Uniqueness for inhomogeneous Dirichlet problem for elliptic-parabolic equations

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Abstract

We show the L^1 contraction and comparison principle for weak (and, more generally, renormalized) solutions of the elliptic-parabolic problem $j(v)_t - \operatorname{div}\Big(\nabla w + F(w)\Big) = f(t,x)$, $w = \varphi(v)$ in $(0,T)\times\Omega \subset \mathbb{R}^+ \times \mathbb{R}^N$ with inhomogeneous Dirichlet boundary datum $g \in L^2(0,T;W^{1,2}(\Omega))$ for w (which is taken in the sense $w - g \in L^2(0,T;H^1_0(\Omega))$) and initial datum $j_o \in L^1(\Omega)$ for $j(v)$. Here φ, j are nondecreasing, and we assume F just continuous.

Our proof consists in doubling of variables in the interior of Ω as introduced by J.Carrillo [9] (Arch. Rational Mech. Anal., vol.147, 1999), and in a careful treatment of the flux term near the boundary of Ω. For this last argument, the result is restricted to the linear dependence on ∇w of the diffusion term. The proof allows for a wide class of domains Ω, including e.g. domains of finite perimeter with uniform exterior cone condition or even domains with cracks.

We obtain the corresponding results for the associated stationary problem and discuss on generalization of our technique to the case of nonlinear diffusion operators.

Key-words: nonlinear Stefan problems, L^1 contraction principle, inhomogeneous Dirichlet data, non-lipschitz boundary.

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Introduction

Let $T > 0$, and $\Omega \subset \mathbb{R}^N$ be a bounded domain. Consider the problem

$$(P_g(f, j_o)) \quad \begin{cases} j(v)_t - \operatorname{div} a(w, \nabla w) = f, \\ \qquad\qquad\qquad w = \varphi(v) \quad \text{in } Q = (0,T) \times \Omega \\ w = g \quad \text{on } \Sigma = (0,T) \times \partial\Omega \\ j(v)|_{t=0} = j_o \quad \text{in } \Omega, \end{cases}$$

where $j, \varphi : \mathbb{R} \to \mathbb{R}$ are continuous nondecreasing functions, and $a : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is continuous and monotone in the second variable. Note that the Dirichlet problem for the equation $u_t - \operatorname{div} a(w, \nabla w) = f$, $w \in \beta(u)$, where β is an arbitrary maximal monotone graph on $\mathbb{R}$, reduces to $(P_g(f, j_o))$ (cf. e.g. [9] and [4]).

The aim of this paper is to show the uniqueness (of $j(v)$) and, more generally, the L^1 contraction and comparison result for $(P_g(f, j_o))$ with respect to the data $f \in L^1(Q)$ and $j_o \in L^1(\Omega)$. The boundary condition in $(P_g(f, j_o))$ is understood in the sense $(w - g) \in L^p(0, T; W_0^{1,p}(\Omega))$; we assume $g \in L^p(0, T; W^{1,p}(\Omega))$. For a sufficiently smooth domain Ω, one can consider $g \in L^p(0, T, W^{1-1/p}(\partial\Omega))$ and then take an order-preserving extension of g in $L^p(0, T, W^{1,p}(\Omega))$.

In the main part of the paper, we only consider the quasilinear equation with convection (in this case, p equals 2): it is assumed that

$(H0)$ $\quad a(w, \nabla w) = \nabla w + F(w)$, where $F : \mathbb{R} \mapsto \mathbb{R}^N$ is continuous.

Results for more general fluxes are discussed in Section 6.2. Since we are not concerned with the problem of existence of weak solutions, we do not require explicitly the usual growth assumptions on the convection F.

The uniqueness of weak solutions to $(P_g(f, j_o))$ is well known in case a is Lipschitz or, more generally, Hölder continuous (of order $1/2$, for fluxes $(H0)$) with respect to the first argument (see Alt, Luckhaus [1] and Otto [22]). Note that if g is independent of t and the existence for the stationary problem is known, this kind of result can be somewhat easier obtained using the tools of the nonlinear semigroup theory, as in Bénilan,Wittbold [6] where the uniqueness of a mild solution is shown as $g \equiv 0$. Previous results on mild solutions were obtained by Simondon [25] and Bénilan, Touré [5].

A uniqueness result for the homogeneous case $g \equiv 0$ and the flux a of the form $(H0)$ without any assumption on the modulus of continuity of F is contained in the paper of Kobayasi [17] (see [4] for a simpler and more general proof). The main tool in this case is the Kruzhkov's doubling of variables techniques adapted to parabolic problems, as introduced by Carrillo [9]. Carrillo proves, for the general elliptic-parabolic-hyperbolic problem, the uniqueness of entropy solutions. As shown in [17, 4], they coincide with weak solutions for the case we are interested in. For $g \equiv 0$

and a wide class of fluxes $a(w, \nabla w)$ including simply continuous convections, the uniqueness of renormalized solutions is shown by Carrillo, Wittbold [10] (another approach for renormalized solutions, which works for Lipschitz convections and avoids the doubling of variables in space, is presented in [8]). In all the works [9, 10, 17, 4], the treatment of the homogeneous boundary condition is carried out through the Carrillo's elegant choice of test functions. Thus the technical difficulty of considering boundary traces which existence is not clear is bypassed (compare with Rouvre, Gagneux [24] for the explicit argument in one case where the strong trace is well defined). Unfortunately, it does not seem straightforward to adapt the Carrillo approach to non-constant Dirichlet boundary condition g. Such an adaptation was recently carried by Ammar, Carrillo, Wittbold [2], for continuous on $\partial\Omega$ boundary data g (more general g are also treated in [2], but in a quite different way).

Uniqueness of entropy solutions for inhomogeneous Dirichlet problem was first addressed by Mascia, Porretta, Terracina [20], for the parabolic-hyperbolic problem. These authors use the approach of Otto [23] (see also [19]) and Chen, Frid [11, 12] which gives sense to the normal trace of the flux on the boundary. The main effort in [20] was made to treat the difficulties due to the possible hyperbolic behavior of the problem, and the simplifying assumptions that the boundary data are regular and F is Lipschitz continuous have been introduced. Another technique for the same problem, which is particularly useful for analysis of convergence of finite volume schemes, was developed by Michel, Vovelle [21]. Also in this paper, regularity assumptions on boundary data and on F are required. Note that for the purely hyperbolic problem, the general uniqueness result is achieved by Ammar, Carrillo, Wittbold [2].

In the present article, we give a proof of uniqueness of weak solutions for the inhomogeneous Dirichlet problem for the elliptic-parabolic equation $(P_g(f, j_o))$ with the flux $(H0)$, without the Lipschitz or Hölder assumptions on the convection term, and for a wide class of domains. We first perform the standard doubling of variables in the interior of Ω (see [9, 15, 17, 4]). In order to generate the boundary term, we consider test functions that truncate in a neighborhood of $\partial\Omega$, in the same spirit as in [24],[20] (see in particular Remark 1.5), [17] and [4]. Under some mild assumptions on $\partial\Omega$ (see $(H1), (H2)$ in Section 1), we construct the test functions such that the "boundary" term coming from the comparison of two solutions has a sign, the fact which was implicit in the Carrillo argument. Note that our argument requires the flux $a(w, \nabla w)$ to be linear in ∇w, its generalisation to, e.g., diffusions of the p-laplacian type $|\nabla w|^{p-2} \nabla w$ is an open question (see Section 6.2).

The paper is organized as follows. In the Section 1, we give definitions and state the results for the evolution problem. In Section 2, we deduce the corresponding results for the associated stationary problem. Section 3 sketches the doubling of variables argument in the interior of Ω (see [4] and the references therein for a more detailed exposition). The explicit treatment of the boundary terms is carried out in Section 4. Section 5 contains the proof of the main result. In Section 6, we prove the L^1 contraction and comparison principle for renormalized solutions of $(P_g(f, j_o))$ (for related works, see [7, 10, 8, 16, 3] and references therein). Then we discuss on extensions of Theorems 1.3 and 2.2 to more general elliptic-parabolic problems.

1 Definitions and results

Definition 1.1 *Let $p \in (1, +\infty)$, $p' = p/(p-1)$, and $g \in L^p(0,T; W^{1,p}(\Omega))$. An a.e. defined measurable function $v : Q \mapsto \overline{I\!R}$ is called weak solution of $(P_g(f, j_o))$ if $j(v) \in L^1(\Omega)$, the function $w = \varphi(v)$ satisfies $w \in g + L^p(0,T; W_0^{1,p}(\Omega))$ and $a(w, \nabla w) \in L^{p'}(Q)^N$, and the distributional derivative $j(v)_t$ can be identified with $\chi \in L^{p'}(0,T; W^{-1,p}(\Omega)) + L^1(Q)$ such that*

$$\int_0^T < \chi, \xi > + \int\int_Q a(w, \nabla w) \cdot \nabla \xi = \int\int_Q f\xi \qquad (1.1)$$

for all test function $\xi \in L^p(0,T; W_0^{1,p}(\Omega)) \cap L^\infty(Q)$, and

$$\int_0^T < \chi, \xi > = - \int\int_Q j(v)\xi_t - \int_\Omega j_o(x)\xi(0,x) \qquad (1.2)$$

for all test function $\xi \in L^p(0,T; W_0^{1,p}(\Omega)) \cap L^\infty(Q)$ such that $\xi_t \in L^\infty(Q)$ and $\xi(T, \cdot) = 0$.

Here and in the sequel, we denote by $< \cdot, \cdot >$ the duality pairing between $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ and its dual. Except in Section 6.2, we will always asume $p = 2$ and write H_0^1 for $W_0^{1,2}$.

In order to formulate the assumptions on Ω, let us require that $\Omega \supset \widetilde{\Omega}$ for all open $\widetilde{\Omega} \subset \overline{\Omega}$ such that $H_0^1(\widetilde{\Omega}) = H_0^1(\Omega)$. Assume that

$(H1)$ the $(N-1)$-dimensional Hausdorff measure of $\partial\Omega$ is finite.

Assume the following Poincaré-Friedrichs property :

$(H2)$ There exists a constant $\mathcal{M}$, independent of h, such that for all $x_o \in \partial\Omega$ one has for all $W \in H_0^1(\Omega)$,
$$\int_{B_h(x_o)\cap\Omega} |W|^2 \leq \mathcal{M}\, h \int_{B_h(x_o)\cap\Omega} |\nabla W|^2.$$

Here $B_h(x_o)$ denotes the N–dimensional ball of radius h centered at x_o.

Remark 1.2 *It is easy to see that $(H2)$ is verified in case Ω is weakly Lipschitz (that is, each point $x \in \partial\Omega$ possesses a neighborhood U_x such that $\Omega \cap U_x$ can be mapped on a half-ball of $I\!R^n$ by a bilipschitz homeomorphism).*
More generally, assume that , for $d = N$, one has

$(H2'(d))$ $\inf_{h>0, x_o\in\partial\Omega} \frac{1}{h^d} |B_h(x_o) \setminus \Omega| > 0,$

where $|\cdot|$ denotes the N–dimensional Lebesgue measure. Then $(H2)$ holds (cf. e.g. [26, Theorem 3.11.1]). Note that, for instance, the uniform exterior cone condition implies the condition $(H2'(N))$, and thus $(H2)$. In both aforementioned cases, the inequality in $(H2)$ actually holds with h replaced by h^2. A sharper sufficient condition for $(H2)$ to hold can be formulated in terms of the Bessel capacity $B_{1,2}$ (cf. e.g. [26]):

$$\left|\begin{array}{l} \inf_{h>0, x_o\in\partial\Omega} \frac{1}{h} B_{1,2}(\mathcal{N}_h(x_o)) > 0, \\ \text{where } \mathcal{N}_h(x_o) = \{\frac{1}{h}(x - x_o) \,|\, x \in B_h(x_o) \setminus \Omega\}. \end{array}\right. \tag{1.3}$$

This condition permits, in particular, to include domains with cracks. For the proof, it is sufficient to map $B_h(x_o)$ on the unit ball of $I\!R^N$ and apply [26, Corollary 4.5.3].
For $N \geq 3$, $B_{1,2}(\mathcal{N}_h(x_o))$ in (1.3) can be replaced by the Newtonian capacity of $\mathcal{N}_h(x_o)$ in $I\!R^N$ (cf. [26, ex.2.8]) and then by $|\mathcal{N}_h(x_o)|^{\frac{N-2}{N}} = \frac{1}{h^{N-2}}|B_h(x_o) \setminus \Omega|^{\frac{N-2}{N}}$ (cf. e.g. [13, Theorem 4.7.2]). Thus, $(H2)$ still holds, if $N \geq 3$ and the aforementioned condition $(H2'(d))$ is fulfilled with $d = N(N-1)/(N-2)$.

Let us state the main result of this paper. Denote by $\mathrm{sign}^+(\cdot)$ the maximal monotone extension of the function $\mathrm{sign}_0^+ : r \in I\!R \mapsto \left\{\begin{array}{ll} 0, & r \leq 0 \\ 1, & r > 0 \end{array}\right.$.

Theorem 1.3 *Assume $(H1)$ and $(H2)$. Let $v, \widehat{v}$ be weak solutions of $(P_g(f, j_o))$, $(P_{\widehat{g}}(\widehat{f}, \widehat{j}_o))$, respectively, with the flux given by $(H0)$ and $p = 2$. Assume $g \leq \widehat{g}$. Then there exists $\eta : Q \to I\!R$, $\eta \in \mathrm{sign}^+(j(v) - j(\widehat{v}))$ a.e. on Q, such that for a.a. $t \in (0, T)$, one has*

$$\int_\Omega \big(j(v) - j(\widehat{v})\big)^+(t) \leq \int_\Omega \big(j_o - \widehat{j}_o\big)^+ + \int_0^t \int_\Omega \eta \left(f - \widehat{f}\right). \tag{1.4}$$

In particular, if $\widehat{g} \geq g, \widehat{f} \geq f$ a.e. on Q and $\widehat{j}_o \geq j_o$ a.e. on Ω, then $j(\widehat{v}) \geq j(v)$ a.e. on Q.

The uniqueness result for $j(v)$ follows readily:

Corollary 1.4 *Assume $(H0), (H1), (H2)$. For all $g \in L^2(0, T, W^{1,2}(\Omega))$, $f \in L^1(Q)$ and $j_o \in L^1(\Omega)$, there exists at most one function $j(v) \in L^1(Q)$ such that v is a weak solution to $(P_g(f, j_o))$. In particular, if j is injective, there exists at most one weak solution v to $(P_g(f, j_o))$.*

2 The stationary problem

We also consider weak solutions to the associated "stationary" elliptic problem with $f \in L^1(\Omega)$:

$$(S_g(f)) \quad \left\{\begin{array}{l} j(v) - \mathrm{div}\, a(w, \nabla w) = f, \quad w = \varphi(v) \quad \text{in } \Omega \\ w = g \quad \text{on } \partial\Omega \end{array}\right.$$

in the sense of the following definition :

Definition 2.1 *Let $p \in (1,+\infty)$, $p' = p/(p-1)$, and $g \in W^{1,p}(\Omega)$. An a.e. defined measurable function $v : \Omega \mapsto \overline{I\!\!R}$ is called weak solution of $(S_g(f))$ if $j(v) \in L^1(\Omega)$, the function $w = \varphi(v)$ is such that $w \in g + W_0^{1,p}(\Omega)$ and $a(w, \nabla w) \in L^{p'}(\Omega)^N$, and*

$$\int_\Omega j(v)\xi + \iint_Q a(w, \nabla w)\cdot \nabla\xi = \iint_Q f\xi \tag{2.1}$$

for all test function $\xi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$.

Theorem 2.2 *Assume $(H1)$ and $(H2)$. Let $v, \widehat{v}$ be weak solutions of $(S_g(f))$,$(S_{\widehat{g}}(\widehat{f}))$, respectively, with the flux given by $(H0)$ and $p = 2$. Assume $g \le \widehat{g}$. Then there exists $\eta : \Omega \to I\!\!R$, $\eta \in \mathrm{sign}^+(j(v) - j(\widehat{v}))$ a.e. on Ω, such that*

$$\int_\Omega \big(j(v) - j(\widehat{v})\big)^+ \le \int_\Omega \eta\,\big(f - \widehat{f}\big). \tag{2.2}$$

In particular, if $\widehat{g} \ge g$ and $\widehat{f} \ge f$ a.e. on Ω, then $j(\widehat{v}) \ge j(v)$ a.e. on Ω.

Proof : Let v be a weak solution of $(S_g(f))$. Set $\tilde{v}(t) \equiv v$; then $\tilde{v}$ is a weak solution of $(P_g(\widetilde{f}, \widetilde{j_o}))$ corresponding to the data $\widetilde{j_o} = j(v)$ and $\tilde{f} = f - v$. Hence the result follows readily by Theorem 1.3. □

The corresponding uniqueness result for $j(v)$ follows:

Corollary 2.3 *Assume $(H0), (H1), (H2)$. For all $g \in W^{1,2}(\Omega)$ and $f \in L^1(\Omega)$, there exists at most one function $j(v) \in L^1(\Omega)$ such that v is a weak solution to $(S_g(f))$. In particular, if j is injective, there exists at most one weak solution v to $(S_g(f))$.*

Note that for the stationary problem, we are able to extend Theorem 2.2 to nonlinear fluxes of the form $a(w, \nabla w) = b(\nabla w) + F(w)$, in case $p = 2$ and under an additional structure assumption on $b : \mathbb{R}^N \to \mathbb{R}^N$ (see Section 6.2).

3 The doubling of variables

Let $v, \widehat{v}$ be two weak solutions of $(P_g(f, j_o))$ and $(P_{\widehat{g}}(\widehat{f}, \widehat{j_o}))$, respectively. We have the following comparison principle in the interior of Ω.

Lemma 3.1 *Let $v, \widehat{v}$ be the two weak solutions in Theorem 1.3; let $w = \varphi(v), \widehat{w} = \varphi(\widehat{v})$. There exists $\eta : Q \to I\!\!R$, $\eta \in \mathrm{sign}^+(j(v) - j(\widehat{v}))$ a.e. on Q, such that for all $\xi \in H_0^1(\Omega)$, one has for a.a. $t \in (0, T)$,*

$$\begin{aligned} \int_\Omega \big(j(v)(t) - j(\widehat{v})(t)\big)^+ \xi \;-\; \int_\Omega \big(j_o - \widehat{j}_o\big)^+ \xi \\ - \int_0^t\!\!\int_\Omega \eta\,(f - \widehat{f})\,\xi \;\;\le\;\; - \int_0^t\!\!\int_\Omega \nabla(w - \widehat{w})^+ \cdot \nabla\xi \\ - \int_0^t\!\!\int_\Omega \mathrm{sign}_0^+(w - \widehat{w})(F(w) - F(\widehat{w}))\cdot \nabla\xi. \end{aligned} \tag{3.1}$$

In order to prove Lemma 3.1, one needs the "entropy inequalities" (3.2) below:

Lemma 3.2 *Let $\xi \in H_0^1(\Omega)$. Let v be a weak solution of $P_g(f, j_o)$; then*

$$\begin{aligned} &\iint_Q -(j(v)-j(k))^+\,\xi\,\psi_{\,t} \\ &\quad -\int_\Omega \xi\,\psi(0)(j_o-j(k))^+ - \iint_Q f\,\xi\,\psi\,\mathrm{sign}_0^+(v-k) \\ &\quad \le -\iint_Q \Big(\nabla\varphi(v)+F(\varphi(v))-F(\varphi(k))\Big)\cdot\nabla\xi\,\psi\,\mathrm{sign}_0^+(v-k) \end{aligned} \tag{3.2}$$

for all $k \in \mathbb{R}$ and all $\psi \in \mathcal{D}(-\infty, T)$.

The outline of the proof of Lemma 3.2 is given in [4, Lemmas 1,2]. The original proofs can be found in [9, 15, 17]. We omit the details in order to avoid the unnecessary duplication of arguments.

Proof of Lemma 3.1 First fix $\xi \in \mathcal{D}(\Omega)$. Take any Lipschitz domain B of $\mathbb{R}^N$ such that $\mathrm{supp}\,\xi \subset B \subset \Omega$ (one can always choose B polygonal). Using the entropy inequalities (3.2) applied on B and the Carrillo's adaptation of the Kruzhkov's doubling of variables method (cf. [9, Theorem 9] and [18], respectively), one deduces (3.1). The result for general $\xi \in H_0^1(\Omega)$ follows by the density argument. □

4 An approach for the boundary flux

Note that heuristically, the limit, as ξ converges to 1 on Ω, of the right-hand side of (3.1) should be the boundary term

$$\int_0^t\int_{\partial\Omega} \frac{\partial}{\partial n}(w-\widehat{w})^+ + \int_0^t\int_{\partial\Omega} \mathrm{sign}_0^+(w-\widehat{w})(F(w)-F(\widehat{w}))\cdot n, \tag{4.1}$$

where n denotes the exterior unit normal to $\partial\Omega$. Moreover, if (4.1) can be understood in the pointwise sense (e.g., for Ω and $w, \widehat{w}$ regular enough), then it is non-positive since $(w-\widehat{w})^+ \ge 0$ in Ω and $(w-\widehat{w})^+ = 0$ on $\partial\Omega$, within the assumptions of Theorem 1.3.

In this section, we search for test functions ξ_h, $h > 0$, such that $\xi_h \to 1$ as $h \to 0$ a.e. on Ω, and that would permit to pass to the limit in the right-hand side of (3.1), generating non-positive "boundary" terms. In the rest of the paper, we denote by $\mathcal{M}$ a generic constant that may depend on $w, \widehat{w}, \Omega, T$, on coefficients and the data of the problem, but is independent of h.

4.1 Assumptions on Ω

Denote by Ω_h the h-neigbourhood of $\partial\Omega$ in Ω: $\Omega_h = \{x \in \Omega \,|\, \mathrm{dist}\,(x, \partial\Omega) < h\}$. Denote by $|\Omega_h|$ its N-dimensional Lebesgue measure. We need the following assumptions on Ω.

$$(H^\Omega)\quad \left|\begin{array}{l}\text{There exists a constant } \mathcal{M}, \text{ independent of } h, \text{ such that}\\ (i)\quad |\Omega_h| \le \mathcal{M}h \quad \text{for all } h \text{ sufficiently small;}\\ (ii)\quad \text{the following Poincaré-Friedrichs inequality in } \Omega_h \text{ holds :}\\ \qquad \displaystyle\int_{\Omega_h} |W|^2 \le \mathcal{M}h \int_{\Omega_{4h}} |\nabla W|^2, \quad \text{for all } W \in H^1_0(\Omega).\end{array}\right.$$

Note that introducing Ω_{4h} in $(H^\Omega)(ii)$ is only due to the convenience of stating $(H2)$ for balls $B_h(x_o)$, rather then for general convex neighbourhoods of x_o .

The assumptions $(H1), (H2)$ given in the Introduction are sufficient for $(H^\Omega)(i), (ii)$ to hold. More exactly, we have

Lemma 4.1 *(i) Assume $(H1)$. Then $(H^\Omega)(i)$ holds true.*
(ii) Assume $(H2)$. Then $(H^\Omega)(ii)$ holds true.

Proof *(i)* Take a countable covering $\mathcal{C}$ of $\partial\Omega$ by balls B_i of radii r_i, $r_i < h$; the balls of the same centers and of radii $2r_i$ cover Ω_h. Hence $|\Omega_h| \le \sum_{B_i\in\mathcal{C}} c_N (2r_i)^N$, where c_N is the measure of the unit ball of $\mathbb{R}^N$. By definition, $\lim_{h\to 0} \inf_{\mathcal{C}} c_N \sum_{B_i\in\mathcal{C}} r_i^{N-1}$ is equal to $\mathcal{H}^{N-1}(\partial\Omega)$, up to a normalizing factor. We deduce that $|\Omega_h| \le \mathcal{M}h$.

(ii) Take a finite covering $\{B_{2h}(x_i)\}$ of $\partial\Omega$ by balls of radius $2h$ centered at points $x_i \in \partial\Omega$. The balls of the same centers and of radius $4h$ cover Ω_h; moreover, if $\operatorname{dist}(x_i, x_j) < h$, we can omit one of the balls $B_{4h}(x_i), B_{4h}(x_j)$ in this covering. This implies that each point of Ω_{4h} belongs to at most $\mathcal{L}$ different balls $B_{4h}(x_i)$, with $\mathcal{L}$ that only depends on the dimension N. Applying $(H2)$ to each of $B_{4h}(x_i)$, we get $(H^\Omega)(ii)$ with $\mathcal{M}$ replaced by $\mathcal{LM}$. □

4.2 Construction of ξ_h

Lemma 4.2 *Assume (H^Ω). Let $w, \widehat{w} \in L^2(0,T; W^{1,2}(\Omega))$ such that $(w-\widehat{w})^+ \in L^2(0,T, H^1_0(\Omega))$. Assume in addition that $w, \widehat{w} \in L^\infty(Q)$. Then there exists a sequence $(\xi_{h_m})_{m\in\mathbb{N}} \subset H^1_0(\Omega)$ such that $0 \le \xi_{h_m} \le 1$ and $\xi_{h_m} \to 1$ as $m \to \infty$ a.e. on Ω, and*

$$\lim_{m\to\infty} \int_0^t \int_\Omega \nabla(w-\widehat{w})^+ \cdot \nabla\xi_{h_m} \ge 0 \quad \textit{for all } t \in (0,T), \tag{4.2}$$

$$\lim_{m\to\infty} \iint_Q \operatorname{sign}^+_0(w-\widehat{w}) |F(w) - F(\widehat{w})| \, |\nabla\xi_{h_m}| = 0. \tag{4.3}$$

Lemma 4.2 is a direct consequence of Lemmas 4.4,4.5 below. In Lemma 4.4, we give a construction ensuring (4.2) together with the additional properties (4.4), which are needed for the proof of (4.3) given in Lemma 4.5.

Remark 4.3 *Note that as a straightforward choice, one could take for ξ_h the distance-to-the-boundary functions $\xi^o_h = \frac{1}{h}\min\{h, \operatorname{dist}(x, \partial\Omega)\}$.*

Proving Lemma 4.2 with ξ_h^o seems to require more smoothness on $\partial\Omega$. See Remark 6.5 in Section 6.2 for a further discussion of this issue.

Lemma 4.4 *There exists a family $(\xi_h)_{h>0} \in H_0^1(\Omega)$ and a constant $\mathcal{M} > 0$ such that*

(i) $0 \le \xi_h \le 1$, *and* $\xi_h \to 1$ *a.e. in* Ω *as* $h \to 0$;

(ii) for all positive $W \in H_0^1(\Omega)$, $\displaystyle\int_\Omega \nabla W \cdot \nabla \xi_h \ge 0$;

(iii) under the assumption $(H^\Omega)(i)$, *one has*

$$\left|\begin{array}{l} 1/\mathcal{M} \le \displaystyle\int_\Omega |\nabla \xi_h| \le \mathcal{M}, \quad \int_\Omega |\nabla \xi_h|^2 \le \mathcal{M}/h, \\ \textit{and} \quad \operatorname{supp} \nabla\xi_h \textit{ is included in } \Omega_h. \end{array}\right. \tag{4.4}$$

Proof For h small enough, for $x \in \Omega_h$, set $u_h^o(x) = \operatorname{dist}(x, \partial\Omega)$. Denote by u_h^* the variational solution $u \in u_h^o + H_0^1(\Omega)$ of the problem

$$\left\{\begin{array}{ll} \Delta u = 0 & \text{in } \Omega_h \\ u = u_h^o & \text{on } \partial\Omega_h. \end{array}\right. \tag{4.5}$$

Set $\xi_h = \frac{2}{h}\min\{u_h^*, h/2\}$. Extending ξ_h by the value 1 on $\Omega \setminus \Omega_h$, we see that (i) holds.

Let us show (iii). By construction, $\operatorname{supp} \nabla\xi_h \subset \Omega_h$. For h sufficiently small, the Friedrichs inequality yields the lower bound on $\displaystyle\int_\Omega |\nabla\xi_h|$:

$$\int_\Omega |\nabla \xi_h| \ge \frac{1}{\mathcal{M}} \int_\Omega |\xi_h| \ge \frac{|\Omega|}{2\mathcal{M}}.$$

Finally, we have $|\nabla u_h^o| \le 1$ in Ω_h, because u_h^o is Lipschitz continuous with the Lipschitz constant equal to 1. Using the variational interpretation of (4.5), the Hölder inequality and $(H^\Omega)(i)$, we deduce

$$\int_\Omega |\nabla\xi_h|^2 \le \frac{4}{h^2}\int_{\Omega_h} |\nabla u_h^*|^2 \le \frac{4}{h^2}\int_{\Omega_h} |\nabla u_h^o|^2 \le \frac{4|\Omega_h|}{h^2} \le \frac{\mathcal{M}}{h},$$

$$\int_\Omega |\nabla\xi_h| \le \int_{\Omega_h} |\nabla\xi_h| \le \Big(|\Omega_h| \int_{\Omega_h} |\nabla\xi_h|^2\Big)^{1/2} \le \mathcal{M}.$$

Let us show that (ii) holds. Denote $\Omega_h^* = \{x \in \Omega_h \,|\, u_h^* < 1/2\}$, and $\partial^{int}\Omega_h^* = \partial\Omega_h^* \setminus \partial\Omega$; $\partial^{int}\Omega_h^*$ is the 1/2-level set of u_h^*. Let n denote the exterior unit normal vector to $\partial^{int}\Omega_h^*$. By the interior regularity result for the problem (4.5), it follows that $\nabla u_h^* \cdot n|_{\partial^{int}\Omega_h^*}$ exists in the pointwise sense; moreover, it is nonnegative, by definition of Ω_h^*. Thus ξ_h solves the Laplace equation in Ω_h^* with the zero Dirichlet condition on $\partial\Omega$ and a nonnegative Neumann condition on $\partial^{int}\Omega_h^*$. Taking W as a test function, we get

$$0 \le \int_{\partial^{int}\Omega_h^*} W\, \nabla\xi_h \cdot n = \int_{\Omega_h^*} \nabla W \cdot \nabla \xi_h = \int_\Omega \nabla W \cdot \nabla\xi_h. \qquad \square$$

Lemma 4.5 *Assume $(H^\Omega)(ii)$. Let $(\xi_h)_{h>0} \subset H_0^1(\Omega)$ be a family of functions satisfying (4.4), and $h_m = \frac{1}{m}$. Then (4.3) holds.*

Proof Set $M=\max\{\|w\|_\infty,\|\widehat{w}\|_\infty\}$. There exists a continuous concave function $\Psi:[0,2M]\to\mathbb{R}^+$, $\Psi(0)=0$, such that

$$|F(z)-F(\widehat{z})|\leq\Psi(r)\text{ for all } z,\widehat{z}\in\mathbb{R}\text{ with } |z-\widehat{z}|\leq r\text{ and } |z|,|\widehat{z}|\leq M. \tag{4.6}$$

Using (4.6) and applying the Jensen inequality for Ψ with respect to the measure on Q given by $|\nabla\xi_h|\,dxdt$, we get

$$\begin{aligned}\iint_Q \mathrm{sign}_0^+(w-\widehat{w})|F(w)-F(\widehat{w})|\,|\nabla\xi_{h_m}| &\leq \iint_Q \Psi\big((w-\widehat{w})^+\big)|\nabla\xi_{h_m}|\\ &\leq \Big(\iint_Q|\nabla\xi_h|\Big)\,\Psi\Big(\frac{1}{\iint_Q|\nabla\xi_h|}\iint_Q(w-\widehat{w})^+|\nabla\xi_h|\Big).\end{aligned} \tag{4.7}$$

Denote the left-hand side of (4.7) by I_h. Using the Cauchy-Schwartz inequality and the properties (4.4), we deduce

$$I_h\leq\mathcal{M}\,\Psi\Big(\mathcal{M}\Big(\frac{1}{h}\iint_{(0,T)\times\Omega_h}|(w-\widehat{w})^+|^2\Big)^{1/2}\Big).$$

Finally, note that since $g\leq\widehat{g}$ and $w-g,\widehat{w}-\widehat{g}\in L^2(0,T,H_0^1(\Omega))$, we have $(w-\widehat{w})^+(t)\in H_0^1(\Omega)$ for a.a. $t\in(0,T)$. Hence the Poincaré-Friedrichs inequality $(H^\Omega)(ii)$ yields

$$I_h\leq\mathcal{M}\,\Psi\Big(\mathcal{M}^{3/2}\Big(\iint_{(0,T)\times\Omega_{4h}}|\nabla(w-\widehat{w})|^2\Big)^{1/2}\Big).$$

Thus I_h converges to zero as $h\to 0$, which ends the proof. □

5 Proof of Theorem 1.3

L^∞ case: assume that $w,\widehat{w}$ are bounded on Q. Then the claim of the theorem follows readily by Lemmas 3.1,4.1,4.2 and the dominated convergence theorem.

General case: let us reduce the general case to the L^∞ one. We proceed as in [16], taking advantage of the homogeneity of the (linear) diffusion term div ∇w. Let $(S_M)_{M\in\mathbb{N}}$ be a sequence of $\mathcal{C}^1(\mathbb{R},\mathbb{R})$ functions such that $S_M(z)=1$ for $|z|\leq M-1$, $S_M(z)=0$ for $|z|\geq M$, and $\max_{\mathbb{R}}|S_M'|\leq 2$. Let v be a weak solution to $(P_g(f,j_o))$, $w=\varphi(v)$. Let $v_o:\Omega\to\mathbb{R}$ be a measurable function such that $j(v_o)=j_o$. Let us define the functions

$$j_M(r)=\int_0^r S_M(\varphi(z))dj(z),\quad j_{M,o}=j_M(v_o),\quad \varphi_M(r)=\int_0^{\varphi(r)}S_M(z)dz,$$

$$g_M=\int_0^g S_M(z)dz,\quad f_M=fS_M(w)-((\nabla w+F(w))\cdot\nabla w)S_M'(w),$$

$$F_M=(FS_M)\circ H,\quad\text{where}\quad H(r)=\begin{cases}\min\{s\mid\int_0^s S_M(z)\,dz=r\},\ r\geq 0\\ \max\{s\mid\int_0^s S_M(z)\,dz=r\},\ r\leq 0.\end{cases}$$

Remark that we actually have $F_M(w_M)=F(w)S_M(w)$, and F_M is a correctly defined continuous function. Further, $g_M\in L^2(0,T;W^{1,2}(\Omega))$ and

$|w_M - g_M| = \Big|\displaystyle\int_g^w S_M(z)\,dz\Big| \le |w-g|$, so that $w_M - g_M \in L^2(0,T;H_0^1)$.
Note that $\widehat{g} \ge g$ implies $\widehat{g}_M \ge g_M$. Finally, $f_M \to f$ in $L^1(Q)$ and $j_{M,o} \to j_o$ in $L^1(\Omega)$ as $M \to \infty$, by the dominated convergence theorem.
Take an admissible test function ξ in Definition 1.1. Then $\xi \frac{1}{h}\int_t^{t+h} S_M(w)$ is still an admissible test function. Passing to the limit as $h \to 0$, by the chain rule lemma (see [1, 22] and [10], for the version we use), we find that v is also a weak solution to the auxiliary problem

$$(P^M_{g_M}(f_M, j_{M,o}))\quad \begin{cases} j_M(v)_t - \operatorname{div}\,(\,\nabla w_M + F_M(w_M)) = f_M(t,x), \\ \qquad\qquad\qquad w_M = \varphi_M(v) \quad \text{in } Q=(0,T)\times\Omega \\ w_M = g_M \quad \text{on } \Sigma = (0,T)\times\partial\Omega \\ j_M(v)|_{t=0} = j_{M,o} \quad \text{in } \Omega. \end{cases}$$

(formally, this point of view corresponds to multiplying the equation in $(P_g(f,j_o))$ by $S_M(w)$).
Now we have $|w_M| \le M$. Applying the same construction to the solution $\widehat{v}$, we find ourselves in the L^∞ case for the problems $(P^M_{g_M}(f_M, j_{M,o}))$, $(P^M_{\widehat{g_M}}(\widehat{f}_M, \widehat{j}_{M,o}))$. Thus (1.4) holds with $f, \widehat{f}$ and $j, j_o, \widehat{j_o}$ replaced by $f_M, \widehat{f_M}$ and $j_M, j_{M,o}, \widehat{j}_{M,o}$, respectively. As $M \to \infty$, by the dominated convergence theorem we deduce the claim of the theorem. □

6 Generalizations

Let us give different extensions of our results and discuss the limitations of our techniques.

6.1 Extension to renormalized solutions

The reduction to the case $w \in L^\infty(Q)$ used in the proof of Theorem 1.3 is inspired by the technique of Igbida, Wittbold [16] developed for renormalized solutions. In this section, we further use it in order to extend the result of Theorem 1.3 to renormalized solutions of $(P_g(f,j_o))$. Note that the result of Theorem 2.2 adapts to the case of renormalized solutions of the stationary problem $(S_g(f))$ in the same way.

Let us first recall the notion of renormalized solutions of elliptic-parabolic problems (see e.g. [7, 10, 8] for the motivation). For $k>0$ we denote by T_k the truncation function defined by $T_k : r \in \mathbb{R} \mapsto \operatorname{sign} r \,\min\{k, |r|\}$.

Definition 6.1 *Let $p \in (1,+\infty)$, $p' = p/(p-1)$. Assume $(f,g,j_o) \in L^1(Q)\times L^1(Q)\times L^1(\Omega)$, and $T_k(g) \in L^2(0,T;W^{1,p}(\Omega))$ for all $k>0$. An a.e. defined measurable function $v : Q \mapsto \overline{\mathbb{R}}$ is called renormalized solution of $(P_g(f,j_o))$ if $j(v) \in L^1(\Omega)$, the function $w = \varphi(v)$ is such that $T_k(w) \in L^p(0,T;W^{1,p}(\Omega))$ with $T_k(w-g) \in L^p(0,T;W_0^{1,p}(\Omega))$, $a(w, \nabla T_k(w)) \in L^{p'}(Q)^N$, and* *(i) for any compactly supported* $S \in \mathcal{C}^1(I\!R; I\!R)$ *the distributional derivative* $\Big(\displaystyle\int_0^{j(v)} S(z)\,dz\Big)_t$ *can be identified with* $\chi_S \in L^{p'}(0,T;W^{-1,p'}(\Omega)) + L^1(Q)$ *such that*

$$\int_0^T < \chi_S, \xi > + \iint_Q a(w,\,\nabla w)\cdot\nabla(S(w)\xi) = \iint_Q f S(w)\xi$$

for all test function $\xi \in L^p(0,T;W_0^{1,p}(\Omega)) \cap L^\infty(Q)$*, and*

$$\int_0^T < \chi_S, \xi > = -\iint_Q \Big(\int_0^{j(v)} S(z)\,dz\Big)\xi_t - \int_\Omega \Big(\int_0^{j_o(x)} S(z)\,dz\Big)\xi(0,x)$$

for all test function $\xi \in L^p(0,T;W_0^{1,p}(\Omega)) \cap L^\infty(Q)$ *such that* $\xi_t \in L^\infty(Q)$ *and* $\xi(T,\cdot) = 0$.

(ii) $\displaystyle\iint_{\{(t,x)\in Q \mid M-1 \le |w(t,x)| \le M\}} a(w,\nabla w)\cdot\nabla w \to 0 \quad$ *as* $M \to \infty$.

For the homogeneous Dirichlet problem and under various restrictions on the flux a, the existence of renormalized solutions is known (see [7, 8, 16, 3] and references therein). Uniqueness for the homogeneous Dirichlet problem is shown in [7, 10, 8]. The theorem below extends these last results to the inhomogeneous case, for fluxes of the form $(H0)$.

Theorem 6.2 *The statement of Theorem 1.3 remains true if we assume that* $v, \widehat{v}$ *are renormalized solutions of* $(P_g(f,j_o))$.

Theorem 6.2 follows from the fact that a renormalized solution of $(P_g(f,j_o))$ is also a weak solution of the problem $(P^M_{g_M}(f_M, j_{M,o}))$ with the corresponding functions S_M,j_M,$j_{M,o}$,φ_M, g_M, F_M and f_M (see Section 5). Using (ii) of Definition 6.1, we deduce that $f_M \to f$ in $L^1(Q)$ as $M \to \infty$, and then conclude the proof as in Section 5.

6.2 On more general fluxes

Remark 6.3 *One can allow for a quite general dependency of* a *on* (t,x) *when* a *is Hölder continuous in* w *of order* $1/2$ *(cf. [1, 22]) or Lipschitz continuous in* w *(cf. [8]). For less regular convections, the method of doubling of variables remains essential. This method imposes important restrictions on the dependence of* a *on* x*, especially for the case of non Lipschitz convection. However, one can extend the result of Lemma 3.1 and then the ones of Theorems 1.3,2.2 to the fluxes of the form*

$$a(t,x,w,\,\nabla w) = \nabla w + F(w) + G(w)q(t,x)$$

with $G : I\!R \to I\!R$ *continuous and* $q : Q \to I\!R^N$ *such that* $\mathrm{div}_x q = 0$ *and* $q \in L^\infty(Q)$ *(cf. [14]).*

Remark 6.4 *The same kind of idea gives an approach to the uniqueness for the stationary problem* $(S_g(f))$ *in the case of nonlinear diffusion of the form*

$$(H0') \quad \left| \begin{array}{l} a(w,\,\nabla w) = b(\,\nabla w) + F(w) \\ \text{with monotone continuous } \ b : I\!R^N \to I\!R^N. \end{array}\right.$$

In this case, one still can obtain the stationary analogue of Lemma 3.1 (cf. e.g. [10]).

In order to avoid the unnecessary complications, let us assume that either the functions $w, \widehat{w}$ are bounded, or b is homogeneous (i.e., $b(\lambda\,\xi) = |\lambda|^{p-1}\lambda\, b(\xi)$; this includes linear elliptic problems and the p-laplacian). As in Lemma 4.2 and Theorem 1.3, the L^1 contraction and comparison property (2.2) would follow, if instead of (4.2) we show that

$$\limsup_{h\to 0} \int_\Omega \operatorname{sign}_0^+(w-\widehat{w})(b(\nabla w)-b(\nabla \widehat{w}))^+ \cdot \nabla \xi_h \geq 0 \qquad (6.1)$$

for an appropriate choice of ξ_h satisfying (4.4). To this end, it suffices to assume

$$(H0'') \quad \left|\begin{array}{l} b = \nabla\Phi \ \text{ for } \ \Phi \in \mathcal{C}^2(\mathbb{R},\mathbb{R}) \\ \text{with the Hessian matrix } D^2\Phi \ \text{ satisfying } \ 1/\mathcal{M} \leq D^2\Phi \leq \mathcal{M}, \end{array}\right.$$

and replace the auxiliary problem (4.5) in the proof of Lemma 4.4 by the appropriate adjoint problem:

$$\begin{cases} \operatorname{div}(P(\cdot)\,\nabla u) = 0 \ \text{ in } \Omega_h, \\ u = u_h^o \ \text{ on } \partial\Omega_h, \end{cases} \qquad P = \int_0^1 D^2\Phi\,(\theta\,\nabla w + (1-\theta)\,\nabla\widehat{w})\, d\theta. \qquad (6.2)$$

Indeed, with the notation of the proof of Lemma 4.4, the left-hand side of (6.1) can be rewritten as

$$\int_\Omega \nabla(w-\widehat{w})^+ \cdot P(x)\,\nabla\xi_h = \int_{\partial^{int}\Omega_h^*} (w-\widehat{w})^+\, P(x)\,\nabla\xi_h\cdot n,$$

which is nonnegative, because $\nabla\xi_h = |\nabla\xi_h| n$ a.e. on $\partial^{int}\Omega_h^*$.

This extends the results of Theorem 2.2 to solutions of $(S_g(f))$ with flux a satisfying $(H0'), (H0'')$. In the same way we easily obtain the extension of both Theorems 1.3 and 2.2 to the case of linear elliptic problems (i.e., for $b(\xi) = A\xi$ with a positive definite matrix A).

Note that this kind of proof would not work for the evolution problem $(P_g(f, j_o))$ with nonlinear diffusion, since ξ_h would depend on t through $w, \widehat{w}$. It should be pointed out that for the homogeneous Dirichlet problem $(P_0(f, j_o))$ with flux a of the form $(H0')$, using the approach of [9] one can prove the L^1 contraction and comparison principle, provided that $\partial\Omega$ can be locally represented by a graph of a continuous function (see [4]).

Remark 6.5 *As mentioned in Remark 4.3, the choice of the distance-to-the-boundary functions $\xi_h^o = \frac{1}{h}\min\{h, \operatorname{dist}(x,\partial\Omega)\}$ readily yields (4.4); then (4.3) follows, by Lemma 4.5. Moreover, (4.2) can also be shown, for a sufficiently regular (say, piecewise $\mathcal{C}^2$) domain Ω. For instance, in the case of a flat portion of the boundary $\{0\}\times U$, $U \subset \mathbb{R}^{N-1}$, (4.2) with $\xi_h = \xi_h^o$ reduces to the inequality*

$$\frac{1}{h}\int_0^h \Big(\int_U \frac{\partial}{\partial x_N}(w-\widehat{w})^+\, dx_1 \dots dx_{N-1}\Big)\, dx_N \geq 0, \quad \text{for a.a. } t \in (0,T).$$

This inequality holds true, thanks to the Newton-Leibnitz formula.

In the same spirit, proving (6.1) for $\xi_h = \xi_h^o$ *would permit to generalize Lemma 4.2. Note that in the assumptions* $(H2)$,$(H^\Omega)(ii)$, *we only have to replace* $H_0^1(\Omega)$ *by* $W_0^{1,p}(\Omega)$ *and substitute the power* p *for* 2 *in the corresponding integrands.*

Thus the results of Theorems 1.3,2.2 can be extended to general nonlinear fluxes of the form $(H0')$ *if the following question is answered positively. For simplicity, we state it in the case of localized flat boundary and the p-laplacian operator.*

> *Let* $w, \widehat{w} \in W^{1,p}(I\!R^+ \times I\!R^{N-1})$ *with coinciding traces on* $\{0\} \times I\!R^{N-1}$ *and compact support. Denote by* e_1 *the unit vector* $(1, 0, ..., 0)$.
> *Is it true that*

$$\lim_{h\to 0} \frac{1}{h} \int_0^h \int_{I\!R^{N-1}} \mathrm{sign}_0^+(w - \widehat{w}) \Big(|\nabla w|^{p-2} \nabla w - |\nabla \widehat{w}|^{p-2} \nabla \widehat{w} \Big) \cdot e_1 \geq 0 \ \ ?$$

It should be pointed out that, thanks to the results of [12], one can assume that a trace of $\mathrm{sign}_0^+(w-\widehat{w}) \big(b(\nabla w) - b(\nabla \widehat{w})\big) \cdot e_1$ *on the boundary exists in the weak sense. Indeed, entropy inequalities and an analogue of Lemma 3.1 can be obtained with fluxes* $(H0')$. *They imply in particular that the distribution* $\mathrm{div}_{(t,x)} \Big((j(v) - j(\widehat{v}))^+ , \mathcal{F} \Big)$ *with*

$$\mathcal{F} = \mathrm{sign}_0^+(w-\widehat{w}) \Big(b(\nabla w) - b(\nabla \widehat{w})\Big) + \mathrm{sign}_0^+(w-\widehat{w}) \Big(F(w) - F(\widehat{w})\Big)$$

is a Radon measure; furthermore, the two terms in the right-hand side of the above formula can be dissociated, thanks to the appropriate version of (4.4) and Lemma 4.5. For the stationary problem in case $N = 1$, *this reasoning eventually leads to the positive answer to the question. Indeed,* $\mathrm{sign}_0^+(w - \widehat{w}) \big(b(w_x) - b(\widehat{w}_x)\big)$ *is a function of bounded variation in this case, therefore its strong limit exists, as* $x \to 0^+$. *It is easily seen that this limit cannot be negative, because* $b(\cdot)$ *is monotone.*

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