Hele-Shaw Type Problems with Dynamical Boundary Conditions

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Abstract

In this paper, we study the nonlinear evolution equation of Hele-Shaw type with dynamical boundary conditions. That is, the equation $u_t = \Delta w + f$ where $u \in H(w)$ and H is the Heaviside function, with boundary condition $\mu(x, w)\, \partial_t w + k\, \nabla w \cdot \nu = g$, where ν denotes the outward normal vector of the fixed boundary of the domain. We prove existence, uniqueness and some qualitative properties of the solution.
keywords : Hele-Shaw problem, dynamical boundary condition, Neuman boundary condition, evolution problem, nonlinear semigroup theory.

1 Introduction and main results

An equation of the Hele-Shaw type is a nonlinear pde of the form

$$u_t = \Delta w + f \quad \text{with } u \in H(w) \tag{1.1}$$

where H is the multivalued Heaviside function defined by

$$H(r) = \begin{cases} 0 & \text{if } r < 0 \\ [0,1] & \text{if } r = 0 \\ 1 & \text{if } r > 0 \ . \end{cases}$$

This equation appears in the study of the weak formulation of the mathematical model of the so called Hele-Shaw problem (cf. [8], [7] and [10]) . The equation (1.1) stated in a bounded domain Ω of $\mathbb{R}^N$, with $N \geq 1$, needs to be completed with boundary conditions on $\partial\Omega$, the boundary of Ω. As far as we know, the Hele-Shaw problem was studied with prescribed static Neumann boundary condition, i.e.

$$\nabla w \cdot \nu = g \quad \text{on } \Gamma \ := \partial\Omega$$

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(cf. [8], [18] and [15]). It was studied also with prescribed Dirichlet boundary condition, i.e. $p = h$ on Γ (cf. [9]). But, in some practical situations it may be not possible to prescribe or to control the exact value of w on Γ. In [20], the authors consider the case of an evolutionary condition of nonlocal type, assuming that w has a constant but unknown value along Γ, they prescribed the condition

$$\mu \frac{d}{dt} w + \int_\Gamma \nabla w \cdot \nu = g(t),$$

where $\mu \in \mathbb{R}$ and g are given. In this paper, we are interested in the case of local evolutionary boundary conditions where the value of w and the flux on Γ has unknown values related by the equation

$$\mu(x, w)\, \partial_t w + \nabla w \cdot \nu = g(t, x) \quad \text{for } x \in \Gamma, \tag{1.2}$$

with $\mu : \Omega \times \mathbb{R}^+ \to \mathbb{R}^+$ and $g : (0, T) \times \Omega \to \mathbb{R}^+$ are given measurable functions. Notice that μ may vanishes on a part of Γ, so that the boundary condition is static on a part and dynamic on the remaining one. Denoting $\rho(., r) = \int_0^r \mu(., s) ds$, the formulation (1.2) is equivalent to

$$\partial_t \rho(x, w) + \nabla w \cdot \nu = g \quad \text{on } \Gamma. \tag{1.3}$$

In other words the boundary condition (1.2) means that w is related to the flux by

$$\rho(x, w(t)) + \int_0^t \nabla w(x, s) \cdot \nu \, ds = \lambda(t, x),$$

for a given function λ depending on the initial data of w and possibly reactions terms on Γ. This kind of boundary condition is called dynamical boundary one, they appear in numerous problems (cf. [19], [5], [1], [14], [13], [22] and the references therein).

So, taking into acount the initial data for the problem, the weak formulation of the Hele-Shaw problem with dynamical boundary condition reads

$$E(u_0, z_0, f, g) \qquad \begin{cases} \partial_t u - \Delta w = f(t, x), \quad u \in H(w) & \text{in } Q_T = (0, T) \times \Omega \\ \\ \partial_t z + \partial_\nu w = g(t, x), \quad z = \rho(x, w) & \text{on } \Sigma_T = (0, T) \times \Gamma \\ \\ u(0) = u_0 \quad \text{in } \Omega, \quad z(0) = z_0 \quad \text{on } \Gamma, \end{cases}$$

where $\partial\nu$ denotes the outward normal derivative of w, i.e. $\nabla w \cdot \nu$, the functions f and g summarize driving forces terms in Ω and on Γ, respectively, u_0 and z_0 are the initial data for u and z respectively. Our main goal is to study the existence and uniqueness of a solution (u, z), as well as to prove some natural qualitative properties of this solution, like the increasing property of the moving interface and the nondecreasing property of the mushy region. Various results of existence, uniqueness and others properties for linear and nonlinear evolution problem with dynamical boundary condition was proven in this last decade (cf [1], [14], [22], [11] and [16]). The most relevant in the study of this particular case is the fact that the inverse of the graph H is not everywhere defined, the domain of H^{-1} is reduced to $[0, 1]$. Recall that in [16], the existence and uniqueness of solutions for problems of the type

$E(u_0, z_0, f, g)$ was proved for a large class of maximal monotone graph instead of H, with an everywhere defined inverse. For the case of the Heaviside function, the problem is completely different, a necessary condition appears for the existence of a solution. Recall that even in the case where $\rho \equiv 0$, i.e. static boundary condition, existence and uniqueness of a solution for this kind of evolution problem is known to be true only if $\int_\Omega u_0 + \int_0^t (\int_\Omega f + \int_\Gamma g) \in (0, |\Omega|)$ for any $t \in [0, T)$ (cf. [9], [15] and [18]). In the case $\rho \not\equiv 0$, we prove that this condition becomes

$$\int_\Omega u_0 + \int_\Gamma z_0 + \int_0^t (\int_\Omega f + \int_\Gamma g) \in (0, |\Omega| + \int_\Gamma \sup_{r \in \mathbb{R}^+} \rho(., r)). \tag{1.4}$$

The notion of solution of the problem $E(u_0, z_0, f, g)$, we have in mind is naturally defined as follows.

Definition 1 *Let $0 < T \le \infty$, $(u_0, z_0) \in L^2(\Omega) \times L^2(\Gamma)$ and $(f, g) \in L^2_{loc}([0, T); L^2(\Omega)) \times L^2_{loc}([0, T); L^2(\Gamma))$ be given. A solution of $E(u_0, z_0, f, g)$ in $(0, T)$ is a couple (u, z) such that $u \in \mathcal{C}([0, \tau); L^1(\Omega))$, $z \in \mathcal{C}([0, \tau); L^1(\Gamma)) \cap L^2(\Sigma_T)$, $0 \le u \le 1$ a.e. in Q_T, $u(0) = u_0$ a.e. in Ω, $z(0) = z_0$ a.e. in Γ and there exists $w \in L^2_{loc}(0, T; H^1(\Omega))$ such that $u \in H(w)$ a.e. in Q_T, $z = \rho(., w)$ a.e. on Σ_T and*

$$\frac{d}{dt} \int_\Omega u\, \xi + \frac{d}{dt} \int_\Gamma z\, \xi + \int_\Omega \nabla w . \nabla \xi = \int_\Omega f\, \xi + \int_\Gamma g\, \xi \quad \text{in } \mathcal{D}'(0, T) \tag{1.5}$$

for any $\xi \in \mathcal{C}^1(\overline{\Omega})$.

Throughout the paper, we denote by $⨍_\Omega f$ the average of f in Ω, given by $\frac{1}{|\Omega|} \int_\Omega f$. For $1 \le p \le \infty$, $L^p(\Omega)^+$ is the cone of nonneagtive functions of $L^p(\Omega)$. We assume that $\rho : \Gamma \times \mathbb{R} \to \mathbb{R}$ is a caratheodory function, such that $\rho(x, r)$ is nondecreasing in r,

$$\rho(., 0) = 0, \quad \text{and} \quad \rho(., r) \le a(.)\, |r| + b(.) \quad \text{a.e. in } \Gamma \text{ and for any } r \in \mathbb{R}^+ \tag{1.6}$$

with $a,\ b\ \in L^\infty(\Omega)$. Moreover, setting $\overline{\rho}(x) = \sup_{r \in \mathbb{R}} \rho(x, r)$ a.e. $x \in \Gamma$, we assume that, either

$$\overline{\rho}(x) = +\infty \text{ a.e. } x \in \Gamma \quad \text{or} \quad \overline{\rho} \in L^1(\Gamma). \tag{1.7}$$

Theorem 1 *Let $f \in L^2_{loc}([0, T); L^2(\Omega)^+)$, $g \in L^2_{loc}([0, T); L^2(\Gamma)^+)$, $u_0 \in L^\infty(\Omega)^+$, $0 \le u_0 \le 1$ and $z_0 \in L^1(\Gamma)^+$ such that $\int_0^{z_0} \rho(., r)\, dr \in L^1(\Omega)$,*

$$z_0(x) \in \overline{Im(\rho(x, .))} \quad \text{a.e. } x \in \Gamma \tag{1.8}$$

and, set

$$\mu(t) = ⨍_\Omega u_0 + \frac{1}{|\Omega|} \int_\Gamma z_0 + \int_0^t \Big(⨍_\Omega f + \frac{1}{|\Omega|} \int_\Gamma g \Big)$$

and

$$T_0 = \max\Big\{t \in [0,T) \ ; \ \mu(t) < 1 + \frac{1}{|\Omega|}\int_\Gamma \overline{\rho}(x)dx\Big\}. \tag{1.9}$$

Then, there exists a unique triplet (u, z, τ)*, such that* $\tau \leq T_0$*,* (u, z) *is the solution of* $E(u_0, z_0, f, g)$ *in* $[0, \tau)$*,* $⨍_\Omega u(t) < 1$ *for any* $t \in [0, \tau)$ *and* $u(t) \equiv 1$ *in* Ω*, for any* $t \in [\tau, T_0)$*. Moreover, we have*

1. *For any* $t \in [0, \tau)$*,*
$$⨍_\Omega u(t) + \frac{1}{|\Omega|}\int_\Gamma z(t) = \mu(t). \tag{1.10}$$

2. *If* (u_i, z_i) *are two solutions of* $E(u_{0i}, z_{0i}, f_i, g_i)$*,* $i = 1,\ 2$*, then*
$$\frac{d}{dt}\Big\|\Big(u_1(t) - u_2(t)\Big)^+\Big\|_{L^1(\Omega)} + \frac{d}{dt}\Big\|\Big(z_1(t) - z_2(t)\Big)^+\Big\|_{L^1(\Gamma)}$$
$$\leq \Big\|\Big(f_1 - f_2\Big)^+\Big\|_{L^1(\Omega)} + \Big\|\Big(g_1 - g_2\Big)^+\Big\|_{L^1(\Gamma)}$$
in $\mathcal{D}'(0, \tau)$*.*

3. *For any* $0 \leq t_1 \leq t_2 \leq \tau$*,*
$$u(t_1) \leq u(t_2) \quad a.e. \ in \ \Omega.$$

4. *For any* $0 \leq t_1 \leq t_2 \leq \tau$*,*
$$\Big[\ u(t_2) < 1\ \Big] \subseteq \Big[\ u(t_2) = u(t_1) + \int_{t_1}^{t_2} f(t)\ \Big] \subseteq \Big[\ u(t_1) < 1\ \Big].$$

Since we are considering the case of nonegative driving forces f and g, then the problem corresponds to the well posed case of the Hele-Shaw problem ; in the sense that there exists nonegative couple (u, z) solution of $E(u_0, z_0, f, g)$. Recal that otherwize, i.e. for negative or changing sign driving forces f and/or g, the problem is ill-posed (cf. [8]), one may loose the existence of nonegative solution. On the other hand, in the case of static boundary condition, we know (cf. [9], [18] and [15]) that the problem is well posed up to T_0 given by (1.9) for which the domain is full and the model breaks down. For dynamical boundary condition, the situation is different. The model turns out to hold on even if the domain is full, with an evolution problem on the boundary up to T_0. More precisely the time τ for which the domain is full may be different from T_0 the time for which the model breaks down. This is the case in the following theorem.

Theorem 2 *Under the assumptions of Theorem 1, we assume moreover that* $\rho(x, r) = \rho(r)$*, with* ρ *convex. Let* (u, z, τ) *be the solution of* $E(u_0, z_0, f, g)$ *given by Theorem 1. Then,* $u(t) \equiv 1$ *inn* Ω *for any* $t \in [\tau, T)$ *and* z *is the unique solution of*

$$\begin{cases} -\Delta w = f(t,x), & in\ (\tau, T) \times \Omega \\ \partial_t z + \partial_\nu w = g(t,x), \quad z = \rho(w) & on\ (\tau, T) \times \Gamma. \end{cases} \tag{1.11}$$

in the sense that $z \in \mathcal{C}([\tau,\infty))$, $z(\tau) = \overline{\rho}$ *and there exists* $w \in L^2_{loc}(\tau,\infty; H^1(\Omega))$ *such that* $z = \rho(w)$ *a.e. on* Σ_T *and*

$$\frac{d}{dt}\int_\Gamma z\,\xi + \int_\Omega \nabla w.\nabla\xi = \int_\Omega f\,\xi + \int_\Gamma g\,\xi \quad \text{in } \mathcal{D}'(\tau, T_0),$$

for any $\xi \in \mathcal{C}^1(\overline{\Omega})$.

Remark 1 *1. In particular, Theorem 2 shows that even if the domain is full at the time* $\tau < T_0$, *the model holds on for* $t \in [\tau, T_0)$, *with* w *satisfying the evolution problem (1.11). This is a particular case of evolution problem with an elliptic equation in the interior of* Ω *and an evolution one on the boundary. Theorem 2 solves this kind of question for convex* ρ.

2. *In general we do not know whenever* $\tau = T_0$.

3. *In terms of the Hele-Shaw problem, the property* 4 *of Theorem 1 reflects the fact that the free boundary increases in times. This is du the injection property of the boundary condition (1.2) and the driving forces terms.*

4. *The property* 4 *describes the evolution of the set* $\Big[0 < u(t) < 1\Big]$, *the so called mushy region. In particluar,* 4) *implies that it is nondecreasing in time. In particular, this shows that if* $f \equiv 0$ *and* $u_0 = \chi_{\Omega_0}$ *with* $\Omega_0 \subset \Omega$, *then there exists* $\Big(\Omega(t)\Big)_{0\le t\le\tau}$ *such that* $\Omega(t_1) \subseteq \Omega(t_2)$ *for any* $t_1 \le t_2$, $u(t) = \chi_{\Omega(t)}$ *for any* $t \in [0,\tau]$, $\Omega(0) = \Omega_0$ *and* $\Omega(\tau) = \Omega$.

We will use nonlinear semigroup theory to study the problem $E(u_0, z_0, f, g)$. For this we need to study the existence and contraction property for the associate stationary problem ; this is the aim of the next section. Then, we show the existence of a solution in the sense of Crandall-Ligget exponential formula, and use it to show Theorem 1 and Theorem 2. In the last section we prove the qualitative properties 3. and 4.. At last, in the Appendix, we give the proof of a more or less known existence result for an elliptic problem that we need for the proof of our result.

2 The stationary problem

To begin with, let us consider the elliptic problem

$$S_\lambda(f,g) \qquad \begin{cases} v - \lambda\,\Delta w = f,\ v \in H(w) & \text{in } \Omega \\ \\ z + \lambda\,\partial_\nu w = g,\ z = \rho(x,w) & \text{on } \Gamma, \end{cases}$$

where $\lambda > 0$.

Definition 2 *For $f \in L^1(\Omega)$ and $g \in L^1(\Gamma)$, we say that (v, w, z) is a solution of $S_\lambda(f, g)$ if $v \in L^1(\Omega)$, $w \in W^{1,1}(\Omega)$, $z \in L^1(\Gamma)$, $v \in H(w)$ a.e. in Ω, $z = \rho(x, w)$ a.e. on Γ and*

$$\lambda \int_\Omega \nabla w \cdot \nabla \xi = \int_\Omega (f - v)\xi + \int_\Gamma g\xi,$$

for any test function $\xi \in \mathcal{C}^1(\overline{\Omega})$.

Proposition 1 *For any $f_1,\ f_2 \in L^1(\Omega)$ and $g_1,\ g_2 \in L^1(\Gamma)$, if (v_i, w_i, z_i) is a solution of $S_\lambda(f_i, g_i)$ for $i = 1,\ 2$, then*

$$\int_\Omega (v_1 - v_2)^+ + \int_\Gamma (z_1 - z_2)^+ \leq \int_\Omega (f_1 - f_2)^+ + \int_\Gamma (g_1 - g_2)^+$$

and

$$\int_\Omega |v_1 - v_2| + \int_\Gamma |z_1 - z_2| \leq \int_\Omega |f_1 - f_2| + \int_\Gamma |g_1 - g_2|\,.$$

Proof : The proof of this proposition follows in the same way as Proposition C of [2]. We omit the details of the proof to avoid to repeat unnecessary the same arguments. ∎

Corollary 1 *For any $\lambda > 0$, $f \in L^1(\Omega)$ and $g \in L^1(\Gamma)$, $S_\lambda(f, g)$ has at most one solution.*

Setting

$$\mathcal{R} = \Big[0, 1 + \frac{1}{|\Omega|} \int_\Gamma \overline{\rho}(x)dx\Big[,$$

we have the following existence result

Theorem 3 *Let $f \in L^1(\Omega)$ and $g \in L^1(\Gamma)$ be nonnegative. If*

$$\fint_\Omega f + \frac{1}{|\Omega|} \int_\Gamma g \in \mathcal{R} \tag{2.1}$$

then, $S_\lambda(f, g)$ has a unique solution.

Recall that in the case $g \equiv 0$ and ρ independent of x, Theorem 3 is a particular case of [2]. For the case where $g \in L^1(\Omega)$ and ρ satisfies (1.7), we will construct the solution of $S_\lambda(f, g)$ as a limit, as $m \to \infty$, of the solution of the following elliptic equation

$$v = \Delta v^m + f \text{ on } \Omega, \quad \frac{\partial v^m}{\partial n} + \rho(x, v^m) = g \quad \text{on } \partial\Omega. \tag{2.2}$$

Thanks to Proposition A.1 in the Appendix, for any $f \in L^1(\Omega)^+$ and $g \in L^1(\Gamma)^+$, the problem (2.2) has a unique solution v_m in the sense that $v_m \in L^1(\Omega)^+$, ${v_m}^m \in W^{1,1}(\Omega)$, $z_m := \rho(x, {v_m}^m) \in L^2(\Gamma)^+$ and

$$\int_\Omega \nabla {v_m}^m \cdot \nabla \xi = \int_\Omega (f - v_m)\, \xi + \int_\Gamma (g - z_m)\, \xi,$$

for any test function $\xi \in \mathcal{C}^1(\overline{\Omega})$. Moreover, according to [2],

$$\|v_m\|_{L^1(\Omega)} + \|z_m\|_{L^1(\Gamma)} \leq \|f\|_{L^1(\Omega)} + \|g\|_{L^1(\Gamma)}, \tag{2.3}$$

$$\|{v_m}^m - \fint_\Omega {v_m}^m\|_{W^{1,q}(\Omega)} \leq C\Big(\|f\|_{L^1(\Omega)} + \|g\|_{L^1(\Gamma)}\Big) \tag{2.4}$$

for any $1 \leq q < \frac{N}{N-1}$ and, for any $\Omega' \subset\subset \Omega$, we have

$$\lim_{|y|\to 0} \sup_{m>0} \int_{\Omega'} |v_m(x+y) - v_m(x)| = 0. \tag{2.5}$$

Lemma 1 *$\big\{v_m\big\}_{m\geq 1}$ is relatively compact in $L^1(\Omega)$.*

Proof : Thanks to (2.3), (2.4) and Lemma A.1 in the Appendix, for $1 < q < \infty$ fixed, we have

$$\begin{aligned}
\|{v_m}^m\|_{L^q(\Omega)} &\leq \left(\frac{2}{|\Omega|}\|v_m\|_{L^1(\Omega)}\right)^m |\Omega|^{\frac{1}{q}} + C\,\|\nabla {v_m}^m\|_{L^q(\Omega)} \\
&\leq \left(\frac{2}{|\Omega|}(\|f\|_{L^1(\Omega)} + \|g\|_{L^1(\Gamma)})\right)^m |\Omega|^{\frac{1}{q}} + C\,(\|f\|_{L^1(\Omega)} + \|g\|_{L^1(\Gamma)})
\end{aligned}$$

so that,

$$\begin{aligned}
\|v_m\|_{L^q(\Omega)} &\leq \|{v_m}^m\|_{L^q(\Omega)}^{\frac{1}{m}} |\Omega|^{\frac{m-1}{mq}} \\
&\leq C_1\,(C_2^m + 1)^{\frac{1}{m}}
\end{aligned}$$

where C_1 and C_2 are independent of m. Obviously, this implies that v_m is bounded in $L^q(\Omega)$ and v_m is weakly relatively compact in $L^1(\Omega)$. So, by using (2.5), we deduce that v_m is relatively compact in $L^1(\Omega)$. ∎

Lemma 2 *If (2.1) is fulfilled, then as $m \to \infty$, $v_m \to v$ in $L^1(\Omega)$, ${v_m}^m \to w$ in $W^{1,1}(\Omega)$-weak, $\rho(., {v_m}^m) \to z$ in $L^1(\Gamma)$ and (v, w, z) is the unique solution of $S_\lambda(f, g)$.*

Proof : By using Lemma 1, the result of the lemma follows exactly in the same way as in Theorem B of [2]. ∎

Lemma 3 *If*

$$\fint_\Omega f + \frac{1}{|\Omega|}\int_\Gamma g \geq 1 + \frac{1}{|\Omega|}\int_\Gamma \overline{\rho}(x)dx \tag{2.6}$$

then, as $m \to \infty$,

$$v_m \to \fint_\Omega f + \frac{1}{|\Omega|}\int_\Gamma (g - \overline{\rho}) \quad \text{in } L^1(\Omega)$$

and

$$z_m \to \overline{\rho} \quad \text{in } L^1(\Gamma).$$

Proof : It is clear that if (2.6) is fulfilled then, the assumption (1.7) implies that $\overline{\rho} \in L^1(\Omega)^+$. Due to the contraction property of the solutions of (2.2), it is enough to prove the result for $f \in L^2(\Omega)^+$, $g \in L^2(\Gamma)^+$ and satisfying

$$\fint_\Omega f + \frac{1}{|\Omega|}\int_\Gamma g > 1 + \frac{1}{|\Omega|}\int_\Gamma \overline{\rho}(x)dx \ . \tag{2.7}$$

Using Lemma 1, there exits $m_k \to \infty$, such that $v_k := v_{m_k} \to v$ in $L^1(\Omega)$, and using (2.4) we have

$$\tilde{w}_k := {v_{m_k}}^{m_k} - C_{m_k} \to \tilde{w}_\infty \text{ in } W^{1,1}(\Omega) - \text{weak} \tag{2.8}$$

where $C_m = \fint_\Omega {v_m}^m$. It is clear that

$$\begin{aligned} \fint_\Omega v_k &= \fint_\Omega f + \frac{1}{|\Omega|}\int_\Gamma g - \frac{1}{|\Omega|}\int_\Gamma \rho(.,v_k) \\ &> 1 + \frac{1}{|\Omega|}\int_\Gamma (\overline{\rho} - \rho(.,v_k)) \\ &> 1, \end{aligned}$$

so that, by using Jensen's inequality, we have

$$C_{m_k} = \fint_\Omega {v_k}^{m_k} \geq \left(\fint_\Omega v_k\right)^{m_k} \qquad \to \infty. \tag{2.9}$$

Then, (2.4) implies that

$${v_k}^{m_k} \to \infty \quad \text{ a.e. } \overline{\Omega} \tag{2.10}$$

and

$$\rho(x, {v_k}^{m_k}) \to \overline{\rho}(x) \quad \text{ a.e. } x \in \Gamma \tag{2.11}$$

and, since $\rho(., {v_k}^{m_k})$ is bounded above by $\overline{\rho}$ which is in $L^1(\Gamma)$, then

$$\rho(., {v_k}^{m_k}) \to \overline{\rho} \quad \text{ in } L^1(\Gamma).$$

On the other hand, thanks to (2.8) and (2.9), we have $\dfrac{\tilde{w}_k}{C_{m_k}} \to 0$ a.e. in Ω and

$$\left(\frac{{v_k}^{m_k}}{C_{m_k}}\right)^{\frac{1}{m_k}} = \left(1 + \frac{\tilde{w}_k}{C_{m_k}}\right)^{\frac{1}{m_k}} \to 1 \text{ a.e. in } \Omega$$

so that $v = \lim_{m_k \to \infty} (C_{m_k})^{\frac{1}{m_k}}$ is constant in Ω. At last, since $\int_\Omega v_m + \int_\Gamma \rho(., {v_m}^m) = \int_\Omega f + \int_\Gamma g$, then by passing to the limit, we deduce that v is equal to $\fint_\Omega v = \fint_\Omega f + \frac{1}{|\Omega|}\int_\Gamma (g - \overline{\rho})$. ∎

Now, one sees that the natural space to study $E(u_0, z_0, f, g)$ is $X = L^1(\Omega)^+ \times L^1(\Gamma)^+$ provided with the natural norm

$$\|(f,g)\| = \|f\|_{L^1(\Omega)} + \|g\|_{L^1(\Gamma)}, \text{ for } (f,g) \in X.$$

Equipped with the usual partial ordering $(f,g) \leq (\tilde{f}, \tilde{g})$ if and only if $f \leq \tilde{f}$ a.e. in Ω and $g \leq \tilde{g}$ a.e. in Γ_N, X is a Banach lattice. In X, we define the multivalued operator A, by $(f,g) \in A(v,z)$ if and only if $v,\ f \in L^1(\Omega)^+,\ g,\ z \in L^1(\Gamma)^+,\ \int_\Omega f = \int_\Gamma g$ and either

$$(2.12) \quad \begin{cases} v \equiv \mu \text{ with } \mu \in \mathbb{R},\ \mu \geq 1 \quad \text{and } z = \overline{\rho} \ \text{ a.e. in } \Gamma \\ \text{or, there exists } w \in W^{1,1}(\Omega),\ v \in H(w) \text{ a.e. in } \Omega, \\ z = \rho(.,w) \text{ a.e. in } \Gamma \text{ and } \displaystyle\int_\Omega \nabla w \cdot \nabla \xi = \int_\Omega f\,\xi + \int_\Gamma g\,\xi\ , \quad \forall\, \xi \in \mathcal{C}^1(\overline{\Omega}). \end{cases}$$

Lemma 4 *A is m-T-accretive in X, i.e. for each $\lambda > 0$, $(I + \lambda A)^{-1}$ is a T-contraction everywhere defined in X.*

Proof : With A being defined as above, for $(f,g) \in X$, we have $(v,z) + A(v,z) \ni (f,g)$ if and only if $v \in L^1(\Omega)^+$, $z \in L^1(\Gamma)^+$, $\int_\Omega v + \int_\Gamma z = \int_\Omega f + \int_\Gamma g$ and either

$$\begin{cases} v \equiv \displaystyle\fint_\Omega f + \frac{1}{|\Omega|}\int_\Gamma (g - \overline{\rho}) \geq 1 \text{ in } \Omega \text{ and } z = \overline{\rho} \text{ a.e. in } \Gamma \\ \text{or, there exists } w \text{ such that } (v,w,z) \text{ is the solution of } S_1(f,g). \end{cases}$$

So, according to Lemma 2 and Lemma 3, there exists a unique solution (v,z) of $(v,z) + A(v,z) \ni (f,g)$ and

$$(v,z) = X - \lim_{m\to\infty} (v_m, \rho(x, {v_m}^m)),$$

where v_m is the solution of (2.2). ∎

Moreover, we have

Proposition 2 *The closure of the domain of A in X is given by*

$$\overline{\mathcal{D}(A)} = D_A := \ D_1 \cup D_2,$$

where

$$D_1 = \Big\{(u,z) \in X\ ;\ |u| \leq 1 \text{ a.e. in } \Omega \text{ and } z(x) \in \overline{Im(\rho(x,.)} \text{ a.e. } x \in \Gamma\Big\}$$

and

$$D_2 = \Big\{(\mu, \overline{\rho})\ ;\ \mu \in \mathbb{R},\ \mu \geq 1\Big\}$$

Proof : By definition of A, we see easily that $\overline{\mathcal{D}(A)} \subseteq D_A$ and $D_2 \subseteq \overline{\mathcal{D}(A)}$. So, it remains to prove that $D_1 \subseteq \overline{\mathcal{D}(A)}$. For this, it is enough to prove that $\overline{\mathcal{D}(A)} \supseteq K$, where

$$K = \Big\{ (u,z) \in L^\infty(\Omega)^+ \times L^\infty(\Gamma)^+ \ ; \ u \leq 1 \text{ a.e. in } \Omega \text{ and } z(x) \in \ Im(\rho(x,.)) \text{ a.e. } x \in \Gamma \Big\}.$$

Let $(u,z) \in K$ and consider $(u_\varepsilon, w_\varepsilon, z_\varepsilon)$ the solution of $S_\varepsilon(u,z)$. It is clear that $(u_\varepsilon, z_\varepsilon) \in \mathcal{D}(A)$. On the other hand, since $z \in \mathrm{Im}(\rho(x,.)$, then, thanks to Proposition 1, one proves exactly in the same way of Proposition 4 of [16] that z_ε is bounded in $L^\infty(\Gamma)$, u_ε is bounded in $L^\infty(\Omega)$ and shows that $u_\varepsilon \to u$ in $L^1(\Omega)$ and $z_\varepsilon \to z$ in $L^1(\Gamma)$, as $\varepsilon \to 0$, which ends up the proof of the proposition. ∎

3 The evolution problem

Now, let us consider the evolution problem

$$CP(U_0, H) \qquad \begin{cases} U_t + AU \ni H & \text{in } (0,T) \\ \\ U(0) = U_0, \end{cases}$$

with $U_0 = (u_0, z_0) \in D_A$ and $H = (f,g) \in L^1_{loc}([0,T); X)$. In order to define the notion of mild solution of $CP(U_0,H)$ in $(0,T)$, for $\varepsilon > 0$, we consider a subdivision $t_0 = 0 < t_1 < ... < t_{n-1} < T = t_n$ with $t_i - t_{i-1} = \varepsilon$, $f_1, ...f_n \in L^2(\Omega)$, $g_1, ...g_n \in L^2(\Gamma)$, $z_{0\varepsilon} \in L^2(\Gamma)$ and

$$\|z_0 - z_{0\varepsilon}\|_{L^1(\Gamma)} + \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \Big(\|f(t) - f_i\|_{L^1(\Omega)} + \|g(t) - g_i\|_{L^1(\Gamma)} \Big) dt \leq \varepsilon.$$

Thanks to Lemma 4, there exists a unique solution $(u_i, z_i) \in X$ of the time discretize scheme associated with (CP), i.e.

$$(3.1) \quad (u_i, z_i) + \varepsilon\ A(u_i, z_i) = \varepsilon\ (f_i, g_i) + (u_{i-1}, z_{i-1}) \quad \text{for } i = 1,\ 2, ..., n \ \text{ and } z_0 = z_{0\varepsilon}\ ;$$

so that, we can define the $\varepsilon-$ approximate solution $U_\varepsilon = (u_\varepsilon, z_\varepsilon)$, by

$$(3.2) \qquad \begin{cases} u_\varepsilon(0) = u_0, \quad z_\varepsilon(0) = z_{0\varepsilon}, \\ \\ u_\varepsilon(t) = u_i, \quad z_\varepsilon(t) = z_i, \quad \text{for } t \in \,]t_{i-1}, t_i],\ i = 1, \ldots n \end{cases} .$$

By using the nonlinear semigroup theory (cf. [2], [6], [12] and [21]) and thanks to Lemma 4, $CP(U_0,H)$ has a unique mild solution $U = (u,z) \in \mathcal{C}([0,T); X)$, such that $U(0) = U_0$, $u(t) = L^1 - \lim u_\varepsilon(t)$ and $z(t) = L^1 - \lim z_\varepsilon(t)$ uniformly for $t \in [0,T)$. So, we have

Corollary 2 *We may define the mapping* $S : (U_0, H) \in D_A \times L^1_{loc}([0,T); X) \to U \in \mathcal{C}([0,T); X)$, *by* $S(U_0, H)$ *is the mild solution of* $CP(U_0, H)$. *Moreover, we have*

1. *If* $(u, z) = S(u_0, z_0, f, g)$, *then*

$$\int_\Omega u(t) + \int_\Gamma z(t) = \int_\Omega u_0 + \int_\Gamma z_0 + \int_0^t \Big(\int_\Omega f + \int_\Gamma g \Big),$$

for any $t \in [0, T)$.

2. *For any* $0 \le t_1 \le t_2 < T$, *we have*

$$S(u_0, z_0, f, g)(t_2) = S\Big(S(u_0, z_0, f, g)(t_1), f(.+t_1), g(.+t_1)\Big)(t_2 - t_1).$$

3. *The* L^1-*comparison principle holds. More precisely, if for* $i = 1,\ 2$, $(u_{0i}, z_{0i}, f_i, g_i) \in D_A \times L^1_{loc}([0,T); X)$ *and* $(u_i, z_i) = S(u_{0i}, z_{0i}, f_i, g_i)$, *then*

$$\begin{aligned} &\frac{d}{dt}\int_\Omega (u_1(t) - u_2(t))^+ + \frac{d}{dt}\int_\Gamma (z_1(t) - z_2(t))^+ \\ &\le \int_\Omega (f_1(t) - f_2(t))^+ + \int_\Gamma (g_1(t) - g_2(t))^+ \end{aligned} \tag{3.3}$$

in $\mathcal{D}'(0, T)$.

Now, the basic idea of the proof of Theorem 1 is to show that $S(u_0, z_0, f, g)$ is the unique solution of $E(u_0, z_0, f, g)$, whenever u_0, z_0, f and g satsify the assumptions of the Theorem. First, let us introduce the intervals

$$I = \Big\{ t \ge 0 \ ; \ \mu(t) < 1 + \frac{1}{|\Omega|}\int_\Gamma \overline{\rho} \Big\} \quad \text{and} \quad J = \Big\{ t \ge 0 \ ; \ \mu(t) \ge 1 + \frac{1}{|\Omega|}\int_\Gamma \overline{\rho} \Big\}.$$

We begin by to give a description of $S(u_0, z_0, f, g)(t)$ for $t \in J$.

Lemma 5 *Let* $(u_0, z_0, f, g) \in D_A \times L^1_{loc}([0,T); X)$. *For any* $t \in J$, *we have*

$$S(u_0, z_0, f, g)(t) = \Big(\mu(t) - \frac{1}{|\Omega|}\int_\Gamma \overline{\rho}, \overline{\rho} \Big).$$

Proof : Set $(u, z) := S(u_0, z_0, f, g)$. By Definition of $S(u_0, z_0, f, g)$, we know that $S(u_0, z_0, f, g)(t) \in D_A$, for each $t \in [0, T)$. On the other hand, thanks to Corollary 2, we know that $\displaystyle⨍_\Omega u(t) + \frac{1}{|\Omega|}\int_\Gamma z(t) = \mu(t)$. Then $\mu(t) \notin \mathcal{R}$ implies that $(u(t), z(t)) \in D_2$, so that, for any $t \in J$, $z(t) = \overline{\rho}$, on Γ, $u(t)$ is a constant function in Ω and necessarly it is equal to $\displaystyle\mu(t) - \frac{1}{|\Omega|}\int_\Gamma \overline{\rho}$. ∎

For the description of $S(u_0, z_0, f, g)$, for $t \in I$, we need the following technical result.

Lemma 6 *Let $u \in \mathcal{C}([0,T], L^1(\Omega)^+)$ such that $\displaystyle -\!\!\!\!\!\!\int_\Omega u(t) < 1$ for any $t \in [0,T]$, and let $u_\varepsilon \in \mathcal{C}([0,T], L^1(\Omega)^+)$ such that $u_\varepsilon \to u$ in $\mathcal{C}([0,T], L^1(\Omega))$. There exists a constant C, independent of ε (and t), such that for any $w_\varepsilon \in L^2(0,T; H^1(\Omega))$ satisfying $u_\varepsilon \in H(w_\varepsilon)$ a.e. in $[0,T] \times \Omega$, and for any $\varepsilon > 0$, we have*

$$\|w_\varepsilon(t)\|_{L^2(\Omega)} \leq C\, \|\nabla w_\varepsilon(t)\|_{L^2(\Omega)} \qquad a.e.\ t \in [0,T].$$

Proof : First, one sees that by using Poincaré's inequality, for any $K \subseteq \Omega$ and $w \in H^1(\Omega)$, we have

$$\Big| -\!\!\!\!\!\!\int_\Omega w \Big|\, |K|^{\frac{1}{2}} \leq C\Big(\|\nabla w\|_{L^2(\Omega)} + \|w\|_{L^2(K)}\Big) \tag{3.4}$$

where C is a real constant depending only on N and Ω. Using the assumptions of the lemma, there exists $0 < \delta < 1$, such that

$$\max_{t \in [0,T]} -\!\!\!\!\!\!\int_\Omega u(t) < \delta,$$

so that $K(t) := [u(t) < \delta]$ is such that $|K(t)| > 0$, for any $t \in [0,T]$ and

$$\inf_{t \in [0,T]} |K(t)|^{\frac{1}{2}} =: M > 0. \tag{3.5}$$

Now, let us denote by $K_\varepsilon(t) = [u_\varepsilon(t) < \delta]$. Since $u_\varepsilon(t) \to u(t)$ in $L^1(\Omega)$, then

$$|K(t)| \leq \liminf_{\varepsilon \to 0} |K_\varepsilon(t)| \qquad \text{for any } t \in [0,T]. \tag{3.6}$$

Applying (3.4) with $w = w_\varepsilon(t)$, $K = K_\varepsilon(t)$ and using (3.5) and (3.6), we deduce that, for ε small enough, we have

$$\Big| -\!\!\!\!\!\!\int_\Omega w_\varepsilon(t) \Big| \leq \frac{C}{M}\Big(\|\nabla w_\varepsilon(t)\|_{L^2(\Omega)} + \|w_\varepsilon(t)\|_{L^2(K_\varepsilon(t))}\Big), \qquad \text{for any } t \in [0,T]. \tag{3.7}$$

Since $w_\varepsilon(t) = 0$ a.e. in $K_\varepsilon(t)$, then (3.7) implies that

$$\Big| -\!\!\!\!\!\!\int_\Omega w_\varepsilon(t) \Big| \leq C_1 \|\nabla w_\varepsilon(t)\|_{L^2(\Omega)} \qquad \text{for any } t \in [0,T]$$

and the result of the lemma follows by using Poincaré inequality again. ∎

Lemma 7 *Assume that $T < \infty$, $f \in L^2(0,T; L^2(\Omega)^+)$, $g \in L^2(0,T; L^2(\Gamma)^+)$ and $(u_0, z_0) \in D_1$ is such that $\displaystyle\int_0^{z_0} \rho(.,r)\, dr \in L^1(\Gamma)$,*

$$\mu(t) \in \mathcal{R} \quad \textit{and} \ -\!\!\!\!\!\!\int_\Omega u_0 < 1 \quad \textit{for any } t \in [0,T]. \tag{3.8}$$

Then, the curve $(u,z) := S(u_0, z_0, f, g)$ is the unique solution of $E(u_0, z_0, f, g)$ in $(0,T)$.

Proof : Let us come back to the definition of a mild solution and consider the time discretize scheme and the $\varepsilon-$approximate solution of $CP(u_0, z_0, f, g)$. Thanks to our hypothesis, we assume, moreover, that

$$(3.9)\quad \begin{cases} z_{0\varepsilon}(x) \in (\mathrm{Im}(\rho(x,.)) \text{ a.e. } x \in \Gamma, \ \displaystyle\int_0^{z_{0\varepsilon}} \rho(.,r)\, dr \in L^1(\Gamma) \text{ and} \\ \displaystyle⨍_\Omega u_{0\varepsilon} + \frac{1}{|\Omega|}\int_\Gamma z_{0\varepsilon} + \sum_{i=1}^{j}\int_{t_{i-1}}^{t_i} \Big(⨍_\Omega f_i + \frac{1}{|\Omega|}\int_\Gamma g_i\Big) < 1 + \frac{1}{|\Omega|}\int_\Gamma \overline{\rho} \quad \text{for any } 1 \le i \le n. \end{cases}$$

So, by definition of A, for any $i = 1, 2, ..n$, the solution (u_i, z_i) of (3.1) is such that there exists $w_i \in W^{1,1}(\Omega)$ satisfying the equations

$$(3.10)\quad \begin{cases} u_i - \varepsilon\Delta w_i = u_{i-1} + \varepsilon f_i, \ u_i \in H(w_i) & \text{in } \Omega, \\ z_i + \varepsilon\partial_\nu w_i = z_{i-1} + \varepsilon g_i, \ z_i = \rho(x, w_i) & \text{on } \Gamma. \end{cases}$$

Moreover, since f_i, g_i, $u_{0\varepsilon}$ and $z_{0\varepsilon}$ are assumed to be nonnegative L^2 function, then $w_i \in H^1(\Omega)$ and, thanks to (1.6), $z_i \in L^2(\Gamma)^+$. Taking w_i as a test function in (3.10) and using the facts that

$$\int_\Omega (u_i - u_{i-1})w_i \ge 0$$

and

$$\int_\Gamma (z_i - z_{i-1})w_i \ge \int_\Gamma \psi(., z_i) - \int_\Gamma \psi(., z_{i-1}),$$

where $\psi(.,r) = \displaystyle\int_0^r \rho(.,s)\, ds$, we get

$$(3.11)\quad \begin{aligned} \int_\Gamma \psi(., z_i) + \varepsilon\int_\Omega |\nabla w_i|^2 &\le \varepsilon\bigg(\int_\Omega f_i w_i + \int_\Gamma g_i w_i\bigg) + \int_\Gamma \psi(., z_{i-1}) \\ &\le \varepsilon\bigg(\|f_i\|_{L^2(\Omega)} + \|g_i\|_{L^2(\Gamma)}\bigg)\|w_i\|_{H^1(\Omega)} + \int_\Gamma \psi(., z_{i-1}). \end{aligned}$$

Adding (3.11) for $i = 0, ...n$, we deduce that w_ε defined by $w_\varepsilon(t) = w_i$ for $t \in]t_{i-1}, t_i]$, $i = 1, \ldots n$, satisfies

$$(3.12)\quad \int_\Gamma \psi(., z_\varepsilon(T)) + \int_0^T\int_\Omega |\nabla w_\varepsilon|^2 \le \int_0^T \bigg(\|f_\varepsilon\|_{L^2(\Omega)} + \|g_\varepsilon\|_{L^2(\Gamma)}\bigg)\|w_\varepsilon\|_{H^1(\Omega)} + \int_\Gamma \psi(., z_0).$$

Now, since $\displaystyle\sup_{t\in[0,T]} ⨍_\Omega u(t) < 1$, then by using Lemma 6, (3.12) implies that

$$\int_\Gamma \psi(., z_\varepsilon) + \int_0^T\int_\Omega |\nabla w_\varepsilon|^2 \le C\|\nabla w_\varepsilon\|_{L^2(Q_T)}\int_0^T (\|f_\varepsilon\|_{L^2(\Omega)} + \|g_\varepsilon\|_{L^2(\Gamma)}) + \int_\Gamma \psi(., z_{0\varepsilon}).$$

By Young's inequality, we deduce hat $|\nabla w_\varepsilon|$ is bounded in $L^2(Q)$, and by using again Lemma 6, we deduce that w_ε is bounded in $L^2(0,T;H^1(\Omega))$. So, there exists a subsequence, that we denote again by w_ε, such that

$$w_\varepsilon \to w \quad \text{weakly in } L^2(0,T;H^1(\Omega)) \quad \text{as } \varepsilon \to 0$$

and

$$w_\varepsilon \to w \quad \text{weakly in } L^2(\Sigma_T), \quad \text{as } \varepsilon \to 0.$$

Since, $u_\varepsilon \to u$ in $L^1(Q_T)$ and $z_\varepsilon \to z$ in $L^1(\Sigma_T)$, then, by classical monotonicity argument (see for instance [2]), we deduce that $0 \leq u \leq 1$, $u \in H(w)$ a.e. in Q and $z = \rho(.,w)$ a.e. in Σ_T. At last, let $\tilde{u}_\varepsilon$ and $\tilde{z}_\varepsilon$ be the functions from $[0,T]$ into $L^1(\Omega)$ and $L^1(\Gamma)$ respectively, defined by $\tilde{u}_\varepsilon(t_i) = u_i$, $\tilde{z}_\varepsilon(t_i) = z_i$ and $(\tilde{u}_\varepsilon, \tilde{z}_\varepsilon)$, is linear in $[t_{i-1}, t_i]$. For $\xi \in \mathcal{C}^1\overline{\Omega})$, we have

$$\frac{d}{dt}\int_\Omega \tilde{u}_\varepsilon \xi + \frac{d}{dt}\int_\Gamma \tilde{z}_\varepsilon \xi + \int_\Omega \nabla w_\varepsilon \cdot \nabla \xi = \int_\Omega f_\varepsilon \xi + \int_\Gamma g_\varepsilon \xi.$$

Passing to the limit we get that (u,w,z) satisfies (1.5). As to the uniquness, this follows exactly in the same way as in [16], we omitt the details of the proof here. ∎

Lemma 8 *Let* $f \in L^2(Q_T)^+$, $g \in L^2(\Sigma_T)^+$, $(u_0, z_0) \in D_A$ *such that* $\psi(., z_0) \in L^1(\Gamma)$ *and* $(u,z) = S(u_0, z_0, f, g)$. *For any* $0 \leq t \leq T$, *we have*

$$\Big[\ u(t) < 1\ \Big] \subseteq \Big[\ u(t) = u_0 + \int_0^t f(s)ds\ \Big] \tag{3.13}$$

Proof : Using again the definition of S, we come back to the time discretize scheme associated with $CP(u_0, z_0, f, g)$ and we consider the $\varepsilon-$ approximate solution $(u_\varepsilon, z_\varepsilon)$ given by (3.1) by repalcing T by t. We prove that

$$\Big[\ u_\varepsilon(t) < 1\ \Big] \subseteq \Big[\ u_\varepsilon(t) = u_0 + \int_0^t f_\varepsilon(s)ds\ \Big]. \tag{3.14}$$

It is clear that, for $i = 1, 2, ...n$, $\Delta w_i = 0$ and $u_i = u_{i-1} + \varepsilon\, f_i$ a.e. in $\Big[u_i < 1\Big]$, so that

$$[u_i < 1] \subseteq \Big[u_i = u_{i-1} + \varepsilon\, f_i \text{ and } u_{i-1} + \varepsilon\, f_i < 1\Big].$$

Moreover, since $u_{i-1} + \varepsilon\, f_i > u_{i-1}$, then

$$\begin{aligned}\Big[u_i = u_{i-1} + \varepsilon\, f_i \text{ and } u_{i-1} + \varepsilon\, f_i < 1\Big] &\subseteq \Big[u_{i-1} < 1\Big] \\ &\subseteq \Big[u_{i-1} = u_{i-2} + \varepsilon\, f_{i-1} \text{ and } u_{i-2} + \varepsilon\, f_{i-1} < 1\Big],\end{aligned}$$

so that,

$$[u_i < 1] \subseteq \Big[u_j = u_{j-1} + \varepsilon\, f_j \text{ and } u_{j-1} + \varepsilon\, f_j < 1\Big], \quad \text{for each } 1 \leq j \leq i \leq n$$

and

$$\Big[u_\varepsilon(\tau) < 1\Big] = \Big[u_n < 1\Big] \subseteq \Big[u_{i+1} = u_i + \varepsilon\, f_{i+1} \text{ and } u_i + \varepsilon\, f_{i+1} < 1\Big],$$

for any $0 \le i \le n-1$, which implies (3.14). At last, since $u_\varepsilon \to u$ in $L^1(Q)$ then (3.13) follows by letting $\varepsilon \to 0$ in (3.14). ∎

Corollary 3 *Under the assumptions of Lemma 8, we have*

$$u(t_1) \le u(t_2) \quad \text{a.e. in } \Omega, \tag{3.15}$$

for any $0 \le t_1 \le t_2 \le T$.

Proof : To prove (3.15), we see that, since $f \ge 0$, then (3.13) implies that $u(t_2) \ge u(t_1)$ a.e. in Ω. Indeed, if $u(t_2, x) = 1$ then, it is clear that $u(t_1, x) \le u(t_2, x)$. Otherwise, thanks to Lemma 8, we have

$$u(t_1, x) \le u(t_2, x) = u(t_1, x) + \int_{t_1}^{t_2} f \le 1,$$

which implies that $u(t_1, x) = u(t_2, x)$, a.e. $x \in \Omega$. ∎

Proposition 3 *Let* $f \in L^2(Q_T)^+$, $g \in L^2(\Sigma_T)^+$, $(u_0, z_0) \in D_A$ *such that* $\psi(., z_0) \in L^1(\Gamma)$ *and* $(u, z) = S(u_0, z_0, f, g)$.

1. *If* $(u_0, z_0) \in D_2$, *i.e.* $u_0 = \mu \in (1, \infty)$ *and* $z = \overline{\rho}$, *then for any* $t \in [0, T)$,

$$u(t) = \mu(t) - \frac{1}{|\Omega|} \int_\Gamma \overline{\rho} \quad \text{and } z(t) = \overline{\rho}.$$

2. *If* $(u_0, z_0) \in D_1$, *i.e.* $0 \le u_0 \le 1$ *and* $z_0 \in L^2(\Gamma)$, *then for any* $t \ge T_0$,

$$u(t) = \mu(t) - \frac{1}{|\Omega|} \int_\Gamma \overline{\rho} \quad \text{and } z(t) = \overline{\rho},$$

   and there exists $\tau \in [0, T_0]$, *such that* (u, z) *is the unique solution of* $E(u_0, z_0, f, g)$ *in* $(0, \tau)$ *and* $u(t) \equiv 1$ *in* Ω *for any* $t \in (\tau, T_0)$.

Proof : Since $f \ge 0$ and $g \ge 0$, then it is clear that $t \to \mu(t)$ is nondecreasing. So, if $(u_0, z_0) \in D_1$, then $J = (0, T)$ and the first part follows by Lemma 5. As to the second part, it is clear that $I = (0, T_0)$ and $J = (T_0, \infty)$. Thanks to lemma 5, for any $t \ge T_0$, $u(t) = \mu(t) - \int_0^t \frac{1}{|\Omega|} \int_\Gamma \overline{\rho}$ and $z(t) = \overline{\rho}$. Now, let τ be defined by

$$\tau = \inf \Big\{ t \in (0, T_0)\ ;\ ⨍_\Omega u(t) = 1 \Big\}.$$

Thanks to Corollary 3, for any $t \in (\tau, T_0)$, $u(t) \equiv 1$ in Ω. At last, since $⨍_\Omega u(t) < 1$ for any $t \in (0, \tau)$, then Lemma 7 implies that (u, z) is the unique solution of $E(u_0, z_0, f, g)$ in $(0, \tau)$. ∎

Lemma 9 *Asumme that* $\rho(x,r) = \rho(r)$, *with* ρ *convex,* $T < \infty$, $f \in L^2(0,T;L^2(\Omega)^+)$, $g \in L^2(0,T;L^2(\Gamma)^+)$ *and* $z_0 \in L^1(\Gamma)^+$ *such that* $\int_0^{z_0} \rho(.,r)\,dr \in L^1(\Omega)$ *satisfying (1.8) and*

$$(3.16) \qquad 1 + \frac{1}{|\Omega|}\int_\Gamma z_0 + \int_0^t \Big(-\!\!\!\!\!\!\int_\Omega f + \frac{1}{|\Omega|}\int_\Gamma g \Big) \in \mathcal{R} \quad \text{for any } t \in [0,T].$$

If $(u,z) = S(1,z_0,f,g)$, *then* $u(t) \equiv 1$, *in* Ω *for any* $t \in [0,T)$ *and* $(1,z)$ *is the unique solution of* $E(1,z_0,f,g)$, *i.e. there exists* $w \in L^2_{loc}([0,T),H^1(\Omega))$ *such that* $w \geq 0$, $z = \rho(.,w)$ *a.e. in* $(0,T)\times\Gamma$ *and*

$$(3.17) \qquad \frac{d}{dt}\int_\Gamma z\,\xi + \int_\Omega \nabla w.\nabla\xi = \int_\Omega f\,\xi + \int_\Gamma g\,\xi \quad \text{in } \mathcal{D}'(0,T),$$

for any test function $\xi \in \mathcal{D}(\overline{\Omega})$.

Proof : We take again the $\varepsilon-$approximate solution $(u_\varepsilon, z_\varepsilon)$ of $CP(1,z_0,f,g)$. Thanks to the assumpyions of the lemma, we assume, moreover, that

$$\begin{cases} z_{0\varepsilon}(x) \in (\mathrm{Im}(\rho) \text{ a.e. } x \in \Gamma, \ \int_0^{z_{0\varepsilon}} \rho(r)\,dr \in L^1(\Gamma) \text{ and} \\ \\ 1 + \frac{1}{|\Omega|}\int_\Gamma z_{0\varepsilon} + \sum_{i=1}^{j}\int_{t_{i-1}}^{t_i} \Big(-\!\!\!\!\!\!\int_\Omega f_i + \frac{1}{|\Omega|}\int_\Gamma g_i \Big) \in \mathcal{R} \quad \text{for any } 1 \leq i \leq n. \end{cases}$$

So, by definition of A, for any $i = 1,2,..n$, the solution (u_i, z_i) of (3.1) is such that there exists $w_i \in W^{1,1}(\Omega)$ satisfying the equations

$$\begin{cases} u_i - \varepsilon\Delta w_i = u_{i-1} + \varepsilon f_i, \ u_i \in H(w_i) & \text{in } \Omega, \\ \\ z_i + \varepsilon\partial_\nu w_i = z_{i-1} + \varepsilon g_i, \ z_i = \rho(w_i) & \text{on } \Gamma. \end{cases}$$

Since $w_i = 0$ a.e. in $[u_i < 1]$, then $u_i = u_{i-1} + \varepsilon\, f_i$ a.e. in $[u_i < 1]$. So, since $u_0 \equiv 1$ and $f_i \geq 0$ for any $i = 1,2,..N$, then $u_i \equiv 1$ for any $i = 1,2,...N$, and (3.10) is reduced to

$$(3.18) \qquad \begin{cases} -\varepsilon\Delta w_i = \varepsilon f_i, & \text{in } \Omega, \\ \\ z_i + \varepsilon\partial_\nu w_i = z_{i-1} + \varepsilon g_i, \ z_i = \rho(x,w_i) & \text{on } \Gamma. \end{cases}$$

Now, thanks to Jensen inequality, we have

$$\frac{1}{|\Gamma|}\int_\Gamma w_\varepsilon \leq \rho^{-1}\left(\frac{1}{|\Gamma|}\int_\Gamma z_\varepsilon\right),$$

which implies that $\frac{1}{|\Gamma|}\int_\Gamma w_\varepsilon$ is bounded, and by Poincaré inequality we deduce that

$$\|w_\varepsilon\|_{L2(\Omega)} \leq C\,\Big(\|\nabla w_\varepsilon\|_{L2(\Omega)} + 1\Big),$$

and the proof completes exactely in the same way of the proof of Lema7. ∎

4 Appendix

In this appendix, we prove the existence of a solution for the elliptic problem

$$(A1)\qquad \begin{cases} v - \Delta\varphi(v) = f & \text{in } \Omega \\ \\ \partial_\nu \varphi(u) + \rho(x, \varphi(u)) = g & \text{on } \Gamma, \end{cases}$$

where $\varphi\ : \mathbb{R} \to \mathbb{R}$ is an increasing continuous function in $\mathbb{R}$, ρ satisfies the assumptions of Section 1, $f \in L^1(\Omega)$ and $g \in L^1(\Gamma)$. Before we give the existence result, let us prove the following technical Lemma.

Lemma A.1 *There exists $C = C(N, \Omega)$ such that for any $1 \le q < \infty$ and $z \in L^1(\Omega)$ such that $\varphi(z) \in W^{1,q}(\Omega)$, we have*

$$(A2)\qquad \|\varphi(u)\varphi(u)\|_{L^q(\Omega)} \le \overline{\varphi}\left(\frac{2}{|\Omega|}\|z\|_{L^1(\Omega)}\right) |\Omega|^{\frac{1}{q}} + C\|\nabla\, \varphi(z)\|_{L^q(\Omega)},$$

where $\overline{\varphi}(r) = \max\limits_{|s|\le r}(\varphi(s)$, for any $r \in \mathbb{R}$.

Proof : We set $w = \varphi(z)$ and we denote by C every constant depending only on N and Ω. Using Poincaré inequality, we have

$$\Big|⨍_\Omega w\Big| \le \frac{1}{|K|^{1/q}}\Big(C\|\nabla\, w\|_{L^q(\Omega)} + \|w\|_{L^q(K)}\Big),$$

for any $K \subseteq \Omega$ with $|K| \neq 0$ and we have

$$\begin{aligned} \|w\|_{L^q(\Omega)} &\le C\|\nabla\, w\|_{L^q(\Omega)} + |\Omega|^{1/q}\Big|⨍_\Omega w\Big| \\ &\le C\Big(\Big(1 + \Big(\frac{|\Omega|}{|K|}\Big)^{1/q}\Big)\ \|\nabla\, w\|_{L^q(\Omega)} + \Big(\frac{|\Omega|}{|K|}\Big)^{1/q}\ \|w\|_{L^q(K)} \\ &\le C\left(\frac{|\Omega|}{|K|}\right)^{1/q}\Big(\|\nabla\, w\|_{L^q(\Omega)} + \|w\|_{L^q(K)}\Big) \end{aligned}$$

Taking $K = [|z| < \lambda]$, and using the fact that

$$\begin{aligned} |K| &= |\Omega| - |[|z| \ge \lambda]| \\ &\ge |\Omega| - \frac{1}{\lambda}\|z\|_{L^1(\Omega)}, \end{aligned}$$

we get

$$\|\varphi(z)\|_{L^q(\Omega)} \le \frac{|\Omega|}{|\Omega| - \frac{1}{\lambda}\|z\|_{L^1(\Omega)}}\, C\ \Big(\|\nabla\, w\|_{L^q(\Omega)} + \overline{\varphi}(\lambda)|\Omega|^{1/q}\Big)$$

for all $\lambda > \dfrac{1}{|\Omega|}\|z\|_{L^1(\Omega)}$. Then, taking for instance $\lambda = \dfrac{2}{|\Omega|}\|z\|_{L^1(\Omega)}$, the result follows. ∎

Proposition A.1 *For any $f \in L^1(\Omega)$ and $g \in L^1(\Gamma)$, there exists a unique solution of (A1) in the sense that $v \in L^1(\Omega)$, $\varphi(v) \in W^{1,1}(\Omega)$, $\rho(x,\varphi(v)) \in L^1(\Gamma)$ and*

$$\int_\Omega \nabla\varphi(u)\cdot\nabla\xi = \int_\Omega (f-v)\,\xi + \int_\Gamma (g-\gamma(x,\varphi(v))\,\xi$$

for any test function $\xi \in \mathcal{C}^1(\Omega)$.

Proof : Thanks to Theorem 23 of [3], we know that for any $\varepsilon > 0$, there exists a unique $v_\varepsilon \in L^1(\Omega)$, $w_\varepsilon := \varphi(v_\varepsilon) \in W^{1,1}(\Omega)$, $z_\varepsilon := \gamma(x,\varphi(v_\varepsilon)) \in L^1(\Gamma)$ and

$$\int_\Omega \nabla w_\varepsilon\cdot\nabla\xi = \int_\Omega (f-v_\varepsilon-\varepsilon\, w_\varepsilon)\,\xi + \int_\Gamma (g-z_\varepsilon)\,\xi$$

for any test function $\xi \in \mathcal{C}^1(\Omega)$. It is enough to prove that $u_\varepsilon = v_\varepsilon + \varepsilon\, w_\varepsilon$, w_ε and z_ε are relatively compact in $L^1(\Omega)$, in $W^{1,1}(\Omega)$-weak and in $L^1(\Gamma)$, respectively. Recall that (cf. [2])

$$(A3)\qquad \begin{aligned} \|u_\varepsilon\|_{L^1(\Omega)} &= \varepsilon\,\|w_\varepsilon\|_{L^1(\Omega)} + \|v_\varepsilon\|_{L^1(\Omega)} \\ &\le \|f\|_{L^1(\Omega)} + \|g\|_{L^1(\Gamma)}, \end{aligned}$$

$$(A4)\qquad \|w_\varepsilon - ⨍ w_\varepsilon\|_{W^{1,q}(\Omega)} \le C\left(\|f\|_{L^1(\Omega)} + \|g\|_{L^1(\Gamma)}\right)$$

for any $1 \le q < \frac{N}{N-1}$, and

$$(A5)\qquad \lim_{|y|\to 0}\sup_{\varepsilon>0}\int_{\Omega'} |u_\varepsilon(x+y)-u_\varepsilon(x)| = 0.$$

It is clear that (13) and (A4) implies that$\|v_\varepsilon\|_{L^1(\Omega)}$ and $\|\nabla w_\varepsilon\|_{L^q(\Omega)}$ are bounded, so that Lemma A.1 implies that w_ε is bounded in $W^{1,1}(\Omega)$. So, w_ε is relatively compact in $W^{1,1}(\Omega)$-weak and in $L^1(\Gamma)$. Moreover, by using the continuity of $r \to \rho(x,r)$ a.e. $x \in \Gamma$ and (1.6), we deduce that z_ε is relatively compact in $L^1(\Gamma)$. For the precompacteness of u_ε is $L^1(\Omega)$, let us assume for the moment that $f \in L^\infty(\Omega)$ and $g \in L^\infty(\Gamma)$. We know, that $v_\varepsilon \in L^2(\Omega)$, $w_\varepsilon := \varphi(v_\varepsilon) \in H^1(\Omega)$, $z_\varepsilon := \rho(x,\varphi(v_\varepsilon)) \in L^2(\Gamma)$, so that we can take w_ε as a test function and we get

$$\begin{aligned} \int_\Omega v_\varepsilon w_\varepsilon + \int_\Omega |\nabla w_\varepsilon|^2 &\le \int_\Omega f_\varepsilon\, w_\varepsilon + \int_\Gamma g_\varepsilon\, w_\varepsilon \\ &\le C(\|f\|_{L^\infty(\Omega)} + \|g\|_{L^\infty(\Gamma)})\,\|w_\varepsilon\|_{W^{1,1}(\Omega)} \end{aligned}$$

where C is independent of ε. So, that we deduce that $\displaystyle\int_\Omega v_\varepsilon w_\varepsilon$ is bounded by a constant M independent of ε. Now, since

$$|v_\varepsilon| \le \frac{1}{\min(\varphi(k),-\varphi(-k))} v_\varepsilon\varphi(v_\varepsilon) + k$$

for any $k > 0$, then

$$\int_E |v_\varepsilon| \le \frac{1}{\min(\varphi(k), -\varphi(-k))} \int_\Omega v_\varepsilon \varphi(v_\varepsilon) + k\,|E| \le \frac{M}{\min(\varphi(k), -\varphi(-k))} + k\,|E|$$

for any $k > 0$ and measurable $E \subseteq \Omega$. Thus, $\int_E |v_\varepsilon| \to 0$, uniformly in ε, as $|E| \to 0$, and by using (A4), we deduce that the relative compactness of u_ε in $L^1(\Omega)$. For the case where f and g are just L^1, the relative compactness of u_ε in $L^1(\Omega)$ follows by the contraction property. ∎

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