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SOME COMPETITION PHENOMENA IN EVOLUTION EQUATIONS

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Abstract. This paper is concerned with some nonlinear evolution equations governed by tow competitive operators. We treat the problem in a general setting and show how to apply the results to the particular situation of reaction diffusion equation with large reaction and/or large diffusion.

keywords : Singular limit, large reaction, large diffusion, elliptic-parabolic problem, L^1 theory, Semigroup of contraction, competition, nonlinear evolution equation, weak solution.

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1 Introduction.

In general, the evolution of biological and physical problem uses different processes. May be the most known processes are diffusion, reaction and convection. Sometimes, they differ very strongly from one part of the physical system to the other, so that some kind of competition between them appears. The aim of this paper is to describe the asymptotic behavior of evolution problem governed by two (or more) competitive processes like reaction/diffusion or convection/diffusion..

Several useful models for this type of problems may be described by evolution problem governed by the sum of two (or many) operators. These operators are connected in the equation by rates which strongly changes. For instance, let us consider the evolution problem

$$u_t + d_k Au + r_k Bu = f \qquad \text{in } (0,T), \tag{1}$$

where A and B are two operators describing the considered processes with rates $d_k \in \mathbb{R}^+$ and $r_k \in \mathbb{R}^+$, and f is a source term. The variation of the processes lies in the parameter $k \in I\!R$. We are interested to the asymptotic behavior of the solution u_k of evolution problem of type (1) when each one of the operators A and B acts strongly. In other words, we assume that $\dfrac{r_k}{d_k} \to +\infty$ or $\dfrac{r_k}{d_k} \to 0$, as $k \to \infty$, and we study the asymptotic behavior of u_k. First, we study the problem within an abstract general framework. Then, we show how to apply the result to the particular situation (1) and also to concrete examples like the reaction-diffusion equation.

Note that this problem is a particular case of an overall program of studying the so called singular limit for nonlinear partial differential equations. That is a perturbation problem where the perturbed problem is of totally different character than the unperturbed one. In a Banach space X, let us consider a family of evolution equations

$$u_t + A_k u \ni f \quad \text{ in } (0,\infty), \quad u(0) = u_0 \tag{2}$$

governed by the family of operators $(A_k)_{k\in I\!N}$ being such that (2) has a solution $u_k \in \mathcal{C}([0,T),X)$ and such that A_k converges to A in the graph sense, as $k \to \infty$. It is known that (see for instance [16]) if $u_0 \in \overline{\mathcal{D}(A)}$, then $u_k \to u$ in $\mathcal{C}([0,T),X)$ and u is the solution of

$$u_t + Au \ni f \quad \text{ in } (0,\infty), \quad u(0) = u_0. \tag{3}$$

But, if $u_0 \in X \setminus \overline{\mathcal{D}(A)}$, then (3) is not well posed and in general the limit of u_k may not exist. However, for a large class of concrete problems the limit u exists and there exists $\underline{u}_0 \in \overline{\mathcal{D}(A)}$, such that u is the solution of

$$u_t + Au \ni f \quad \text{ in } (0,\infty), \quad u(0) = \underline{u}_0.$$

But the characterization of $\underline{u}_0$ is not clear yet in general. For instance, if X is a Hilbert space and A_k is the Yoshida approximation of A assumed to be maximal monotone in X, then (cf. [14]) $\underline{u}_0$ is the projection of u_0 on the closure of the domain of A. It is again the projection, in some particular case of $k-$homogeneous accretive operators A_k ; i.e. $A_k(\lambda\, u) = \lambda^k\, A(u)$ (cf. [9]). But, in general it is not the projection (see [9] and [11]). Paper [10] treats the case where $A_k = B_k + F$ which B_k being a family of accretive operators, and F being nondecreasing bounded and continuous. In [10], it is proved that $\underline{u}_0$ is given by $\lim_{t\to 0}\lim_{k\to\infty} e^{-tB_k}u_0$, where e^{-tB_k} is the semigroup generated by B_k. In this paper, we treat the case where a rescaling makes the limit regular. More precisely, we assume that there exists $m(k)\to 0$, such that $m(k)A_k \to \tilde{A}$ and $u_0 \in \overline{\mathcal{D}(\tilde{A})}$. As well as the reaction-diffusion problems is concerned (cf. section 3.), this condition is fulfilled in a large field of applications like models of the type (1) (cf. section 2).

To give a brief description of our main results, let us consider $\Omega \subseteq \mathbb{R}^N$ a bounded domain with smooth boundary $\partial\Omega$, $u_0 \in L^\infty(\Omega)$ and $f \in L^\infty(Q)$. In Ω, we consider the Reaction-Diffusion problem of the form

$$P^{d,r}(u_0,f)\quad \begin{cases} u_t - d\,\Delta w + r\,g(u) = f, \quad u = \beta(w) & \text{in } Q := \Omega\times(0,T) \\ \partial_{\vec{n}} w = 0 & \text{in } \Sigma := \partial\Omega\times(0,T) \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases}$$

where $d>0$, $r\geq 0$, $\beta:\mathbb{R}\to\mathbb{R}$ is a nondecreasing continuous function such that $\beta(0)=0$,

$$(H_1) \qquad Im(\beta) = \mathbb{R}$$

and

$$(H_g) \qquad g:\mathbb{R}\to\mathbb{R} \text{ is continuous, nondecreasing with } g(0)\equiv 0.$$

It is known that $P^{d,r}(u_0,f)$ is well posed in the sense of weak solution. More precisely (see for instance [31] and the references therein), there exists a unique $u\in L^\infty(Q)$, such that there exists $w \in L^2\Big(0,T;H^1(\Omega)\Big)$, $u=\beta(w)$ a.e. in Q and

$$d\int_0^\tau\int_\Omega DwD\xi + r\int_0^\tau\int_\Omega g(u)\xi = \int_0^\tau\int_\Omega f\xi + \int_0^\tau\int_\Omega u\xi_t + \int_\Omega u_0\xi(0),$$

for any $\xi \in \mathcal{C}^1([0,\tau]\times\overline{\Omega})$ such that $\xi(.,\tau)\equiv 0$. For $P^{d,r}(u_0,f)$, d and r represent respectively the diffusion and the reaction rate. We are interested in the asymptotic behavior of the solution as d and/or r being very large. Concretely this situation can be found in models combining reactive and diffusive processes acting strongly and creating asome competition between the processes (see [26], [15], [18], [21], [12], [23], [13], [24], [19], [35], [27] and the references therein for concrete applications).

If $r=0$ (resp. $d=0$) the limit of the solution as $d\to\infty$ (resp. $r\to\infty$) is also given by the large time behavior of the solution of $P^{d,0}(u_0,f)$ (resp. $P^{0,r}(u_0,f)$). Indeed, it is enough to consider the rescaling $\tau = d\,t$ (resp . $\tau = r\,t$). In these cases, the problem is well understood (see [2], [1], [34], [32], [31] and the references therein). Our main interest lies in the case where d and r are both non null.

If we assume that $d>0$ and $r\to\infty$, then the problem is the reaction-diffusion equation with large reaction (cf. [15], [21], [12], [13], [24], [19], [35] and [27]) . Formally, we see that the limiting problem is

$$\begin{cases} u_t - d\Delta w + G(u) = f, \quad u=\beta(w) & \text{in } Q \\ \partial_{\vec{n}} w = 0, & \text{in } \Sigma \end{cases} \tag{4}$$

where, G is the maximal monotone graph given by

$$G(r) = \begin{cases} 0 & \text{if} \quad m_0 < r < M_0 \\ [0, +\infty) & \text{if} \quad r = M_0 \\ (-\infty, 0] & \text{if} \quad r = m_0 \end{cases}$$

where m_0 and M_0 are given by $g^{-1}\{0\} = [m_0,\, M_0]$. In other words the limiting problem is

$$\begin{cases} \left.\begin{array}{l} m_0 \le u \le M_0, \quad u = \beta(w) \\ (u_t - d\,\Delta w - f)(u - m_0)(u - M_0) = 0 \\ (u_t - d\,\Delta w - f) \ge 0 \text{ in } [u = m_0] \\ (u_t - d\,\Delta w - f) \le 0 \text{ in } [u = M_0] \end{array}\right\} & \text{in } Q \\ \partial_{\vec{n}} w = 0 & \text{on } \Sigma, \end{cases}$$

which is the so called obstacle problem. Indeed, an instantaneous new distribution of the spatial inhomogeneities appears in the limiting problem. The solution is forced to transfer between M_0 and m_0. Compatible initial data for (4) are functions living in $[m_0, M_0]$, so that the limit of the solution of $P^{d,r}(u_0, f)$ is singular. In the sense that a boundary layer appears in the passage to the limit and it could be interesting to identify the compatible initial data for (4) associated with u_0. As a consequence of the main results of this paper, the corresponding compatible initial data for the limiting problem is given by the limit, as $t \to \infty$, of the solution of the ode

$$\begin{cases} z_t + r\, g(z) = 0 & \text{in } (0, T) \\ z(0) = u_0, \end{cases}$$

which is equal to $m_0 \vee (u_0 \wedge M_0)$.

If $r > 0$ and $d \to \infty$, then the problem describe a reaction-diffusion problem with large diffusion (cf. [3], [38], [26] and [30]). Using the fact that g is nondecreasing and continuous this is a particular case of [30] and the limit of the solution is given by

$$\begin{cases} c_t + r\, g(c) = \displaystyle -\!\!\!\!\!\!\int_\Omega f & \text{in} \quad (0, T) \\ c(0) = \displaystyle -\!\!\!\!\!\!\int_\Omega u_0. \end{cases} \tag{5}$$

Assuming that $d \to \infty$ and $r \to \infty$, the problem models a reaction diffusion problem with large reaction and large diffusion. In this case, a competition between the reaction and the diffusion appears and we need to distinguish between the cases where one of the rates is more important than the other. So, assume that $d = d(k)$ and $r = r(k)$, with $\lim_{k\to\infty} d(k) = \lim_{k\to\infty} r(k) = \infty$. In the limiting problem, the solution is forced to be constant in space and transfers between M_0 and m_0. Precisely, the limiting problem is the following ode

$$c_t + G(c) \ni \fint_\Omega f \quad \text{in} \quad (0,T).$$

For the identification of the corresponding initial data, and as a consequence of the competition phenomena, we treat separately cases $\lim_{k\to\infty} \frac{d(k)}{r(k)} = \infty$ and $\lim_{k\to\infty} \frac{d(k)}{r(k)} = 0$. Actually, we prove that

1. if $\lim_{k\to\infty} \frac{d\,(k)}{r(k)} = 0$, then $c(0) = m_0 \vee (M_0 \wedge \fint_\Omega u_0)$,
2. if $\lim_{k\to\infty} \frac{d\,(k)}{r(k)} = \infty$, then $c(0) = \lim_{t\to\infty} z(t)$ where z is the solution of the obstacle problem P_∞^d with $z(0) = m_0 \vee (u_0 \wedge M_0)$, a.e. Ω.

Concerning the corresponding initial data for the limiting problem, we remark that it is given by the large time behavior of the equation stated only with the most competitive process. In this direction, our main results (cf. Theorem 2.2, Theorem 2.3 and Theorem 2.5) also allow us to describe the large time behavior for some evolution problems by adding an artificial regularizing processes (cf. Corollary 2.7). A concrete situation for hyperbolic equations will be treated in details separately in forthcoming papers.

In the following section, we give in the first part, some preliminaries on nonlinear semigroup theory. In the second part, we state and prove our main result in a general abstract framework. We also show how to apply this result to the particular situation (1). At last, section 3 is devoted to the proof of the results for concrete situation $P^{d,r}(u_0, f)$.

2 Abstract framework.

2.1 Preliminaries.

Throughout this section $(X, |.|)$ is a Banach space, $[.,.]$ is the bracket defined as follow

$$[x, y] = \inf_{\lambda>0} \frac{|x + \lambda y| - |x|}{\lambda} \qquad \forall x, y \in X.$$

Recall that an accretive operator A is a function (possibly multi-valued) from X to $\mathcal{P}(X)$ with nonexpansive resolvent ; i.e. $\mathcal{J}_\lambda = (I + \lambda A)^{-1}$. It is known also, that this is equivalent to say that for any $v_1 \in Au_1$ and $v_2 \in Au_2$, then

$$[u_1 - u_2, v_1 - v_2] \geq 0.$$

In connection with the Hilbert case, i.e. X is a Hilbert space, an operator is accretive if and only if it is monotone. If A is m-accretive, i.e. A is accretive and the resolvent is everywhere defined, then, for any $f \in L^1_{loc}(0,T;X)$, and $u_0 \in \overline{\mathcal{D}(A)}$ (the closure of the effective domain $\mathcal{D}(A) = \{x \in X \;;\; Ax \neq \emptyset\}$) the evolution problem

$$\begin{cases} u_t + A\,u \ni f \quad \text{ in } (0,T) \\ \\ u(0) = u_0, \end{cases} \tag{6}$$

has a unique *mild-solution*. It is the solution that we obtain through the discretization of the derivative in (6) by the implicit difference schema. Indeed, for any partition $0 = t_0 < t_1 < ... < t_{n-1} < T \leq t_n$, take the system of difference relations

$$\frac{u_i - u_{i-1}}{\epsilon_{i-1}} + Au_i \ni f_i, \qquad i = 1, 2, ..., n \tag{7}$$

where $\epsilon_{i-1} = t_i - t_{i-1}$ and $f_1,\ f_2, ..., f_n$ are such that

$$\sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} |f - f_i| \leq \epsilon.$$

Using the resolvent of A, the value u_i, are determined successively by

$$u_i = \mathcal{J}_{\epsilon_{i-1}}(u_{i-1} + \epsilon\, f_i), \quad i = 1, 2, ..., n$$

and therefore (7) has a solution if, and only if, $u_i + \epsilon\, f_i \in \mathcal{R}(I + \lambda A)$, for all $i = 1, 2, ..., n$. In particular, this is true for m-accretive operator. The step functions $u\ :\ [0,T] \to X$ defined by $u(0) = u_0$ and $u(t) = u_i$ for $t_{i-1} < t \leq t_i$ is considered to be an approximated solution of (6), converges to a unique function $u \in \mathcal{C}([0,T);X)$ such that $u(0) = u_0$. This function u is called the mild-solution of (6) on $[0,T]$.
In particular, if $f = 0$, then the mild solution is given by the exponential formula

$$u(t) = e^{-tA}u_0 := \lim_{n\to\infty} \Big(I + \frac{t}{n}A\Big)^{-n} u_0. \tag{8}$$

Moreover, the family of operators e^{-tA}, $t > 0$, is a continuous semigroup of nonexpansive self-mappings of $\overline{\mathcal{D}(A)}$.
Many of partial differential equations that can be studied by means of the nonlinear semigroup theory satisfies a “comparison principle”. This fact is a consequence at the order preserving property of the semigroup $(e^{-tA}u_0)_{t\geq 0}$. The operators which generates order-preserving semigroups are the following : Let X be a Banach lattice and A be an operator in X. A is called *T-accretive* if, its resolvents are T-contractions, i.e.,

$$|(\mathcal{J}_\lambda x - \mathcal{J}_\lambda \hat{x})^+| \leq |(x - \hat{x})^+| \quad \text{ for } x,\ \hat{x} \in \mathcal{D}(\mathcal{J}_\lambda).$$

Now, since every T-contraction is order-preserving, then if A is T-accretive then, for each $t > 0$, e^{-tA} is order-preserving. In general, T-accretivity does not implies accretivity, but in some Banach spaces T-accretivity implies accretivity, this remains true for the case of $L^p(\Omega)$ spaces, with $1 \leq p \leq \infty$.

To end up these preliminaries, remember that if we replace the function f by a continuous perturbation $F(t,u)$; i.e. we consider in X, the evolution problem

$$\begin{cases} u_t + A\,u \ \ni F(t,u) & \text{in } (0,T) \\ \\ u(0) = u_0, \end{cases} \tag{9}$$

then the definition of the mild solution remains the same by replacing $f(.)$ by $F(.,u(.))$, and we know that

Proposition 2.1 *(Cf. [10]) Let A be m-accretive in X, $u_0 \in \overline{\mathcal{D}(A)}$ and $F : [0,T] \times \overline{\mathcal{D}(A)} \longrightarrow X$ be a Caratheodory function ; i.e. $F(t,x)$ is measurable in t and continuous with respect to x, such that*

$$(\mathcal{F}_1) \qquad \begin{pmatrix} [x-y, F(t,x) - F(t,y)] \leq \alpha(t)|x-y| \quad \textit{for any } \ x,y \in \overline{\mathcal{D}(A)} \\ \\ \textit{and a.e. } t \in (0,T) \textit{ with } \alpha \in L^1_{loc}([0,T]), \end{pmatrix}$$

$$(\mathcal{F}_2) \qquad |F(t,x)| \leq c(t) \quad \textit{with } c \in L^1_{loc}([0,T)),$$

then (9) has a unique mild solution.

2.2 Main results.

Now, in X let us consider $(\mathcal{A}_k)_{k\geq 1}$ a sequence of m-accretive operator, such that

$$(\mathcal{H}_1) \qquad \mathcal{A}_k \longrightarrow \mathcal{A}_\infty \qquad \text{as } k \longrightarrow +\infty, \quad \text{in the sense of the resolvent,}$$

i.e. $(I + \lambda\,\mathcal{A}_k)^{-1}$ converges, in X, to $(I + \lambda\,\mathcal{A}_\infty)^{-1}$. Recall that, since $\mathcal{A}_k$ is assumed to be m-accretive, then $(\mathcal{H}_1)$ is equivalent to the convergence in the graph sense ; i.e. for $v_k \in \mathcal{A}_k\,u_k$, if $u_k \to u$ and $v_k \to v$, then $v \in \mathcal{A}_\infty\,u$.

We are interested to the asymptotic behavior, as $k \to \infty$, of mild solution u_k of

$$(\mathcal{P}_k) \qquad \begin{cases} u_t + \mathcal{A}_k u \ni F(t,u) & \textit{in } (0,T) \\ \\ u(0) = u_0, \end{cases}$$

where $u_0 \in \overline{\mathcal{D}(\mathcal{A}_k)}$ and $F \ : \ (0,T) \times \overline{\mathcal{D}(\mathcal{A}_k)} \to X$ is a Caratheodory function satisfying $(\mathcal{F}_1)$ and $(\mathcal{F}_2)$. In general, $\overline{\mathcal{D}(\mathcal{A}_\infty)} \neq \cap_{k\geq 1}\overline{\mathcal{D}(\mathcal{A}_k)}$, and

$$\begin{cases} u_t + \mathcal{A}_\infty u \ni F(t,u) & \textit{in } [0,T) \\ \\ u(0) = u_0 \end{cases}$$

has a solution if and only if $u_0 \in \overline{\mathcal{D}(\mathcal{A}_\infty)}$. If $u_0 \in \overline{\mathcal{D}(\mathcal{A}_\infty)}$, we know (Theorem of Trotter-Kato-Brezis-Pazy [16] and [28]), that $u_k \longrightarrow u \quad$ in $\ \mathcal{C}([0,T),X)$ and u is the unique mild solution of

$$\begin{cases} u_t + \mathcal{A}_\infty u \ni F(t,u) & \textit{in } [0,T) \\ \\ u(0) = u_0. \end{cases}$$

But, if $u_0 \notin \overline{\mathcal{D}(\mathcal{A}_\infty)}$ and the limit of u_k exists, then the limit is singular, a boundary layer at $t = 0$ appears in the passage to the limit. In fact, there exists a modified initial data $\underline{u}_0$ (depending on u_0 and $\mathcal{A}_\infty$) such that the limit of u_k is the solution of

$$\begin{cases} u_t + \mathcal{A}_\infty u \ni F(t,u) & \textit{in } [0,T) \\ \\ u(0) = \underline{u}_0. \end{cases}$$

The characterization of $\underline{u}_0$ is not well understood in general. In this paper we are interested to characterization of $\underline{u}_0$, in the case where $\mathcal{A}_k$ is such that there exist $m : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\lim_{k\to\infty} m(k) = 0$,

$$(\mathcal{H}_2) \qquad \widetilde{\mathcal{A}}_k := m(k)\mathcal{A}_k \longrightarrow \widetilde{\mathcal{A}}_\infty \qquad \text{as } k \longrightarrow \infty,$$

and

$$(\mathcal{H}_3) \qquad u_0 \in \overline{\mathcal{D}(\widetilde{\mathcal{A}}_\infty)}.$$

In other words, we assume that there exists a rescaling for which the limit is not singular.

Theorem 2.2 *Let $(\mathcal{A}_k)_{k\in\mathbb{N}}$ be a sequence of m - accretive operators in X satisfying $(\mathcal{H}_1)$ and $(\mathcal{H}_2)$, $F : [0,T]\times\overline{\mathcal{D}(\mathcal{A}_k)} \longrightarrow X$ be a Caratheodory function satisfying $(\mathcal{F}_1)$ and $(\mathcal{F}_2)$, $u_0 \in \bigcap_{k\geq 1}\overline{\mathcal{D}(\mathcal{A}_k)}$ and u_k be the mild solution of $(\mathcal{P}_k)$. If, u_0 satisfies $(\mathcal{H}_3)$ and*

$$\lim_{t\longrightarrow+\infty} e^{-t\widetilde{\mathcal{A}}_\infty} u_0 =: \underline{u}_0 \in \overline{\mathcal{D}(\mathcal{A}_\infty)},$$

then

$$u_k \longrightarrow \underline{u} \textit{ in } \mathcal{C}((0,T],X) \qquad \textit{as } k \longrightarrow +\infty,$$

where $\underline{u}$ is the unique mild solution of

$$(\mathcal{P}_\infty) \qquad \begin{cases} \underline{u}_t + \mathcal{A}_\infty \underline{u} \ni F(t,\underline{u}) & \textit{in } (0,T) \\ \\ \underline{u}(0) = \underline{u}_0. \end{cases}$$

Proof : Thanks to Theorem 4.1. of [28], it is enough to prove the result for $F \equiv 0$. In this case, it is known that the mild solution is given by the exponential formula. So, for any $t, \tau > 0$ and $k \in \mathbb{N}$ such that $\tau m(k) < t$, we have

$$\begin{aligned} |u^k(t) - \underline{u}(t)| &= |e^{-t\mathcal{A}_k} u_0 - e^{-t\mathcal{A}_\infty}\underline{u}_0| \\ &\leq |e^{-t\mathcal{A}_k} u_0 - e^{-(t-\tau m(k))\mathcal{A}_k}\underline{u}_0| + |e^{-(t-\tau m(k))\mathcal{A}_\infty}\underline{u}_0 - e^{-t\mathcal{A}_\infty}\underline{u}_0| \\ &+ |e^{-(t-\tau m(k))\mathcal{A}_k}\underline{u}_0 - e^{-(t-\tau m(k))\mathcal{A}_\infty}\underline{u}_0|. \end{aligned} \tag{10}$$

Using the additive property of the semigroup $e^{-t\mathcal{A}_k}$, the first term of the right hand of (10) satisfies

$$\begin{aligned}
|e^{-t\mathcal{A}_k}u_0 - e^{-(t-\tau m(k))\mathcal{A}_k}\underline{u}_0| &= |e^{-(t-\tau m(k))\mathcal{A}_k}e^{-\tau m(k)\mathcal{A}_k}u_0 - e^{-(t-\tau m(k))\mathcal{A}_k}\underline{u}_0| \\
&\leq |e^{-\tau m(k)\mathcal{A}_k}u_0 - \underline{u}_0| = |e^{-\tau\widetilde{\mathcal{A}}_k}u_0 - \underline{u}_0| \\
&\leq |e^{-\tau\widetilde{\mathcal{A}}_k}u_0 - e^{-\tau\widetilde{\mathcal{A}}_\infty}u_0| + |e^{-\tau\widetilde{\mathcal{A}}_\infty}u_0 - \underline{u}_0|.
\end{aligned}$$

The contraction property of $e^{-\mathcal{A}_\infty}$ implies that the second term (10) is such that

$$|e^{-(t-\tau m(k))\mathcal{A}_\infty}\underline{u}_0 - e^{-t\mathcal{A}_\infty}\underline{u}_0| \leq |e^{-\tau m(k)\mathcal{A}_\infty}\underline{u}_0 - \underline{u}_0|.$$

For the last term of the right hand of (10), we have

$$|e^{-(t-\tau m(k))\mathcal{A}_k}\underline{u}_0 - e^{-(t-\tau m(k))\mathcal{A}_\infty}\underline{u}_0| \leq \sup_{s\in[0,T]}|e^{-s\mathcal{A}_k}\underline{u}_0 - e^{-s\mathcal{A}_\infty}\underline{u}_0|.$$

So

$$\begin{aligned}
|u^k(t) - \underline{u}(t)| &\leq |e^{-\tau\widetilde{\mathcal{A}}_k}u_0 - e^{-\tau\widetilde{\mathcal{A}}_\infty}u_0| + |e^{-\tau\widetilde{\mathcal{A}}_\infty}u_0 - \underline{u}_0| \\
&+ \sup_{s\in[0,T]}|e^{-s\mathcal{A}_k}\underline{u}_0 - e^{-s\mathcal{A}_\infty}\underline{u}_0| + |e^{-\tau m(k)\mathcal{A}_\infty}\underline{u}_0 - \underline{u}_0|
\end{aligned}$$

Using the fact that $u_0 \in \overline{\mathcal{D}(\widetilde{\mathcal{A}}_\infty)}$, $\underline{u}_0 \in \overline{\mathcal{D}(\mathcal{A}_\infty)}$, $\lim_{k\to\infty} m(k) = 0$ and [16], we deduce that

$$\limsup_{k\to+\infty}|u^k(t) - \underline{u}(t)| \leq |e^{-\tau\widetilde{\mathcal{A}}_\infty}u_0 - \underline{u}_0|.$$

At last, letting $\tau \longrightarrow +\infty$ we obtain

$$\lim_{k\longrightarrow\infty}|u^k(t) - \underline{u}(t)| = 0.$$

■

Still in the abstract setting, the main applications we have in mind is the study of evolution problems governed by two competitive operators. More precisely, consider the evolution problem

$$(\mathcal{P}'_k) \qquad \begin{cases} u_t + d(k)Au + r(k)Bu \ni F(t,u) & in\ (0,T) \\ u(0) = u_0 \end{cases}$$

where $r\ ,d\ : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ are two functions such that

$$\lim_{k\to\infty}\frac{r(k)}{d(k)} = +\infty, \tag{11}$$

A, B, $d(k)A + r(k)B$ and $\mathcal{H}$ are accretive operators such that $\mathcal{D}(A) \subseteq \mathcal{D}(B)$, F satisfies $(\mathcal{F}_1) - (\mathcal{F}_2)$ and $u_0 \in \overline{\mathcal{D}(A)}$. We assume that

$$d(k)A + r(k)B \longrightarrow \mathcal{H}, \qquad \text{as } k \to \infty. \tag{12}$$

As a consequence of Theorem 2.2, we have

Theorem 2.3 *Let u_k be the mild solution of $(\mathcal{P}'_k)$. Assume that*

$$\varepsilon\, A + B \to \tilde{B}, \qquad \text{as } \varepsilon \to 0 \tag{13}$$

and $u_0 \in \overline{\mathcal{D}(\tilde{B})}$. If

$$\lim_{t\to+\infty} e^{-t\tilde{B}} u_0 := \underline{u}_0 \in \overline{\mathcal{D}(\mathcal{H})}, \tag{14}$$

then, as $k \longrightarrow \infty$, $u_k \longrightarrow u$ in $\mathcal{C}((0,T),X)$, and u is the mild solution of

$$(\mathcal{P}'_\infty) \qquad \begin{cases} u_t + \mathcal{H}u \ni F(t,u) & \text{in } (0,T) \\ u(0) = \underline{u}_0. \end{cases}$$

Proof. It is enough to apply Theorem 2.2 with $\mathcal{A}_k := d(k)A + r(k)B$ and $m(k) = [r(k)]^{-1}$. ■

Remark 2.4 *1. In general, $\tilde{B} \neq B$. For instance, let $X = \mathbb{R}$ and A be the maximal monotone graph defined by*

$$A(r) = \begin{cases} 0 & \text{if } |r| < 1 \\ [0,\infty) & \text{if } r = 1 \\ (-\infty, 0] & \text{if } r = -1. \end{cases}$$

Then, it is clear that for any $\varepsilon > 0$, $\varepsilon\, A = A$ and $\varepsilon\, A + B = A + B = \tilde{B}$.

2. If $\mathcal{D}(A) = \mathcal{D}(B)$ and A is strictly accretive, i.e. $[u_1 - u_2, v_2 - v_1] \leq 0$ for any $v_1 \in Au_1, v_2 \in Au_2$, then $\tilde{B} = B$. Indeed, set

$$[x,y]_s = -[x,-y] := \sup_{\lambda<0} \frac{|x + \lambda y| - |x|}{\lambda},$$

so that A is strictly accretive is equivalent to $[u_1 - u_2, v_1 - v_2]_s \geq 0$, for any $v_1 \in Au_1$, $v_2 \in Au_2$. For $f \in X$, let $\widetilde{u}_\varepsilon$ be the solution of the problem

$$\widetilde{u}_\varepsilon + \varepsilon\, A\widetilde{u}_\varepsilon + B\widetilde{u}_\varepsilon \ni f,$$

and $\widetilde{u}$ *is the solution of*

$$\widetilde{u} + B\widetilde{u} \ni f.$$

Since B *is accretive in* X *then for* $\widetilde{w}_\varepsilon \in B\widetilde{u}_\varepsilon$, $\widetilde{w} \in B\widetilde{u}$ *we have*

$$[\widetilde{u}_\varepsilon - \widetilde{u},\, \widetilde{w}_\varepsilon \; - \; \widetilde{w}] \geq 0$$

and

$$[\widetilde{u}_\varepsilon - \widetilde{u}, -\widetilde{u}_\varepsilon - \varepsilon\, \widetilde{v}_\varepsilon + \widetilde{u}] \geq 0 \qquad \textit{where } \; \widetilde{v}_\varepsilon \in A\widetilde{u}_\varepsilon,$$

which implies that

$$\begin{aligned} |\widetilde{u}_\varepsilon - \widetilde{u}| &\leq [\widetilde{u}_\varepsilon - \widetilde{u}, -\varepsilon\, \widetilde{v}_\varepsilon] \\ &\leq [\widetilde{u}_\varepsilon - \widetilde{u}, -\varepsilon\, (\widetilde{v}_\varepsilon - \widetilde{v}) - \varepsilon\, \widetilde{v}] \\ &\leq \varepsilon\, [\widetilde{u}_\varepsilon - \widetilde{u}, -\widetilde{v}_\varepsilon + \widetilde{v}] + [\widetilde{u}_\varepsilon - \widetilde{u}, -\varepsilon\, \widetilde{v}]. \end{aligned}$$

This implies that,

$$[\widetilde{u}_\varepsilon - \widetilde{u}, -\varepsilon\, \widetilde{v}] \leq \varepsilon\, |\widetilde{v}|$$

and

$$\varepsilon\, [\widetilde{u}_\varepsilon - \widetilde{u}, \widetilde{v}_\varepsilon - \widetilde{v}]_s + |\widetilde{u}_\varepsilon - \widetilde{u}| \leq \varepsilon\, |\widetilde{v}| \; \textit{for} \;\; \widetilde{v} \in A\widetilde{u}$$

and, Since A *is strictly accretive, we deduce that*

$$|\widetilde{u}_\varepsilon - \widetilde{u}| \leq \varepsilon\, |\widetilde{v}|$$

so that, letting $\varepsilon \longrightarrow 0$, *we obtain* $\widetilde{u}_\varepsilon \longrightarrow \widetilde{u}$, *in* X.

3. *It is clear by the preceding remarks that an s-accretive operator is accretive, but conversely an accretive operator is not necessarily s-accretive. If* $X = L^1(\Omega)$, *then*

$$[u, v] = \int_\Omega v \; sign_0(u) + \int_{[u=0]} |v|.$$

Then, it is not difficult to see that an accretive operator A *is strictly accretive if and only if*

$$\int_{[u_1=u_2]} |v_1 - v_2| \leq \Big| \int_\Omega (v_1 - v_2)\; sign_0(u_1 - u_2) \Big|.$$

For instance, if γ *is a nondecreasing continuous function in* $\mathbb{R}$, *then* A *defined in* $L^1(\Omega)$ *by* $Au = \gamma(u)$ *and* $\mathcal{D}(A) = \Big\{ z \in L^1(\Omega)\; ;\; \gamma(z) \in L^1(\Omega) \Big\}$ *is strictly accretive.*

In some practical situation the condition (14) may not be fulfilled (as for example $P^{d,r}(u_0, f)$ in the case of large reaction and diffusion), in this case we need to work moreover with the limit of the modified operator $A + [d(k)]^{-1} r(k) B$. More precisely, we have

Theorem 2.5 *Assume (12) and (13) are fulfilled,* $u_0 \in \overline{\mathcal{D}(\tilde{B})}$ *and let* u_k *be the mild solution of* $(\mathcal{P}'_k)$*. Let* G *be given by*

$$A + r(k)\,[d(k)]^{-1}B \longrightarrow G, \qquad \text{as } k \to \infty.$$

If,

$$\lim_{t\longrightarrow+\infty} e^{-t\tilde{B}}u_0 := \underline{u}_0 \in \overline{\mathcal{D}(G)}$$

and

$$\lim_{t\longrightarrow+\infty} e^{-tG}\underline{u}_0 := \underline{\underline{u}}_0 \in \overline{\mathcal{D}(\mathcal{H})}, \tag{15}$$

then, as $k \longrightarrow \infty$,

$$u_k \longrightarrow u \quad \text{in} \quad \mathcal{C}((0,T), X)$$

where u *is the mild solution of*

$$(\mathcal{P}'_\infty) \qquad \begin{cases} u_t + \mathcal{H}u \ni F(t,u) & \text{in } (0,T) \\ u(0) = \underline{\underline{u}}_0 \ . \end{cases}$$

Proof. Let $\mathcal{H}_k := d(k)A + r(k)B$, $G_k := A + [d(k)]^{-1}r(k)B$, $\gamma(k) = [r(k)]^{-1} + [d(k)]^{-1}$, for any $t, \tau > 0$ and $k \in \mathbb{N}$ such that $\tau < \gamma(k)t$, we have

$$\begin{aligned} |u^k(t) - u(t)| &= |e^{-t\mathcal{H}_k}u_0 - e^{-t\mathcal{H}}\underline{\underline{u_0}}| \\ &\le |e^{-t\mathcal{H}_k}u_0 - e^{-(t-\tau\gamma(k))\mathcal{H}_k}\underline{\underline{u_0}}| + |e^{-(t-\tau\gamma(k))\mathcal{H}}\underline{\underline{u_0}} - e^{-t\mathcal{H}}\underline{\underline{u_0}}| \\ &+ |e^{-(t-\tau\gamma(k))\mathcal{H}_k}\underline{\underline{u_0}} - e^{-(t-\tau\gamma(k))\mathcal{H}}\underline{\underline{u_0}}|. \end{aligned} \tag{16}$$

It is clear that the first term of the right hand of (16) satisfies

$$\begin{aligned} |e^{-t\mathcal{H}_k}u_0 - e^{-(t-\tau\gamma(k))\mathcal{H}_k}\underline{\underline{u}}_0| &\le |e^{-t\mathcal{H}_k}u_0 - e^{-(t-\frac{\tau}{r(k)})\mathcal{H}_k}\underline{u}_0| \\ &+ |e^{-(t-\frac{\tau}{r(k)})\mathcal{H}_k}\underline{u}_0 - e^{-(t-\tau\gamma(k))\mathcal{H}_k}\underline{\underline{u}}_0| \\ &\le |e^{-\frac{\tau}{r(k)}\mathcal{H}_k}u_0 - \underline{u}_0| + |e^{-\frac{\tau}{d(k)}\mathcal{H}_k}\underline{u}_0 - \underline{\underline{u}}_0|. \end{aligned}$$

The second one is such that

$$|e^{-(t-\tau\gamma(k))\mathcal{H}}\underline{\underline{u}}_0 - e^{-t\mathcal{H}}\underline{\underline{u}}_0| \le |e^{-\tau\gamma(k)\mathcal{H}}\underline{\underline{u}}_0 - \underline{\underline{u}}_0|.$$

The last term of the right hand of (16) satisfies

$$|e^{-(t-\tau\gamma(k))\mathcal{H}_k}\underline{\underline{u}}_0 - e^{-(t-\tau\gamma(k))\mathcal{H}}\underline{\underline{u}}_0| \le \sup_{s\in[0,T]} |e^{-s\mathcal{H}_k}\underline{\underline{u}}_0 - e^{-s\mathcal{H}}\underline{\underline{u}}_0|$$

So that

$$
\begin{aligned}
|u^k(t) - u(t)| \;&\leq\; |e^{-\frac{\tau}{r(k)}\mathcal{H}_k} u_0 - \underline{u}_0| + |e^{-\frac{\tau}{d(k)}\mathcal{H}_k} \underline{u}_0 - \underline{\underline{u}}_0| \\
&+\; |e^{-\tau\gamma(k)\mathcal{H}} \underline{\underline{u}}_0 - \underline{\underline{u}}_0| + \sup_{s\in[0,T]} |e^{-s\mathcal{H}_k} \underline{\underline{u}}_0 - e^{-s\mathcal{H}} \underline{\underline{u}}_0| \;.
\end{aligned}
$$

Thanks to (12), we have

$$
\limsup_{k\to+\infty} |e^{-\frac{\tau}{r(k)}\mathcal{H}_k} u_0 - \underline{u}_0| = |e^{-\tau\tilde{B}} u_0 - \underline{u}_0|
$$

Using the fact that $u_0 \in \overline{\mathcal{D}(B)}$, $\underline{\underline{u}}_0 \in \overline{\mathcal{D}(\mathcal{H})}$ and the Theorem of [16] , we get

$$
\limsup_{k\to+\infty} |u^k(t) - u(t)| \leq |e^{-\tau\tilde{B}} u_0 - \underline{u}_0| + |e^{-\tau G} \underline{u}_0 - \underline{\underline{u}}_0|
$$

and by letting $\tau \longrightarrow +\infty$ we obtain

$$
\lim_{k\longrightarrow\infty} |u^k(t) - u(t)| = 0.
$$

■

Remark 2.6 *The passage to the limit in the problem $(\mathcal{P}_k)$ leads to a singular limit. An instantaneous change of initial data is necessary. If $F \equiv 0$, then, in order to describe the behavior of the solution for small $t > 0$ and large value for k, it is sufficient to work with v_k given by $v_k(t) = u_k(t/m(k))$. Indeed, it is clear that v_k is the mild solution of*

$$
\begin{cases}
v_t + \tilde{\mathcal{A}}_k v \ni 0 \qquad \text{in } (0,\infty) \\[2ex]
v(0) = u_0.
\end{cases}
$$

So, thanks to Theorem 2.3, the compatible initial data for the limiting problem of $(\mathcal{P}_k)$ is given by $\lim\limits_{t\to\infty}\lim\limits_{k\to\infty} v_k(t)$. In other words

$$
\lim_{t\to 0}\lim_{k\to\infty} u_k(t) = \lim_{t\to\infty}\lim_{k\to\infty} v_k(t).
$$

At last, we note that in some practical situation, the previous result may also describe the large time behavior of the mild solution of evolution problems. In the following corollary, we give one consequence of Theorem 2.3 in this direction. Concrete situation will be given in details in forthcoming papers.

Corollary 2.7 *Let A be an accretive operator and $u_0 \in \overline{\mathcal{D}(A)}$. If $\underline{u_0} := \lim\limits_{t\to\infty} e^{-tA} u_0$ exists then*

$$
\underline{u_0} = \lim_{t\to 0}\lim_{k\to\infty} u_k,
$$

where u_k is the mild solution of

$$\begin{cases} u_t + k\,Au + Bu \ni 0 & in\ (0,T) \\ \\ u(0) = u_0, \end{cases} \tag{17}$$

<u>and B is an accretive</u> operator such that $A + \varepsilon\, B$ converges to A, as $\varepsilon \to 0$, and $\underline{u}_0 \in$ <u>$\mathcal{D}(\liminf_{k\to\infty}(kA+B))$</u>.

In Corollary 2.7, B is an arbitrary artificial process. The interest of B may be in the fact that we can choose it such that the solution of (17) is regular. In particular this could be interesting for numerical analysis for $\lim_{t\to\infty} e^{-tA}u_0$.

3 Applications.

Throughout this section, $\Omega \subseteq \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$, $\beta$and g are nondecreasing continuous functions such that $\beta(0) = g(0) = 0$ and $\mathrm{Im}(\beta) = \mathbb{R}$. In Ω, we consider the problem $P^{d,r}(u_0, f)$, where $u_0 \in L^\infty(\Omega)$ and $f \in L^\infty(Q)$. Our aim in this section is to study the competition between reaction r and diffusion d in the reaction diffusion problem $P^{d,r}(u_0, f)$. The main result of this section is the following Theorem 3.1.

Theorem 3.1 *Let $f \in L^\infty(Q)$, $u_0 \in L^\infty(\Omega)$ and denote by $u^{d,r}$ the solution of $P^{d\,,r}(u_0, f)$.*

1. *<u>Large diffusion</u> : For any $r > 0$, as $d \to \infty$, $u^{d,r} \to c$ in $\mathcal{C}((0,T), L^1(\Omega))$, where $c \in \mathcal{C}^1([0,T))$ is the unique solution of the ode,*

$$\begin{cases} c_t + r\,g(c) = \displaystyle⨍_\Omega f & in \quad (0,T) \\ \\ c(0) = \displaystyle⨍_\Omega u_0. \end{cases} \tag{18}$$

2. *<u>Large reaction</u> : For any $d > 0$, as $r \to \infty$, $u^{d,r} \to u$ in $\mathcal{C}((0,T), L^1(\Omega))$, where u is the unique weak solution (see Proposition 3.10 for the definition) of the obstacle problem (4) with $u(0) = m_0 \vee (u_0 \wedge M_0)$ a.e. in Ω.*

3. *<u>Large diffusion and reaction</u> : If $d = d\,(k)$ and $r = r(k)$, with $\lim_{k\to\infty} d\,(k) = \lim_{k\to\infty} r(k) = \infty$, then*

$$u^{d,r} \to c \quad in\ \mathcal{C}((0,T), L^1(\Omega)$$

where $c \in \mathcal{C}^1([0,T))$ is the solution of the ode

$$c_t + G(c) \ni ⨍_\Omega f \quad in \quad (0,T),$$

with

(a) if $\lim_{k\to\infty} \dfrac{d\,(k)}{r(k)} = 0$, *then* $c(0) = m_0 \vee (M_0 \wedge \fint_\Omega u_0)$,

(b) if $\lim_{k\to\infty} \dfrac{d\,(k)}{r(k)} = \infty$, *then* $c(0) = \lim_{t\to\infty} z(t)$ *where* z *is the solution of the obstacle problem* P^d_∞ *with* $z(0) = m_0 \vee (u_0 \wedge M_0)$, *a.e.* Ω.

It is known that the weak solution of $P^{d,r}(u_0, f)$ is given by nonlinear semigroup theory. Indeed, we know that the weak solution is the mild solution of the Cauchy problem

$$CP^{d,r}(u_0, f) \begin{cases} u_t + \mathcal{A}^{d,r} u = f & \text{in } (0, T) \\ \\ u(0) = u_0, \end{cases}$$

where, $\mathcal{A}^{d,r}$ is the m-T-accretive operator defined in $L^1(\Omega)$ by

$$f = \mathcal{A}^{d,r} v \Leftrightarrow \begin{cases} v, f \in L^1(\Omega),\ g(v) \in L^1(\Omega),\ \exists\, w \in W^{1,1}(\Omega),\ v = \beta(w) \text{ a.e. in } \Omega \text{ and} \\ \\ d \displaystyle\int_\Omega Dw D\xi + r \int_\Omega g(v)\xi = \int_\Omega f\xi \quad \text{for any } \xi \in W^{1,\infty}(\Omega). \end{cases}$$

and, $\overline{\mathcal{D}(\mathcal{A}^{d,r})} = L^1(\Omega)$. More precisely (see for instance [31]) we have

Proposition 3.2 *If* $f \in L^\infty(Q)$ *and* $u_0 \in L^\infty(\Omega)$, *then the mild solution* u *of* $CP^{d,r}(u_0, f)$ *is the unique solution of* $P^{d,r}(u_0, f)$ *in the sense that* $u \in L^\infty(Q)$, *there exists* $w \in L^2\Big(0, T; H^1(\Omega)\Big)$ *such that,* $u = \beta(w)$ *a.e. in* Q, *and*

$$d \int_0^\tau \int_\Omega Dw D\xi + r \int_0^\tau \int_\Omega g(u)\xi = \int_0^\tau \int_\Omega f\xi + \int_0^\tau \int_\Omega u\xi_t + \int_\Omega u_0 \xi(0), \tag{19}$$

for any $\xi \in \mathcal{C}^1([0, \tau] \times \overline{\Omega})$ *such that* $\xi(., \tau) \equiv 0$. *Moreover, for any* $\tau \geq 0$

$$||u(\tau)||_{L^\infty(Q)} \leq ||u_0||_{L^\infty(\Omega)} + T||f||_{L^\infty(Q)}, \tag{20}$$

$$\int_\Omega |u(\tau)| + r \int_0^\tau \int_\Omega |g(u)| \ \leq \ \int_0^\tau \int_\Omega |f| + \int_\Omega |u_0| \tag{21}$$

and

$$\int_\Omega j(u(\tau)) + d \int_0^\tau \int_\Omega |Dw|^2 \ \leq \ \int_\Omega j(u_0) + \int_0^\tau \int_\Omega fw, \tag{22}$$

where $j : \mathbb{R} \longmapsto [0, \infty]$ *is a proper convex s.c.i function such that* $j(\beta(q)) = \displaystyle\int_0^q s d\beta(s)$.

Now, the study of the asymptotic behavior of the solution of $P^{d,r}(u_0, f)$ with respect to d and r is closely connected to the limit of its associate stationary equation

$$S^{d,r}(f) \begin{cases} v - d\Delta w + rg(v) = f, \quad v = \beta(w) & \text{in } \Omega \\ \\ \partial_{\vec{n}} w = 0, & \text{in } \partial\Omega\ . \end{cases}$$

Recall (cf. [7]) that for any $f \in L^1(\Omega)$, there exists a unique (u, w) solution of $S^{d,r}(f)$ in the sense that

$$\begin{cases} v \in L^1(\Omega),\ g(v) \in L^1(\Omega), \exists\ w \in W^{1,1}(\Omega),\ v = \beta(w) \text{ a.e. in } \Omega \text{ and} \\ d \displaystyle\int_\Omega DwD\xi + r \int_\Omega g(v)\xi = \int_\Omega (f - v)\xi \quad \text{for any } \xi \in W^{1,\infty}(\Omega). \end{cases}$$

In addition, for $f_1, f_2 \in L^1(\Omega)$, if (v_i, w_i) is the solution of $S^{d,r}(f_i))$ for $i = 1, 2$, then

$$\int_\Omega (v_1 - v_2)^+ + r \int_\Omega (g(v_1) - g(v_2))^+ \leq \int_\Omega (f_1 - f_2)^+ \tag{23}$$

and

$$\int_\Omega |v_1 - v_2| + r \int_\Omega |g(v_1) - g(v_2)| \leq \int_\Omega |f_1 - f_2|. \tag{24}$$

Moreover, if $f \in L^\infty(\Omega)$ then the solution $(v, w) \in L^\infty(\Omega) \times H^2(\Omega)$ and one has the following estimates:

$$||v||_{L^\infty(\Omega)} \leq ||f||_{L^\infty(\Omega)} \quad \text{and} \quad ||r\, g(v)||_{L^\infty(\Omega)} \leq ||f||_{L^\infty(\Omega)}. \tag{25}$$

Indeed, taking c_r such that $c_r + rg(c_r) = \|f\|_{L^\infty(\Omega)}$, then it is clear that c_r is the solution of $S^{d,r}(\|f\|_\infty)$ and using (23), we get

$$v \leq c_r \leq \|f\|_{L^\infty(\Omega)} \quad \text{and } rg(v) \leq rg(c_r) \leq \|f\|_{L^\infty(\Omega)},$$

where we used the fact that $c_r \geq 0$ and $g(c_r) \geq 0$. In the same way, we prove

$$v \geq -\|f\|_{L^\infty(\Omega)} \quad \text{and} \quad rg(v) \geq -\|f\|_{L^\infty(\Omega)}.$$

Thanks to (H_1), we have

$$\|w\|_{L^\infty(\Omega)} \leq \max\left(\beta^{-1}(\|f\|_{L^\infty(\Omega)}) \cup -\beta^{-1}(-\|f\|_{L^\infty(\Omega)})\right) =: C. \tag{26}$$

and

$$d \int_\Omega |Dw|^2 \leq C' ||f||_{L^\infty(\Omega)}, \tag{27}$$

where C' is a constant which depends only on Ω and $||f||_{L^1(\Omega)}$.

Theorem 3.3 *Let $f \in L^1(\Omega)$ and let us denote by $v^{d,r}$ the solution of $S^{d\,,r}(f)$.*

1. *If $d = d\,(k)$ and $\lim_{k\to\infty} d\,(k) = \infty$, then*

$$v^{d,r} \longrightarrow (I\!I_\mathbb{R} + r\, g)^{-1}(\fint_\Omega f) \quad \text{in } L^1(\Omega).$$

2. *If $r = r(k)$ and $\lim_{k\to\infty} r(k) = \infty$, then $v^{d,r} \to v$ in $L^1(\Omega)$ and v is the unique solution of the elliptic problem*

$$\begin{cases} v - d\Delta w + G(v) \ni f, \quad v = \beta(w) & \text{in } \Omega \\ \partial_{\vec{n}} w = 0, & \text{in } \partial\Omega \end{cases}$$

in the sense that $v \in L^\infty(\Omega)$ there exists $w \in W^{1,1}(\Omega)$, $u = \beta(w)$ a.e. in Ω, there exists $\eta \in L^1(\Omega)$, $\eta \in G(v)$ a.e. in Ω and

$$d\int_\Omega DwD\xi + \int_\Omega \eta\,\xi = \int_\Omega (f - v)\,\xi$$

for any $\xi \in W^{1,\infty}(\Omega)$. Moreover,

$$\|v\|_{L^\infty(\Omega)} \le \|f\|_{L^\infty(\Omega)} \quad \text{and} \quad \|\eta\|_{L^\infty(\Omega)} \le \|f\|_{L^\infty(\Omega)} \tag{28}$$

3. *If $d = d(k)$ and $r = r(k)$, with $\lim_{k\to\infty} d(k) = \lim_{k\to\infty} r(k) = \infty$, then*

$$v^{d,r} \longrightarrow (I + G)^{-1}(\fint_\Omega f) \quad \text{in } L^1(\Omega).$$

Proof : Since in each part of the theorem, d and/or r depends on k, then, throughout the proof, $v^{d,r}$, $w^{d,r}$ and $rg(v^{d,r})$ are denoted by v_k, w_k and η_k respectively. First, let us assume that $f \in L^\infty(\Omega)$. Thanks to (25), (26) and (27), v_k and w_k are bounded in $L^\infty(\Omega)$ and moreover w_k is bounded in $H^1(\Omega)$. So that, w_k is weakly relatively compact in $H^1(\Omega)$, and since β is continuous then v_k is relatively compact in $L^1(\Omega)$. Then, there exists a subsequence that we denote again by k, such that $v_k \to v$ in $L^1(\Omega)$, $w_k \to w$ in $L^1(\Omega)$ and weakly in $H^1(\Omega)$, and $u = \beta(w)$ a.e in Ω. Moreover, $h_k := v_k + r(k)g(v_k)$ is relatively compact in $L^1(\Omega)$. Indeed, thanks to Lemma F. of [7] we have

$$\lim_{|y|\to 0} \sup_k \int_{\Omega'} |h_k(x+y) - h_k(x)| = 0,$$

and moreover, the sequence is uniformly integrable. Therefore

$$r(k)g(v_k) \to \eta \text{ in } L^1(\Omega).$$

Now, in order to characterize u, w and η, we treat separately each case of the theorem.

1. Thanks to (27), we have

$$d(k)\int_\Omega |Dw_k|^2 \le C'||f||_\infty,$$

so that

$$\int_\Omega |Dw|^2 \le \liminf_{k\to\infty} \int_\Omega |Dw_k|^2 = 0,$$

which implies that w and v are constant functions. Taking $\xi \equiv 1$ as a test function and passing to the limit we deduce that the constant v satisfies $v + rg(v) = \fint_\Omega f$. This ends up the proof of the first part of the Theorem.

2. Since $g(0) = \beta(0) = 0$, then (24) implies that

$$r(k)|g(v_k)|_{L^1(\Omega)} \leq |f|_{L^1(\Omega)}.$$

So, $g(v) \equiv 0$ and $m_0 \leq v \leq M_0$ a.e. in Ω. Recall that $r(k)g$ converges in the graph sense to G, then by standard monotone arguments $\eta \in G(v)$ a.e. in Ω. As to (28), it follows from the estimates (25). And the proof of the second part finishes.

3. Since

$$\int_\Omega |Dw_k|^2 \leq \frac{C_1(\Omega, N)}{d(k)} \|f\|_{L^\infty(\Omega)},$$

then w and v are constants. Moreover, since $r(k)g$ converges in the graph sense to G, then $\eta \in G(v)$ a.e. in Ω. Integrating the equation over Ω and letting $k \to \infty$, we get

$$v + G(v) \ni ⨍_\Omega f.$$

At last, we see that the application $f \to \lim_{k\to\infty} v_k$ as defined in each cases of the theorem is well defined and is a contraction from in $L^1(\Omega)$ into $L^1(\Omega)$, so the convergence result for $f \in L^1(\Omega)$ follows by density of $L^\infty(\Omega)$ in $L^1(\Omega)$. ■

As an immediate consequence, we have

Corollary 3.4 *1. For any $r > 0$, as $d \to \infty$, then the operator $\mathcal{A}^{d,r}$ converges to the T-accretive operator $\mathcal{A}^{\infty,r}$ defined, in $L^1(\Omega)$, by*

$$f \in \mathcal{A}^{\infty,r} v \Leftrightarrow f \in L^1(\Omega),\ v \equiv c,\ c \in \mathbb{R} \text{ and } r\, g(c) = ⨍_\Omega f.$$

2. For any $d > 0$, as $r \to \infty$, then $\mathcal{A}^{d,r}$ converges to the T-accretive operator $\mathcal{A}^{d,\infty}$ defined, in $L^1(\Omega)$, by

$$f \in \mathcal{A}^{d,\infty} v \Leftrightarrow \begin{cases} v, f \in L^1(\Omega), \exists\ w \in W^{1,1}(\Omega), \exists\ \eta \in L^1(\Omega) \\ v = \beta(w),\ \eta \in G(v)\ \text{ a.e. on } \Omega \text{ and} \\ d\int_\Omega DwD\xi + \int_\Omega \eta\xi = \int_\Omega f\xi \quad \text{for any } \xi \in W^{1,\infty}(\Omega) \end{cases}$$

3. If $d = d(k)$ and $r = r(k)$, with $\lim_{k\to\infty} d(k) = \lim_{k\to\infty} r(k) = \infty$, then $\mathcal{A}^{d,r}$ converges to the T-accretive operator $\mathcal{A}^\infty$, defined by

$$f \in \mathcal{A}^\infty v \Leftrightarrow f \in L^1(\Omega),\ v \equiv c,\ c \in \mathbb{R} \text{ and } ⨍_\Omega f \in G(c).$$

It is not difficult to see that

$$\mathcal{A}^{d,r} = dA + rB \tag{29}$$

where $A = \mathcal{A}^{1,0}$ and $B\ :\ L^1(\Omega) \to L^1(\Omega)$ is defined by $Bu = g(u)$ with

$$\mathcal{D}(B) = \Big\{ u \in L^1(\Omega);\ g(u) \in L^1(\Omega) \Big\},$$

and $\mathcal{D}(A) \subseteq \mathcal{D}(B)$.

Lemma 3.5 *As $\varepsilon \to 0$, $\varepsilon\, A + B \to B$, in the sense of resolvent.*

Proof : Since, $\varepsilon\beta^{-1}$ converges to the graph $N \equiv 0$, then the proof is a simple consequence of [7].

■

Proposition 3.6 *1. $\overline{\mathcal{D}(\mathcal{A}^{d,\infty})} = \Big\{ z \in L^1(\Omega)\ ;\ m_0 \le z \le M_0\ a.e.\ \Omega \Big\}$.*

2. $\overline{\mathcal{D}(\mathcal{A}^{\infty,r})} = \mathbb{R}$.

3. $\overline{\mathcal{D}(\mathcal{A}^{\infty})} = [m_0, M_0]$.

Proof :

1. Clearly, by the definition of $\mathcal{A}^{d,\infty}$ we have $\overline{\mathcal{D}(\mathcal{A}^{d,\infty})} \subseteq \Big\{ z \in L^1(\Omega)\ ;\ m_0 \le z \le M_0 \text{ a.e. } \Omega \Big\}$. To prove the converse part, let $u \in L^1(\Omega)$ be such that $m_0 \le u \le M_0$ a.e. in Ω and consider u_ε the solution of

$$\begin{cases} u_\varepsilon - \varepsilon \triangle w_\varepsilon = u, \quad u_\varepsilon = \beta(w_\varepsilon) & \text{in } \Omega \\ \\ \partial_{\vec{n}} w_\varepsilon = 0, & \text{in } \partial\Omega \end{cases}$$

   It is clear that $m_0 \le u_\varepsilon \le M_0$ a.e. in Ω, so that $u_\varepsilon \in \mathcal{D}(\mathcal{A}^{d,\infty})$, for each $\varepsilon > 0$. Since, $\varepsilon\beta^{-1}$ converges to 0 in the graph sense, then by [7], we deduce that $u_\varepsilon \to u$ in $L^1(\Omega)$, as $\varepsilon \to 0$, and the proof completes.

2. This is a simple consequence of the fact that $\mathcal{D}(g) = \mathbb{R}$. Indeed, for any $c \in \mathbb{R}$, $rg(c)$ is well defined and there exists $f \in L^1(\Omega)$, such that $\displaystyle⨍_\Omega f = rg(c)$, so that $f \in \mathcal{A}^{\infty,r}(c)$ and $\mathcal{A}^{\infty,r}(c) \neq \emptyset$.

3. It is clear that $\overline{\mathcal{D}(\mathcal{A}^{\infty})} \subseteq [m_0, M_0]$. Now, for $c \in [m_0, M_0]$, $G(c)$ is not empty and for any $\alpha \in G(c)$, there exists $f \in L^1(\Omega)$, such that $\displaystyle⨍_\Omega f = \alpha$, so that $f \in \mathcal{A}^{\infty}(c)$ and we deduce that $\mathcal{A}^{\infty}(c) \neq \emptyset$, and $c \in \mathcal{D}(\mathcal{A}^{\infty})$.

■

3.1 Large Diffusion.

In this subsection, we begin by studying the first part of Theorem 3.1, i.e. the case of large diffusion. So, we fix $r > 0$ and we let $d \to \infty$. In order to apply Theorem 2.3, we consider first the Cauchy problem

$$\begin{cases} u_t + Au = 0 \quad \text{in } (0,T) \\ \\ u(0) = u_0. \end{cases} \tag{30}$$

It is known (see for instance [31] and [34] and the references therein) that, if $u_0 \in L^\infty(\Omega)$, then the mild solution u of (30) is the unique weak solution of

$$\begin{cases} u_t - \Delta w = 0 \qquad u = \beta(w) & \text{in } Q \\ \\ \partial_{\vec{n}} w = 0, & \text{on } \Sigma \\ \\ u(0) = u_0 & \text{in } \Omega \end{cases} \tag{31}$$

and

$$\lim_{t\to\infty} u(t) = \fint_\Omega u_0.$$

Lemma 3.7 *For any $u_0 \in L^\infty(\Omega)$ and $f \in L^1(\Omega)$, the mild solution of*

$$\begin{cases} u_t + B\ u = f \quad \text{in } (0,T) \\ \\ u(0) = u_0 \end{cases} \tag{32}$$

is the solution of the ode

$$\begin{cases} u_t + r\ g(u) = f \qquad \text{in} \quad (0,T) \\ \\ u(0) = u_0 \end{cases}$$

in the sense that $u \in W^{1,1}((0,T), L^1(\Omega))$ and the equation is satisfied a.e. in Q with $u(0) = u_0$. Moreover, if $f \equiv 0$, then

$$\lim_{t\to\infty} u(t) = m_0 \vee (u_0 \wedge M_0) \qquad \text{a.e. in } \Omega.$$

Proof. Thanks to Lemma 3.5, we know that $\varepsilon A + B$ converges to B, so the limit of the mild solution u_ε of

$$\begin{cases} u_t + \varepsilon Au + Bu = 0 \quad \text{in } (0,T) \\ \\ u(0) = u_0, \end{cases}$$

converges to u (the mild solution of (32)). Using Proposition 3.2, $u_\varepsilon \in L^\infty(Q)$, there exists $w_\varepsilon \in L^2(0,T;H^1(\Omega))$, $u_\varepsilon = \beta(w_\varepsilon)$ a.e. in Q and

$$\varepsilon \int_0^\tau \int_\Omega Dw_\varepsilon D\xi + r\int_0^\tau \int_\Omega g(u_\varepsilon)\xi = \int_0^\tau \int_\Omega f\xi + \int_0^\tau \int_\Omega u_\varepsilon \xi_t + \int_\Omega u_0\xi(0). \tag{33}$$

Moreover, u_ε, w_ε and $rg(u_\varepsilon)$ are bounded in $L^\infty(\Omega)$ and $\varepsilon w_\varepsilon$ is bounded in $L^2(0,T;H^1(\Omega))$, so that

$$\varepsilon w_\varepsilon \to 0 \quad \text{weakly in } L^2(0,T;H^1(\Omega)).$$

Passing to the limit in (33), we deduce that $g(u) \in L^\infty(Q)$ and

$$r\int_0^\tau \int_\Omega g(u)\,\xi = \int_0^\tau \int_\Omega f\,\xi + \int_0^\tau \int_\Omega u\,\xi_t + \int_\Omega u_0\,\xi(0).$$

Since, $f,\ g(u) \in L^1(Q)$ then $u \in W^{1,1}(0,T,L^1(\Omega))$ and, for a.e. $x \in \Omega$, $u_t(x) + rg(u(x)) = f(x)$ in $(0,T)$. This ends up the proof of the first part of the lemma.
Assuming that $f \equiv 0$, there exists a measurable function u_∞, such that, as $t \to \infty$, $u(t) \to u_\infty$, a.e. in Ω and $g(u_\infty) = 0$, a.e. in Ω ; i.e. $m_0 \leq u_\infty \leq M_0$, p.p. in Ω. Moreover, since u is bounded in $L^\infty(Q)$, then the convergence is in $L^1(\Omega)$. Let us prove that $u_\infty = m_0 \vee (u_0 \wedge M_0)$. First, note that, if $m_0 \leq u_0(x) \leq M_0$, then, $u(t,x) = u_0(x)$, for any $t \geq 0$, and $u_\infty(x) = u_0(x)$. If $u_0(x) \geq M_0$, then it is clear that $t \longmapsto u(t,x)$ is nonincreasing. So,

$$u_\infty(x) \leq u_0(x),$$

and we deduce that $u_\infty(x) = M_0$. In a similar way, $u_\infty(x) = m_0$, a.e.$x \in \Omega$, such that $u_0 \leq m_0$, and the proof of the lemma finishes. ■

Now, applying Theorem 2.3, we get the first part of Theorem 3.1. More precisely,

Proposition 3.8 *For any $r > 0$, as $d \to \infty$,*

$$u^{d,r} \to c \quad \text{in } \mathcal{C}((0,T), L^1(\Omega)),$$

where $c \in \mathcal{C}^1([0,T))$ is the unique solution of the ode

$$\begin{cases} c_t + r\,g(c) = \displaystyle -\!\!\!\!\!\!\int_\Omega f & \text{in} \quad (0,T) \\ c(0) = \displaystyle -\!\!\!\!\!\!\int_\Omega u_0. \end{cases}$$

Proof : Recall that $u^{d,r}$ is the mild solution of $CP^{d,r}(u_0,f)$, $\mathcal{A}^{d,r} = dA + rB$. Moreover, thanks to Lemma 3.5, A and B satisfy the assumption (13), with $\tilde{B} = B$, and $\lim\limits_{t\to\infty} e^{-tA}u_0 = -\!\!\!\!\!\!\int_\Omega u_0$. Since $-\!\!\!\!\!\!\int_\Omega u_0 \in \overline{\mathcal{D}(\mathcal{A}^{\infty,r})}$, then applying Theorem 2.3 we deduce that $u^{d,r} \to u$ in $\mathcal{C}((0,T);L^1(\Omega))$, where u is the mild solution of

$$\begin{cases} u_t + \mathcal{A}^{\infty,r}u \ni f & \text{in } (0,T) \\ u(0) = \displaystyle -\!\!\!\!\!\!\int_\Omega u_0. \end{cases} \tag{34}$$

It is not difficult to see that the mild solution of (34) is the mild solution of

$$\begin{cases} u_t + Bu = \displaystyle -\!\!\!\!\!\!\int_\Omega f & \text{in} \quad (0,T) \\ \\ u(0) = \displaystyle -\!\!\!\!\!\!\int_\Omega u_0, \end{cases}$$

and, thanks to Lemma 3.7, the proof is finished. ■

3.2 Large Reaction.

Assume now, that $d > 0$ is fixed and that r is very large ; i.e. $r \to \infty$.

Lemma 3.9 *As $r \to \infty$, $u^{d,r} \to u$ in $\mathcal{C}((0,T); L^1(\Omega))$, where u is the mild solution of*

$$\begin{cases} u_t + \mathcal{A}^{d,\infty} u \ni f & \text{in } (0,T) \\ \\ u(0) = m_0 \vee (u_0(x) \wedge M_0). \end{cases} \tag{35}$$

Proof : Thanks to Lemma 3.7,

$$\lim_{t\to\infty} e^{-tB} u_0 = m_0 \vee (u_0(x) \wedge M_0) \quad \text{ a.e. } x \in \Omega.$$

On the other hand, we have $\mathcal{A}^{d,r(k)} \to \mathcal{A}^{d,\infty}$ and $m_0 \vee (u_0 \wedge M_0) \in \overline{\mathcal{D}(\mathcal{A}^{d,\infty})}$, then the lemma is a simple consequence of Theorem 2.3. ■

At last, the proof of the second part of Theorem 3.1 follows by characterizing the mild solution of

$$\begin{cases} u_t + \mathcal{A}^{d,\infty} u \ni f & \text{in } (0,T) \\ \\ u(0) = v_0 \end{cases} \tag{36}$$

Proposition 3.10 *Let $f \in L^\infty(Q)$ and $v_0 \in L^\infty(\Omega)$ such that $m_0 \le v_0 \le M_0$ a.e. in Ω. Then, the mild solution of (36) is the unique solution of the obstacle problem*

$$\begin{cases} u_t - d\Delta w + G(u) = f, \quad u = \beta(w) & \text{in } Q \\ \\ \partial_{\vec{n}} w = 0, & \text{in } \Sigma \\ \\ u(0) = v_0 & \text{on } \Omega. \end{cases}$$

That is : $u \in \mathcal{C}([0,T), L^1(\Omega))$, $u(0) = v_0$, there exists $w \in L^2(0,T; H^1(\Omega))$, $u = \beta(w)$, there exists $\eta \in L^2(0,T; L^2(\Omega))$ such that $\eta \in G(u)$ a.e. in Q and

$$d \int_0^\tau \int_\Omega DwD\xi + \int_0^\tau \int_\Omega \eta\xi = \int_0^\tau \int_\Omega f\xi + \int_0^\tau \int_\Omega u\xi_t + \int_\Omega v_0 \xi(0)$$

for any $\xi \in \mathcal{C}^1([0,\tau] \times \overline{\Omega})$ such that $\xi(.,\tau) \equiv 0$.

Proof : The proof of this proposition is standard by now (see for instance [31]). For completeness let us give the arguments. For $t \in [0,\tau]$, consider a subdivision $t_0 = 0 < t_1 < ... < t_{n-1} < \tau \leq t_n$, with $t_i - t_{i-1} = \varepsilon$, $f_1, ..., f_2 \in L^\infty(\Omega)$ with $\displaystyle\sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} ||f(t) - f_i||_{L^1(\Omega)} \leq \varepsilon$ and $\displaystyle\sum_{i=1}^{n} \varepsilon ||f_i||_{L^\infty(\Omega)} \leq \int_0^T \|f||_{L^\infty(\Omega)}$. By definition of the mild solution, u is given by

$$u(t) = L^1 - \lim_{\varepsilon \to 0} u_\varepsilon(t)$$

uniformly for $t \in [0,T)$, where u_ε is the approximate solution by $u_\varepsilon(0) = u_0$, $u_\varepsilon(t) = u_i$ for $t \in]t_{i-1}, t_i]$, $i = 1, ..., n$, where u_i satisfies $u_i + \varepsilon \mathcal{A}^{d,\infty} u_i = \varepsilon f_i + u_{i-1}$. That is, there exists $w_i \in H^1(\Omega)$ and $\eta_i \in L^\infty(\Omega)$ such that

$$\begin{cases} u_i - \varepsilon d \Delta w_i + \varepsilon \eta_i = u_{i-1} + \varepsilon f_i, \quad u_i = \beta(w_i),\ \eta_i \in G(u_i) & \text{in } \Omega \\ \partial_{\vec{n}} w_i = 0, & \text{in } \partial\Omega \end{cases} \tag{37}$$

Thanks to (24) and (25), it follows that

$$\int_\Omega |u_i| + \varepsilon \int_\Omega |\eta_i| \leq \int_\Omega |u_{i-1}| + \varepsilon \int_\Omega |f_i| \tag{38}$$

and

$$||u_i||_{L^\infty(\Omega)} + \varepsilon ||\eta_i||_{L^\infty(\Omega)} \leq 2\Big(||u_0||_{L^\infty(\Omega)} + \int_0^T \|f||_{L^\infty(\Omega)} \Big) =: M,$$

so that,

$$||u_\varepsilon(t)||_{L^\infty(\Omega)} + \int_0^T ||\eta_\varepsilon(t)||_{L^\infty(\Omega)} dt \leq M \quad \forall t \in [0,T], \tag{39}$$

where $f_\varepsilon,\ \eta_\varepsilon\ : [0,\tau] \longrightarrow L^1(\Omega)$, with $f_\varepsilon(t) = f_i$ and $\eta_\varepsilon = \eta_i$, for any $t \in]t_{i-1}, t_i], i = 1, ..., n$. Taking $\xi = w_i$ as a test function in (37) and using the fact that

$$\int_\Omega (u_{i-1} - u_i) w_i \leq \int_\Omega j(u_{i-1}) - \int_\Omega j(u_i)$$

we deduce that

$$\int_\Omega j(u_i) + \varepsilon d \int_\Omega |Dw_i|^2 + \varepsilon \int_\Omega \eta_i w_i \leq \varepsilon \int_\Omega f_i w_i + \int_\Omega j(u_{i-1}), \tag{40}$$

so that, adding for $i = 1, ..., n$, we get

$$\int_\Omega j(u_\varepsilon(\tau)) + d \int_0^\tau \int_\Omega |Dw_\varepsilon|^2 + \int_0^\tau \int_\Omega \eta_\varepsilon w_\varepsilon \leq \int_\Omega j(u_0) + \int_0^\tau \int_\Omega f_\varepsilon w_\varepsilon. \tag{41}$$

where $w_\varepsilon\ : [0,\tau] \longrightarrow H^1(\Omega)$ with $w_\varepsilon = w_i$, for any $t \in]t_{i-1}, t_i], i = 1, ..., n$. Since $\text{Im}(\beta) = \mathbb{R}$, then we deduce that w_ε is bounded in $L^\infty((0,\tau) \times \Omega)$ and, using the fact that $j \geq 0$, $\eta_\varepsilon w_\varepsilon \geq 0$ a.e. in $[0,\tau] \times \Omega$, it follows that w_ε bounded in $L^2(0,\tau, H^1(\Omega))$.

So, let $w \in L^2(0,\tau,H^1(\Omega))$ and $\eta \in L^2((0,\tau)\times\Omega)$, such that $w_{\varepsilon_k} \longrightarrow w$, weakly in $L^2(0,\tau,H^1(\Omega))$ and $\eta_\varepsilon \to \eta$ weakly in $L^2((0,\tau)\times\Omega)$, where ε_k is a sequence such that $\varepsilon_k \longrightarrow 0$. Using the monotony of β and G, we have $u=\beta(w)$ and $\eta \in G(w)$ a.e. in Q. Now, let us consider $\tilde{u}_\varepsilon$ the function from $[0,\tau]$ into $L^1(\Omega)$, defined by $\tilde{u}_\varepsilon(t_i)=u_i$ and $\tilde{u}_\varepsilon$ linear in $[t_{i-1},t_i]$, then (37) implies that

$$d\int_0^\tau\int_\Omega Dw_\varepsilon D\xi+\int_0^\tau\int_\Omega \eta_\varepsilon\xi=\int_0^\tau\int_\Omega f_\varepsilon\xi+\int_0^\tau\int_\Omega \tilde{u}_\varepsilon\xi_t+\int_\Omega u_0\xi(0). \tag{42}$$

for any $\xi \in \mathcal{C}^1([0,\tau]\times\overline{\Omega})$, s.t $\xi(\tau)=0$. Letting $\varepsilon \to 0$ in (42) , we get

$$d\int_0^\tau\int_\Omega DwD\xi+\int_0^\tau\int_\Omega \eta\xi=\int_0^\tau\int_\Omega f\xi+\int_0^\tau\int_\Omega u\xi_t+\int_\Omega u_0\xi(0). \tag{43}$$

for any $\xi \in \mathcal{C}^1([0,\tau]\times\overline{\Omega})$, s.t $\xi(\tau)=0$. ∎

3.3 Large Reaction and Diffusion.

To finish the proof of Theorem 3.1, we consider the case where both the reaction and the diffusion rates are very large. So, we assume that $d=d(k)$ and $r=r(k)$, with $\lim_{k\to\infty} d(k)=\lim_{k\to\infty} r(k)=\infty$. First let us consider the case where

$$\lim_{k\to\infty}\frac{d(k)}{r(k)}=0, \tag{44}$$

in other words the reaction is more competitive than the diffusion. Thanks to Theorem 2.5, we need to consider the operator

$$A+\frac{r(k)}{d(k)}B=:A_k.$$

Since $\dfrac{r(k)}{d(k)}\to\infty$, then by Theorem 3.3, we deduce that

$$A_k \to \mathcal{A}^{1,\infty}.$$

On the other hand, thanks to Lemma 3.7, remember that $\lim_{t\to\infty} e^{-tB}u_0 = m_0\vee(u_0\wedge M_0)$, a.e. Ω. Since, $m_0\vee(u_0\wedge M_0)\notin\overline{\mathcal{D}(\mathcal{A}^\infty)}$, then, having in mind Theorem 2.5, the boundary layer of the limit of $u^{d,r}$ is given by the large time behavior of the solution of the Cauchy problem

$$\begin{cases} u_t+\mathcal{A}^{1,\infty}u\ni 0 & \text{in } (0,\infty)\\ \\ u(0)=m_0\vee(u_0\wedge M_0). \end{cases} \tag{45}$$

To this aim, we consider the set

$$\mathcal{K}=\{z\in[m_0,M_0];\ \exists c\in\mathbb{R},\ z=\beta(c)\}.$$

It is not difficult to see that $\mathcal{K}$ is a nonempty closed subset of $L^1(\Omega)$ and, moreover, $\mathcal{K}$ is contained in the set of stationary solution of (45), i.e. for any $z\in\mathcal{K}$, $e^{-t\mathcal{A}^{1,\infty}}z=z$, for any $t\geq 0$.

Proposition 3.11 *For any* $\underline{u}_0 \in L^1(\Omega)$*, such that* $m_0 \leq \underline{u}_0 \leq M_0$ *a.e. in* Ω*, there exists a unique* $\underline{\underline{u}}_0 \in \mathcal{K}$*, such that*

$$e^{-t\mathcal{A}^{1,\infty}}\underline{u}_0 \to \underline{\underline{u}}_0 \quad in \;\; L^1(\Omega), \quad as \;\; t \to \infty.$$

Proof : The proof of this proposition follows the same line of [28]. To summarize the proof we give just the main lines. For more details one can see [28]. Indeed, using the results of [7], we prove first that the resolvents are relatively compact from $L^\infty(\Omega)$ into $L^1(\Omega)$, i.e. if B is a bounded subset of $L^\infty(\Omega)$, then $(I+\epsilon\mathcal{A}^{1,\infty})^{-1}(B)$ is relatively compact in $L^1(\Omega)$ so that, the orbit $\gamma(\underline{u}_0) = \{e^{-t\mathcal{A}^{1,\infty}}\underline{u}_0 \,:\, t \geq 0\}$ is relatively compact in $L^1(\Omega)$. In addition, it is not difficult to prove that

$$\int_0^\infty \int_\Omega |Dw|^2 \leq \int_\Omega j(\underline{u}_0),$$

which implies that the $\omega-$limit set is contained in $\mathcal{K}$. Then, by using the additive semi-group property, we deduce that the $\omega-$ limit set is a singleton and the proof finished. ∎

Remark 3.12 *The identification of* $\underline{\underline{u}}_0$ *is an open problem. In this direction, the reader can see the papers [29] and [31].*

Lemma 3.13 *Under the assumption (44),* $u^{d,r} \to c$ *in* $\mathcal{C}((0,T);L^1(\Omega))$*, where* c *is the mild solution of the Cauchy problem*

$$\begin{cases} c_t + \mathcal{A}^\infty c \ni f & in \ (0,T) \\ \\ c(0) = \underline{\underline{u}}_0 \end{cases} \tag{46}$$

and $\underline{\underline{u}}_0$ *is given by Proposition 3.11.*

Proof : Recall that we have $d(k)A + r(k)B \to \mathcal{A}^\infty$, $\lim_{k\to\infty} \dfrac{r(k)}{d(k)} = \infty$, $\lim_{t\to\infty} e^{-tB}u_0 = m_0 \vee (u_0 \wedge M_0)$ and $\underline{\underline{u}}_0 = \lim_{t\to\infty} e^{-t\mathcal{A}^{1,\infty}} m_0 \vee (u_0 \wedge M_0) \in \overline{\mathcal{D}(\mathcal{A}^\infty)}$. Then, the result of the lemma is a consequence of Theorem 2.3.

∎

Proposition 3.14 *For any* $c_0 \in [m_0, M_0]$*, the mild solution of*

$$\begin{cases} c_t + \mathcal{A}^\infty c \ni f & in \ (0,T) \\ \\ c(0) = c_0 \end{cases} \tag{47}$$

is the unique solution of the ode

$$\begin{cases} c_t + G(c) \ni \fint_\Omega f \quad in \quad (0,T), \\ \\ c(0) = c_0, \end{cases}$$

in the sense that $c \in W^{1,1}(0,T)$*, there exists* $\eta \in L^1(0,T)$*, such that* $c(0) = c_0$ *and* $c_t + \eta = \fint_\Omega f$*, a.e. in* $(0,T)$*.*

Proof Since, for any $f \in L^1(\Omega)$, $(I + \lambda\, \mathcal{A}^\infty)^{-1}(f) = (I + \lambda\, \mathcal{A}^\infty)^{-1}(\fint_\Omega f) = (I + \lambda\, G)^{-1}(\fint_\Omega f)$, then the mild solution of (47) is the mild solution of

$$\begin{cases} c_t + G(c) \ni \fint_\Omega f \quad \text{in } (0,T) \\ \\ c(0) = c_0. \end{cases} \tag{48}$$

Thus the proof is a simple application the classical theory of nonlinear semigroup governed by maximal monotone graphs in Hilbert space (cf. [14]). ■

To end up the proof of Theorem 3.1, we consider the case

$$\lim_{k\to\infty} \frac{d\,(k)}{r(k)} = \infty. \tag{49}$$

i.e.; the diffusion is more competitive than the reaction.

Lemma 3.15 *Under the assumption (49),* $u^{d,r} \to c$ *in* $\mathcal{C}((0,T); L^1(\Omega))$*, where* c *is the mild solution of the Cauchy problem*

$$\begin{cases} c_t + G(c) \ni f \quad in\ (0,T) \\ \\ c(0) = m_0 \vee (\fint_\Omega u_0 \wedge M_0). \end{cases} \tag{50}$$

Proof : Thanks to [7], it is not difficult to see that $A + \varepsilon\, B \to A$, as $\varepsilon \to 0$, in the sense of resolvent. So, using Theorem 2.5, we need to consider the operator

$$\frac{d\,(k)}{r(k)}\, A + B =: B_k.$$

Since $\dfrac{d\,(k)}{r(k)} \to \infty$, then Corollary 3.4 implies that

$$B_k \to \mathcal{A}^{\infty,1}.$$

On the other hand, remember that $\lim_{t\to\infty} e^{-tA}u_0 = \fint_\Omega u_0$ and $\fint_\Omega u_0 \notin \overline{\mathcal{D}(\mathcal{A}^\infty)}$. So, we need to consider the Cauchy problem

$$\begin{cases} u_t + \mathcal{A}^{\infty,1}u \ni 0 & \text{in } (0,\infty) \\ u(0) = \fint_\Omega u_0. \end{cases} \tag{51}$$

Since the mild solution of (51) is the mild solution of

$$\begin{cases} c_t + g(c) = 0 & \text{in } (0,\infty) \\ c(0) = \fint_\Omega u_0, \end{cases}$$

then, using Lemma 3.7, the proof is complete. ■

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