq 0.$$

Since $\psi_{m+1,n}(r) \leq \psi_{m,n}(r)$ and $\psi_{m+1,n}$ is strictly increasing, then for all $m,n>0$

$$\varphi(v_{m,n}) \leq \varphi(v_{m+1,n}) \text{ a.e. on } Q.$$

The same reasoning implies that for all $m,n>0$ $\varphi(v_{m,n}) \geq \varphi(v_{m,n+1})$ a.e. on Q. Therefore, thanks to the monotone convergence theorem

$$\varphi(v_{m,n}) \uparrow_m w_n \downarrow_n w \text{ in } L^1(Q),$$

where $w_n, w : Q\to\mathbb{R}$ are measurable functions, finite a.e. on Q. Here and in the sequel, we use the notation $\uparrow_m$ respectively $\downarrow_m$, to denote convergence of a sequence which is monotone increasing, respectively decreasing, in m.

Applying the diagonal procedure, we may assume that, for some sequence $m(n)$, $\varphi(v_n) := \varphi(v_{m(n),n}) \to w$ in $L^1(Q)$.

Extracting a subsequence if necessary, we may therefore assume that

$$T_k\varphi(v_n) \rightharpoonup T_k w \text{ weakly in } L^2(0,T;H_0^1(\Omega)) \quad \text{for all } k>0 \tag{5.17}$$

and

$$\varphi(v_n) \to w \quad \text{a.e. on } Q.$$

As v_n is a mild solution of $\frac{dv}{dt} + A_{m(n),n}v \ni f, v(0) = v_0$

$$u_n := j(v_n) \to u \quad \text{in} \quad L^\infty(0,T;L^1(\Omega)).$$

Since $\varphi(v_n)$ converges weakly in $L^2(Q)$, and since $\varphi \circ j^{-1}$ is a maximal monotone operator (in $L^2(Q)$), we deduce that

$$w \in \varphi \circ j^{-1}(u),$$

whence there exits $\tilde{u} \in j^{-1}(u)$ such that $w = \varphi(\tilde{u})$. Then we set

$$v = ((\varphi + j)^{-1})_0(u + w) = ((\varphi + j)^{-1})_0(\varphi(\tilde{u}) + j(\tilde{u})).$$

Obviously, v is a measurable function and we have $u = j(v)$ and $w = \varphi(v)$. We may assume that for some sequence $(m(n))_n$, we have (with $f_n = f_{m(n),n}$, $v_n^0 = v^0_{m(n),n}, \psi_n = \psi_{m(n),n}$)

$$\begin{aligned} &f_n \to f \quad \text{in } L^1(Q),\\ &j(v_n^0) \to u_0 \quad \text{in } L^1(\Omega) \end{aligned}$$

and the weak solution v_n of $E'(v_n^0, f_n, \psi_n)$ satisfies

$$\varphi(v_n) \to \varphi(v) \quad \text{a.e. on } Q$$

and

$$j(v_n) \to u \quad \text{in } L^\infty(0,T;L^1(\Omega)), \quad \text{a.e. on } Q.$$

The task now is to prove that

$$|\nabla T_k\varphi(v_n)|^2 \to |\nabla T_k\varphi(v)|^2 \quad \text{in } L^1(Q) \quad \text{as } n \to \infty. \tag{5.18}$$

For this we need to recall the following definition of a time regularization of $T_k(u)$, which was first introduced in [24], and used in several papers afterward (see e.g. [2,3,5,6,12]). Let $\nu > 0$ and $(w_\nu^0)_\nu$ be a sequence of functions such that

$$\begin{cases} w_\nu^0 \in H_0^1(\Omega) \cap L^\infty(\Omega)\\ \|w_\nu^0\|_{L^\infty(\Omega)} \le k\\ w_\nu^0 \to T_k\varphi(v(0)) \quad \text{a.e. on } \Omega \text{ as } \nu \to \infty\\ \dfrac{1}{\nu}\|w_\nu^0\|_{H_0^1(\Omega)} \to 0 \quad \text{as } \nu \to \infty. \end{cases} \tag{5.19}$$

Then, for all $k, \nu > 0$, we denote by $(T_k\varphi(v))_\nu$ the unique solution of the problem

$$\begin{cases} \dfrac{\partial (T_k\varphi(v))_\nu}{\partial t} = \nu\,(T_k\varphi(v) - (T_k\varphi(v))_\nu) & \text{on } Q\\ (T_k\varphi(v))_\nu(0,\cdot) = w_\nu^0 & \text{on } \Omega. \end{cases} \tag{5.20}$$

Then $(T_k\varphi(v))_\nu \in L^2(0,T;H_0^1(\Omega)) \cap L^\infty(Q)$, $\frac{\partial (T_k\varphi(v))_\nu}{\partial t} \in L^2(0,T;H_0^1(\Omega)) \cap L^\infty(Q)$, and up to a subsequence, we can assume that

$$\begin{aligned}&(T_k\varphi(v))_\nu \to T_k\varphi(v) \;\text{ strongly in } L^2(0,T;H_0^1(\Omega)),\\ &(T_k\varphi(v))_\nu(t) \to T_k\varphi(v)(t) \;\text{ a.e. on } \Omega \text{ for a.e. } t\end{aligned}$$

and

$$\|(T_k\varphi(v))_\nu\|_{L^\infty(Q)} \le k \quad \forall \nu > 0.$$

Let $\sigma \in \mathcal{D}_+(0,T)$ and $h_l(r) = (l+1-|r|)^+ \wedge 1, l \in \mathbb{N},\; l > k$. We prove that, for any fixed $k > 0$,

$$\liminf_{l\to\infty} \liminf_{\nu\to\infty} \lim_{n\to\infty} \int_Q \sigma\, \nabla\varphi(v_n) \cdot \nabla \left(h_l(\varphi(u_n))(T_k\varphi(v_n) - (T_k\varphi(v))_\nu)\right) \le 0. \tag{5.21}$$

To this end, consider $\sigma h_l(u_{m,n})(T_k\varphi(v_n)) - (T_k\varphi(v))_\nu)$ as a test function in (5.15) and pass to the limit with $n \to \infty$ in each term. We use the same techniques as in [3, Proof of Theorem 2.4] to prove that

$$\liminf_{\nu\to\infty} \lim_{n\to\infty} \langle j(v_n)_t, \sigma h_l(\varphi(v_n))(T_k\varphi(v_n) - (T_k\varphi(v))_\nu)\rangle \ge 0,$$

where $\langle\cdot,\cdot\rangle$ denotes the paring between $L^2(0,T;H_0^1(\Omega))$ and $L^2(0,T;H^{-1}(\Omega))$. As in (3.5), we show that

$$\lim_{\nu\to\infty} \lim_{n\to\infty} \int_Q \sigma F(j(v_n),\varphi(v_n)) \cdot \nabla \left(h_l(\varphi(v_n))(T_k\varphi(v_n) - (T_k\varphi(v))_\nu)\right) = 0.$$

It is clear that

$$\lim_{\nu\to\infty} \lim_{n\to\infty} \int_Q \psi_n(\varphi(v_n))\sigma h_l(\varphi(v_n))(T_k\varphi(v_n) - (T_k\varphi(v))_\nu) = 0$$

and

$$\lim_{\nu\to\infty} \lim_{n\to\infty} \int_Q f_n \sigma h_l(\varphi(v_n))(T_k(\varphi(v_n)) - (T_k\varphi(v))_\nu) = 0. \tag{5.22}$$

An equivalent formulation of (5.21) is

$$\begin{aligned}\limsup_{\nu\to\infty} \limsup_{n\to\infty} \Bigg(&\int_Q \sigma h_l(\varphi(v_n))\nabla\varphi(v_n) \cdot \nabla \left(T_k\varphi(v_n) - (T_k\varphi(v))_\nu\right)\\ &+ \int_{\{l<|\varphi(v_n)|<l+1\}} \sigma h_l'(\varphi(v_n))(T_k\varphi(v_n) - (T_k\varphi(v))_\nu)\nabla\varphi(v_n)\cdot\nabla\varphi(v_n)\Bigg)\\ \le 0.&\end{aligned} \tag{5.23}$$

The choice of h_l and $l > k$ implies

$$\begin{aligned}&\int_{\{l<|\varphi(v_n)|<l+1\}} \sigma h_l'(\varphi(v_n))(T_k\varphi(v_n) - (T_k\varphi(v))_\nu)\nabla\varphi(v_n)\cdot\nabla\varphi(v_n)\\ &\quad \ge -2k \int_{\{l<|\varphi(v_n)|<l+1\}} \sigma|\nabla\varphi(v_n)|^2.\end{aligned}$$

Choose the test function $\sigma\phi(\varphi(v_n))$, where $\phi_l(r) = \mathrm{sign}_0(r)(|r| - l)^+ \wedge 1$, we get

$$\lim_{l\to\infty} \sup_n \int_{\{l<|\varphi(v_n)|<l+1\}} |\nabla\varphi(v_n)|^2 \le 0.$$

Further

$$\int_{\{\varphi(v_n)\ge k\}} \sigma h_l(\varphi(v_n))\nabla\varphi(v_n)\cdot\nabla T_k\varphi(v_n) = 0,$$

$$\limsup_{\nu\to\infty}\limsup_{n\to\infty}\int_{\{\varphi(v_n)\ge k\}} \sigma h_l(\varphi(v_n))\nabla\varphi(v_n)\cdot\nabla(T_k\varphi(v))_\nu \le \int_{\{|w|\ge k\}} \sigma h_l(w)\nabla T_{k+1}w\cdot\nabla T_k w = 0.$$

Hence, as $l\to\infty$, it results from (5.23) that

$$\limsup_{\nu\to\infty}\limsup_{n\to\infty}\int_Q \sigma\nabla T_k\varphi(v_n)\cdot\nabla(T_k\varphi(v_n) - (T_k\varphi(v))_\nu) \le 0.$$

As a further consequence,

$$\limsup_{\nu\to\infty}\limsup_{n\to\infty}\int_Q \sigma(\nabla T_k\varphi(v_n) - \nabla(T_k\varphi(v))_\nu)\cdot\nabla(T_k\varphi(v_n) - (T_k\varphi(v))_\nu) = 0.$$

By a diagonal principle, there exists a sequence $n(\nu)$ such that the function $\sigma|\nabla T_k\varphi(v_n) - \nabla(T_k\varphi(v))_\nu|^2$ converges to zero strongly in $L^1(Q)$ as $\nu\to\infty$. We deduce that

$$\nabla T_k\varphi(v_{n(\nu)})\cdot\nabla(T\varphi(v_{n(\nu)}) - (T_k\varphi(v))_\nu) \to 0 \ \text{ weakly in } L^1(Q)$$

and then, by using the fact that $\nabla T_k\varphi(v_{n(\nu)}) \rightharpoonup \nabla T_k\varphi(v)$ weakly in $L^1(Q)$ as $\nu\to\infty$, that

$$\sigma|\nabla T_k\varphi(v_{n(\nu)})|^2 \to \sigma|\nabla T_k\varphi(v)|^2 \ \text{ weakly in } L^1(Q) \ \text{ as } \nu\to\infty.$$

Estimate (5.18) then follows.

Now, let us pass to the limit in (4.13) with $n\to\infty$. Take $h(\varphi(v_n))\xi$, where $h\in C_c^1(\mathbb{R}), \xi\in C_c^\infty([0,T)\times\Omega)$ as a test function in inequality (4.13), and pass to the limit with n in each term. By means of the dominated convergence theorem, we conclude that

$$\lim_{n\to\infty}\int_Q f_n h(\varphi(v_n))\xi = \int_Q f h(w)\xi \tag{5.24}$$

and

$$\lim_{n\to\infty}\int_Q \psi_n(\varphi(v_n))h(\varphi(v_n))\xi = 0. \tag{5.25}$$

Lemma 2.1 implies

$$\int_Q j(v_n)_t h(\varphi(v_n))\xi = -\int_Q \xi_t \int_{j(v_n^0)}^{j(v_n)} h(\varphi\circ j_0^{-1})(r)dr,$$

and by means of the dominated convergence theorem again, we have

$$\lim_{n\to\infty}\int_Q \xi_t \int_{j(v_n^0)}^{j(v_n)} h(\varphi\circ j_0^{-1})(r)dr = -\int_Q \xi_t \int_{u_0}^{u} h(\varphi\circ j_0^{-1})(r)dr. \tag{5.26}$$

From (5.18), and the fact that $j(v_n)\to u$ a.e. on Q and $\varphi(v_n)\to w$ a.e. on Q we deduce that

$$\begin{aligned}&\lim_{n\to\infty}\int_Q (\nabla\varphi(v_n)+F(j(v_n),\varphi(v_n)))\cdot\nabla(h(\varphi(v_n))\xi)\\ &=\int_Q(\nabla w+F(u,w))\cdot\nabla(h(w)\xi).\end{aligned} \tag{5.27}$$

Remains to prove that u satisfies (2.4). For this aim, take $T_{l+1}(\varphi(v_n)) - T_l(\varphi(v_n))$ as a test function in (5.15). Thanks again to Lemma 2.1 and the monotonicity of the function ψ_n, we have

$$\begin{aligned}&\int_{Q\cap\{l\le|\varphi(v_n)|\le l+1\}}\left\{|\nabla\varphi(v_n)|^2+F(j(v_n),\varphi(v_n))\cdot\nabla\varphi(v_n)\right\}\\ &\le\int_{Q\cap\{|\varphi(v_n)|\ge l\}}|f_n|+\int_{\{|v_n^0|\ge l\}}|v_n^0|.\end{aligned}$$

Passing to the limit as $n\to\infty$ and arguing as for (3.5) to prove that $\int_{\{l<|\varphi(v_n)|<l+1\}} F(j(v_n),\varphi(v_n))\cdot\nabla\varphi(v_n)=0$, we get

$$\limsup_{n\to\infty}\int_{Q\cap\{l\le|\varphi(v_n)|\le l+1\}}|\nabla\varphi(v_n)|^2\le\int_{Q\cap\{|w|\ge l\}}|f|+\int_{\Omega\cap\{|v_0|\ge l\}}|v_0|.$$

So, since $|\nabla\varphi(v_n)|^2\,\chi_{\{l<|\varphi(v_n)|<l+1\}} = |\nabla(T_{l+1}\varphi(v_n)-T_l\varphi(v_n))|^2$ and $T_{l+1}\varphi(v_n)-T_l\varphi(v_n)\rightharpoonup T_{l+1}w-T_lw$ weakly in $L^2(0,T;H_0^1(\Omega))$, then

$$\int_{Q\cap\{l\le|w|\le l+1\}}|\nabla w|^2\le\int_{Q\cap\{|w|\ge l\}}|f|+\int_{\Omega\cap\{|v_0|\ge l\}}|v_0|$$

and, letting $l\to\infty$, we obtain

$$\iint_{\{l\le|w|\le l+1\}}|\nabla w|^2\to 0\ \text{ as } l\to\infty. \tag{5.28}$$

Finally, collecting together all limits (5.24)–(5.28) we conclude on existence of a renormalized solution of Problem $E(u_0,f)$ for all $f\in L^1(Q)$ and $u_0\in L^1(\Omega)$. □

Proof of Theorem 2.1: **Existence part.**

Let G be the map from $[0,T)\times L^1(\Omega)$ into $L^1(\Omega)$ defined by

$$G(t,u)=g(t,\cdot,u),$$

and A be the operator in $L^1(\Omega)$, defined by

$$Az=-\Delta\varphi(z)-\nabla F(z,w),\ w\in\beta(z)\ \text{ in } \mathcal{D}'(\Omega)$$

and

$$D(A)=\{z\in L^\infty(\Omega);\ w\in H_0^1(\Omega),\ Az\in L^1(\Omega)\}.$$

Thanks to [15], we know that A is T-accretive in $L^1(\Omega)$ and $\overline{A}$ is m-accretive in $L^1(\Omega)$, and, moreover, $\overline{D(A)} = L^1(\Omega)$.
Thanks to $i)$ and $ii)$ of Assumption (H_3), G is integrable in $t \in (0,T)$ for any $u \in L^1(\Omega)$ and continuous in $u \in L^1(\Omega)$ for a.e. $t \in (0,T)$. Moreover, using $ii)$ of Assumption (H_3) we see that $CI - G(t,.)$ is accretive in $L^1(\Omega)$. Then (see for instance [9], Lemma 1) there exists a unique mild solution of

$$\frac{du}{dt} + Au = G(\cdot, u) \ \text{ on } (0,T), \quad u(0) = u_0,$$

which is also a mild solution of

$$\frac{du}{dt} + Au \ni f,\ u(0) = u_0$$

with $f = g(\cdot, u)$. By Proposition 4.2, u is a renormalized solution of $E(u_0, f)$ and thus u is a renormalized solution of $E(u_0, g)$. □

6. The elliptic problem

At the end of this paper, let us give some consequences of the previous results for the stationary problem

$$\begin{cases} u - \Delta w - \nabla F(u,w) = f,\ w \in \beta(u) & \text{in} \quad \Omega \\ w = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{$S(f)$}$$

by assuming that Assumptions (H_1)–(H_2) are fulfilled.

Proposition 6.1. *Let $f \in L^1(\Omega)$. Then, there exists a unique renormalized solution u of $S(f)$ in the sense that*

(i) $u \in L^1(\Omega)$
(ii) $T_k w \in H_0^1(\Omega)$ *for any* $k > 0$
(iii) *for all* $\xi \in \mathcal{D}(\Omega)$ *and* $h \in \mathcal{C}_c^1(\mathbb{R})$

$$\int_\Omega (u - f) h(w)\xi + \int_\Omega (\nabla w + F(u,w)) \cdot \nabla(h(w)\xi) = 0,$$

and moreover

$$\int_{\Omega \cap \{n \le |w| \le n+1\}} |\nabla w|^2 \to 0 \quad \text{as } n \to \infty.$$

Moreover, for any $f_i \in L^1(\Omega)$ and u_i a renormalized solution of $S(f_i)$, $i = 1, 2$, we have

$$\left\|(u_1 - u_2)^+\right\|_1 \le \left\|(f_1 - f_2)^+\right\|_1.$$

Proof. The uniqueness follows from the fact that if u is a renormalized solution of $S(f)$ then $\tilde{u}(t) \equiv u$ is a renormalized solution of $E(\tilde{u}_0, \tilde{g})$ with $\tilde{u}_0 = u$ and $\tilde{g}(\cdot, u) = f(\cdot) - u$.

To prove existence, we consider a sequence f_n in $L^\infty(\Omega)$ such that f_n converges to f in $L^1(\Omega)$ as $n \to \infty$. It follows from [15] that there exists a unique u_n solution of

$$\begin{cases} u_n \in L^\infty(\Omega),\ w_n \in \beta(u_n) \in H_0^1(\Omega) \quad \text{and} \\ u_n = \nabla\cdot(\nabla w_n + F(u_n, w_n)) + f_n \ \text{ in } \ \mathcal{D}'(\Omega); \end{cases}$$

moreover, we have

$$\|u_n - u_m\|_{L^1(\Omega)} \le \|f_n - f_m\|_{L^1(\Omega)} \qquad \text{for any } n, m \ \in \mathbb{N}.$$

This implies that $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^1(\Omega)$ and there exists $u \in L^1(\Omega)$ such that $u_n \to u$ in $L^1(\Omega)$ as $n \to \infty$. To prove that u is a renormalized solution of $S(f)$, note that u_n is also a renormalized solution of the evolution problem $E(u_n, f_n - u_n)$; therefore, passing to the limit as $n \to \infty$, the result follows. □

Corollary 6.1. *The closure of the operator A in $L^1(\Omega)$ satisfies*

$$\overline{A} = \big\{(z,h) \in L^1(\Omega)\times L^1(\Omega)\ ;\ z \textit{ is a renormalized solution of } S(u+h)\big\} =: \mathcal{A}.$$

Proof. Since a weak solution is also a renormalized solution, we have $A \subseteq \mathcal{A}$. On the other hand, using Theorem 5.1, we deduce that $\mathcal{A}$ is m-accretive in $L^1(\Omega)$, so that $\mathcal{A}$ is closed in $L^1(\Omega)$, and

$$\overline{A} \subseteq \mathcal{A}. \tag{6.29}$$

Thanks to [15], we know that A is accretive and $R(I + A) \supseteq L^\infty(\Omega)$, then $\overline{A}$ is m-accretive in $L^1(\Omega)$, and (6.29) implies that $\overline{A} = \mathcal{A}$. □

7. Appendix

Lemma 7.1. *Let $h \in W^{1,\infty}(\mathbb{R})$, $h \ge 0$, $u_0 \in L^1(\Omega)$, $u \in L^1(Q)$ such that $T_k w \in L^2(0,T;H_0^1(\Omega))$ for any $k > 0$ and $G \in L^2(0,T;H^{-1}(\Omega)) + L^1(Q)$. Suppose that*

$$\int_Q \xi_t \int_{u_0}^u h(\varphi_0(s))\, ds = \int_0^T \langle G, h(w)\,\xi\rangle \tag{7.30}$$

for any nonnegative $\xi \in \mathcal{D}([0,T)\times\Omega)$. Then,

$$\iint_Q \xi_t \left\{\int_{u_0}^u H_\varepsilon\left(T_k\varphi_0(s) - \varphi(z)\right) h(\varphi_0(s))\mathrm{d}s\right\} \le \int_0^T \langle G, H_\varepsilon(T_k w - \varphi(z))\, h(w)\xi\rangle\,. \tag{7.31}$$

for all $\xi \in L^2(0,T;H^1(\Omega)) \cap W^{1,1}(0,T;L^\infty(\Omega)) \cap L^\infty(Q)$ such that $\xi \ge 0$, $\xi(T,.) = 0$ a.e. in Ω, and for any $z \in L^1(Q)$ such that $\varphi(z)\xi \in L^2(0,T;H_0^1(\Omega))$.

Proof. We extend u onto $\mathbb{R}\times\Omega$ by 0 if $t>T$ and by u_0 if $t<0$ and we consider

$$\Phi = H_\varepsilon\left(T_k w - \varphi(z)\right)\,\xi.$$

It is clear that $\Phi \in L^2\left(0,T;H_0^1(\Omega)\right)$ and, for any $\delta>0$, $\Phi^\delta(t) = \frac{1}{\delta}\int_t^{t+\delta}\Phi(s)\,\mathrm{d}s$ is an admissible test function in the problem (7.30) and

$$\iint_Q \Phi_t^\delta \int_{u_0}^{u} h(\varphi_0(s))ds = \int_0^T \left\langle G, \Phi^\delta\, h(w)\right\rangle. \tag{7.32}$$

We see that

$$\iint_Q \Phi_t^\delta \int_{u_0}^{u} h(\varphi_0(s))ds = \iint_Q \frac{\Phi(t+\delta)-\Phi(t)}{\delta} \int_{u_0}^{u} h(\varphi_0(s))ds$$
$$= \iint_Q \Phi(t)\,\frac{1}{\delta}\int_{u(t)}^{u(t-\delta)} h(\varphi_0(s))ds$$

and, since for any $r,\hat{r},w\in\mathbb{R}$,

$$H_\varepsilon\left(T_k\varphi_0(r)-\varphi(w)\right)\int_r^{\hat{r}} h(\varphi_0(s))ds \le \psi_w^\varepsilon(\hat{r}) - \psi_w^\varepsilon(r)\ ,$$

where $\psi_w^\varepsilon(r) = \int_w^r H_\varepsilon\left(T_k\varphi_0(s)-\varphi(w)\right)\,h(\varphi_0(s))\mathrm{d}s$, it follows that

$$\iint_Q \Phi_t^\delta \int_{u_0}^{u} h(\varphi_0(s))ds \le \iint_Q \frac{\psi^\varepsilon_{z(t,x)}(u(t-\delta,x)) - \psi^\varepsilon_{z(t,x)}(u(t,x))}{\delta}\,\xi(t,x)\,\mathrm{d}t\,\mathrm{d}x$$
$$\le \iint_Q \left(\psi^\varepsilon_{z(t,x)}(u(t,x)) - \psi^\varepsilon_{z(t,x)}(u_0(x))\right)\,\frac{\xi(t+\delta,x)-\xi(t,x)}{\delta}\,\mathrm{d}t\,\mathrm{d}x.$$

Consequently, we have

$$\liminf_{\delta\to 0}\iint_Q \Phi_t^\delta \int_{u_0}^{u} h(\varphi_0(s))ds \le \iint_Q \left(\psi_z^\varepsilon(u)-\psi_z^\varepsilon(u_0)\right)\,\xi_t$$
$$\le \iint_Q \xi_t\,\left\{\int_{u_0}^{u} H_\varepsilon\left(T_k\varphi_0(s)-\varphi(z)\right)\,\mathrm{d}s\right\}.$$

Since $h(T_k w)\Phi^\delta \to H_\varepsilon(T_k w-\varphi(z))\,h(w)\xi$ in $L^2(0,T;H_0^1(\Omega))$, then from (7.32) and the preceding estimate, (7.31) follows. □

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N. Igbida
LAMFA, CNRS-UMR 6140,
Université de Picardie Jules Verne,
33 rue Saint Leu,
80038 Amiens,
France
e-mail: noureddine.igbida@u-picardie.fr

K. Sbihi
Laboratoire de Mathématiques,
Université de Franche-comté,
16 route de Gray,
25030 Besancon,
France
e-mail: karima.sbihi@univ-fcomte.fr

P. Wittbold
Technische Universität Berlin,
Strasse des 17. Juni 136,
10623 Berlin,
Germany.
e-mail: wittbold@math.tu-berlin.de

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