A Nonlinear Diffusion Problem with Localized Large Diffusion

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Dedicated to the Memory of Philippe Bénilan

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Abstract

This paper is concerned with a nonlinear parabolic problem, with nonlinear boundary conditions, for which the diffusion coefficient becomes very large in a sub-region of the physical domain.
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1. Introduction

There is a large class of physical problems for which the behavior is described by the study of the perturbation of the evolution, or stationary, equations. In these equations, there appears some parameters that vary strongly, and for a large class of problems, the perturbed equation is of totally different character then the unperturbed one : this a *singular limit* phenomena. For instance, take the reaction-diffusion system $u_t - d\,\Delta u + k\,g(u) = 0$, where $u = (u_1, u_2, ..., u_n)$, $d = (d_1, d_2, ..., d_N)$ is a N-uplet of the diffusion coefficients and k is the reaction rate. The limit becomes singular when the diffusivity d_i of the component species u_i and/or the reaction rate k is very large (cf. [25], [19], [24], [22], [27], [15] and [16]). Other examples appear in the the study of the extreme cases of the porous medium equation $u_t = \frac{1}{m}\Delta u^m$, i.e. the very slow diffusion : $m \to \infty$ (cf. [18], [7], [9, 10, 11], [29, 30]) and the very fast diffusion : $m \to 0$ (cf. [37] and [28]). Also, the study of the sandpile model exhibits the study of the singular limit of the p-laplacien equation $u_t = \Delta_p u$, as $p \to \infty$ (cf. [23]).

In this paper, we will be concerned with a nonlinear diffusion equation in a bounded domain Ω, for which the diffusion coefficient vary strongly in a sub-domain of Ω. Concretely this situation can be found in models of diffusive process for which the diffusion is very large in a subregion. For example, in chemical engineering, the heat diffusion properties

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in a composite material differ very strongly from one part to another one, or in population dynamics when one species diffuses and/or reacts much faster than the others in some determined regions, and many other situations. In connection with the examples cited above, this is a situation where the singular limit phenomena is localized in space, so that the passage to the limit will exhibit some instantaneous local redistribution of the spatial inhomogeneities of the solution.

To give a brief description of our main results, let us consider, for instance, the model

$$\begin{cases} \partial_t u = \nabla\cdot\Big(a_\varepsilon\,\nabla\varphi(u)\Big) + f & \text{in } Q=(0,T)\times\Omega \\ a_\varepsilon\,\nabla\varphi(u)\cdot\overrightarrow{n} = g & \text{on } \Sigma=(0,T)\times\partial\Omega \\ u(0)=u_0 & \text{in } \Omega \end{cases} \tag{1.1}$$

where φ is an increasing continuous function in $\mathbb{R}$, $f\in L^2(Q)$, $g\in L^2(\Sigma)$, $\overrightarrow{n}$ denotes the outward normal vector of the boundary $\partial\Omega$ and a_ε is a regular function in Ω such that, for any $\varepsilon>0$, $0<m\le a_\varepsilon\le M_\varepsilon$. It is not difficult to prove that (1.1) has a unique weak solution u_ε, i.e. $u_\varepsilon\in L^2(Q)$, $\varphi(u_\varepsilon)\in L^2(0,T;H^1(\Omega))$, and satisfies the equation in a weak sense. Now, let Ω_1 be a sub-domain of Ω, such that for any $K\subset\subset\Omega_1$, $\inf\limits_{x\in K} a_\varepsilon(x)\to\infty$, as $\varepsilon\to 0$ and $a_\varepsilon\to a_0$ uniformly in $\Omega_0=\Omega\setminus\Omega_1$. We are interested in the asymptotic behavior of u_ε, as $\varepsilon\to 0$. In the case where $\Omega_0=\emptyset$, it is known that a very rapid redistribution of the spatial inhomogeneities of u and therefore u converges to a space constant function, which is equal to $\displaystyle ⨍ u_0+\frac{1}{|\Omega|}\int_0^t\Big(\int_\Omega f+\int_\Gamma g\Big)$, for each instant $t>0$. As to the case where $\Omega_1=\emptyset$, obviously $u_\varepsilon\to u$ and u is a solution of

$$u_t=\nabla\cdot\Big(a_0\,\nabla\varphi(u)\Big)+f \tag{1.2}$$

with the corresponding boundary condition

$$a_0\,\nabla\varphi(u)\cdot\overrightarrow{n}=g. \tag{1.3}$$

So, if Ω_0 and Ω_1 are not empty, then one expects that inside Ω_0, u_ε converges to a function u which satisfies (1.2) in Ω_0 and (1.3) on $(0,T)\times(\partial\Omega\cap\partial\Omega_0)$. As to inside Ω_1, u_ε approaches u_{Ω_1} a constant function in Ω_1. This constant is not arbitrary and must take into account the limit of boundary value of $\partial\varphi(u_\varepsilon)$ on $\partial\Omega_1$. In fact, integrating over Ω_1 and using the inward normal $\overrightarrow{n}_1$ in the integration by parts, we obtain

$$\int_{\Omega_1}(u_\varepsilon)_t+\int_{\partial\Omega_1}a_\varepsilon\,\nabla\varphi(u_\varepsilon)\cdot\overrightarrow{n}_1=\int_{\Omega_1}f$$

and formally taking the limit and dividing by $|\Omega_1|$ we obtain

$$(u_{\Omega_1})_t+\frac{1}{|\Omega_1|}\int_{\partial\Omega_1}a_0\,\nabla\varphi(u)\cdot\overrightarrow{n}_1=\frac{1}{|\Omega_1|}\int_{\Omega_1}f\quad\text{ in }(0,T)\times\Omega_1.$$

So, taking into account the boundary condition and the initial data, the limiting problem should be

$$(1.4)\quad \begin{cases} \partial_t u = \nabla \cdot \Big(a_0 \, \nabla w \Big) + f, \quad w = \varphi(u), & \text{in } (0,T)\times\Omega_0 \\ u(t) \text{ is constant on } \Omega_1 & \text{for } t \in (0,T) \\ \dfrac{d}{dt} u(t) + \dfrac{1}{|\Omega_1|} \displaystyle\int_{\partial\Omega_1} a_0 \, \nabla w \cdot \overrightarrow{n}_1 = \dfrac{1}{|\Omega_1|} \displaystyle\int_{\Omega_1} f(t) & \text{in } (0,T)\times\Omega_1 \\ a_0 \, \nabla w \cdot \overrightarrow{n} = g & \text{on } (0,T)\times(\partial\Omega \setminus \partial\Omega_1) \\ u(0) = u_0. \end{cases}$$

Recall that, if $\varphi = \mathrm{Id}_{\mathbb{R}}$ and Ω_1 is interior to Ω, i.e. $\partial\Omega \cap \partial\Omega_0 = \partial\Omega$, then the problem is a particular case of [3]. In this case, (1.1) and (1.4) are semi-linear equations in $L^p(\Omega)$, with $p > 1$, for which the authors used the results of [38] for the associate elliptic equation and the results of [26] and [2] for semi-linear equations in abstract form. Among the results of [3], it is proved that for any initial data $u_0 \in L^2(\Omega)$ such that u_0 is constant in Ω_1, (1.4) has a unique solution u which is the limit of u_ε. Moreover, u solves (1.4) in a strong sense, more precisely u and u_t are in $\mathcal{C}([0,T);X)$ where X is some fractional power space, so that the normal derivative of w and $\nabla \cdot (a_0 \, \nabla \, w)$ are integrable functions and can be used explicitly in the formulation of the solution (in other words the equations in (1.4) are pointwise satisfied). In the nonlinear case, i.e. $\varphi \neq Id_{\mathbb{R}}$, this kind of formulation turns out to be useful, since even for (1.1), with $\varepsilon > 0$, the existence of strong solutions is not true in general, a weak formulation of the solution is needed. In this paper, we introduce the notion of weak solution for problems of type (1.4) which coincides with the strongest one whenever u and w are regular enough. Moreover, we prove that (1.4) is well posed via this notion and that these solutions are limits of weak solutions of (1.1). On the other hand, one sees that functions u_0 which are constant on Ω_1 are the natural initial datums for (1.4). However, we will prove that even with an initial data u_0 not constant on Ω_1, the problem (1.4) still has a unique solution which is also the solution with the initial data

$$(1.5)\quad \underline{u}_0(x) = \begin{cases} \dfrac{1}{|\Omega_1|} \displaystyle\int_{\Omega_1} u_0 & \text{for any } x \in \Omega_1 \\ u_0(x) & \text{a.e. } x \in \Omega_0. \end{cases}$$

Though, it must be born in mind that, in this case the limit of u_ε is singular and, in the passage to the limit, there appears a boundary layer at time $t = 0$ in the domain Ω_1 given by $\dfrac{1}{|\Omega_1|} \displaystyle\int_{\Omega_1} u_0$.

In fact, in this paper, we consider the equation

$$(1.6)\quad \beta(w)_t = \nabla \cdot \Big(a_\varepsilon \, \nabla w + \sigma(x,w) \Big) + F(t,x,w) \quad \text{in } (0,T)\times\Omega$$

where $T > 0$, Ω is a regular bounded domain, a_ε is the diffusion coefficient, the function β is assumed to be nondecreasing, the convection σ is assumed to be dissipatif and the reaction term F is a caratheodory function satisfying assumptions that we precise in section 2. This equation is considered with nonlinear boundary condition of the type

$$(1.7)\quad \Big(a_\varepsilon \, \nabla w + \sigma(x,w) \Big) \cdot \overrightarrow{n} + \gamma(x,w) = g(x,t)$$

on a part of the boundary of Ω and homogeneous Dirichlet boundary condition in the remaining one. The problem (1.6)-(1.7) appears as a model for a large class of evolution physical problems, as for instance, diffusion in porous medium ([5]), dynamical population (cf. [35]) and many others. Recall that in particular β may be such that β is increasing in $(-\infty, 1]$ and β is constant and equals, for example, to 1 in $[1, \infty)$. In this case (1.1) is of elliptic-parabolic type and models problems of evolution equation for which the evolution is null in the region $[u \geq 1]$ and the equation becomes elliptic. Existence and uniqueness of a weak solution for (1.1)-(1.2) is well known by now when the boundary conditions are homogeneous (cf. [36] and [12] and the references therein). In this paper, we extend slightly this results, by treating the case of boundary conditions of the type (1.7). Our main interest lies in the study of the asymptotic behavior of the solutions of (1.6)-(1.7), as $\varepsilon \to 0$, by assuming that $a_\varepsilon \to \infty$, in a connected sub-domain Ω_1 of Ω. We identify the limiting problem that we call the Shadow problem : it consists of an evolution PDEs for which the solution tends to be a space constant function in Ω_1. We introduce the notion of weak solution for this kind of problems and prove the existence and the uniqueness. Moreover, although a solution of (1.6)-(1.7) converges to a solution of the Shadow problem, we prove that the limit is singular and a boundary layer appears in Ω_1, for small time $t > 0$.

In the following section, we give the assumptions that will be hold throughout the paper and after a formal derivation of the limiting problem (the shadow problem), we state our main results concerning the existence, the uniqueness and the convergence. Section 3 and Section 4 are devoted to preparatory results for the proof of the mains theorem. In Section 3, we prove the existence of a solution for the problem (1.1)-(1.2) and the Shadow problem. For the existence of a solution of (1.1)-(1.2), we use the nonlinear semigroup theory, as to the Shadow problem, we obtain the existence by proving the convergence of solutions of (1.1)-(1.2). In a first time, we prove the convergence directly by showing the relative compactness of the solutions of (1.1)-(1.2). Then in the concern of showing uniqueness and exhibiting the singular limit phenomena, we improve the convergence by using nonlinear semigroup theory. Section 4 is devoted to the proof of uniqueness. We use the concept of integral sub/super solution for (1.1)-(1.2) and also for the Shadow problem. We show that weak solutions are integral solutions and, thus, unique too. At last, in Section 5, we gather the main results of the Section 3 and Section 4 to complete the proofs of the main theorems given in Section 2.

At last, we notice that in order to simplify the presentation we are treating only the case where Ω is a connected domain. However, one will see that all our results may be stated in the case where Ω_1 is an union of connected domain.

2. Assumptions and main results

Throughout the paper Ω is a bounded regular open connected domain of $\mathbb{R}^N$ and Γ is the boundary of Ω with $\Gamma = \Gamma_N \cup \Gamma_D$ is a partition of Γ, such that Γ_D is nonempty. The diffusion coefficient $a_\varepsilon(x)$ is such that

$$a_\varepsilon \in \mathcal{C}^1(\Omega) \text{ and } 0 < m_0 \le a_\varepsilon(x) \le M_\varepsilon, \tag{2.1}$$

for every $x \in \Omega$ and $0 < \varepsilon \le \varepsilon_0$. The convection term σ is a function defined from $\Omega \times \mathbb{R}$ in $\mathbb{R}^N$ such that $\sigma(x,r)$ is Lipschitz in $r \in \mathbb{R}$ and belongs to $W^{1,\infty}(\Omega)$ in x and

$$\nabla_x \cdot \sigma(x,r) \le 0 \quad \text{ a.e. } x \in \Omega \text{ and for any } r \in \mathbb{R}. \tag{2.2}$$

We assume moreover, that σ satisfies the structure condition

$$\sigma(x,r) = \tilde{\sigma}(x,\beta(r)) \quad \text{for each } (x,r) \in \Omega \times \mathbb{R} \tag{2.3}$$

where $\tilde{\sigma}(x,s)$ is continuous in s. To simplify the notation, for $0 < \varepsilon \le \varepsilon_0$, we set

$$A_\varepsilon(x,r,\eta) = a_\varepsilon(x)\,\eta + \sigma(x,r) \quad \text{ for any } (x,r,\eta) \in \Omega \times \mathbb{R} \times \mathbb{R}^N,$$

and we consider the problem

$$P_\varepsilon(u_0,F,g) \quad \begin{cases} \partial_t u - \nabla \cdot A_\varepsilon(x,w,\nabla w) = F(t,x,u), \quad u = \beta(w) \quad \text{ in } Q = (0,T) \times \Omega \\ \partial_{\overrightarrow{n}_\varepsilon} w + \rho(x,w) = g \quad \text{ on } \Sigma_N = (0,T) \times \Gamma_N \\ w = 0 \quad \text{ on } \Sigma_D = (0,T) \times \Gamma_D \\ u(0) = u_0 \quad \text{ in } \Omega \end{cases}$$

where $\partial_{\overrightarrow{n}_\varepsilon}$ denotes the conormal derivative relative to the diffusion operator $\nabla \cdot (a_\varepsilon \nabla w + \sigma(x,w))$, i.e.

$$\partial_{\overrightarrow{n}_\varepsilon} w = \Big\langle a_\varepsilon \nabla w + \sigma(x,w), \overrightarrow{n} \Big\rangle,$$

and $\overrightarrow{n}$ denotes the outward normal vector of the related boundary. The nonlinearity $\beta : \mathbb{R} \to \mathbb{R}$ is a nondecreasing Lipschitz continuous function and $\rho : \Gamma_N \times \mathbb{R} \to \mathbb{R}$ is measurable in $x \in \Gamma_N$ and nondecreasing continuous in $r \in \mathbb{R}$. We assume that $\beta(0) = \rho(x,0) = 0$ a.e. $x \in \Gamma_N$ and

$$|\rho(x,r)| \le a(x)|r| + b(x) \quad \text{ a.e. } x \in \Gamma_N \tag{2.4}$$

with $a,\ b \in L^\infty(\Gamma_N)$. The reaction term $F(t,x,u)$ is continuous in u, measurable in $(t,x) \in Q$ and satisfies the following

$$\left(\begin{array}{ll} i) & \dfrac{\partial F}{\partial u}(t,x,u) \le K \text{ in } \mathcal{D}'(\mathbb{R}),\ K \in \mathbb{R}^+ \\ ii) & |F(t,x,u)| \le K_1(t,x)|u| + K_2(t,x) \end{array} \right. \tag{2.5}$$

with $K_i \in L^\infty(Q)$, for $i = 1,\ 2$. The initial data $u_0 \in L^1(\Omega)$ satisfies

$$u(x) \in \overline{\mathrm{Im}(\beta)} = [\beta(-\infty), \beta(+\infty)] \text{ a.e. } x \in \Omega. \tag{2.6}$$

At last, we denote by V the space $V = \Big\{ \ w \in H^1(\Omega)\ ;\ w = 0 \text{ on } \Gamma_D \ \Big\}$ whose dual space is denoted by H^{-1}, and we assume that σ satisfies

$$\int_{\Gamma_N} \left(\int_0^{w(x)} \sigma(x,s)ds\, dx \right) \cdot \overrightarrow{n} \le 0 \text{ for any } w \in V. \tag{2.7}$$

Definition 1 *For $g \in L^1(\Sigma_N)$ and $u_0 \in L^1(\Omega)$, we say that u is a solution of $P_\varepsilon(u_0, F, g)$, if there exists a measurable function $w : Q \to I\!R$ such that*

$$(2.8) \qquad \begin{cases} u \in L^2(Q),\ w \in L^2(0,T;V),\ u = \beta(w) \ a.e.\ in\ Q \\ \displaystyle\iint_Q \Big(-u\,\xi_t + A_\varepsilon(x, w, \nabla w)\cdot\nabla\xi \Big) = \int_\Omega u_0\,\xi(0) + \iint_Q F(.,u)\,\xi \\ \quad + \displaystyle\iint_{\Sigma_N} \Big(g - \rho(.,w) \Big)\,\xi,\ \forall\,\xi \in \mathcal{C}^1(\overline{Q}) \cap L^2(0,T;V) s.t.\ \xi(T) \equiv 0. \end{cases}$$

Theorem 1 *For any $g \in L^2(\Sigma_N)$ and $u_0 \in L^2(\Omega)$ satisfying (2.6), there exists a unique u solution of $P_\varepsilon(u_0, F, g.)$ Moreover, $u \in \mathcal{C}([0,T); L^1(\Omega))$, $u(0) = u_0$ and if for $i = 1,\ 2$, $u_{0i} \in L^2(\Omega)$ satisfies (2.6), $g_i \in L^2(\Sigma_N)$, F_i satisfies (2.5) and u_i is a solution of $P_\varepsilon(u_{0i}, F_i, g_i)$ then*

$$(2.9) \qquad \begin{aligned} &\frac{d}{dt}\int_\Omega \Big(u_1(t) - u_2(t)\Big)^+ \le \int_{[u_1>u_2]} \Big(F_1(.,u_1) - F_2(.,u_2)\Big) \\ &+ \int_{[u_1=u_2]} \Big(F_1(.,u_1) - F_2(.,u_2)\Big)^+ + \int_{\Gamma_N} \Big(g_1 - g_2\Big)^+ \end{aligned}$$

in $\mathcal{D}'(0,T)$.

Let $\Omega_1 \subseteq \Omega$ be an open connected domain of Ω and denote by Γ_1 its boundary, $\Omega_0 = \Omega \setminus \overline{\Omega}_1$ and Γ_0 the boundary of Ω_0. Assume that the diffusion becomes very large on Ω_1, as $\varepsilon \to 0$; i.e. we assume that as $\varepsilon \to 0$, we have

$$a_\varepsilon(x) \to \begin{cases} a_0(x) \text{ uniformly on } \Omega_0 \\ +\infty \text{ uniformly on compact subsets of } \Omega_1. \end{cases}$$

If we formally take the limit in $P_\varepsilon(u_0, F, g)$, we expect that inside Ω_0, the solution u satisfies

$$u_t - \nabla\cdot(a_0\nabla w + \sigma(x,w)) = F(t,x,u) \quad \text{with } u = \beta(w) \quad \text{in } Q_0 := (0,T)\times\Omega_0.$$

As to inside Ω_1, we intuitively guess that for small values of ε, the solution of $P_\varepsilon(u_0, F, g)$ should be approximatively constant and satisfies

$$\frac{d}{dt}u(t) + \frac{1}{|\Omega_1|}\int_{\Gamma_1} \partial_{\overrightarrow{n}_0} w(t) = \frac{1}{|\Omega_1|}\int_{\Omega_1} F(t,.,u(t)) \quad u = \beta(w), \quad \text{in } Q_1 := (0,T)\times\Omega_1.$$

Taking into account the boundary condition and the initial data, the limiting problem, or what we call "Shadow " problem should be

$$Sh(u_0,F,g) \quad \begin{cases} \partial_t u - \nabla\cdot A_0(x,w,\nabla w) = F(t,x,u), \quad u = \beta(w) \quad \text{in } Q_0 \\ w(t) \text{ is constant on } \Omega_1, \quad \text{for } t \in (0,T) \\ \displaystyle\frac{d}{dt}u(t) + \frac{1}{|\Omega_1|}\int_{\Gamma_1} \partial_{\overrightarrow{n}_0} w(t) = \frac{1}{|\Omega_1|}\int_{\Omega_1} F(t,.,u(t)), \quad u = \beta(w), \quad \text{in } Q_1 \\ \partial_{\overrightarrow{n}_0} w + \rho(x,w) = g, \quad \text{on } (0,T)\times(\Gamma_N \setminus \Gamma_1) \\ w = 0, \quad \text{on } \Sigma_D \\ u(0) = u_0 \quad \text{in } \Omega, \end{cases}$$

where $A_0(x,r,\eta) = a_0(x)\,\eta + \sigma(x,r)$. As in [3], we see that the natural spaces for the study of the Shadow problem are $L^p_{\Omega_1} = \Big\{ u \in L^p(\Omega)\ ;\ u \text{ is a constant on } \Omega_1 \Big\}$ endowed with the natural norm L^P, for $1 \le p \le \infty$, and we denote $H^1_{\Omega_1} = V \cap L^2_{\Omega_1}$.

Definition 2 *For $g \in L^1(\Sigma_N)$ and $u_0 \in L^1(\Omega)$, we say that u is a solution of $Sh(u_0,F,g)$, if there exists a measurable function $w\ :\ Q \to I\!R$ such that*

$$
(2.10)\qquad \left(\begin{array}{l} u \in L^2(Q),\ w \in L^2(0,T;H^1_{\Omega_1}),\ u = \beta(w) \ a.e.\ in\ Q\ and \\ \displaystyle\iint_Q \Big(- u\,\xi_t A_0(x,w,\nabla w)\cdot\nabla\xi \Big) = \int_\Omega u_0\,\xi(0) + \iint_Q F(.,u)\,\xi \\ \displaystyle\quad + \iint_{\Sigma_N} \Big(g - \rho(.,w) \Big)\xi,\ \forall\ \xi \in \mathcal{C}(\overline{Q}) \cap L^2(0,T;H^1_{\Omega_1})\ s.t.\ \xi(T) \equiv 0. \end{array} \right.
$$

Theorem 2 *For any $g \in L^2(\Sigma_N)$ and $u_0 \in L^2(\Omega)$ satisfying (2.6), there exists a unique u solution of $Sh(u_0,F,g)$. Moreover, $u \in \mathcal{C}([0,T);L^1_{\Omega_1})$, $u(0) = u_{0\Omega_1}$, and if for $i = 1,\ 2$, $u_{0i} \in L^2(\Omega)$ satisfies (2.6), $g_i \in L^2(\Sigma_N)$, F_i satisfies (2.5) and u_i is a solution of $Sh(u_{0i},F_i,g_i)$ then (2.9) is fulfilled in $\mathcal{D}'(0,T)$.*

Remark 1 *Without abusing, we will say that (u,w) is a solution of $P_\varepsilon(u_0,F,g)$ (resp. $Sh(u_0,F,g)$), if (u,w) satisfies (2.8) (resp. (2.10)). Though it must be born in mind that u is unique ; as to the function w, in general, we don not know if it is unique or not.*

Corollary 1 *A solution (u,w) of $Sh(u_0,F,g)$, with $u_0 \in L^1(\Omega)$ and F satisfying (2.5) is also a solution of $Sh(u_{0\Omega_1},F_{\Omega_1},g_{\Omega_1})$, where $u_{0\Omega_1} = u_0\,\chi_{\Omega_0} + \dfrac{\chi_{\Omega_1}}{|\Omega_1|}\displaystyle\int_{\Omega_1} u_0(x)\,dx$, $F_{\Omega_1} = F\,\chi_{\Omega_0} + \dfrac{\chi_{\Omega_1}}{|\Omega_1|}\displaystyle\int_{\Omega_1} F(.,y,.)\,dy$ and $g_{\Omega_1} = g\,\chi_{\Gamma_0} + \dfrac{\chi_{\Gamma_1}}{|\Omega_1|}\displaystyle\int_{\Gamma_1} g(x)\,dx$.*

Remark 2 *By an appropriate choice of a test function ξ, one sees that the formulation (2.10) implies that $Sh(f,g)$ is pointwise satisfied whenever u, w, Γ_0 and Γ_1 are regular enough.*

At last, we close this section of main results by given the theorem of convergence of solutions of P_ε to those one of Sh.

Theorem 3 *Let $u_0 \in L^2(\Omega)$ satisfying (2.6) and $g \in L^2(\Sigma_N)$. If $K_1 \equiv 0$, then there exists a solution $(u_\varepsilon, w_\varepsilon)$ of $P_\varepsilon(u_0,F,g)$, such that, as $\varepsilon \to 0$,*

$$
(2.11)\qquad u_\varepsilon \to u \quad in\ \mathcal{C}([\delta,T);L^1(\Omega))
$$

for any $\delta > 0$, and, by taking subsequences if necessary,

$$
(2.12)\qquad w_\varepsilon \to w \quad in\ L^2(0,T;V) - weak
$$

where (u,w) is a solution of $Sh(u_0,F,g)$. Moreover, if $u_0 \in L^2_{\Omega_1}$, then (2.11) and (2.12) remains true even if $K_1 \not\equiv 0$; and also, $\delta = 0$ is admissible in (2.11).

Remark 3 *The assumptions (2.2) and (2.7) means that the convection σ is dissipatif, while (2.3) is a structure condition. We emphasize that our results can be obtained under relaxed assumptions. However, since we are working with nonhomogeneous boundary conditions, we decided to study the problem under such assumptions only to simplify the proofs of existence of a solutions of $P_\varepsilon(u_0, F, g)$ (which is not the aim of this paper). For discussions in this direction, we refer the readers to the papers [1], [13], [32] and the references therein.*

3. Existence of solutions and convergence results

3..1 Existence of a solution for P_ε

We introduce the following notation that will be used throughout the paper:

$$\mathrm{Sign}^+(s) = \begin{cases} 1 & \text{if } s > 0 \\ [0,1] & \text{if } s = 0 \\ 0 & \text{if } s < 0 \end{cases}, \quad \mathrm{Sign}_0^+(s) = \begin{cases} 1 & \text{if } s > 0 \\ 0 & \text{if } s \le 0 \end{cases}$$

and and for $\epsilon > 0$, $H_\epsilon(s) = \inf(s^+/\varepsilon, 1)$, for any $s \in \mathbb{R}$. We will treat the problem in the context of nonlinear semigroup theory. Through the implicit discretization in time arising in this theory, the study of P_ε is closely connected to the associate stationary problem which is

$$St_\varepsilon(f,g) \qquad \begin{cases} v - \nabla \cdot A_\varepsilon(x, w, \nabla w) = f, \quad v = \beta(w) & \text{in } \Omega \\ \partial_{\overrightarrow{n}_\varepsilon} w + \rho(x, w) = g & \text{on } \Gamma_N, \\ w = 0 & \text{in } \Gamma_D. \end{cases}$$

We say that a couple of function (v, w) is a solution of $St_\varepsilon(f, g)$, if $v \in L^2(\Omega)$, $w \in V$, $v = \beta(w)$ a.e. Ω and $\int_\Omega (v - f)\, \xi + \int_\Omega A_\varepsilon(x, w, \nabla w) \cdot \nabla \xi = \int_{\Gamma_N} (g - \rho(x, w))\, \xi$, for any $\xi \in V$.

Proposition 1 *For any $f \in L^2(\Omega)$ and $g \in L^2(\Gamma_N)$, there exists a unique v and there exists w such that (v, w) is a solution of $St_\varepsilon(f, g)$. Moreover, if (v_i, w_i) is a solution of $St_\varepsilon(f_i, g_i)$ for $i = 1,\ 2$, with $f_i \in L^2(\Omega)$ and $g_i \in L^2(\Gamma_N)$, then*

$$\int_\Omega (v_1 - v_2)^+ \le \int_\Omega (f_1 - f_2)^+ + \int_{\Gamma_N} (g_1 - g_2)^+ \tag{3.1}$$

and

$$\int_\Omega |v_1 - v_2| \le \int_\Omega |f_1 - f_2| + \int_{\Gamma_N} |g_1 - g_2|\,. \tag{3.2}$$

The result of this proposition is quite standard. However, we did not find such statement in the literature ; we will prove it at the end of this subsection.

Now, since we are considering time dependent boundary conditions, then we will study P_ε in $X = L^1(\Omega) \times L^1(\Gamma)$ endowed with the natural norm $|(f, g)|_X = \|f\|_{L^1(\Omega)} + \|g\|_{L^1(\Gamma_N)}$, for any $(f, g) \in X$. Moreover, X becomes a Banach lattice if equipped with the usual partial ordering $(f, g) \leq (\tilde{f}, \tilde{g})$ iff $f \leq \tilde{f}$ a.e. in Ω and $g \leq \tilde{g}$ a.e. in Γ_N. In X, we define the operator $\mathcal{A}_\varepsilon$ by, $(h, g) = \mathcal{A}_\varepsilon(u, 0)$ if and only if

$$\begin{cases} u \in L^2(\Omega),\ h \in L^2(\Omega),\ g \in L^2(\Gamma_N),\ \exists\, w \in H^1(\Omega),\ u = \beta(w) \text{ a.e. } \Omega, \\ \displaystyle\int_\Omega A_\varepsilon(x, w, \nabla w) \cdot \nabla \xi = \int_\Omega h\, \xi + \int_{\Gamma_N} (g - \rho(x, w))\, \xi\ ,\ \forall\, \xi \in H^1(\Omega), \end{cases}$$

and we consider the Cauchy problem :

$$CP_\varepsilon(u_0, f, g) \qquad \begin{cases} U_t + \mathcal{A}_\varepsilon U \ni (f, g) & \text{in } (0, T) \\ U(0) = (u_0, 0). \end{cases}$$

Thanks to Proposition 1, we see that $(I + \lambda\, \mathcal{A}_\varepsilon)^{-1}$, the resolvent of the operator $\mathcal{A}_\varepsilon$, is an order preserving contraction in X, in other words $\mathcal{A}_\varepsilon$ is T-accretive in X. Moreover, $R(I + \lambda \mathcal{A}_\varepsilon) \supseteq L^2(\Omega) \times L^2(\Gamma_N)$, so that $\overline{\mathcal{A}_\varepsilon}^X$ the closure of $\mathcal{A}_\varepsilon$ in X is m-T-accretive in X. Then thanks to Crandall-Ligget Theorem (cf. [20]), $\mathcal{A}_\varepsilon$ generates a nonlinear semigroup of order preserving contractions in X.

Lemma 1 *Setting* $D = \Big\{u \in L^1(\Omega)\ ;\ u(x) \in \overline{Im(\beta)} \text{ a.e. } x \in \Omega\Big\}$, *we have* $\overline{\mathcal{D}(\mathcal{A}_\varepsilon)} = D \times \Big\{0\Big\}$.

Proof : Clearly, by density and the definition of $\mathcal{A}_\varepsilon$ we have $\overline{\mathcal{D}(\mathcal{A}_\varepsilon)} \subseteq D \times \Big\{0\Big\}$. To prove that $D \times \Big\{0\Big\} \subseteq \overline{\mathcal{D}(\mathcal{A}_\varepsilon)}$, it is enough to prove that $\Big(D \cap L^\infty(\Omega)\Big) \times \Big\{0\Big\} \subseteq \overline{\mathcal{D}(\mathcal{A}_\varepsilon)}$. So, let $u \in D \cap L^\infty(\Omega)$ and consider (u_λ, w_λ) the solution of

$$\begin{cases} u_\lambda - \lambda\, \nabla \cdot A_\varepsilon(x, w_\lambda, \nabla w_\lambda) = u, \quad u_\lambda = \beta(w_\lambda) & \text{in } \Omega \\ \partial_{\overrightarrow{n}_\varepsilon} w_\lambda + \rho(x, w_\lambda) = 0 & \text{on } \Gamma_N, \\ w_\lambda = 0 & \text{in } \Gamma_D. \end{cases}$$

Since $(u_\lambda, 0) \in D(\mathcal{A}_\varepsilon)$, for each $\lambda > 0$, then it is enough to prove that, by choosing a subsequence if necessary, $u_\lambda \to u$ in $L^1(\Omega)$, as $\lambda \to 0$. Since $u \in \overline{Im(\beta)}$ a.e. $x \in \Omega$, then it is clear that $\lambda\, w_\lambda$ is bounded in V, $\|u_\lambda\|_{L^\infty(\Omega)} \leq \|u\|_{L^\infty(\Omega)}$ and $\|u_\lambda\|_{L^2(\Omega)} \leq \|u\|_{L^2(\Omega)}$. So that, $\lambda\, w_\lambda \to 0$ in $H^1(\Omega)-$weak and $u_\lambda \to u$, in $L^2(\Omega)-$weak, as $\lambda \to 0$. Moreover, since $\|u_\lambda\|_{L^2(\Omega)} \leq \|u\|_{L^2(\Omega)}$, then we deduce that $u_\lambda \to u$ in $L^2(\Omega)$ and the convergence in $L^1(\Omega)$ follows. ∎

Using nonlinear semigroup theory for abstract evolution problem (cf. [8] and [21]), we have the following :

Corollary 2 *For any* $u_0 \in D$, $f \in L^1(Q)$ *and* $g \in L^1(\Sigma_N)$, $CP_\varepsilon(u_0, f, g)$ *has a unique mild solution* $U = (u, 0)$. *Moreover, we may define a mapping* $S_\varepsilon\ : (u_0, f, g) \in D \times L^1(Q) \times L^1(\Sigma_N) \to u \in \mathcal{C}([0, T); L^1(\Omega))$ *such that the* L^1-*comparison principle holds.*

More precisely, if for $i = 1\ 2$, $u_{0i} \in D$, $f_i \in L^1(Q)$, $g_i \in L^1(\Sigma_N)$ *and* $u_i = S_\varepsilon(u_{0i}, f_i, g_i)$, *then*

$$\frac{d}{dt}\int_\Omega (u_1(t)-u_2(t))^+ \leq \int_{[u_1(t)>u_2(t)]} (f_1(t)-f_2(t)) + \int_{[u_1(t)=u_2(t)]} (f_1(t)-f_2(t))^+ + \int_{\Gamma_N} (g_1-g_2)^+ \tag{3.3}$$

in $\mathcal{D}'(0,T)$.

It is quite natural to ask in what sense the mild solution $U = (u, 0)$ and then u satisfies the pde $P_\varepsilon(u_0, f, g)$. Under the general assumptions of Corollary 2, i.e. u_0, f and g are just L^1, one expects that u satisfies this pde in a renormalized sense. For the case of homogeneous Dirichlet boundary conditions, one can see [14] and [31]. As to the case of general boundary conditions, the problem is still open. In this paper, we will not use this notion of solution, we restrict ourself to the case where u should satisfies the pde in the usual weak sense. So, we will assume additional assumptions on u_0, f and g.

Proposition 2 *Let* $f \in L^2(Q)$, $g \in L^2(\Sigma_N)$, $u_0 \in L^2(\Omega)$ *satisfying (2.6) and* $u = S_\varepsilon(u_0, f, g)$. *Then, there exists* w *such that* (u, w) *is a solution of* $P_\varepsilon(u_0, f, g)$ *in the sense of (2.8). Moreover, we have*

$$\int_\Omega j(u(t)) + \int_0^t\int_\Omega a_\varepsilon |\nabla w|^2 \leq \int_0^t \Big(\int_\Omega f\, w + \int_{\Gamma_N} g\, w\Big) + \int_\Omega j(u_0) \tag{3.4}$$

for any $t \in [0, T)$, *where* $j(r) = \displaystyle\int_0^r s\, d\beta(s)$, *for any* $r \in I\!R$.

Proof : By definition of $S_\varepsilon(u_0, f, g)$, $u \in \mathcal{C}([0,T); L^1\Omega))$. Moreover, $u(t) = L^1 - \lim_{\epsilon\to 0} u_\lambda(t)$ uniformly in $t \in [0, \tau]$, for any $0 < \tau \leq T$, where for $\lambda > 0$, u_λ is a $\lambda-$approximate solution of $CP_\varepsilon(u_0, f, g)$ corresponding to a subdivision $t_0 = 0 < t_1 < ... < t_{n-1} < \tau \leq t_n$, with $t_i - t_{i-1} = \lambda$, $f_1, ... f_n \in L^2(\Omega)$ and $g_1, ... g_n \in L^2(\Gamma_N)$ with $\displaystyle\sum_{i=1}^n \int_{t_{i-1}}^{t_i} \Big(\|f(t) - f_i\|_{L^1(\Omega)} + \|g(t) - g_i\|_{L^1(\Gamma_N)}\Big) dt \leq \lambda$. This approximate solution is defined by $u_\lambda(t) = u_i$, for $t \in]t_{i-1}, t_i]$, $i = 1, ...n$, where u_i is such that $U_i := (u_i, 0) \in L^2(\Omega) \times L^2(\Gamma_N)$ satisfies $U_i - U_{i-1} + \lambda\, \mathcal{A}_\varepsilon U_i = (\lambda f_i, \lambda g_i)$ with $U_0 = (u_0, 0)$. That is, there exists $w_i \in V$ such that (u_i, w_i) is a solution of

$$\begin{cases} u_i - \lambda\, \nabla \cdot A_\varepsilon(x, w_i, \nabla w_i) = u_{i-1} + \lambda f_i, \quad u_i = \beta(w_i) & \text{in } \Omega, \\ \partial_{\overrightarrow{n}_\varepsilon} w_i + \rho(x, w_i) = g_i & \text{on } \Gamma_N, \\ w_i = 0 & \text{on } \Gamma_D. \end{cases} \tag{3.5}$$

Taking w_i as a test function in (3.5) and using the fact that $\displaystyle\int_\Omega (u_{i-1} - u_i) w_i \leq \int_\Omega j(u_{i-1}) - \int_\Omega j(u_i)$, we have

$$\int_\Omega j(u_i) + \lambda \int_\Omega a_\varepsilon |\nabla w_i|^2 + \lambda \int_\Omega \sigma(x, w_i) \cdot \nabla w_i \leq \int_\Omega j(u_{i-1}) + \lambda \int_\Omega f_i w_i + \lambda \int_{\Gamma_N} (g_i - \rho(x, w_i)) w_i. \tag{3.6}$$

Adding (3.6) from $i = 1$ to n, we get

$$\begin{aligned}(3.7)\quad &\int_\Omega j(u_\lambda(\tau)) + \int_0^\tau\int_\Omega a_\varepsilon\,|\nabla w_\lambda|^2 + \int_0^\tau\int_\Omega \sigma(x,w_\lambda)\cdot\nabla w_\lambda \leq \int_\Omega j(u_0) + \int_0^\tau\int_\Omega f_\lambda w_\lambda \\ &\quad + \int_0^\tau\int_{\Gamma_N}(g_\lambda - \rho(x,w_\lambda))w_\lambda\end{aligned}$$

where $w_\lambda\ :\ [0,\tau]\to V$, $f_\lambda\ :\ [0,\tau]\to L^1(Q)$ and $g_\lambda\ :\ [0,\tau]\to L^1(\Sigma_N)$ with $w_\lambda(t) = w_i$, $f_\lambda(t) = f_i$, and $g_\lambda(t) = g_i$ for any $t\in\,]t_{i-1},t_i]$, $i = 1,...n$. Thanks to (2.2) and (2.7), we have

$$(3.8)\quad \int_\Omega \sigma(x,w_\lambda)\cdot\nabla w_\lambda = \int_\Omega \nabla\cdot(\int_0^{w_\lambda}\sigma(x,r)dr) - \int_\Omega\int_0^{w_\lambda}\nabla_x\cdot\sigma(x,r)dr \quad \geq 0,$$

so, since $j\geq 0$ and $\rho(x,w_\lambda)w_\lambda\geq 0$ on Σ_N, then (3.7) implies

$$m_0\int_0^\tau\int_\Omega|\nabla w_\lambda|^2 \leq \int_\Omega j(u_0) + C\Big(\|f_\lambda\|_{L^2(Q)} + \|g_\lambda\|_{L^2(\Sigma_N)}\Big)\|\nabla w_\lambda\|_{L^2(Q)}$$

which implies that ∇w_λ is bounded in $L^2(Q)$, w_λ is bounded in $L^2(0,\tau;V)$ and, by (2.4), $\rho(x,w_\lambda)$ is bounded in $L^2(\Sigma_N)$. Let $w\in L^2(0,\tau;V)$, $\chi\in L^2(\Sigma_N)$, and $\lambda_k\to 0$, such that $w_{\lambda_k}\to w$ in $L^2(0,\tau;V)-$weak and $\rho(x,w_{\lambda_k}\to\chi$ in $L^2(\Sigma_N)$-weak. Since, for each $t>0$, $u_{\lambda_k}(t) = \beta(w_{\lambda_k}(t))$ a.e. in Q and $u_{\lambda_k}(t)\to u(t)$ in $L^1(\Omega)$, then by monotonicity arguments we deduce that $u(t) = \beta(w(t))$ a.e. in Ω and by (2.3), $\sigma(.,w_{\lambda_k})\to\sigma(.,w)$ in $L^2(Q)$. Now, we consider $\tilde u_\lambda$ the function from $[0,\tau]$ into $L^1(\Omega)$, defined by $\tilde u_\lambda(t_i) = u_i$ and $\tilde u_\lambda$ linear in $[t_{i-1},t_i]$, then (3.5) implies that

$$\begin{aligned}(3.9)\quad \int_0^\tau\int_\Omega(-\tilde u_\lambda\,\xi_t + A_\varepsilon(x,w_\lambda,\nabla w_\lambda)\cdot\nabla\xi) &= \int_\Omega \xi(0)\,u_0 + \int_0^\tau\int_\Omega f_\lambda\,\xi \\ &\quad + \int_0^\tau\int_{\Gamma_N}(g_\lambda - \rho(x,w_\lambda))\,\xi\ ,\end{aligned}$$

for any $\xi\in\mathcal{C}^1([0,\tau]\times\overline{\Omega})\cap L^2(0,\tau;V)$, s.t. $\xi(\tau)\equiv 0$. Letting $\lambda\to 0$, we get

$$(3.10)\int_0^\tau\int_\Omega(-u\,\xi_t + A_\varepsilon(x,w,\nabla w)\cdot\nabla\xi) = \int_\Omega\xi(0)\,u_0 + \int_0^\tau\int_\Omega f\,\xi + \int_0^\tau\int_{\Gamma_N}(g-\chi)\,\xi\ .$$

for any $\xi\in\mathcal{C}^1([0,\tau]\times\overline{\Omega})\cap L^2(0,\tau;V)$, s.t. $\xi(\tau)\equiv 0$. To end up the proof of the proposition, it remains to show that $\chi = \rho(x,w)$ a.e. in Σ_N. Since, $\rho(x,w_\lambda)\to\chi$ and $w_\lambda\to w$ in $L^2(\Sigma_N)-$weak, then, by using Minty Lemma, it is enough to prove that

$$\liminf_{\lambda\to 0}\iint_{\Sigma_N}\rho(x,w_\lambda)\,w_\lambda \leq \iint_{\Sigma_N}\xi\,\chi.$$

To this aim, we see that by letting $\lambda\to 0$ in (3.7), we have

$$\begin{aligned}\liminf_{\lambda\to 0}\iint_{\Sigma_N}\rho(x,w_\lambda)\,w_\lambda\ &\leq\ \iint_Q f\,w + \iint_{\Sigma_N} g\,w - \int_\Omega j(u(\tau)) - \iint_Q a_\varepsilon\,|\nabla w|^2 \\ &\quad - \iint_Q\sigma(x,w)\cdot\nabla w + \int_\Omega j(u_0).\end{aligned}$$

On the other hand, since u and w satisfies (3.10), then by using the chain rule Lemma (cf. Lemma 1 of [1]), we deduce that

$$\int_\Omega j(u(\tau)) + \iint_Q a_\varepsilon\, |\nabla w|^2 + \iint_Q \sigma(x,w)\, w = \iint_Q f\, w + \iint_{\Sigma_N} (g-\chi)\, w + \int_\Omega j(u_0),$$

and the result follows. At last, by using again (3.8) and passing to the limit in (3.7), we get (3.4). ∎

In order to treat the problem P_ε with a reaction term $F(t,x,u)$ satisfying (2.5), we will use the general Lemma 1 of [10]. Thus,

Proposition 3 *For any $u_0 \in D$ and $g \in L^1(\Sigma_N)$, there exists a unique $u \in \mathcal{C}([0,T);L^1(\Omega))$ such that $u = S_\varepsilon(u_0, F(.,u(.)), g)$. If moreover, $u_0 \in L^2(\Omega)$ and $g \in L^2(\Sigma_N)$, then there exists w such that (u,w) is a solution of $P_\varepsilon(u_0,F,g)$.*

Proof : Fix $g \in L^1(\Sigma_N)$ and let H be the application from $[0,T)\times X$ into X defined by $H(t,U) = (F(t,.,u_1), g)$ for any $U=(u_1,u_2) \in X$. Thanks to (2.5), $H(.,U) \in L^1(0,T;X)$ for any $U \in X$ and $U \to H(t,U)$ is continuous for a.e. $t \in (0,T)$. Moreover, using again (2.5) we see that $CI - H(t,.)$ is accretive in X. Then (see for instance [10], Lemma 1), for any $u_0 \in D$, there exists a unique mild solution $U=(u,0)$ of

$$\frac{dU}{dt} + \mathcal{A}_\varepsilon U = H(.,U) \text{ on } (0,T), \quad U(0) = U_0 := (u_0, 0). \tag{3.11}$$

Since, by definition of a mild solution of (3.11), U is also the unique mild solution of $CP_\varepsilon(u_0,f,g)$ with $f = F(.,u(.))$, then we deduce that $u = S_\varepsilon(u_0, F(.,u(.)), g)$ and by uniqueness of a mild solution of (3.11) we deduce that u is unique. As to the second part of the proposition is a simple consequence of Proposition 2. ∎

Now, we end up this subsection by proving Proposition 1.

Proof of Proposition 1. Uniqueness : It is clear that (3.2) and the uniqueness follows directly from (3.1). To prove (3.1), we substrate the equations satisfied by (v_1,w_1) and (v_2,w_2). Setting $V = v_1 - v_2$, $W = w_1 - w_2$, $F = f_1 - f_2$, $G = g_1 - g_2$, and taking $\xi = H_\epsilon(w_1 - w_2)$, we get

$$\begin{aligned}\int_\Omega V\, H_\epsilon(W) + \int_\Omega a_\varepsilon |\nabla W|^2 H'_\varepsilon(W) + \int_{\Gamma_N} \Big(\rho(x,w_1) - \rho(x,w_2)\Big) H_\epsilon(W) \\ + \int_\Omega \Big(\sigma(x,w_1) - \sigma(x,w_2)\Big)\cdot \nabla W\, H'_\varepsilon(W) = \int_\Omega F\, H_\epsilon(W) + \int_{\Gamma_N} G\, H_\epsilon(W).\end{aligned}$$

On the other hand, since $\sigma(x,w)$ is Lipschitz in w then

$$\Big|\int_\Omega (\sigma(x,w_1)) - \sigma(x,w_2))\cdot \nabla W H'_\varepsilon(W)\Big| \leq C \int_{[0\leq |W|\leq \varepsilon]} |\nabla W| \quad \longrightarrow 0 \text{ as } \varepsilon \to 0.$$

So, since $(\rho(x,w_1) - \rho(x,w_2))H_\epsilon(W) \geq 0$ on Γ_N, then letting $\varepsilon \to 0$ and using the fact that $(v_1 - v_2)\mathrm{Sign}_0^+(w_1 - w_2) = (v_1 - v_2)^+$, we get (3.1).

Existence : Let $\mathcal{B} : V \to V^*$ be defined by $\Big\langle \mathcal{B}w, \xi \Big\rangle = \int_\Omega \beta(w)\xi + \int_\Omega A_\varepsilon(x,w,\nabla w)\cdot\nabla\xi + \int_{\Gamma_N} (\rho(x,w)-g)\,\xi$. Since β and $\sigma(x,.)$ are Lipschitz, then by Poincaré inequality, it is not difficult to see that $\mathcal{B}$ is bounded and is weakly continuous. Moreover, thanks to (2.2) and (2.7), we have $\int_\Omega \sigma(x,w)\cdot\nabla w \geq 0$, so that, $\dfrac{\Big\langle \mathcal{B}w, w\Big\rangle}{\|w\|_V} \geq m\|w\|_V - C\|g\|_{L^2(\Gamma_N)} \to \infty$ as $\|w\|_V \to \infty$, which implies that $\mathcal{B}$ is coercive. So, thanks to [34] (Cf. Chap. 2, Theorem 2.1 and Remark 2.1), we conclude that for $f \in V^*$ (and in particular for $f \in L^2(\Omega)$), there exists $w \in V$, such that $\Big\langle \mathcal{B}w, \xi\Big\rangle = \Big\langle f, \xi\Big\rangle$ for any $\xi \in V$, which ends up the proof of existence. ∎

3..2 Convergence results : existence for Sh

As said in the introduction, we will construct a solution of the Shadow problem as limit of a solution of the problem P_ε, by letting $\varepsilon \to 0$. This is the aim of the following proposition.

Proposition 4 *For $0 < \varepsilon < \varepsilon_0$, let $f_\varepsilon \in L^2(Q)$, $g_\varepsilon \in L^2(\Sigma_N)$, $u_{0\varepsilon} \in L^2(\Omega)$ satisfying (2.6) and denote by $(u_\varepsilon, w_\varepsilon)$ the solution of $P_\varepsilon(u_{0\varepsilon}, f_\varepsilon, g_\varepsilon)$ given by Proposition 2. Assume that f_ε, g_ε and $u_{0\varepsilon}$ are weakly convergente, respectively in $L^2(Q)$, $L^2(\Sigma_N)$ and $L^2(\Omega)$. Then, there exists $u \in L^2(0,T;L^2_{\Omega_1})$ and $w \in L^2(0,T;H^1_{\Omega_1})$ such that, $u = \beta(w)$ a.e. Q, and by taking subsequences if necessary, we have $u_\varepsilon \to u$ in $L^1(Q)$ and $w_\varepsilon \to w$ in $L^2(0,T;V)-$ weak. Moreover, (u,w) satisfies*

$$(3.12)\quad \int_\Omega \iint_Q \Big(-u\xi_t + A_0(x,w,\nabla w)\cdot\nabla\xi\Big) = u_0\,\xi(0) + \iint_Q f\,\xi + \iint_{\Sigma_N} \Big(g - \rho(.,w)\Big)\,\xi$$

for any $\xi \in \mathcal{C}^1\overline{Q}) \cap L^2(0,T;H^1_{\Omega_1})$ such that $\xi(T) \equiv 0$, where f, g and u_0 are the weak limits respectively of f_ε, g_ε and $u_{0\varepsilon}$.

Proof : Recall that (3.4) with (2.1) and Poincarré inequality implies that

$$(3.13)\qquad \int_\Omega j(u_\varepsilon(t)) + \iint_Q a_\varepsilon\,|\nabla w_\varepsilon|^2 \leq C, \quad \text{a.e. } t \in (0,T),$$

where C is a constant independent of ε, so that w_ε is bounded in $L^2(0,T;V)$, there exists $w \in L^2(0,T;V)$ and a subsequence that we denote again by $\{\varepsilon\}$, such that $w_\varepsilon \to w$ in $L^2(0,T;V)$-weak and in $L^2(\Sigma_N)$-weak . Moreover, since $a_\varepsilon \to a_0$ uniformly in Ω_0, then $a_\varepsilon\,\nabla w_\varepsilon \to a_0\,\nabla w$ in $L^2((0,T)\times\Omega_0)$-weak. On the other hand, since $\inf_{x\in K} a_\varepsilon(x) \int_0^T\int_K |\nabla w_\varepsilon|^2 \leq \int_0^T\int_\Omega a_\varepsilon\,|\nabla w_\varepsilon|^2 \leq C$ and $\inf_{x\in K} a_\varepsilon(x) \to \infty$ as $\varepsilon \to 0$, for any $K \subset\subset \Omega_1$, then

$$\int_0^T\int_K |\nabla w|^2 \leq \liminf_{\varepsilon\to 0} \int_0^T\int_K |\nabla w_\varepsilon|^2 = 0, \qquad \text{for any } K \subset\subset \Omega_1.$$

Since Ω_1 is connected, we conclude that $w(t)$ is constant in Ω_1, for each $t > 0$, so that $w \in L^2(0,T;H^1_{\Omega_1})$.

Now, let us prove that, as $\varepsilon \to 0$, $u_\varepsilon \to \beta(w) =: u$ in $L^1(Q)$. Since for any $k > 0$ and $r \in \mathbb{R}$, $|\beta(r)| \le \frac{1}{k} j(r) + \sup\{-\beta(-k), \beta(k)\}$, then, (3.13) implies that for any measurable set $\tilde{Q} \subseteq Q$ and $k > 0$, we have

$$\iint_{\tilde{Q}} |u_\varepsilon| \le \frac{C}{k} + |\tilde{Q}| \, \sup\{-\beta(-k), \beta(k)\},$$

which implies that $\{u_\varepsilon\}$ is equi-integrable in $L^1(Q)$. On the other hand, since β is Lipschitz, $u_\varepsilon = \beta(w_\varepsilon)$ and w_ε is bounded in $L^2(0,T;V)$, then $\{u_\varepsilon\}$ is also bounded in $L^2(0,T;V)$ and $\lim_{h\to 0} \int_0^T \int_\omega |u_\varepsilon(t,x+h) - u_\varepsilon(t,x)| = 0$, for each $\omega \subset\subset \Omega$. So, thanks to Theorem 2 of [33], we deduce that $\{u_\varepsilon\}$ is relatively compact in $L^1(Q)$. Then, there exists $u \in L^1(Q)$, such that $u_\varepsilon \to u$ in $L^1(Q)$, and by monotonicity argument we deduce that $u = \beta(w)$ a.e. in Q. This ends up the proof of the first part of the proposition. To prove that (v,w) satisfies (3.12), we consider ξ as in the statement of (3.12) and take it as a test function in (2.8), i.e.

$$\iint_Q \Big(-u\, \xi_t + A_\varepsilon(x, w_\varepsilon, \nabla w_\varepsilon) \cdot \nabla \xi \Big) = \int_\Omega u_{0\varepsilon}\, \xi(0) + \iint_Q f_\varepsilon\, \xi + \iint_{\Sigma_N} \Big(g_\varepsilon - \rho(., w_\varepsilon) \Big)\, \xi.$$

Obviously, $\int_0^T \int_\Omega A_\varepsilon(x, w_\varepsilon, \nabla w_\varepsilon) \cdot \nabla \xi = \int_0^T \int_{\Omega_0} A_\varepsilon(x, w_\varepsilon, \nabla w_\varepsilon) \cdot \nabla \xi$, so, using (2.3) and letting $\varepsilon \to 0$, we get $\iint_Q \Big(-u\xi_t + A_0(x,w,\nabla w) \cdot \nabla \xi \Big) = \int_\Omega u_0 \xi(0) + \iint_Q f\xi + \iint_{\Sigma_N} \Big(g - \chi \Big) \xi$, where χ is a weak limit in $L^2(\Sigma_N)$ of $\rho(x, w_\varepsilon)$. By using monotonicity arguments (Minty Lemma) exactly in the same way as in the proof of Proposition 2, we deduce that $\chi = \rho(x,w)$ a.e. in Σ_N, and complete the proof of the proposition. ∎

Now, in the concern of showing uniqueness (cf. Section 4) and exhibiting the singular limit phenomena, we improve the convergence by using nonlinear semigroup theory. We will be interest in the limit, as $\varepsilon \to 0$, of the semigroup generated by $\mathcal{A}_\varepsilon$. For this, let us introduce the elliptic problem associated with $Sh(u_0, f, g)$, that is

$$S_0(f,g) \qquad \begin{cases} v - \nabla \cdot A_0(x, w, \nabla w) = f, \quad v = \beta(w) & \text{in } \Omega_0 \\ w \text{ is constant} & \text{in } \Omega_1 \\ \partial_{\overrightarrow{n_0}} w + \rho(x,w) = g & \text{on } \Gamma_N \cap \Gamma_0 \\ w = 0 & \text{on } \Gamma_D. \end{cases}$$

Proposition 5 *Let $f \in L^2(\Omega)$, $g \in L^2(\Gamma_N)$, $(v_\varepsilon, w_\varepsilon)$ a solution of $St_\varepsilon(f,g)$. As, $\varepsilon \to 0$, we have $v_\varepsilon \to v$ in $L^1(\Omega)$ and, by taking subsequences if necessary, $w_\varepsilon \to w$ in $H^1(\Omega)-$weak. The couple (v,w) solves $S_0(f,g)$ in the following sense : $v \in L^2(\Omega)$, $w \in H^1_{\Omega_1}$, $v = \beta(w)$ a.e. Ω and $\int_\Omega (v - f)\, \xi + \int_\Omega A_0(x, w, \nabla w) \cdot \nabla \xi = \int_\Gamma (g - \rho(x,w))\, \xi$, for any $\xi \in H^1_{\Omega_1}$. Moreover, v is unique.*

Proof : The proof of uniqueness follows in the same way for the problem $St_\varepsilon(f,g)$. Indeed, subtracting the equations satisfying by two solutions (v_1, w_1) and (v_2, w_2), one sees that $\xi = H_\epsilon(w_1 - w_2)$ is an admissible test function and the proof follows in in the same way of Proposition 1. To prove the convergence of $(v_\varepsilon, w_\varepsilon)$ to (v, w), observe that $(v_\varepsilon, w_\varepsilon)$ is also a solution of $P_\varepsilon(v_\varepsilon, f - v_\varepsilon, g)$, so that, by using (3.4), we deduce that w_ε is bounded in $H^1(\Omega)$ and, since β is Lipschitz, then v_ε is bounded in $L^2(\Omega)$. So, applying Proposition 4 the convergence results follow. ∎

Applying Proposition 5, we define in X the limiting operator $\mathcal{A}_0$, by $(h, g) \in \mathcal{A}_0(u, 0)$ if and only if

$$\begin{cases} u \in L^2_{\Omega_1},\ h \in L^2(\Omega),\ g \in L^2(\Gamma_N),\ \exists\, w \in H^1_{\Omega_1},\ u = \beta(w) \text{ a.e. } \Omega, \\ \displaystyle\int_\Omega A_0(x, w, \nabla w) \cdot \nabla \xi = \int_\Omega h\, \xi + \int_{\Gamma_N} (g - \rho(x, w))\, \xi,\ \forall\, \xi \in H^1_{\Omega_1}. \end{cases}$$

Denoting by J^ε_λ (resp. J^0_λ) the resolvent of $\mathcal{A}_\varepsilon$ (resp. $\mathcal{A}_0$), we have

Corollary 3 *As, $\varepsilon \to 0$, $J^\varepsilon_\lambda(f, g) \to J^0_\lambda(f, g)$ for any $f \in L^2(\Omega)$, $g \in L^2(\Gamma_N)$ and $\lambda > 0$.*

This means that the operator $\mathcal{A}_\varepsilon$ converges to the operator $\mathcal{A}_0$ in the sense of resolvent. So, $\mathcal{A}_0$ is T-accretive and we have $\mathcal{R}(I + \lambda\, \mathcal{A}_0) \supseteq L^2(\Omega) \times L^2(\Gamma_N)$, which implies that $\mathcal{A}_0$ generates a nonlinear semigroup of order preserving contractions in X. Moreover, we have

Lemma 2 $\overline{\mathcal{D}(\mathcal{A}_0)} = \left(D \cap L^1_{\Omega_1}\right) \times \left\{0\right\}$.

Proof : We follow the same idea of the proof of Lemma 1. By density and the definition of $\mathcal{A}_0$ we have $\overline{\mathcal{D}(\mathcal{A}_0)} \subseteq \left(D \cap L^1_{\Omega_1}\right) \times \left\{0\right\}$. To prove that $\left(D \cap L^1_{\Omega_1}\right) \times \left\{0\right\} \subseteq \overline{\mathcal{D}(\mathcal{A}_0)}$, it is enough to prove that $\left(D \cap L^\infty_{\Omega_1}\right) \times \left\{0\right\} \subseteq \overline{\mathcal{D}(\mathcal{A}_0)}$. So, let $u \in D \cap L^\infty_{\Omega_1}$ and consider (u_λ, w_λ) the solution of

$$\begin{cases} u_\lambda - \lambda\, \nabla \cdot A_0(x, w_\lambda \nabla w_\lambda) = u, \quad u_\lambda = \beta(w_\lambda) & \text{in } \Omega \\ w \text{ is constant} & \text{in } \Omega_1 \\ \partial_{\overrightarrow{n_0}} w + \rho(x, w) = 0 & \text{on } \Gamma_N \cap \Gamma_0 \\ w_\lambda = 0 & \text{on } \Gamma_D. \end{cases}$$

Since $u \in \overline{Im(\beta)}$ a.e. $x \in \Omega$, and (cf. Proposition 5) u_λ is obtained as limit of a solution of $St_\varepsilon(f, 0)$, as $\varepsilon \to 0$, then we deduce that $\lambda\, w_\lambda$ is bounded in V, $\|u_\lambda\|_{L^\infty(\Omega)} \leq \|u\|_{L^\infty(\Omega)}$ and $\|u_\lambda\|_{L^2(\Omega)} \leq \|u\|_{L^2(\Omega)}$. And, the convergence $u_\lambda \to u$ in $L^1(\Omega)$ follows exactelly in the same way of Lemma 1. ∎

As in the previous subsection, considering the Cauchy problem in X

$$CP_0(u_0, f, g) \qquad \begin{cases} U_t + \mathcal{A}_0\, U = (f, g) & \text{in } (0, T) \\ U(0) = (u_0, 0) =: U_0, \end{cases}$$

we have

Corollary 4 *1. For any $u_0 \in D \cap L^1_{\Omega_1}$, $f \in L^1(Q)$ and $g \in L^1(\Sigma_N)$, $CP_\varepsilon(u_0, f, g)$ has a unique mild solution $U = (u, 0)$. Moreover, we may define a mapping $S_0 : (u_0, f, g) \in \left(D \cap L^1_{\Omega_1}\right) \times X \to u \in \mathcal{C}([0,T); L^1_{\Omega_1})$ such that the L^1-comparison principle holds.*

2. For any $u_0 \in D \cap L^1_{\Omega_1}$ and $g \in L^1(\Sigma_N)$, there exists a unique $u \in \mathcal{C}([0,T); L^1_{\Omega_1})$ such that $u = S_0(u_0, F(., u(.)), g)$.

Now, applying Corollary 3 with classical theorems for regular semigroup perturbation (cf. [17] and also [10]), we have

Corollary 5 *1. For $\varepsilon > 0$, let $u_{0\varepsilon} \in D$, $g_\varepsilon \in L^1(\Sigma_N)$ and $u_\varepsilon = S_\varepsilon(u_{0\varepsilon}, F(., u_\varepsilon(.)), g_\varepsilon)$. If, letting $\varepsilon \to 0$, $g_\varepsilon \to g$ in $L^1(\Sigma_N)$ and $u_{0\varepsilon} \to u_0$ in $L^1(\Omega)$, with $u_0 \in L^1_{\Omega_1}$, then $u_\varepsilon \to u$ in $\mathcal{C}([0,T); L^1(\Omega))$, where $u = S_0(u_0, F(., u(.)), g)$.*

2. If, moreover, $u_{0\varepsilon} \in D \cap L^2_{\Omega_1}$, $g_\varepsilon \in L^2(\Sigma_N)$ and w_ε is the function given by Proposition 2, then, by taking subsequence if necessary, we have $w_\varepsilon \to w$ in for any $\xi \in H^1_{\Omega_1} L^2(0,T;V)$-weak and (u, w) is a solution of $Sh(u_0, F, g)$.

4. Uniqueness

To prove uniqueness we use the concept of integral (sub/super) solution, which is well known in the context of the abstract Cauchy problem (cf. [6], [4] and [8]). This concept was previously used in [12] for the proof of uniqueness of weak solution for elliptic-parabolic problems, with homogeneous Dirichlet boundary conditions. Before to give a definition of this notion in our context, let us first introduce the following stationary problems associated with P_ε and Sh respectively :

$$(4.1) \qquad \begin{cases} -\nabla \cdot A_\varepsilon(x, \eta, \nabla \eta) = h & \text{in } \Omega \\ \partial_{\overrightarrow{n}_\varepsilon} \eta + \sigma(x, \eta) = l & \text{on } \Gamma_N \\ \eta = 0 & \text{on } \Gamma_D \end{cases}$$

and

$$(4.2) \qquad \begin{cases} -\nabla \cdot A_0(x, \eta, \nabla \eta) = h & \text{in } \Omega_0 \\ \eta \text{ is constant} & \text{in } \Omega_1 \\ \partial_{\overrightarrow{n}_0} \eta + \rho(x, \eta) = l & \text{on } \Gamma_N \cap \Gamma_0 \\ \eta = 0 & \text{on } \Gamma_D. \end{cases}$$

For $h \in L^2(\Omega)$ and $l \in L^2(\Gamma_N)$ we say that η is a solution of (4.1) (resp. of (4.2)), if $\eta \in V$ (resp. $\eta \in H^1_{\Omega_1}$) and $\int_\Omega A_\varepsilon(x, \eta, \nabla\eta) \cdot \nabla\xi = \int_{\Gamma_N} (l - \rho(x, \eta))\, \xi + \int_\Omega h\, \xi$ for any $\xi \in V$ (resp. $\int_{\Omega_1} A_0(x, \eta, \nabla\eta) \cdot \nabla\xi = \int_{\Gamma_N} (l - \rho(x, \eta))\, \xi + \int_\Omega h\, \xi$ for any $\xi \in H^1_{\Omega_1}$.)

Definition 3 *Let $u_0 \in D$ (resp. $u_0 \in D \cap L^1_{\Omega_1}$), $f \in L^1(Q)$ and $g \in L^1(\Sigma_N)$.*

i) *An integral subsolution of* $P_\varepsilon(u_0,f,g)$ *(resp.* $Sh(u_0,f,g)$*) is a function* $u\in L^1(Q)$ *(resp.* $u\in L^1(0,T;L^1_{\Omega_1})$*), such that* ess-$\lim_{t\to 0}\|(u(t)-u_0)^+\|_{L^1(\Omega)}=0$, *and moreover, for any* $h\in L^2(\Omega)$, $l\in L^2(\Gamma_N)$ *and* η *solution of (4.1) (resp. (4.2)), we have*

$$\frac{d}{dt}\int_\Omega (u(t)-\beta(\eta))^+ \le \int_{[u(t)>\beta(\eta)]}(f-h)+\int_{[u(t)=\beta(\eta)]}(f-h)^+ + \int_{\Gamma_N}(g-l)^+ \quad \text{in } \mathcal{D}'(0,T). \tag{4.3}$$

ii) *An integral supersolution of* $P_\varepsilon(u_0,f,g)$ *(resp.* $Sh(u_0,f,g)$*) is a function* $u\in L^1(Q)$ *(resp.* $u\in L^1(0,T;L^1_{\Omega_1})$*), such that* ess-$\lim_{t\to 0}\|(u_0-u(t))^+\|_{L^1(\Omega)}=0$, *and, moreover, for any* $h\in L^2(\Omega)$, $l\in L^2(\Gamma_N)$ *and* η *solution of (4.1) (resp. (4.2)), we have*

$$\frac{d}{dt}\int_\Omega (\beta(\eta)-u(t))^+ \le \int_{[\beta(\eta)>u(t)]}(h-f)+\int_{[u(t)=\beta(\eta)]}(h-f)^+ + \int_{\Gamma_N}(l-g)^+ \quad \text{in } \mathcal{D}'(0,T). \tag{4.4}$$

iii) *A function* $u\in L^1(Q)$ *(resp.* $u\in L^1(0,T;L^1_{\Omega_1})$*) is called an integral solution of* $P_\varepsilon(u_0,f,g)$ *(resp.* $Sh(u_0,f,g)$*), if* u *is an integral subsolution and also an integral supersolution.*

First, let us prove the uniqueness of integral solutions. This is the aim of the following proposition.

Proposition 6 *Let* $u_0\in D$ *(resp.* $u_0\in D\cap L^1_{\Omega_1}$*),* $f\in L^1(Q)$ *and* $g\in L^1(\Sigma_N)$*. Suppose that* u *is an integral subsolution of* $P_\varepsilon(u_0,f,g)$ *(resp.* $Sh(u_0,f,g)$*) and* $\hat u$ *is an integral supersolution of* $P_\varepsilon(\tilde u_0,\tilde f,\tilde g)$ *(resp.* $Sh(\tilde u_0,\tilde f,\tilde g)$*). Then* u *and* $\tilde u$ *satisfy the comparison principle; i.e.*

$$\frac{d}{dt}\int_\Omega \Big(u(t)-\tilde u(t)\Big)^+ \le \int_{[u(t)>\tilde u(t)]}\Big(f(t)-\tilde f(t)\Big) + \int_{[u(t)=\tilde u(t)]}\Big(f(t)-\tilde f(t)\Big)^+ + \int_{\Gamma_N}\Big(g(t)-\tilde g(t)\Big)^+ \tag{4.5}$$

in $\mathcal{D}'(0,T)$*. In particular there is uniqueness of an integral solution of* $P_\varepsilon(u_0,f,g)$ *(resp.* $Sh(u_0,f,g)$*).*

Proof : This proposition follows in the same way of Proposition 4.3 of [12]. For completeness, let us give the arguments. It is clear that the uniqueness assertion follows by (4.5). To prove (4.5), recall from [4] that an integral subsolution of $CP_\varepsilon(u_0,f,g)$, with $u_0\in D$, $u_0\in L^1_{\Omega_1}$ if $\varepsilon=0$, and $(f,g)\in L^1(0,T;X)$ is $U=(u,0)$ such that $u\in L^1(Q)$, $u\in L^1(0,T;L^1_{\Omega_1})$ if $\varepsilon=0$, ess-$\lim_{t\to 0}\|(u(t)-u_0)^+\|_{L^1(\Omega)}=0$ and, moreover,

$$\frac{d}{dt}\int_\Omega \Big(u(t)-z\Big)^+ \le \int_{[u(t)>z]}(f-h)+\int_{[u(t)=z]}(f-h)^+ + \int_{\Gamma_N}(g-l)^+ \tag{4.6}$$

in $\mathcal{D}'(0,T)$, for any $(z,h,l) \in L^1(\Omega) \times X$ such that $(h,l) \in \overline{\mathcal{A}_\varepsilon}(z,0)$. Now, one sees easily, that Definition 3.i) implies that (4.6) is satified for any $(h,l) \in \mathcal{A}_\varepsilon(z,0)$. On the other hand, recall from section 2 that $\overline{\mathcal{A}_\varepsilon}$ is m-accretive in X and since the notion of integral solution is invariant under the closure of the operator, then we deduce that (4.6) is satisfied for any $(h,l) \in \overline{\mathcal{A}}_\varepsilon(z,0)$. So, assume for instance, that $\varepsilon > 0$ and let u (resp. $\hat{u}$) be an integral subsolution (resp. supersolution) of $P_\varepsilon(u_0,f,g)$ (resp. $P_\varepsilon(\tilde{u}_0,\tilde{f},\tilde{g})$). Thanks to the first part of the proof, $U = (u,0)$ (resp. $\hat{U} = (\hat{u},0)$) is an integral subsolution (resp. supersolution) in the sense of [4] of the corresponding Cauchy problem $CP_\varepsilon(u_0,f,g)$ (resp. $CP_\varepsilon(\tilde{u}_0,\tilde{f},\tilde{g})$) with the operator $\overline{\mathcal{A}_\varepsilon}$. Thus estimates (4.3) follows directly from Theorem 3 of [4]. The proof remains the same if $\varepsilon = 0$. ∎

Now, it is clear that uniqueness of u such that (u,w) is a weak solution of $P_\varepsilon(u_0,f,g)$ or $Sh(u_0,f,g)$ follows by proving that u is an integral solution.

Proposition 7 *Let $u_0 \in L^2(\Omega) \cap D$ (resp. $u_0 \in L^2_{\Omega_1} \cap D$), $f \in L^2(Q)$ and $g \in L^2(\Sigma_N)$. If $u \in L^2(Q)$ (resp. $u \in L^2(0,T;L^2_{\Omega_1})$)) is such that there exists w such that (u,w) is a solution of $P_\varepsilon(u_0,f,g)$ (resp. $Sh(u_0,f,g)$) in the sense of (2.8) (resp. (2.10)), then u is an integral subsolution of $P_\varepsilon(u_0,f,g)$, (resp. $Sh(u_0,f,g)$).*

Proof : We will give the proof for the problem P_ε, with $\varepsilon > 0$, as to the problem Sh the proof follows exactly in the same way. Let h, l and η be as in the statement of Definition 3 and let $\xi \in \mathcal{D}(-\infty,T)$ with $\xi \geq 0$. First, let us prove that

$$
\begin{aligned}
&-\iint_Q \xi_t \int_{w_0}^{w} H_\epsilon(s-\eta)\, d\beta(s) + \iint_Q \xi\, A_\varepsilon(x,w,\nabla w).\nabla H_\varepsilon(w-\eta) \\
&\quad \leq \iint_Q f\,\xi\, H_\epsilon(w-\eta) + \iint_{\Sigma_N} (g-\sigma(.,w))\,\xi\, H_\epsilon(w-\eta),
\end{aligned} \tag{4.7}
$$

where w_0 is a mesurable function in Ω such that $u_0 = \beta(w_0)$ a.e. in Ω. To this end, for $\delta > 0$, we consider $\psi^\delta(t) = \dfrac{1}{\delta}\displaystyle\int_t^{t+\delta} H_\epsilon(w(s)-\eta)\xi(s)ds$, a.e. Ω, where we extend w onto $\mathbb{R} \times \Omega$ by 0 if $t > T$, by w_0 if $t < 0$. It is clear that ψ^δ is an admissible test function and

$$
\iint_Q \Big(\beta(w) - u_0\Big)\psi_t^\delta = \iint_Q \xi H_\epsilon(w(t)-\eta)\frac{\beta(w(t-\delta) - \beta(w(t))}{\delta}.
$$

So, since $H_\epsilon(p-\eta)\Big(\beta(p)-\beta(q)\Big) \leq \displaystyle\int_p^q H_\epsilon(r-\eta)\, d\beta(r)$ for any $p,\ q\ \in \mathbb{R}$, then we deduce that

$$
\begin{aligned}
&\iint_Q \xi\, A_\varepsilon(x,w,\nabla w).\nabla \psi^\delta - \iint_Q f\,\xi\,\psi^\delta - \iint_{\Sigma_N} (g-\sigma(.,w))\,\xi\,\psi^\delta \\
&\qquad \leq \iint_Q \frac{\xi(t+\delta)-\xi(t)}{\delta} \int_{w_0}^{w(t)} H_\epsilon(r-\eta)\, d\beta(r).
\end{aligned}
$$

Letting $\delta \to 0$ and using the fact that $\psi^\delta \to H_\epsilon(w-\eta)$ in $L^2(0,T;V)$, (4.7) follows. Now, it is clear that, as $\epsilon \to 0$, $\displaystyle\int_\eta^{w(t,x)} H_\epsilon(s-\eta)\, d\beta(s) \to \Big(\beta(w(t,x)) - \beta(\eta)\Big)^+$ a.e.

$(t,x) \in Q$, so that by Lebesgue's dominated convergence theorem, the first term in (4.7) converges to $\iint_Q \Big((u(t,x) - \beta(\eta))^+ - (u_0 - \beta(\eta))^+ \Big) \xi_t$. Obviously, $\iint_Q f\,\xi\, H_\epsilon(w-\eta)$ and $\iint_{\Sigma_N} (g - \sigma(.,w))\xi H_\epsilon(w-\eta)$ converges, respectively, to $\iint_Q f \xi \mathrm{Sign}_0^+(w-\eta)$ and $\iint_{\Sigma_N} (g - \sigma(.,w))\xi \mathrm{Sign}_0^+(w-\eta)$. As to the second term, note that using the definition of a solution of (4.1), we have $\iint_Q \xi\, A_\varepsilon(x,w,\nabla w).\nabla H_\epsilon(w-\eta) = I_\epsilon^1 + I_\epsilon^2$, with $I_\varepsilon^1 = \iint_Q \xi \Big(A_\varepsilon(x,w,\nabla w) - A_\varepsilon(x,\eta,\nabla\eta)\Big) \cdot \nabla H_\epsilon(w-\eta)$ and $I_\epsilon^2 = \iint_Q \xi\, h\, H_\epsilon(w-\eta) + \iint_{\Sigma_N} (l - \rho(x,\eta))\,\xi\, h\, H_\epsilon(w-\eta)$. Clearly, I_ϵ^2 converges to $\iint_Q \xi\, h\, \mathrm{Sign}_0^+(w-\eta) + \iint_{\Sigma_N} \xi\, \mathrm{Sign}_0^+(w-\eta)\,(l - \sigma(x,\eta))$. On the other hand, $\liminf_{\epsilon\to 0} I_\epsilon^1 \geq \lim_{\epsilon \to 0} \iint_Q \xi\, H'_\epsilon(w-\eta)\,(\sigma(x,w) - \sigma(x,\eta)) \cdot \nabla\,(w-\eta) \quad = 0$. So, letting $\epsilon \to 0$ in (4.7), we get

$$
\begin{aligned}
&- \iint_Q \Big((u-\beta(\eta))^+ - (u_0 - \beta(\eta))^+ \Big)\xi_t + \iint_{\Sigma_N} (\rho(.,w) - \rho(.,\eta))^+ \xi \\
&\quad \leq \iint_Q \xi\, \mathrm{Sign}_0^+(w-\eta)\,(f-h) + \iint_{\Sigma_N} \xi\, \mathrm{Sign}_0^+(w-\eta)\,(g - \tilde{g}) \\
&\quad \leq \int_0^T \Big\{ \int_{[u(t)>\beta(\eta)]} \xi\,(f-h) + \int_{[u(t)=\beta(\eta)]} \xi\,(f-h)^+ \Big\} + \iint_{\Sigma_N} (g-l)^+ .
\end{aligned} \tag{4.8}
$$

As $\xi \in \mathcal{D}(-\infty, T)$ was arbitrary, it follows that (4.3) holds. Now, note that (4.8) implies that for any $t \in [0,T)$, we have

$$
\int_\Omega (u(t) - z)^+ \leq \int_\Omega (u_0 - z)^+ + \int_0^t \int_{[u(t)>z]} (f-h) + \int_0^t \int_{[u(t)=z]} (f-h)^+ + \int_0^t \int_{\Gamma_N} (g-l)^+
$$

for any $z \in D$ and $(h,l) \in \overline{\mathcal{A}_\varepsilon}(z,0)$. A consequence, ess-$\lim_{t\to 0} \int_\Omega (u(t)-z)^+ \leq \int_\Omega (u_0 - z)^+$, for any $z \in D$ and $u_0 \in D$ implies that ess-$\lim_{t\to 0} \|(u(t) - u_0)^+\|_{L^1(\Omega)} = 0$. This ends up the proof of the proposition. ∎

Corollary 6 *Let $u_0 \in D$ (resp. $u_0 \in D \cap L^1_{\Omega_1}$) and $g \in L^2(\Sigma_N)$. There is uniqueness of u such that there exists w such that (u,w) satisfies (2.8) (resp. (2.10)).*

Proof : If u_1 and u_2 are such that (u_1, w_1) and (u_2, w_2) satisfy (2.8), then (u_1, w_1) and (u_2, w_2) are two solutions of $P_\varepsilon(u_0, f_1, g)$ and $P_\varepsilon(u_0, f_2, g)$ respectively with $f_1 = F(., u_1(.))$ and $f_2 = F(., u_2(.))$. So, applying Proposition 7 and Proposition 6, we deduce that

$$
\int_\Omega (u_1(t) - u_2(t))^+ \leq \int_0^t \int_{[u_1 \geq u_2]} (F(.,u_1) - F(.,u_2)) \leq C \int_0^t \int_\Omega (u_1 - u_2)^+,
$$

which implies, by Granwall, that $u_1 \leq u_2$. In the same way we can prove that $u_2 \leq u_1$, and deduce that $u_1 = u_2$. The uniqueness of u such that (u,w) satisfies (2.10) in the case where $u_0 \in D \cap L^1_{\Omega_1}$ follows in the same way. ∎

5. Proofs of the main Theorems

Proof of Theorem 1 : Now, it is clear that the existence of (u,w) satisfying (2.8) follows by Proposition 4 and the uniqueness of u follows by Corollary 6. On the other hand, since by Proposition 4, $u = S_\varepsilon(u_0, F(., u(.)), g)$, then $u \in \mathcal{C}([0,T); L^1(\Omega))$, $u(0) = u_0$ and the comparaison principle is satisfied. ∎

Proof of Theorem 2 : The existence assertion of the theorem follows by Corollary 5. As to the uniqueness and the comparaison principle, recall that a solution u of $Sh(u_0, F, g)$ is also a solution of $Sh(u_{0\Omega_1}, F, g)$, which is unique by Corollary 6. Thanks to Corollary 5, u is given by $u = S_0(u_{0\Omega_1}, F(., u(.)), g)$, so that $u \in \mathcal{C}([0,T); L^1(\Omega))$, $u(0) = u_{0\Omega_1}$ and the comparaison principle is satisfied. ∎

Proof of Theorem 3 : Assume that $u_0 \in L^1_{\Omega_1}$. Since $u_\varepsilon = S_\varepsilon(u_0, F(., u_\varepsilon), g)$ and $u = S_0(u_0, F(., u(.)), g)$ then the theorem is a simple consequence of Corollary 5.
Now, in order to prove the theorem in the case where $u_0 \notin L^1_{\Omega_1}$, we begin by to treat the case $F \equiv 0$. In this case $u_\varepsilon = S_\varepsilon(u_0, 0, g)$ and, for any $t \geq \tau > 0$, we have $u_\varepsilon(t) = S_\varepsilon(u_\varepsilon(\tau), 0, g(.+\tau))(t-\tau)$. Applying Proposition 4, we know that there exists $\varepsilon_k \to 0$, such that, for a fixed $\tau > 0$, $u_{\varepsilon_k}(\tau) \to S_0(u_{0\Omega_1}, 0, g(.+\tau))(\tau)$ in $L^1(\Omega)$. Then, applying the first part of the proof in (τ, T), we deduce that $u_\varepsilon \to S_0(S_0(u_{0\Omega_1}, 0, g(.+\tau))(\tau), 0, g)(.-\tau) = S_0(u_{0\Omega_1}, 0, g)$ in $\mathcal{C}([\tau, T); L^1(\Omega))$. Since $\tau > 0$ is arbitrary, then the results of the Theorem follows in the case $F \equiv 0$. To end up the proof, assume that $F \not\equiv 0$. For fixed $\tau > 0$, we set $F_\tau(t) = F(t)\,\chi_{(\tau,T)}(t)$ for $t \in (0,T)$, and we consider $z^\tau_\varepsilon = S_\varepsilon(u_0, F_\tau(., z^\tau_\varepsilon, g)$ and $z^\tau = S_0(u_{0\Omega_1}, F_\tau(., z^\tau), g)$. Using the fact that $u_\varepsilon = S_\varepsilon(u_0, F(., u_\varepsilon), g)$, Corollary 2 and (2.5) implies that

$$\int_\Omega |u_\varepsilon(t) - z^\tau_\varepsilon(t)| \leq \int_0^t \int_\Omega \Big(K_2\, \chi_{(0,\tau)} + K|u_\varepsilon - z^\tau_\varepsilon|\, \chi_{(\tau,T)} \Big),$$

so that, by Granwall, we get $\int_\Omega |u_\varepsilon(t) - z^\tau_\varepsilon(t)| \leq e^{\int_\tau^t K} \int_0^\tau \int_\Omega K_2$ for any $t \in (\tau, T)$. In the same way, since $u = S_0(u_{0\Omega_1}, F(., u(.), g)$, then $\int_\Omega |u(t) - z^\tau(t)| \leq e^{\int_\tau^t K} \int_0^\tau \int_\Omega K_2$ for any $t \in (\tau, T)$. On the other hand, since $z^\tau_\varepsilon(\tau) = S_\varepsilon(u_0, 0, g)(\tau)$ and $z^\tau(\tau) = S_0(u_{0\Omega_1}, 0, g)(\tau)$, then using the second part of the proof we have $z^\tau_\varepsilon(\tau) \to z^\tau(\tau)$ in $L^1(\Omega)$, and, by regular semigroup perturbation thereoms, we deduce that $\sup_{t\in[\tau,T)} \int_\Omega |z_\varepsilon(t) - z^\tau(t)| \to 0$. So, $\lim_{\varepsilon\to 0} \sup_{t\in[t_1,t_2]} \int_\Omega |u_\varepsilon(t) - u(t)| \leq 2\, e^{\int_\tau^t K} \int_0^\tau \int_\Omega K_2$, for any $0 < \tau \leq t_1 < t_2 < T$, and the convergence (2.11) follows. For the convergence of w_ε, this follows by Proposition 4. ∎

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