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Singular limit of changing sign solutions of the porous medium equation

NOUREDDINE IGBIDA AND PHILIPPE BENILAN

Abstract. In this paper, we study the limit as $m \to \infty$ of changing sign solutions of the porous medium equation: $u_t = \Delta : |u|^{m-1}u$ in a domain Ω of $\mathbb{R}^N$, with Dirichlet boundary condition.

1. Introduction and main results

We consider the initial boundary value problem:

$$\begin{cases} u_t = \Delta\ |u|^{m-1}u & \text{in } Q := (0,\infty)\times\Omega \\ u = 0 & \text{on } \Sigma := (0,\infty)\times\Gamma \\ u(0) = u_0 \end{cases} \tag{$\mathrm{P_m}$}$$

where Ω is an open domain of $\mathbb{R}^N$ not necessarily bounded, $m \geq 1$ and $u_0 \in L^1(\Omega)$. Throughout the paper we will use the notation r^m for $|r|^{m-1}r$, for any $r \in \mathbb{R}$. It is well known by now that $(\mathrm{P_m})$ has a unique strong solution u, that is $u \in \mathcal{C}([0,\infty); L^1(\Omega)) \cap L^\infty((\delta,\infty)\times\Omega) \cap W^{1,1}(\delta,\infty; L^1(\Omega))$, $u^m \in L^2(\delta,\infty; H_0^1(\Omega))$, for any $\delta > 0$, $\partial_t u = \Delta u^m$ in $\mathcal{D}'(Q)$ and $u(0) = u_0$ a.e. in Ω. Let us denote this solution by u_m. We are interested in the asymptotic behaviour of u_m, as $m \to \infty$. If $\|u_0\|_{L^\infty(\Omega)} \leq 1$, it is known (cf. [5] and [6]) that

$$u_m \to u_0 \quad \text{in } \mathcal{C}([0,\infty), L^1(\Omega)).$$

But, if $\|u_0\|_{L^\infty(\Omega)} > 1$, then one can prove that u_m is relatively compact in $\mathcal{C}((0,\infty), L^1(\Omega))$, but not in $\mathcal{C}([0,\infty), L^1(\Omega))$, an initial boundary layer appears at $t = 0$ when passing to the limit: the limit is singular. Indeed, since the nonlinearity $\varphi_m(r) = r^m$ in the equation $(\mathrm{P_m})$ converges in the graph sense to the maximal monotone graph φ_∞ given by

$$\varphi_\infty(r) = \begin{cases} 0 & \text{if } |r| < 1 \\ \pm[0,\infty) & \text{if } r = \pm 1, \end{cases}$$

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then, the limiting equation is

$$u_t = \Delta w, \quad w \in \varphi_\infty(u) \quad \text{in} Q \tag{1.1}$$

for which a solution u satisfies $|u| \leq 1$ and compatible initial data should live in $[-1, 1]$.

Our aim in this paper is to describe the limit of u_m, when the initial data $\|u_0\|_{L^\infty(\Omega)} > 1$. This kind of question attracts much attention, by its physical interest, since for large m, $(\mathrm{P_m})$ appears in a variety of physical problems, for instance $m = 3$ for the spreading of a liquid film under gravity [23] and semiconductor fabrication [22] and $m \in (5.5, 6.5)$ in a radiation in ionized gazes [25], and also by its mathematical interest in the study of singular limits of linear and nonlinear semigroups (cf. [2] and [9]). Indeed, in a Banach space X, let us consider a family of m-accretive operator A_n, such that $\overline{\mathcal{D}(A_n)} = X$ and, as $n \to \infty$, $A_n \to A$ in the sense of resolvent with $\overline{\mathcal{D}(A)} \neq X$. For any $u_0 \in X$, the Cauchy problem

$$u_t + A_n u \ni 0 \quad \text{in } (0, \infty), \quad u(0) = u_0$$

has a unique mild solution u_n given by the Crandall-Ligget exponential formula

$$u_n(t) = L^1 - \lim_{k\to\infty} \left(I + \frac{t}{k} A_n\right)^{-k} u_0 \quad =: e^{-tA_n} u_0.$$

Letting $n \to \infty$, it is known that (cf. [11]) if $u_0 \in \overline{\mathcal{D}(A)}$, then $u_n \to u$ in $\mathcal{C}([0, \infty), X)$ and u is the mild solution of

$$u_t + Au \ni 0 \quad \text{in } (0, \infty), \quad u(0) = u_0. \tag{1.2}$$

But, if $u_0 \in X \setminus \overline{\mathcal{D}(A)}$, then (1.2) is not well posed and in general the limit of u_n may not exist. However for a large class of concrete problems the limit exists and it would be interesting to characterize it. In this case, a conjecture is that there exists $\underline{u}_0 \in \overline{\mathcal{D}(A)}$, such that the limit u is the solution of

$$u_t + Au \ni 0 \quad \text{in}(0, \infty), \quad u(0) = \underline{u}_0,$$

but the characterization of $\underline{u}_0$ is not clear yet in general.

Coming back to the problem $(\mathrm{P_m})$, $X = L^1(\Omega)$, and the family of operators A_m is given by

$$A_m u = -\Delta u^m \quad \text{in } \mathcal{D}'(\Omega) \tag{1.3}$$

with

$$\mathcal{D}(A_m) = \{\, z \in L^1(\Omega) \cap L^\infty(\Omega) \ ; \ z^m \in H_0^1(\Omega) \text{and } \Delta z^m \in L^1(\Omega)\}.$$

As $m \to \infty$, we prove (cf. Proposition 2.3) that A_m converges to A_∞ the multivalued operator given by

$$z \in A_\infty v \Leftrightarrow \begin{cases} v,\ z \in L^1(\Omega),\ \exists\, w \in H_0^1(\Omega),\ v \in \mathrm{Sign}(w) \text{a.e. on} \Omega \\ \text{and} \int_\Omega \nabla w.\nabla \xi = \int_\Omega z\,\xi, \quad \forall\, \xi \in H_0^1(\Omega) \cap L^\infty(\Omega) \end{cases} \tag{1.4}$$

with $\overline{\mathcal{D}(A_\infty)} = \{u \in L^1(\Omega) \cap L^\infty(\Omega)\ ;\ \|u\|_{L^\infty(\Omega)} \le 1\}$. It is known that the mild solution u_m is the strong solution of the pde ($\mathrm{P_m}$). It is true that u_m is convergent in $L^1(\Omega)$, but as far as we know, the characterization of $\underline{u}_0$ is completely solved only in the case where $u_0 \ge 0$; it is known that

$$\underline{u}_0 = (I + A_\infty)^{-1} u_0. \tag{1.5}$$

More precisely, it was proved in [3] (see also[14] and [12]) that the limit of u_m is independent of t: it is equal to the mesa of height 1, that is $u_0\, \chi_{[\underline{w}=0]} + \chi_{[\underline{w}>0]}$, where $\underline{w}$ is the unique solution of the so-called "mesa problem"

$$\underline{w} \in H^2(\Omega) \cap H^1_0(\Omega),\, \underline{w} \ge 0,\ 0 \le \Delta\underline{w} + u_0 \le 1,\, \underline{w}(\Delta\underline{w} + u_0 - 1) = 0 \text{ a.e } \Omega,$$

and one verifies easily that $u_0\, \chi_{[\underline{w}=0]} + \chi_{[\underline{w}>0]}$ coincides with $(I + A_\infty)^{-1}u_0$. By the way, let us remark that, in this case, we can say

$$\lim_{m\to\infty} \lim_{k\to\infty} \left(I + \frac{t}{k} A_m\right)^{-k} u_0 = \lim_{k\to\infty} \lim_{m\to\infty} \left(I + \frac{t}{k} A_m\right)^{-k} u_0. \tag{1.6}$$

Recall that the proofs of these results use strongly the fact that $u_m \ge 0$, since they are based on the regularizing effect (cf. [4]) of the type

$$-u_t \le \frac{u}{(m-1)t} \quad \text{a.e. in} Q,$$

which is fulfilled only if $u_0 \ge 0$. In this paper, our approach is completely different. Inspired by the paper [15] where the limit, as $p \to \infty$, of a solution of $u_t = \Delta_p u$ is studied, we characterize $\underline{u}_0$ in the case where u_0 may change sign. Our technics are general, we are using only, and strongly, the homogeneity of the equation in ($\mathrm{P_m}$). Let us notice, that Ph. Bénilan was able to develop most of the arguments we are using in a general setting of abstract nonlinear homogeneous semigroup (see the article [7]).

Before stating our main results, let us mention that the independence of time of the limit of u_m is due to the absence of a reaction in the equation and also to the homogeneity of the boundary conditions. In connection with the Hele Shaw problem, the equation (1.1) is used in the weak formulation of this problem (cf. [13]) and obviously $\underline{u}_0$ corresponds to a stationary solution; this is due to the absence of an injection and/or a suction. In the case of non null reaction (cf. [9, 8, 10]) and/or non null boundary conditions (cf. [20], [24] and [19, 18]), it is proved that the limit may depends on t and is a solution of the equation (1.1) with the corresponding nonhomogeneous terms and the (compatible) initial data $\underline{u}_0$.

Throughout the paper, we assume that Ω is an open domain, not necessarily bounded, $u_0 \in L^1(\Omega) \cap L^\infty(\Omega)$ and u_m is the mild solution of the Cauchy problem

$$u_t + A_m u = 0 \quad \text{in } (0,\infty), \quad u(0) = u_0, \tag{1.7}$$

with A_m given as in (1.3). To simplify the notation, we set

$$a = \begin{cases} 1 & \text{if } \|u_0\|_\infty \le 1 \\ 1/\|u_0\|_\infty & \text{if } \|u_0\|_\infty > 1 \end{cases} \quad \text{and} \quad v_0 = a\, u_0, \quad \text{a.e. in } \Omega.$$

The main idea of the paper is to consider the change of variables

$$z_m(t) = t u_m(t^m/m)$$

and study the limit of z_m, as $m \to \infty$. So, our main result is

THEOREM 1.1. *As $m \to \infty$, we have*

$$z_m \to z \quad in \quad \mathcal{C}([0,\infty); L^1(\Omega))$$

where z is given as follows:

i - $z(t) = t\, u_0$ *for any* $t \in [0, a]$.
ii - z *is the unique mild solution of the evolution problem*

$$\begin{cases} z_t + A_\infty z \ni z/t & in (a,\infty), \\ z(a) = v_0. \end{cases} \tag{1.8}$$

As to the limit of $u_m(t)$, we can deduce now the following result.

COROLLARY 1.2. *As $m \to \infty$, we have*

$$u_m(t) \to z(1) \quad in\ L^1(\Omega),\ \textit{uniformly for } t \textit{ in a compact set of } (0,\infty)$$

where z is given as in Theorem 1.1.

It is clear that Corollary 1.2 implies that the limit of u_m, as $m \to \infty$, is independent of time t. Moreover, if $\|u_0\|_\infty > 1$, the convergence cannot be extended to 0; similarly it does not to ∞. This means that the question arises as to the asymptotic behavior of $u_m(x,t)$ when $t \to 0$ (or $t \to \infty$) as $m \to \infty$. Theorem 1.1, implies that with the new scale $\tau = t^m/m$, the limit of the solution u_m, depends on time and one can describe the asymptotic of the limit as $t \to 0$ or $t \to \infty$. This kind of results was first studied by A. Friedman and Sh. Huang [17] (see also [16] and [14]) in the case where $u_0 \ge 0$. In those papers, the new scale was $\tau = t^m$ suggested by the asymptotic behavior of the Barenblatt solutions, and the authors prove that the limit with the new scale depends on τ and coincides with the mesa of height $1/\tau$. In the case where u_0 is changing sign, the limit is not a mesa in general (see Remark 1.3 below).

REMARK 1.3. 1. Using the results of the appendix of [15], one can prove that, in contrast to the nonnegative case, the limit of u_m in general is not a projection on the closure of the domain of A_∞. In other words, (1.5) and (1.6) are not true in general.
2. By using the results of [9], [8] and [20], one can treat ($\mathrm{P_m}$) with a reaction term and/or nonhomogeneous boundary condition of Dirichlet or Neumann type, exactly in the same way as in this paper.

2. Proofs

The main ingredient we use for the proof of the first part of the theorem is the following lemma.

LEMMA 2.1. *Let $f \in L^1(\Omega) \cap L^\infty(\Omega)$ such that $\|f\|_\infty \leq 1$ and, for $\lambda > 0$, let us consider $f_m = (I + \lambda\, A_m)^{-1} f$. As $m \to \infty$, we have*

$$f_m \to f \quad \text{and} \quad A_m\, f_m \to 0 \quad \text{in } L^1(\Omega).$$

Proof. Since $A_m f_m = f - f_m$, then it is enough to prove that $f_m \to f$ and the conclusion of the lemma follows. We begin by assuming that $\|f\|_{L^\infty(\Omega)} \leq c < 1$. By definition of A_m, f_m satisfies

$$f_m^m \in H_0^1(\Omega) \text{and} - \lambda\, \Delta f_m^m = f - f_m \text{ in } \mathcal{D}'(\Omega); \tag{2.1}$$

and, moreover, we have

$$\|f_m\|_{L^p(\Omega)} \leq \|f\|_{L^p(\Omega)} \text{ for any } 1 \leq p \leq \infty. \tag{2.2}$$

Taking f_m^m as a test function and letting $m \to \infty$, we get

$$\begin{aligned} \lambda \int_\Omega |\nabla f_m^m|^2 &= \int_\Omega (f - f_m)\, f_m^m \\ &\leq 2\, c^m\, \|f\|_1 \ \to 0 \quad \text{as } m \to 0, \end{aligned}$$

so that (2.1) implies that, $f_m \to f$ in $\mathcal{D}'(\Omega)$ and by using (2.2) we deduce that the convergence holds true in $L^1(\Omega)$. Indeed, thanks to (2.2), $f_m \to f$, in $L^p(\Omega)$-weak and $\|f\|_{L^p(\Omega)} = \lim_{m\to\infty} \|f_m\|_{L^p(\Omega)}$, for any $1 < p < \infty$, so that the convergence holds true in $L^p(\Omega)$, for any $1 < p < \infty$. Then, by using again (2.2), with $p = \infty$, and Lebesgue's dominated convergence theorem, we deduce, by choosing a subsequence that we denote again by m, that $f_m \to f$, in $L^1(\Omega)$. At last, if $\|f\|_\infty = 1$, then we consider a sequence $\{f_\varepsilon\}_{\varepsilon>0}$ in $L^1(\Omega)$ such that $\|f_\varepsilon\|_\infty < 1$ for any $\varepsilon > 0$ and, as $\varepsilon \to 0$, $f_\varepsilon \to f$ in $L^1(\Omega)$. Then, thanks to the L^1 contraction property of the operator $(I + \lambda\, A_m)^{-1}$, we deduce, by using the previous part of the proof, that $f_m \to f$ in $L^1(\Omega)$ and the proof is complete. ☐

As a consequence of this lemma, we have

LEMMA 2.2. *As* $m \to \infty$, *we have*

$$u_m(t^m/m) \to u_0 \quad \text{in } L^1(\Omega) \text{ uniformly for } t \in [0, a].$$

Proof. Set $f_m = (I + A_m)^{-1} v_0$ and $\tilde{f}_m = \frac{1}{a} f_m$. It is clear that $f_m \in \mathcal{D}(A_m)$, $\tilde{f}_m \in \mathcal{D}(A_m)$, and, thanks Lemma 2.2, $\tilde{f}_m \to u_0$ and $A_m f_m \to 0$ in $L^1(\Omega)$, as $m \to \infty$. Using the fact that $u_m(t^m/m) = e^{-\frac{t^m}{m} A_m} u_0$, we can write

$$\begin{aligned} |u_m(t^m/m) - u_0| &\leq |e^{-\frac{t^m}{m} A_m} u_0 - e^{-\frac{t^m}{m} A_m} \tilde{f}_m| \\ &\quad + |e^{-\frac{t^m}{m} A_m} \tilde{f}_m - \tilde{f}_m| + |\tilde{f}_m - u_0|, \end{aligned}$$

so that, by using the L^1 contraction property of the semigroup generated by A_m and the homogeneity of A_m, we deduce that

$$\begin{aligned} \|u_m(t^m/m) - u_0\|_1 &\leq 2\, \|u_0 - \tilde{f}_m\|_1 + \frac{t^m}{m}\, \|A_m \tilde{f}_m\|_1 \\ &\leq 2\, \|u_0 - \tilde{f}_m\|_1 + \frac{1}{m}\, (t/a)^m\, \|A_m f_m\|_1. \end{aligned}$$

Letting $m \to \infty$, the second term of the last inequality tends to 0, uniformly for $t \in [0, a]$, and the result of the lemma follows. □

At this stage, one sees that Lemma 2.2 gives the proof of the first part of Theorem 1.1. In other words, it characterizes the limit of $z_m(t)$, as $m \to \infty$, for $t \in [0, a]$. For the remaining part, i.e. for $t \in [a, \infty)$, the main ingredient we use is, in this paper, is the convergence of A_m to A_∞ in the sense of resolvent. Recall that the result is well known by now in the case where $\Omega = \mathbb{R}^N$ (cf. [5]) and also in the case where Ω is a bounded domain of $\mathbb{R}^N$ (cf. [6]). As to the case of an open domain Ω, not equal to $\mathbb{R}^N$ and not necessarily bounded, this is done in [21]. For completeness, we give hereafter the proof.

PROPOSITION 2.3. *For any* $f \in L^1(\Omega) \cap L^\infty(\Omega)$, *as* $m \to \infty$, *we have*

$$(I + A_m)^{-1} f \to (I + A_\infty)^{-1} f \quad \text{in } L^1(\Omega).$$

To simplify the notation, set $u_m = (I + A_m)^{-1} f$, then u_m is the unique solution of

$$u_m^m \in H_0^1(\Omega) \text{ and } -\Delta u_m^m = f - u_m \text{ in } \mathcal{D}'(\Omega)$$

and, recall that (cf. [1]),

$$\|u_m\|_{L^p(\Omega)} \leq \|f\|_{L^p(\Omega)} \text{ for any } 1 \leq p \leq \infty.$$

To prove the proposition, we need to prove that $(u_m)_{m \geq 1}$ is relatively compact in $L^1(\Omega)$ and that $(u_m^m)_{m \geq 1}$ is weakly relatively compact in $H_0^1(\Omega)$. For this, we need the following technical lemma.

LEMMA 2.4. (cf. [21]) *For any* $w \in H_0^1(\Omega)$ *and* $k > 0$, *we have*

$$\|(|w| - k)^+\|_2 \leq C\, |[|w| > k]|^{1/N} \left(\int_{[|w|>k]} |\nabla w|^2 \right)^{1/2}$$

where C *is a constant depending only on* Ω.

LEMMA 2.5. *The sequence* $(u_m^m)_{m\geq 1}$ *is bounded in* $H^1(\Omega)$.

Proof. First, thanks to Lemma 2.4, we see that it is enough to prove that

$$\int_{[|u_m^m|\geq 1]} |\nabla u_m^m|^2 \text{ is bounded.} \tag{2.3}$$

Indeed, it is clear that

$$|[|u_m^m| \geq 1]| = |[|u_m| \geq 1]| \leq \|f\|_1,$$

and

$$\begin{aligned} \int_\Omega |u_m^m|^2 &= \int_{[|u_m^m|\leq 1]} |u_m^m|^2 + \int_{[|u_m^m|\geq 1]} |u_m^m|^2 \\ &\leq \|f\|_1 + (\|(|u_m^m| - 1)^+\|_2 + |[|u_m^m| \geq 1]|^{1/2})^2, \end{aligned}$$

so that, by using Lemma 2.4, and (2.3), we deduce that u_m^m is bounded in $L^2(\Omega)$. As to the boundness of $\|\nabla u_m^m\|_2$, this follows from the fact that

$$\begin{aligned} \int_\Omega |\nabla u_m^m|^2 &= \int_\Omega (f - u_m)\, u_m^m \\ &\leq 2\|f\|_2\, \|u_m^m\|_2. \end{aligned}$$

Now, by taking $(|u_m^m| - 1)^+$ as a test function and using Lemma 2.4, we see that

$$\begin{aligned} \int_{[|u_m^m|\geq 1]} |\nabla u_m^m|^2 &= \int (f - u_m)\, (|u_m^m| - 1)^+ \\ &\leq 2\, \|f\|_2\, \|(|u_m^m| - 1)^+\|_2 \\ &\leq 2\, C\, \|f\|_2 \|f\|_1^{1/N} \left(\int_{[|u_m^m|\geq 1]} |\nabla u_m^m|^2 \right)^{1/2}, \end{aligned}$$

and (2.3) follows. □

LEMMA 2.6. *The sequence* $(u_m)_{m\geq 1}$ *is relatively compact in* $L^1(\Omega)$.

Proof. First, we see that

$$\lim_{|y|\to 0} \sup_{m\geq 1} \int_{\Omega'} |u_m(.+y) - u_m(.)| = 0, \tag{2.4}$$

for any $\Omega' \subset\subset \Omega$. Indeed, for any $\xi \in H_0^1(\Omega')$ and $y \in \mathbb{R}^N$ such that $|y| \leq \text{dist}(\Omega', \Gamma)$, we have

$$\int_\Omega (u_m(.+y) - u_m(.))\,\xi \\ + \int_\Omega \nabla(u_m^m(.+y) - u_m^m(.)) \cdot \nabla\xi = \int_\Omega (f(.+y) - f(.))\,\xi,$$

so that, by using standard arguments (see for instance [6]), we deduce that

$$\int_\Omega |u_m(.+y) - u_m(.)|\,\xi \leq \int_\Omega |u_m^m(.+y) - u_m^m(.)|\,\Delta\xi + \int_\Omega |f(.+y) - f(.)|\,\xi$$

and, by using Lemma 2.5, we get (2.4). This implies that $(u_m)_{m\geq 1}$ is relatively compact in $L^1_{\text{loc}}(\Omega)$. To end up the proof of the lemma, we show that

$$u_{m-} \leq u_m \leq u_{m+} \quad \text{a.e. in } \Omega \tag{2.5}$$

where u_{m-} (resp. u_{m+}) is the solution of

$$u_{m-} - \Delta u_{m-}^m = f^- \quad (\text{resp } u_{m-} - \Delta u_{m+}^m = f^+) \quad \text{ in } \mathbb{R}^N,$$

in the sense that $u_{m-} \in L^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$ (resp. $u_{m+} \in L^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$), $u_{m-}^m \in H^1(\mathbb{R}^N)$ (resp. $u_{m+}^m \in H^1(\mathbb{R}^N)$) and the equation is satisfied in $\mathcal{D}'(\mathbb{R}^N)$. Indeed, since u_{m-} and u_{m+} are relatively compact in $L^1(\mathbb{R}^N)$ (cf. [5]), then (2.5) and the relative compactness of $(u_m)_{m\geq 1}$ in $L^1_{\text{loc}}(\Omega)$, ends up the proof of the lemma. So, let us prove (2.5). It is clear that

$$\int_\Omega (u_m - u_{m+})\,\xi + \int_\Omega \nabla(u_m^m - u_{m+}^m) \cdot \nabla\xi = \int_\Omega (f - f^+)\,\xi$$

for any $\xi \in H_0^1(\Omega)$. On the other hand, it is not difficult to see that, $(u_m^m - u_{m+}^m)^+ \in H_0^1(\Omega)$, so that

$$\int_\Omega (u_m - u_{m+})\,(u_m^m - u_{m+}^m)^+ + \int_\Omega |\nabla(u_m^m - u_{m+}^m)|^2 \\ = \int_\Omega (f - f^+)\,(u_m^m - u_m^m+)^+ \\ \leq 0$$

and we deduce that $u_m \leq u_{m+}$ a.e. in Ω. In the same way one can prove that $u_{m-} \leq u_m$ a.e. in Ω and (2.5) follows. □

Proof of Proposition 2.3. Using Lemma 2.5 and Lemma 2.6, the proposition follows by standard arguments (cf. [5] and [6]). We omit the details of the proof.

Proof of Theorem 1.1. Now, it is clear that the first part of the theorem follows from Lemma 2.1. Let us prove part 2. It is clear that z_m is the mild solution of $\frac{du}{dt} + A_m\, u \ni u/t$ in (a, ∞), so that since $z_m(a) \to a\, u_0$ in $L^1(\Omega)$, as $m \to \infty$, and $a\, u_0 \in \mathcal{D}(A_\infty)$, then, by using Proposition 2.3 with classical theorem for regular perturbation of nonlinear semigroup (cf. [11]), we deduce that $z_m \to z$ in $\mathcal{C}([a;\infty), L^1(\Omega))$ where z is the unique mild solution of (1.8), and the proof of the theorem is complete.

Proof of Corollary 1.2. It is clear that, for any $t > 0$, $u_m(t) = \dfrac{1}{(mt)^{1/m}} z_m((mt)^{1/m})$ and, as $m \to \infty$, $(mt)^{1/m} \to 1$, then the corollary is a simple conseqence of the L^1 convergence of $z_m(t)$ to $z(t)$, uniformly for t in a compact subset of $(0, \infty)$.

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Noureddine Igbida
LAMFA, CNRS-UMR 6140
Université de Picardie Jules Verne
33 rue Saint Leu
80038 Amiens
France
e-mail: noureddine.igbida@u-picardie.fr

Philippe Bénilan
Equipe de Mathématiques
UMR-CNRS 6623
Université de Franche-Comté
Route de Gray
25030 Besançon
France
consequence