A degenerate elliptic-parabolic problem with nonlinear dynamical boundary conditions

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Abstract

In this paper we prove existence and uniqueness of weak solutions for a general degenerate elliptic-parabolic problem with nonlinear dynamical boundary conditions. Particular instances of this problem appear in various phenomena with changes of phase like multiphase Stefan problem and in the weak formulation of the mathematical model of the so called Hele Shaw problem. Also, the problem with non-homogeneous Neumann boundary condition is included.

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1 Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$ and $p > 1$, and let $\mathbf{a} : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ be a Carathéodory function satisfying

(H_1) there exists $\lambda > 0$ such that $\mathbf{a}(x,\xi)\cdot\xi \geq \lambda|\xi|^p$ for *a.e.* $x \in \Omega$ and for all $\xi \in \mathbb{R}^N$,

(H_2) there exists $c > 0$ and $\varrho \in L^{p'}(\Omega)$ such that $|\mathbf{a}(x,\xi)| \leq \sigma(\varrho(x) + |\xi|^{p-1})$ for *a.e.* $x \in \Omega$ and for all $\xi \in \mathbb{R}^N$, where $p' = \frac{p}{p-1}$,

(H_3) $(\mathbf{a}(x,\xi_1) - \mathbf{a}(x,\xi_2))\cdot(\xi_1 - \xi_2) > 0$ for *a.e.* $x \in \Omega$ and for all $\xi_1, \xi_2 \in \mathbb{R}^N$, $\xi_1 \neq \xi_2$.

The hypotheses $(H_1 - H_3)$ are classical in the study of nonlinear operators in divergent form (see [43] or [10]). The model example of function $\mathbf{a}$ satisfying these hypotheses is $\mathbf{a}(x,\xi) = |\xi|^{p-2}\xi$. The corresponding operator is the p-Laplacian operator $\Delta_p(u) = \mathrm{div}(|Du|^{p-2}Du)$.

We are interested in the following degenerate elliptic-parabolic problem with nonlinear dynamical boundary condition

$$P_{\gamma,\beta}(f,g,z_0,w_0) \quad \begin{cases} z_t - \mathrm{div}\,\mathbf{a}(x,Du) = f, \ z \in \gamma(u), & \text{in } Q_T :=]0,T[\times\Omega \\ \\ w_t + \mathbf{a}(x,Du)\cdot\eta = g, \ w \in \beta(u), & \text{on } S_T :=]0,T[\times\partial\Omega \\ \\ z(0) = z_0 \ \text{ in } \Omega, \ w(0) = w_0 \ \text{ in } \partial\Omega, \end{cases}$$

where $T > 0$, the nonlinearities γ and β are maximal monotone graphs in $\mathbb{R}^2$ (see, e.g. [20]) such that $0 \in \gamma(0)$, $\{0\} \neq \mathrm{Dom}(\gamma)$, and $0 \in \beta(0)$, $v_0 \in L^1(\Omega)$, $w_0 \in L^1(\partial\Omega)$, $f \in L^1(0,T;L^1(\Omega))$, $g \in L^1(0,T;L^1(\partial\Omega))$ and η is the unit outward normal on $\partial\Omega$. In particular, γ and β may be multivalued and this allows to include the Dirichlet boundary condition (taking β to be the monotone graph $\mathtt{D} = \{0\}\times\mathbb{R}$), in such a case we are considering, in fact, the following problem with static boundary condition

$$DP_\gamma(f,z_0) \quad \begin{cases} z_t - \mathrm{div}\,\mathbf{a}(x,Du) = f, \ z \in \gamma(u), & \text{in } Q_T \\ \\ u = 0, & \text{on } S_T \\ \\ z(0) = z_0 \ \text{ in } \Omega, \end{cases}$$

and the non-homogeneous Neumann boundary condition (taking β to be the monotone graph $\mathtt{N}$ defined by $\mathtt{N}(r) = 0$ for all $r \in \mathbb{R}$), in such a case we are considering the following problem

$$NP_\gamma(f,g,z_0) \quad \begin{cases} z_t - \mathrm{div}\,\mathbf{a}(x,Du) = f, \ z \in \gamma(u), & \text{in } Q_T \\ \\ \mathbf{a}(x,Du)\cdot\eta = g, & \text{on } S_T \\ \\ z(0) = z_0 \ \text{ in } \Omega, \end{cases}$$

as well as many other nonlinear fluxes on the boundary that occur in some problems in Mechanics and Physics (see, e.g., [27] or [19]). Note also that, since γ may be multivalued, problems of type $P_{\gamma,\beta}(f,g,z_0,w_0)$ appear in various phenomena with changes of phase like multiphase Stefan problem (cf [25]) and in the weak formulation of the mathematical model of the so called Hele Shaw problem (see [26] and [28]). Moreover, if $\gamma = \mathtt{N}$, we consider the following elliptic problem with nonlinear dynamical boundary condition

$$BP_\beta(f,g,w_0) \quad \begin{cases} -\mathrm{div}\,\mathbf{a}(x,Du) = f, & \text{in } Q_T \\ \\ w_t + \mathbf{a}(x,Du)\cdot\eta = g, \ w \in \beta(u), & \text{on } S_T \\ \\ w(0) = w_0 \ \text{ in } \partial\Omega. \end{cases}$$

The dynamical boundary conditions, although not too widely considered in the mathematical literature, are very natural in many mathematical models as heat transfer in a solid in contact with moving fluid, thermoelasticity, diffusion phenomena, the heat transfer in two phase medium (Stefan problem), problems in fluid dynamics, etc. (see [8], [23], [29], [45] and the reference therein).

They appears in the studie of the Stefan problem when the boundary material has a large thermal conductivity and sufficiently small thickness. Hence, the boundary material is regarded as the boundary of the domain. For instance, one considers an iron ball in which water and ice coexists. For more details about above physical considerations one can see for instance [1]. They appears also in the studie of the Hele-Shaw problem. Recall that, in [26] the authors gives the weak formulation of the problem in the form of a non linear degenerate parabolic problem, governed by the Laplace operator and the multivalued heaviside function, with static boundary condition. From the physical point of view they assume that the prescribed value of the flux on the boundary is known. But, in some practical situations it may be not possible to prescribe or to control the exact value of the flux on the boundary. In [44], the authors consider the case of nonlocal dynamical boundary conditions and use variationnal method to solve the problem. In this paper, we are convering the case of general nonlinear diffusion and local dynamical boundary conditions. Another interesting application we have in mind concerns the filtration equation with dynamical boundary conditions (see for instance [46]), which appears for example in the study of rainfall infiltration through the soil, when the accumulation of the water on the ground surfaces caused by the saturation of the surface layer is taken into account. Notice that β may be such that $\mathrm{Im}(\rho) \neq \mathbb{R}$, so that we can cover the case where the boundary conditions are either dynamical or static whith respect to the values of w. For instance, one can think about the situation where the saturation happens only for values of w in a subinterval of $\mathbb{R}$.

In contrast to the case of Dirichlet boundary condition (problem $DP_\gamma(f, z_0)$), which is well known (see [2], [4], [15], [17], [21], [38] and the references therein), at our knowledge there is few literature concerning problems $P_{\gamma,\beta}(f, g, z_0, w_0)$, being the results mostly for particular non linearities and for the Laplace operator. For instance, the problem $NP_\gamma(f, g, z_0)$ is treated by Hulshof in [32] in the particular case where γ is a uniformly Lipschitz continuous function, $\gamma(r) = 1$ for $r \in \mathbb{R}^+$, $\gamma \in C^1(\mathbb{R}^-)$, $\gamma' > 0$ on $\mathbb{R}^-$ and $\lim_{r\downarrow-\infty} \gamma(r) = 0$ and some particular functions g. Kenmochi in [39] considers the same problem in the case $\gamma \in \mathcal{C}(\mathbb{R})$ with $\mathrm{Ran}(\gamma)$ a closed bounded interval. The second author of this paper, in [33] and [35], studies the cases where γ is the Heaviside maximal monotone graph and the case where $\gamma(r) = \exp(r)$, respectively. In one space dimension, much more literature exists (see [16] and [47] and the references therein).

For elliptic-parabolic problems with dynamical boundary conditions, the cases in which γ and β are both linear are well known (see for instance [31], [29], [30], [41], [3] and the references therein). For the general nonlinear case, that is, γ and β maximal monotone graphs, most of the papers in the literature are for the Laplace operator and for γ and β with range equal to $\mathbb{R}$ (see [46], [1] and the references therein). The problem becomes more complicated if one of the ranges of γ and β may not be equal to $\mathbb{R}$ and there are few results in the literature. In [34] the case where β is a continuous nondecreasing function (possibly depending on x) and γ is the Heaviside maximal monotone graph, which corresponds to the Hele-Shaw problem is studied. In [37], the authors consider the homogeneous case, i.e., $f = 0$ and $g = 0$, and γ and β maximal monotone graphs everywhere defined.

Roughly speaking, in contrast to the Dirichlet boundary condition, for the Neumann boundary condition and/or dynamical boundary conditions, the problem is noncoercive and moreover, the conservation of the mass exhibits a necessary condition for the existence of a solution related to the ranges of the nonlinearities γ and β (see (6)). Indeed, prescribing the value of u at some part of the lateral boudary, one can control the Sobolev norm of the solution in the interior of Ω by the L^p norm of the gradient in Ω. This is not possible in the case of purely Neumann boundary condition or dynamical ones, and one has to find some substitute for this kind of arguments. In the case the nonlinearities have ranges equal to $\mathbb{R}$ and assuming additional assumptions on f and g one can obtain L^∞-estimates for the solutions (see for instance [32] and [37]). If one of the ranges is not equal to $\mathbb{R}$, the L^∞-estimates are lost and the existence proof of solutions becomes complicated.

Another main difficulty when dealing with doubly nonlinear parabolic problems is the uniqueness. For the Laplace operator, thanks to the linearity of the operator, the problem can be solved by using suitable test functions with respect to u (see for instance [37]). For nonlinear operators this kind of arguments turns out to be non useful. In [15], for an elliptic-parabolic problem with Dirichlet boundary conditions, it is shown that the notion of integral solution ([9]) is a very useful tool to prove uniqueness (see also [36] for nonhomogeneous and time dependent Neumann boundary conditions). For general non linearities, even for homogeneous Dirichlet boundary condition, the question of uniqueness is more difficult and most of the arguments used in the literature are based on doubling variables methods (see for instance [21], [22], [38], [17], [5] and the references therein). In this paper, we use the notion of integral solution and we show that is a very good technique to prove uniqueness for this kind of problems.

To study the problem we use as a main tool the Nonlinear Semigroup Theory ([13], [48]). So we need to consider the elliptic problem

$$(S^{\gamma,\beta}_{\phi,\psi}) \quad \begin{cases} -\mathrm{div}\ \mathbf{a}(x, Du) + \gamma(u) \ni \phi & \text{in } \Omega \\ \\ \mathbf{a}(x, Du) \cdot \eta + \beta(u) \ni \psi & \text{on } \partial\Omega. \end{cases}$$

In [6], under rather general assumptions, existence of solutions and a contraction principle for the problem $(S^{\gamma,\beta}_{\phi,\psi})$ are obtained. Using these results we prove the existence of mild solutions for the associated Cauchy problem and under some additional natural conditions, we show that mild solution are weak solutions. For the uniqueness, we show that weak solutions are integral solutions.

Let us briefly summarize the contents of the paper. In Section 2 we fix the notation and give some preliminaries; we also give the concept of weak solution for the problem

$P_{\gamma,\beta}(f,g,z_0,w_0)$ and state the existence and uniqueness result for weak solutions of problem $P_{\gamma,\beta}(f,g,z_0,w_0)$ and a contraction principle satisfied by weak solutions. In Section 3 we study the problem from the point of view of Nonlinear Semigroup Theory, which is a tool used to prove our results. In Section 4 we prove the existence of weak solutions and in Section 5 we prove the contraction principle. Finally, in the appendix we give the proof of the characterization of the closure of the domain of the accretive operator associated to our problem.

2 Preliminaries and main result

Throughout the paper, $\Omega \subset \mathbb{R}$ is a bounded domain with smooth boundary $\partial\Omega$, γ and β are maximal monotone graphs in $\mathbb{R}^2$ such that $\mathrm{Dom}(\gamma) \neq \{0\}$ and $0 \in \gamma(0) \cap \beta(0)$ and the Carathéodory function $\mathbf{a} : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ satisfies $(H_1) - (H_3)$.

We denote by $|A|$ the Lebesgue measure of a set $A \subset \mathbb{R}^N$ or its $(N-1)$-Hausdorff measure. For $1 \leq q < +\infty$, $L^q(\Omega)$ and $W^{1,q}(\Omega)$ denotes respectively the standard Lebesgue and Sobolev spaces, and $W_0^{1,q}(\Omega)$ is the closure of $\mathcal{D}(\Omega)$ in $W^{1,q}(\Omega)$. For $u \in W^{1,q}(\Omega)$, we denote by u or $\mathtt{tr}(u)$ the trace of u on $\partial\Omega$ in the usual sense. Recall that $\mathtt{tr}(W^{1,q}(\Omega)) = W^{\frac{1}{q'},q}(\partial\Omega)$ and $\mathrm{Ker}(\mathtt{tr}) = W_0^{1,q}(\Omega)$.

We need to introduce the following sets,

$$V^{1,q}(\Omega) := \Big\{\phi \in L^1(\Omega) : \exists M > 0 \text{ such that}$$

$$\int_\Omega |\phi v| \leq M \|v\|_{W^{1,q}(\Omega)} \ \forall v \in W^{1,q}(\Omega)\Big\}$$

and

$$V^{1,q}(\partial\Omega) := \Big\{\psi \in L^1(\partial\Omega) : \exists M > 0 \text{ such that}$$

$$\int_{\partial\Omega} |\psi v| \leq M \|v\|_{W^{1,q}(\Omega)} \ \forall v \in W^{1,q}(\Omega)\Big\}.$$

$V^{1,q}(\Omega)$ is a Banach space endowed with the norm

$$\|\phi\|_{V^{1,q}(\Omega)} := \inf\{M > 0 : \int_\Omega |\phi v| \leq M \|v\|_{W^{1,q}(\Omega)} \ \forall v \in W^{1,q}(\Omega)\},$$

and $V^{1,q}(\partial\Omega)$ is a Banach space endowed with the norm

$$\|\psi\|_{V^{1,q}(\partial\Omega)} := \inf\{M > 0 : \int_{\partial\Omega} |\psi v| \leq M \|v\|_{W^{1,q}(\Omega)} \ \forall v \in W^{1,q}(\Omega)\}.$$

Observe that, Sobolev embeddings and Trace theorems imply, for $1 \leq q < N$,

$$L^{q'}(\Omega) \subset L^{(Nq/(N-q))'}(\Omega) \subset V^{1,q}(\Omega)$$

and

$$L^{q'}(\partial\Omega) \subset L^{((N-1)q/(N-q))'}(\partial\Omega) \subset V^{1,q}(\partial\Omega).$$

Also,

$$V^{1,q}(\Omega) = L^1(\Omega) \ \text{ and } \ V^{1,q}(\partial\Omega) = L^1(\partial\Omega) \ \text{ when } \ q > N,$$

$$L^q(\Omega) \subset V^{1,N}(\Omega) \ \text{ and } \ L^q(\partial\Omega) \subset V^{1,N}(\partial\Omega) \ \text{ for any } \ q > 1.$$

We say that $\mathbf{a}$ is *smooth* (see [7] and [6]) when, for any $\phi \in L^\infty(\Omega)$ such that there exists a bounded weak solution u of the homogeneous Dirichlet problem

$$(D) \qquad \begin{cases} -\operatorname{div} \mathbf{a}(x, Du) = \phi & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

there exists $\psi \in L^1(\partial\Omega)$ such that u is also a weak solution of the Neumann problem

$$(N) \qquad \begin{cases} -\operatorname{div} \mathbf{a}(x, Du) = \phi & \text{in } \Omega \\ \mathbf{a}(x, Du) \cdot \eta = \psi & \text{on } \partial\Omega. \end{cases}$$

Functions $\mathbf{a}$ corresponding to linear operators with smooth coefficients and p-Laplacian type operators are smooth (see [19] and [42]). In [6], we prove that $\mathbf{a}$ is smooth if and only if for any $\phi \in V^{1,p}(\Omega)$ there exists $\psi = T(\phi) \in V^{1,p}(\partial\Omega)$ such that the weak solution u of (D) is a weak solution of (N). Moreover,

$$\int_\Omega (T(\phi_1) - T(\phi_2))^+ \leq \int_\Omega (\phi_1 - \phi_2)^+,$$

for all $\phi_1, \phi_2 \in V^{1,p}(\Omega)$.

For a maximal monotone graph ϑ in $\mathbb{R} \times \mathbb{R}$ the *main section* ϑ^0 of ϑ is defined by

$$\vartheta^0(s) := \begin{cases} \text{the element of minimal absolute value of } \vartheta(s) \ \text{ if } \vartheta(s) \neq \emptyset, \\ \\ +\infty \qquad \text{if } [s, +\infty) \cap \operatorname{Dom}(\vartheta) = \emptyset, \\ \\ -\infty \qquad \text{if } (-\infty, s] \cap \operatorname{Dom}(\vartheta) = \emptyset. \end{cases}$$

We shall denote $\vartheta_- := \inf \operatorname{Ran}(\vartheta)$ and $\vartheta_+ := \sup \operatorname{Ran}(\vartheta)$. If $0 \in \operatorname{Dom}(\vartheta)$, $j_\vartheta(r) = \int_0^r \vartheta^0(s)ds$ defines a convex l.s.c. function such that $\vartheta = \partial j_\vartheta$. If j_ϑ^* is the Legendre transformation of j_ϑ then $\vartheta^{-1} = \partial j_\vartheta^*$.

In [12] the following relation for $u, v \in L^1(\Omega)$ is defined,

$$u \ll v \ \text{ if}$$

$$\int_\Omega (u-k)^+ \leq \int_\Omega (v-k)^+ \text{ and } \int_\Omega (u+k)^- \leq \int_\Omega (v+k)^- \text{ for any } k > 0,$$

and the following facts are proved.

Proposition 2.1 *Let Ω be a bounded domain in $\mathbb{R}^N$.*

(i) If $u, v \in L^1(\Omega)$ and $u \ll v$, then $\|u\|_q \leq \|v\|_q$ for any $q \in [1, +\infty]$.

(ii) If $v \in L^1(\Omega)$, then $\{u \in L^1(\Omega) : u \ll v\}$ is a weakly compact subset of $L^1(\Omega)$.

As we said in the introduction, our aim is to study the existence and uniqueness of a weak solution for the problem $P_{\gamma,\beta}(f,g,z_0,w_0)$. The concept of weak solution we have in mind is the following.

Definition 2.2 Given $f \in L^1(0,T;L^1(\Omega))$, $g \in L^1(0,T;L^1(\partial\Omega))$, $z_0 \in L^1(\Omega)$ and $w_0 \in L^1(\partial\Omega)$, a *weak solution* of $P_{\gamma,\beta}(f,g,z_0,w_0)$ in $[0,T]$ is a couple (z,w) such that $z \in C([0,T] : L^1(\Omega))$, $w \in C([0,T] : L^1(\partial\Omega))$, $z(0) = z_0$, $w(0) = w_0$ and there exists $u \in L^p(0,T;W^{1,p}(\Omega))$ such that $z \in \gamma(u)$ a.e. in Q_T, $w \in \beta(u)$ a.e. on S_T and

$$\frac{d}{dt}\int_\Omega z(t)\xi + \frac{d}{dt}\int_{\partial\Omega} w(t)\xi + \int_\Omega \mathbf{a}(x,Du(t))\cdot D\xi = \int_\Omega f(t)\xi + \int_{\partial\Omega} g(t)\xi \quad \text{in } \mathcal{D}'(]0,T[) \tag{1}$$

for any $\xi \in C^1(\overline{\Omega})$.

Remark 2.3 Observe that taking $\xi = 1$ in the above definition, we get

$$\int_\Omega z(t) + \int_{\partial\Omega} w(t) = \int_\Omega z_0 + \int_{\partial\Omega} w_0 + \int_0^t \left(\int_\Omega f + \int_{\partial\Omega} g\right) \qquad \forall\, t \in [0,T]. \tag{2}$$

Recall that in the case $\beta = 0$, for the Laplacian operator and γ the multivalued Heaviside function (i.e., for the Hele-Shaw problem), existence and uniqueness of weak solutions for this problem is known to be true only if

$$\int_\Omega z_0 + \int_0^t \left(\int_\Omega f + \int_{\partial\Omega} g\right) \in (0,|\Omega|) \quad \text{for any} \quad t \in [0,T)$$

(see [33] or [39])). For the maximal monotone graphs γ and β, we shall denote

$$\mathcal{R}^+_{\gamma,\beta} := \gamma_+|\Omega| + \beta_+|\partial\Omega|, \qquad \mathcal{R}^-_{\gamma,\beta} := \gamma_-|\Omega| + \beta_-|\partial\Omega|.$$

We will suppose $\mathcal{R}^-_{\gamma,\beta} < \mathcal{R}^+_{\gamma,\beta}$ and we will write $\mathcal{R}_{\gamma,\beta} :=]\mathcal{R}^-_{\gamma,\beta}, \mathcal{R}^+_{\gamma,\beta}[$.

The main results of this paper are the following contraction principle and the following existence and uniqueness theorem.

Theorem 2.4 *Let $T > 0$. For $i = 1,2$, let $f_i \in L^1(0,T;L^1(\Omega))$, $g_i \in L^1(0,T;L^1(\partial\Omega))$, $z_{i0} \in L^1(\Omega)$ and $w_{i0} \in L^1(\partial\Omega)$; let (z_i,w_i) be a weak solution in $[0,T]$ of $P_{\gamma,\beta}(f_i,g_i,z_{i0},w_{i0})$, $i = 1,2$. Then*

$$\begin{aligned}\int_\Omega (z_1(t)-z_2(t))^+ + \int_{\partial\Omega}(w_1(t)-w_2(t))^+ &\le \int_\Omega (z_{10}-z_{20})^+ + \int_{\partial\Omega}(w_{10}-w_{20})^+ \\ &+ \int_0^t\int_\Omega (f_1(\tau)-f_2(\tau))^+\,d\tau + \int_0^t\int_{\partial\Omega}(g_1(\tau)-g_2(\tau))^+\,d\tau\end{aligned} \tag{3}$$

for almost every $t \in]0,T[$.

Theorem 2.5 *Assume $Dom(\gamma)=\mathbb{R}$, $\mathcal{R}^-_{\gamma,\beta}<\mathcal{R}^+_{\gamma,\beta}$ and $Dom(\beta)=\mathbb{R}$ or* **a** *smooth. Let $T>0$. Let $f\in L^{p'}(0,T;L^{p'}(\Omega))$, $g\in L^{p'}(0,T;L^{p'}(\partial\Omega))$, $z_0\in L^{p'}(\Omega)$ and $w_0\in L^{p'}(\partial\Omega)$ such that*

$$\gamma_-\le z_0\le\gamma_+,\quad \beta_-\le w_0\le\beta_+, \tag{4}$$

$$\int_\Omega j^*_\gamma(z_0)+\int_\Gamma j^*_\beta(w_0)<+\infty, \tag{5}$$

and

$$\int_\Omega z_0+\int_{\partial\Omega}w_0+\int_0^t\left(\int_\Omega f+\int_{\partial\Omega}g\right)\in\mathcal{R}_{\gamma,\beta}\qquad\forall\,t\in[0,T]. \tag{6}$$

Then, there exists a unique weak solution (z,w) of problem $P_{\gamma,\beta}(f,g,z_0,w_0)$ in $[0,T]$.

The uniqueness part of Theorem 2.5 follows from Theorem 2.4. To prove Theorem 2.4 and the existence part of Theorem 2.5 we shall use the Theory of Nonlinear Semigroups (c.f. [9], [13] or [24]). We will show the existence of a mild-solution and we will prove that this solution is a weak solution of problem $P_{\gamma,\beta}(f,g,z_0,w_0)$. To prove the contraction principle we will show that weak solutions are integral solutions. For all this we need to rewrite problem $P_{\gamma,\beta}(f,g,z_0,w_0)$ as an abstract Cauchy problem and to use the results obtained in [6] for the associated elliptic problem.

3 Mild solutions

First let us recall some basic facts for the elliptic problem $(S^{\gamma,\beta}_{\phi,\psi})$ given in [6], which will be crucial for the proof of our main results. In [6] the following concept of solution for problem $(S^{\gamma,\beta}_{\phi,\psi})$ is introduced.

Definition 3.1 Let $\phi\in L^1(\Omega)$ and $\psi\in L^1(\partial\Omega)$. A triple of functions $[u,z,w]\in W^{1,p}(\Omega)\times L^1(\Omega)\times L^1(\partial\Omega)$ is a *weak solution* of problem $(S^{\gamma,\beta}_{\phi,\psi})$ if $z(x)\in\gamma(u(x))$ *a.e.* in Ω, $w(x)\in\beta(u(x))$ *a.e.* in $\partial\Omega$, and

$$\int_\Omega \mathbf{a}(x,Du)\cdot Dv+\int_\Omega zv+\int_{\partial\Omega}wv=\int_{\partial\Omega}\psi v+\int_\Omega\phi v, \tag{7}$$

for all $v\in L^\infty(\Omega)\cap W^{1,p}(\Omega)$.

Observe that, if $(S^{\gamma,\beta}_{\phi,\psi})$ has a weak solution then, necessarily ϕ and ψ must satisfy

$$\mathcal{R}^-_{\gamma,\beta}\le\int_{\partial\Omega}\psi+\int_\Omega\phi\le\mathcal{R}^+_{\gamma,\beta}.$$

Indeed, by taking $v=1$ as test function in (7), we get that

$$\int_\Omega z+\int_{\partial\Omega}w=\int_{\partial\Omega}\psi+\int_\Omega\phi.$$

Moreover we have the following existence and uniqueness results on weak solutions of problem $(S^{\gamma,\beta}_{\phi,\psi})$ (see [6]).

Theorem 3.2 *Let $\phi \in L^1(\Omega)$ and $\psi \in L^1(\partial\Omega)$, and let $[u_1, z_1, w_1]$ and $[u_2, z_2, w_2]$ be weak solutions of problem $(S^{\gamma,\beta}_{\phi,\psi})$. Then, there exists a constant $c \in \mathbb{R}$ such that*

$$u_1 - u_2 = c \quad \textit{a.e. in } \Omega,$$

$$z_1 - z_2 = 0 \quad \textit{a.e. in } \Omega$$

and

$$w_1 - w_2 = 0 \quad \textit{a.e. in } \partial\Omega.$$

Moreover, if $c \neq 0$, there exists a constant $k \in \mathbb{R}$ such that $z_1 = z_2 = k$.
(ii) For any $[u_1, z_1, w_1]$ weak solution of problem $(S^{\gamma,\beta}_{\phi_1,\psi_1})$, $\phi_1 \in L^1(\Omega)$ and $\psi_1 \in L^1(\partial\Omega)$, and any $[u_2, z_2, w_2]$ weak solution of problem $(S^{\gamma,\beta}_{\phi_2,\psi_2})$, $\phi_2 \in L^1(\Omega)$ and $\psi_2 \in L^1(\partial\Omega)$, we have that

$$\int_\Omega (z_1 - z_2)^+ + \int_{\partial\Omega} (w_1 - w_2)^+ \leq \int_{\partial\Omega} (\psi_1 - \psi_2)^+ + \int_\Omega (\phi_1 - \phi_2)^+.$$

Theorem 3.2 (ii) is given in [6] in a different way. We want to point out that with the technique used in Section 5 we can get exactly the above result.

Theorem 3.3 *Assume $Dom(\gamma) = \mathbb{R}$. Let $Dom(\beta) = \mathbb{R}$ or* **a** *smooth. For any $\phi \in V^{1,p}(\Omega)$ and $\psi \in V^{1,p}(\partial\Omega)$ with*

$$\int_\Omega \phi + \int_{\partial\Omega} \psi \in \mathcal{R}_{\gamma,\beta}, \tag{8}$$

there exists a weak solution $[u, z, w]$ of problem $(S^{\gamma,\beta}_{\phi,\psi})$. Moreover $z \in V^{1,p}(\Omega)$, $w \in V^{1,p}(\partial\Omega)$ and

$$\int_\Omega \mathbf{a}(x, Du) \cdot Dv + \int_\Omega zv + \int_{\partial\Omega} wv = \int_{\partial\Omega} \psi v + \int_\Omega \phi v,$$

for any $v \in W^{1,p}(\Omega)$.

This results imply that the natural space to study problem $P_{\gamma,\beta}(f, g, z_0, w_0)$ from the point of view of Nonlinear Semigroup Theory is $X = L^1(\Omega) \times L^1(\partial\Omega)$ provided with the natural norm

$$\|(f, g)\| := \|f\|_{L^1(\Omega)} + \|g\|_{L^1(\partial\Omega)}.$$

In this space we define the following operator,

$$\mathcal{B}^{\gamma,\beta} := \Big\{((z, w), (\hat{z}, \hat{w})) \in X \times X \ : \ \exists u \in W^{1,p}(\Omega) \text{ such that}$$

$$[u, z, w] \text{ is a weak solution of } (S^{\gamma,\beta}_{z+\hat{z},w+\hat{w}})\Big\},$$

in other words, $(\hat{z},\hat{w}) \in \mathcal{B}^{\gamma,\beta}(z,w)$ if and only if there exists $u \in W^{1,p}(\Omega)$ such that $z(x) \in \gamma(u(x))$ *a.e.* in Ω, $w(x) \in \beta(u(x))$ *a.e.* in $\partial\Omega$, and

$$\int_\Omega \mathbf{a}(x,Du)\cdot Dv = \int_\Omega \hat{z}v + \int_{\partial\Omega} \hat{w}v \tag{9}$$

for all $v \in L^\infty(\Omega)\cap W^{1,p}(\Omega)$, which allows us to rewrite problem $P_{\gamma,\beta}(f,g,z_0,w_0)$ as the following abstract Cauchy problem in X,

$$\begin{cases} V'(t) + \mathcal{B}^{\gamma,\beta}(V(t)) \ni (f,g) & t\in(0,T) \\ \\ V(0) = (z_0,w_0). \end{cases} \tag{10}$$

A direct consequence of Theorems 3.2 and 3.3 is the following result.

Corollary 3.4 *The operator $\mathcal{B}^{\gamma,\beta}$ is a T-accretive operator in X and, assuming $Dom(\gamma) = \mathbb{R}$, and $Dom(\beta) = \mathbb{R}$ or* **a** *smooth, it satisfies the following range condition,*

$$\left\{(\phi,\psi) \in V^{1,p}(\Omega)\times V^{1,p}(\partial\Omega) : \int_\Omega \phi + \int_{\partial\Omega}\psi \in \mathcal{R}_{\gamma,\beta}\right\} \subset Ran(I+\mathcal{B}^{\gamma,\beta}).$$

Moreover, we can characterize $\overline{D(\mathcal{B}^{\gamma,\beta})}^{L^1(\Omega)\times L^1(\partial\Omega)}$ as follows.

Theorem 3.5 *Under the hypothesis $Dom(\gamma) = \mathbb{R}$, and $Dom(\beta) = \mathbb{R}$ or* **a** *smooth, we have*

$$\overline{D(\mathcal{B}^{\gamma,\beta})}^{L^1(\Omega)\times L^1(\partial\Omega)} = \left\{(z,w)\in L^1(\Omega)\times L^1(\partial\Omega):\ \gamma_- \le z \le \gamma_+,\ \beta_- \le w \le \beta_+\right\}.$$

The proof of this theorem is quite technical and we prove it in the Appendix.

The above results allow us to prove our main result concerning mild solutions.

Theorem 3.6 *Let $T>0$. Under the hypothesis $Dom(\gamma) = \mathbb{R}$, and $Dom(\beta) = \mathbb{R}$ or* **a** *smooth, for every $z_0 \in L^1(\Omega)$, $w_0 \in L^1(\partial\Omega)$ and every $f \in L^1(0,T;L^1(\Omega))$, $g \in L^1(0,T;L^1(\partial\Omega))$, satisfying (4) and (6), there exists a unique mild solution of (10).*

Proof. For $n\in\mathbb{N}$, let $\epsilon = T/n$, and consider a subdivision $t_0 = 0 < t_1 < \cdots < t_{n-1} < T = t_n$ with $t_i - t_{i-1} = \epsilon$, $f_1^\epsilon,\ldots,f_n^\epsilon \in L^{p'}(\Omega)$, $g_1^\epsilon,\ldots,g_n^\epsilon \in L^{p'}(\partial\Omega)$, $w_0^\epsilon \in L^{p'}(\Omega)$ $z_0^\epsilon \in L^{p'}(\partial\Omega)$ with

$$\sum_{i=1}^n \int_{t_{i-1}}^{t_i} \left(\|f(t)-f_i^\epsilon\|_{L^1(\Omega)} + \|g(t)-g_i^\epsilon\|_{L^1(\partial\Omega)}\right)\,dt \le \epsilon$$

and

$$\|z_0^\epsilon - z_0\|_{L^1(\Omega)} + \|w_0^\epsilon - w_0\|_{L^1(\partial\Omega)} \le \epsilon.$$

If we set

$$f_\epsilon(t) = f_i^\epsilon, \quad g_\epsilon(t) = g_i^\epsilon \quad \text{for} \quad t \in]t_{i-1}, t_i], \quad i = 1, \ldots, n,$$

it follows that

$$\int_0^T \left(\|f(t) - f_\epsilon(t)\|_{L^1(\Omega)} + \|g(t) - g_\epsilon(t)\|_{L^1(\partial\Omega)} \right) dt \leq \epsilon. \tag{11}$$

By Theorem 3.3, for n large enough, there exists a weak solution $[u_i^\epsilon, z_i^\epsilon, w_i^\epsilon]$ of

$$\begin{cases} \gamma(u_i^\epsilon) - \epsilon \mathrm{div}\ \mathbf{a}(x, Du_i^\epsilon) \ni \epsilon f_i^\epsilon + z_{i-1}^\epsilon \\ \\ \epsilon \mathbf{a}(x, Du_i^\epsilon) \cdot \eta + \beta(u_i^\epsilon) \ni \epsilon g_i^\epsilon + w_{i-1}^\epsilon, \end{cases} \tag{12}$$

for $i = 1, \ldots, n$. That is, there exists a unique solution $(z_i^\epsilon, w_i^\epsilon) \in X$ of the time discretized scheme associated with (10),

$$(z_i^\epsilon, w_i^\epsilon) + \epsilon \mathcal{B}^{\gamma,\beta}(z_i^\epsilon, w_i^\epsilon) \ni \epsilon(f_i^\epsilon, g_i^\epsilon) + (z_{i-1}^\epsilon, w_{i-1}^\epsilon), \quad \text{for} \quad i = 1, \ldots, n. \tag{13}$$

Observe in fact that, by Theorem 3.3 applied recursively for each $i = 1, \ldots, n$, $[u_i^\epsilon, z_i^\epsilon, w_i^\epsilon] \in W^{1,p}(\Omega) \times V^{1,p}(\Omega) \times V^{1,p}(\partial\Omega)$ and

$$\int_\Omega \mathbf{a}(x, Du_i^\epsilon) \cdot Dv + \int_\Omega \frac{z_i^\epsilon - z_{i-1}^\epsilon}{\epsilon} v + \int_{\partial\Omega} \frac{w_i^\epsilon - w_{i-1}^\epsilon}{\epsilon} v = \int_\Omega f_i^\epsilon v + \int_{\partial\Omega} g_i^\epsilon v \tag{14}$$

for all $v \in W^{1,p}(\Omega)$. Therefore, taking $v = 1$ in (14), we have

$$\int_\Omega z_i^\epsilon + \int_{\partial\Omega} w_i^\epsilon = \epsilon \left(\int_\Omega f_i^\epsilon + \int_{\partial\Omega} g_i^\epsilon \right) + \int_\Omega z_{i-1}^\epsilon + \int_{\partial\Omega} w_{i-1}^\epsilon. \tag{15}$$

From here, it follows that

$$\int_\Omega z_i^\epsilon + \int_{\partial\Omega} w_i^\epsilon = \epsilon \sum_{j=1}^{i} \left(\int_\Omega f_j^\epsilon + \int_{\partial\Omega} g_j^\epsilon \right) + \int_\Omega z_0^\epsilon + \int_{\partial\Omega} w_0^\epsilon,$$

and taking n large enough, condition (8) is always satisfied.

Therefore, if we define $V_\epsilon(t) = (z_\epsilon(t), w_\epsilon(t))$ by

$$\begin{cases} z_\epsilon(0) = z_0^\epsilon, \qquad w_\epsilon(0) = w_0^\epsilon, \\ \\ z_\epsilon(t) = z_i^\epsilon, \qquad w_\epsilon(t) = w_i^\epsilon \quad \text{for} \quad t \in]t_{i-1}, t_i], \quad i = 1, \ldots, n, \end{cases} \tag{16}$$

it is an ϵ-approximate solution of problem (10).

By using now the Nonlinear Semigroup Theory (see [9], [13], [24]), on account of Corollary 3.4 and Theorem 3.5, problem (10) has a unique mild-solution $V(t) = (z(t), w(t)) \in C([0,T] : X)$, $z(t) = L^1(\Omega) - \lim_{\epsilon \to 0} z_\epsilon(t)$ and $w(t) = L^1(\partial\Omega) - \lim_{\epsilon \to 0} w_\epsilon(t)$ uniformly for $t \in [0, T]$. □

In principle, it is not clear how these mild solutions have to be interpreted respect to the problem $P_{\gamma,\beta}(f, g, z_0, w_0)$. We will see they are weak solutions of problem $P_{\gamma,\beta}(f, g, z_0, w_0)$ under the hypothesis of Theorem 2.5, which proves the existence part of Theorem 2.5.

4 Existence of weak solutions

As said in the previous section, the existence part of Theorem 2.5 is shown by proving that the mild solution of problem (10) is a weak solution of problem $P_{\gamma,\beta}(f,g,z_0,w_0)$ whenever the assumptions of Theorem 2.5 are fulfilled. Before giving the proof we need to prove some technical lemmas.

4.1 Preparatory lemmas

We shall use the following integration by parts lemma in the proof of the existence part and in the proof of the contraction principle. We denote by $(.,.)$ the pairing between $(W^{1,p}(\Omega))'$ and $W^{1,p}(\Omega)$.

Lemma 4.1 *Let $(z,w) \in C([0,T] : L^1(\Omega)) \times L^1(\partial\Omega))$ and $F \in L^{p'}(0,T;(W^{1,p}(\Omega))')$, such that*

$$\int_0^T \int_\Omega z(t)\psi_t + \int_0^T \int_{\partial\Omega} w(t)\psi_t = \int_0^T (F(t),\psi(t))dt, \tag{17}$$

for any $\psi \in W^{1,1}(0,T;W^{1,1}(\Omega) \cap L^\infty(\Omega)) \cap L^p(0,T;W^{1,p}(\Omega))$, $\psi(0) = \psi(T) = 0$. Then,

$$\int_0^T \int_\Omega \left(\int_0^{z(t)} H(.,(\gamma^{-1})^0(s))ds \right) \psi_t + \int_0^T \int_{\partial\Omega} \left(\int_0^{w(t)} H(.,(\beta^{-1})^0(s))ds \right) \psi_t$$

$$= \int_0^T (F(t), H(.,u(t))\psi(t))dt,$$

for any $u \in L^p(0,T;W^{1,p}(\Omega))$, $z \in \gamma(u)$ a.e. in Q_T, $w \in \beta(u)$ a.e. in S_T, for any $\psi \in \mathcal{D}(]0,T[\times\mathbb{R}^N)$, and for any $H : \Omega\times\mathbb{R} \to \mathbb{R}$ Caratheodory function such that $H(x,r)$ is nondecreasing in r, $H(.,u) \in L^p(0,T;W^{1,p}(\Omega))$, $\int_0^z H(x,(\gamma^{-1})^0(s))ds \in L^1(Q_T)$ and $\int_0^w H(x,(\beta^{-1})^0(s))ds \in L^1(S_T)$.

Proof. The proof is similar to the one given in [22] for Dirichlet boundary condition. We give it here for the sake of completeness.

Let $\psi \in \mathcal{D}(]0,T[\times\mathbb{R}^N)$, $\psi \geq 0$. And let, for $H_\tau = T_{\frac{1}{\tau}}H$, $\tau > 0$,

$$\eta_\tau(t) = \frac{1}{\tau}\int_t^{t+\tau} H_\tau(.,u(s))\psi(s)ds.$$

Then η_τ can be used as test function in (17) and therefore

$$\int_0^T (F(t),\eta_\tau(t))dt = \int_0^T \int_\Omega z(t)(\eta_\tau)_t + \int_0^T \int_{\partial\Omega} w(t)(\eta_\tau)_t$$

$$= \int_0^T \int_\Omega z(t) \frac{H_\tau(.,u(t+\tau))\psi(t+\tau) - H_\tau(.,u(t))\psi(t)}{\tau}$$

$$+ \int_0^T \int_{\partial\Omega} w(t) \frac{H_\tau(.,u(t+\tau))\psi(t+\tau) - H_\tau(.,u(t))\psi(t)}{\tau}$$

$$= \int_0^T \int_\Omega \frac{z(t-\tau)-z(t)}{\tau} H_\tau(.,u(t))\psi(t) + \int_0^T \int_{\partial\Omega} \frac{w(t-\tau)-w(t)}{\tau} H_\tau(.,u(t))\psi(t).$$

Now, since

$$H_\tau(.,\gamma^{-1}(r)) \subset \partial\left(\int_0^r H_\tau(.,(\gamma^{-1})^0(s))ds\right),$$

$$H_\tau(.,\beta^{-1}(r)) \subset \partial\left(\int_0^r H_\tau(.,(\beta^{-1})^0(s))ds\right),$$

$$H_\tau(.,u(t)) \in H_\tau(.,\gamma^{-1}(z(t))) \quad a.e. \text{ in } \Omega$$

and

$$H_\tau(.,u(t)) \in H_\tau(.,\beta^{-1}(w(t))) \quad a.e. \text{ on } \partial\Omega,$$

we have that

$$(z(t-\tau)-z(t))H_\tau(.,u(t)) \le \int_{z(t)}^{z(t-\tau)} H_\tau(.,(\gamma^{-1})^0(s))ds \quad a.e. \text{ in } \Omega$$

and

$$(w(t-\tau)-w(t))H_\tau(.,u(t)) \le \int_{w(t)}^{w(t-\tau)} H_\tau(.,(\beta^{-1})^0(s))ds \quad a.e. \text{ on } \partial\Omega.$$

Therefore

$$\int_0^T (F(t),\eta_\tau(t))dt \le \int_0^T \int_\Omega \frac{1}{\tau} \int_{z(t)}^{z(t-\tau)} H_\tau(.,(\gamma^{-1})^0(s))ds\psi(t)$$

$$+ \int_0^T \int_{\partial\Omega} \frac{1}{\tau} \int_{w(t)}^{w(t-\tau)} H_\tau(.,(\beta^{-1})^0(s))ds\psi(t)$$

$$= \int_0^T \int_\Omega \int_0^{z(t)} H_\tau(.,(\gamma^{-1})^0(s))ds \frac{\psi(t+\tau)-\psi(t)}{\tau}$$

$$+ \int_0^T \int_{\partial\Omega} \int_0^{w(t)} H_\tau(.,(\beta^{-1})^0(s))ds \frac{\psi(t+\tau)-\psi(t)}{\tau}.$$

Taking limits as $\tau \to 0^+$ we get

$$\int_0^T (F(t), H(., u(t))\psi(t))dt \leq$$

$$\int_0^T \int_\Omega \left(\int_0^{z(t)} H(x, (\gamma^{-1})^0(s))ds \right) \psi_t + \int_0^T \int_{\partial\Omega} \left(\int_0^{w(t)} H(x, (\beta^{-1})^0(s))ds \right) \psi_t.$$

Taking now $\tilde{\eta}_\tau(t) = \frac{1}{\tau} \int_t^{t+\tau} H_\tau(., u(s-\tau))\psi(s)ds$, and arguing as above we get the another inequality. $\square$

To prove the existence of weak solutions we shall also use the following result.

Lemma 4.2 *Let $\{u_n\}_{n\in\mathbb{N}} \subset W^{1,p}(\Omega)$, $\{z_n\}_{n\in\mathbb{N}} \subset L^1(\Omega)$, $\{w_n\}_{n\in\mathbb{N}} \subset L^1(\partial\Omega)$ such that, for every $n \in \mathbb{N}$, $z_n \in \gamma(u_n)$ a.e. in Ω and $w_n \in \beta(u_n)$ a.e. in $\partial\Omega$. Let us suppose that*

(i) if $\mathcal{R}^+_{\gamma,\beta} = +\infty$, there exists $M > 0$ such that

$$\int_\Omega z_n^+ + \int_{\partial\Omega} w_n^+ < M \qquad \forall n \in \mathbb{N},$$

(ii) if $\mathcal{R}^+_{\gamma,\beta} < +\infty$, there exists $M \in \mathbb{R}$ and $h > 0$ such that

$$\int_\Omega z_n + \int_{\partial\Omega} w_n < M < \mathcal{R}^+_{\gamma,\beta} \qquad \forall n \in \mathbb{N}$$

and

$$\max\left\{ \int_{\{x\in\Omega: z_n(x) < -h\}} |z_n|, \int_{\{x\in\partial\Omega: w_n(x) < -h\}} |w_n| \right\} < \frac{\mathcal{R}^+_{\gamma,\beta} - M}{8} \qquad \forall n \in \mathbb{N}.$$

Then, there exists a constant $C = C(M)$ in case (i), $C = C(M, h)$ in case (ii), such that

$$\|u_n^+\|_{L^p(\Omega)} \leq C \left(\|Du_n^+\|_{L^p(\Omega)} + 1 \right) \qquad \forall n \in \mathbb{N}.$$

In order to prove Lemma 4.2, we use the following well known result (see [49]).

Lemma 4.3 *1. There exists a constant $C(\Omega, N, p)$ such that, for any $K \subset \Omega$ with $|K| > 0$,*

$$\|u\|_{L^p(\Omega)} \leq \frac{C(\Omega, N, p)}{|K|^{1/p}} \left(\|Du\|_{L^p(\Omega)} + \|u\|_{L^p(K)} \right), \quad \forall u \in W^{1,p}(\Omega). \tag{18}$$

2. There exists a constant $\hat{C}(\Omega, N, p)$ such that, for any $\Gamma \subset \partial\Omega$ with $|\Gamma| > 0$,

$$\|u\|_{L^p(\Omega)} \leq \frac{\hat{C}(\Omega, N, p)}{|\Gamma|^{1/p}} \left(\|Du\|_{L^p(\Omega)} + \|u\|_{L^p(\Gamma)} \right), \quad \forall u \in W^{1,p}(\Omega). \tag{19}$$

Proof of Lemma 4.2. Consider first that $\mathcal{R}^+_{\gamma,\beta} = +\infty$. Then $\gamma^+ = +\infty$ or $\beta^+ = +\infty$. Let us suppose first that $\gamma^+ = +\infty$. Then, by assumption, there exists $M > 0$ such that

$$\int_\Omega z_n^+ < M \qquad \forall n \in \mathbb{N}.$$

Let $K_n = \left\{ x \in \Omega : z_n^+(x) < \frac{2M}{|\Omega|} \right\}$ for every $n \in \mathbb{N}$. Then

$$0 \le \int_{K_n} z_n^+ = \int_\Omega z_n^+ - \int_{\Omega \setminus K_n} z_n^+ \le M - (|\Omega| - |K_n|) \frac{2M}{|\Omega|} = |K_n| \frac{2M}{|\Omega|} - M.$$

Therefore,

$$|K_n| \ge \frac{|\Omega|}{2},$$

and

$$\|u_n^+\|_{L^p(K_n)} \le |K_n|^{1/p} \sup \gamma^{-1} \left(\frac{2M}{|\Omega|} \right).$$

Then, by Lemma 4.3, for all $n \in \mathbb{N}$,

$$\|u_n^+\|_{L^p(\Omega)} \le C(\Omega, N, p) \left(\left(\frac{2}{|\Omega|} \right)^{1/p} \|Du_n^+\|_{L^p(\Omega)} + \sup \gamma^{-1} \left(\frac{2M}{|\Omega|} \right) \right).$$

If $\beta^+ = +\infty$, we similarly get that, for all $n \in \mathbb{N}$,

$$\|u_n^+\|_{L^p(\Omega)} \le \hat{C}(\Omega, N, p) \left(\left(\frac{2}{|\partial\Omega|} \right)^{1/p} \|Du_n^+\|_{L^p(\Omega)} + \sup \beta^{-1} \left(\frac{2M}{|\partial\Omega|} \right) \right),$$

where $\hat{C}(\Omega, N, p)$ is given in Lemma 4.3.

Let us consider now that $R^+_{\gamma,\beta} < +\infty$. And let $\delta = R^+_{\gamma,\beta} - M$. Then, by assumption,

$$\int_\Omega z_n + \int_{\partial\Omega} w_n < \mathcal{R}^+_{\gamma,\beta} - \delta.$$

Consequently, for every $n \in \mathbb{N}$,

$$\int_\Omega z_n < \gamma^+ |\Omega| - \frac{\delta}{2} \tag{20}$$

or

$$\int_{\partial\Omega} w_n < \beta^+ |\partial\Omega| - \frac{\delta}{2}. \tag{21}$$

For $n \in \mathbb{N}$ such that (20) holds, let $K_n = \left\{ x \in \Omega : z_n(x) < \gamma^+ - \frac{\delta}{4|\Omega|} \right\}$. Then, on the one hand,

$$\int_{K_n} z_n = \int_\Omega z_n - \int_{\Omega \setminus K_n} z_n < -\frac{\delta}{4} + |K_n| \left(\gamma^+ - \frac{\delta}{4|\Omega|} \right),$$

and, on the other hand,

$$\int_{K_n} z_n = -\int_{K_n\cap\{x\in\Omega: z_n<-h\}} |z_n| + \int_{K_n\cap\{x\in\Omega: z_n\geq -h\}} z_n \geq -\frac{\delta}{8} - h|K_n|.$$

Therefore,

$$|K_n|\left(h - \frac{\delta}{4|\Omega|} + \gamma^+\right) \geq \frac{\delta}{8}.$$

Hence $|K_n| > 0$, $h - \frac{\delta}{4|\Omega|} + \gamma^+ > 0$ and

$$|K_n| \geq \frac{\frac{\delta}{8}}{h - \frac{\delta}{4|\Omega|} + \gamma^+}.$$

Consequently,

$$\|u_n^+\|_{L^p(K_n)} \leq |K_n|^{1/p} \sup \gamma^{-1}\left(\gamma^+ - \frac{\delta}{4|\Omega|}\right).$$

Then, by Lemma 4.3,

$$\|u_n^+\|_{L^p(\Omega)} \leq C(\Omega, N, p)\left(\left(\frac{h - \frac{\delta}{4|\Omega|} + \gamma^+}{\frac{\delta}{8}}\right)^{1/p} \|Du_n^+\|_{L^p(\Omega)} + \sup \gamma^{-1}\left(\gamma^+ - \frac{\delta}{4|\Omega|}\right)\right).$$

Similarly, for $n \in \mathbb{N}$ such that (21) holds, we get $\left|\left\{x \in \partial\Omega : w_n(x) < \beta^+ - \frac{\delta}{4|\partial\Omega|}\right\}\right| > 0$, $h - \frac{\delta}{4|\partial\Omega|} + \beta^+ > 0$ and

$$\|u_n^+\|_{L^p(\Omega)} \leq \hat{C}(\Omega, N, p)\left(\left(\frac{h - \frac{\delta}{4|\partial\Omega|} + \beta^+}{\frac{\delta}{8}}\right)^{1/p} \|Du_n^+\|_{L^p(\Omega)} + \sup \beta^{-1}\left(\beta^+ - \frac{\delta}{4|\partial\Omega|}\right)\right),$$

where $\hat{C}(\Omega, N, p)$ is given in Lemma 4.3. $\square$

4.2 Proof of the existence part of Theorem 2.5

Let $T > 0$. Let $f \in L^{p'}(0, T; L^{p'}(\Omega))$, $g \in L^{p'}(0, T; L^{p'}(\partial\Omega))$, $z_0 \in L^{p'}(\Omega)$ and $w_0 \in L^{p'}(\partial\Omega)$ satisfying (4), (5) and (6). Let $V(t) = (z(t), w(t))$ the mild solution of problem (10) given by Theorem 3.6. Our aim is to prove that (z, w) is a weak solution of problem $P_{\gamma,\beta}(f, g, z_0, w_0)$.

Following the proof of the existence of the mild solution (Theorem 3.6) for $n \in \mathbb{N}$, let $\epsilon = T/n$, and consider a subdivision $t_0 = 0 < t_1 < \cdots < t_{n-1} < T = t_n$ with $t_i - t_{i-1} = \epsilon$, $f_1, \ldots, f_n \in L^{p'}(\Omega)$, $g_1, \ldots, g_n \in L^{p'}(\partial\Omega)$ with

$$\sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \left(\|f(t) - f_i\|_{L^{p'}(\Omega)}^{p'} + \|g(t) - g_i\|_{L^{p'}(\partial\Omega)}^{p'}\right) dt \leq \epsilon.$$

Then, it follows that

$$
\begin{aligned}
&z(t) = L^1(\Omega) - \lim\nolimits_{\epsilon\to 0} z_\epsilon(t) \quad \text{ uniformly for } t \in [0,T],\\
&w(t) = L^1(\partial\Omega) - \lim\nolimits_{\epsilon\to 0} w_\epsilon(t) \quad \text{ uniformly for } t \in [0,T],
\end{aligned}
\tag{22}
$$

where $z_\epsilon(t)$ and $w_\epsilon(t)$ are given, for ϵ small enough, by

$$
\begin{cases}
z_\epsilon(t) = z_0, \qquad w_\epsilon(t) = w_0 \quad \text{for } \ t \in]-\infty, 0],\\[2ex]
z_\epsilon(t) = z_i, \qquad w_\epsilon(t) = w_i \quad \text{for } \ t \in]t_{i-1}, t_i], \quad i = 1, \ldots, n,
\end{cases}
\tag{23}
$$

where $[u_i, z_i, w_i] \in W^{1,p}(\Omega) \times V^{1,p}(\Omega) \times V^{1,p}(\partial\Omega)$ sastisfies

$$
\int_\Omega \mathbf{a}(x, Du_i)\cdot Dv + \int_\Omega \frac{z_i - z_{i-1}}{\epsilon} v + \int_{\partial\Omega} \frac{w_i - w_{i-1}}{\epsilon} v = \int_\Omega f_i v + \int_{\partial\Omega} g_i v \tag{24}
$$

for all $v \in W^{1,p}(\Omega)$.

Taking $v = u_i$ as test function in (24), we get

$$
\int_\Omega \mathbf{a}(x, Du_i)\cdot Du_i + \int_\Omega \left(\frac{z_i - z_{i-1}}{\epsilon}\right) u_i + \int_{\partial\Omega} \left(\frac{w_i - w_{i-1}}{\epsilon}\right) u_i = \int_\Omega f_i u_i + \int_{\partial\Omega} g_i u_i. \tag{25}
$$

Since $z_i(x) \in \gamma(u_i(x))$ *a.e.* in Ω and $w_i(x) \in \beta(u_i(x))$ *a.e.* in $\partial\Omega$, we have

$$
u_i(x) \in \gamma^{-1}(z_i(x)) = \partial j_\gamma^*(z_i(x)) \quad \text{ a.e. in } \ \Omega
$$

and

$$
u_i(x) \in \beta^{-1}(w_i(x)) = \partial j_\beta^*(w_i(x)) \quad \text{ a.e. in } \ \partial\Omega.
$$

Hence,

$$
j_\gamma^*(z_{i-1}(x)) - j_\gamma^*(z_i(x)) \geq (z_{i-1}(x) - z_i(x))u_i(x) \quad \text{ a.e. in } \ \Omega
$$

and

$$
j_\beta^*(w_{i-1}(x)) - j_\beta^*(w_i(x)) \geq (w_{i-1}(x) - w_i(x))u_i(x) \quad \text{ a.e. in } \ \partial\Omega.
$$

Therefore,

$$
\begin{aligned}
&\frac{1}{\epsilon}\int_\Omega \left(j_\gamma^*(z_i) - j_\gamma^*(z_{i-1})\right) + \frac{1}{\epsilon}\int_{\partial\Omega} \left(j_\beta^*(w_i) - j_\beta^*(w_{i-1})\right)\\
&\qquad \leq \int_\Omega \left(\frac{z_i - z_{i-1}}{\epsilon}\right) u_i + \int_{\partial\Omega} \left(\frac{w_i - w_{i-1}}{\epsilon}\right) u_i
\end{aligned}
$$

and by (25), we get

$$
\begin{aligned}
\int_\Omega \mathbf{a}(x, Du_i)\cdot Du_i &+ \frac{1}{\epsilon}\int_\Omega \left(j_\gamma^*(z_i) - j_\gamma^*(z_{i-1})\right) + \frac{1}{\epsilon}\int_{\partial\Omega} \left(j_\beta^*(w_i) - j_\beta^*(w_{i-1})\right)\\
&\leq \int_\Omega f_i u_i + \int_{\partial\Omega} g_i u_i.
\end{aligned}
$$

Then, integrating in time and adding in the last inequality, we obtain that

$$\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\int_\Omega \mathbf{a}(x,Du_i)\cdot Du_i+\int_\Omega \left(j_\gamma^*(z_n)-j_\gamma^*(z_0)\right)+\int_{\partial\Omega}\left(j_\beta^*(w_n)-j_\beta^*(w_0)\right)$$
$$\leq \sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\left(\int_\Omega f_iu_i+\int_{\partial\Omega}g_iu_i\right).$$

Consequently, if we set $f_\epsilon(t)=f_i$, $g_\epsilon(t)=g_i$ and $u_\epsilon(t)=u_i$ for $t\in]t_{i-1},t_i]$, $i=1,\ldots,n$, it follows that

$$\begin{aligned}\int_0^T\int_\Omega \mathbf{a}(x,Du_\epsilon(t))\cdot Du_\epsilon(t)\,dt+\int_\Omega \left(j_\gamma^*(z_n)-j_\gamma^*(z_0)\right)+\int_{\partial\Omega}\left(j_\beta^*(w_n)-j_\beta^*(w_0)\right)\\ \leq \int_0^T\int_\Omega f_\epsilon(t)u_\epsilon(t)+\int_0^T\int_{\partial\Omega}g_\epsilon(t)u_\epsilon(t).\end{aligned}\tag{26}$$

Then, having in mind (H$_1$) and (5), we get that there exists a positive constant C_1 such that

$$\lambda\int_0^T\int_\Omega |Du_\epsilon(t)|^p\,dt\leq \int_0^T\int_\Omega \mathbf{a}(x,Du_\epsilon(t))\cdot Du_\epsilon(t)\,dt$$
$$\leq \int_\Omega j_\gamma^*(z_0)+\int_{\partial\Omega}j_\beta^*(w_0)+\int_0^T\int_\Omega f_\epsilon(t)u_\epsilon(t)+\int_0^T\int_{\partial\Omega}g_\epsilon(t)u_\epsilon(t)$$
$$\leq C_1+\int_0^T\|f_\epsilon(t)\|_{L^{p'}(\Omega)}\|u_\epsilon(t)\|_{L^p(\Omega)}\,dt+\int_0^T\|g\|_{L^{p'}(\partial\Omega)}\|u_\epsilon(t)\|_{L^p(\partial\Omega)}\,dt.$$

Therefore, using Young's inequality, for any $\mu>0$ there exists $C_2(\mu)>0$ such that

$$\leq C_2(\mu)+\mu\int_0^T\left(\|u_\epsilon(t)\|^p_{L^p(\Omega)}+\|u_\epsilon(t)\|^p_{L^p(\partial\Omega)}\right)\,dt.$$

From here, by the Trace Theorem, we obtain that for any $\mu>0$ there exists $C_3(\mu)>0$ such that

$$\int_0^T\int_\Omega |Du_\epsilon(t)|^p\,dt\leq C_3+\mu\int_0^T\left(\|u_\epsilon(t)\|^p_{L^p(\Omega)}+\|Du_\epsilon(t)\|^p_{L^p(\Omega)}\right)\,dt.\tag{27}$$

By (22), if $\mathcal{R}^+_{\gamma,\beta}=+\infty$, there exists $M>0$ and $n_0\in\mathbb{N}$, such that

$$\sup_{t\in[0,T]}\int_\Omega z_\epsilon^+(t)+\int_{\partial\Omega}w_\epsilon^+(t)<M\qquad\forall n\geq n_0,$$

and if $\mathcal{R}^+_{\gamma,\beta}<+\infty$, there exists $M\in\mathbb{R}$, $h>0$ and $n_0\in\mathbb{N}$ such that, for all $n\geq n_0$,

$$\sup_{t\in[0,T]}\int_\Omega z_\epsilon(t)+\int_{\partial\Omega}w_\epsilon(t)<M<\mathcal{R}^+_{\gamma,\beta}$$

and

$$\sup_{t\in[0,T]} \max\left\{\int_{\{x\in\Omega: z_\epsilon(t)(x)<-h\}} |z_\epsilon(t)|, \int_{\{x\in\partial\Omega: w_\epsilon(t)(x)<-h\}} |w_\epsilon(t)|\right\} < \frac{\mathcal{R}^+_{\gamma,\beta} - M}{8}.$$

Consequently, from Lemma 4.2, we get that, there exists a constant $C_4 > 0$ such that

$$\|u^+_\epsilon(t)\|_{L^p(\Omega)} \le C_4 \left(\|Du^+_\epsilon(t)\|_{L^p(\Omega)} + 1\right) \qquad \text{for all } \ t \in [0,T]. \tag{28}$$

Similarly, we get that there exists $C_5 > 0$ such that

$$\|u^-_\epsilon(t)\|_{L^p(\Omega)} \le C_5 \left(\|Du^-_\epsilon(t)\|_{L^p(\Omega)} + 1\right) \qquad \text{for all } t \in [0,T]. \tag{29}$$

Consequently, from (27), (28) and (29), choosing μ small enough, we obtain that there exist $C_6 > 0$ such that

$$\int_0^T \int_\Omega |Du_\epsilon(t)|^p \, dt \le C_6. \tag{30}$$

By (30), (28) and (29), we get that $\{u_\epsilon\}$ is bounded in $L^p(0,T;W^{1,p}(\Omega))$. So, there exists a subsequence, denoted equal, such that

$$u_\epsilon \rightharpoonup u \quad \text{ weakly in } \ L^p(0,T;W^{1,p}(\Omega)) \ \text{ as } \ \epsilon \to 0^+ \tag{31}$$

and

$$u_\epsilon \rightharpoonup u \quad \text{ weakly in } \ L^p(S_T) \ \text{ as } \ \epsilon \to 0^+. \tag{32}$$

Since $z_\epsilon \in \gamma(u_\epsilon)$ a.e. in Q_T, $w_\epsilon \in \beta(u_\epsilon)$ a.e. on S_T, $z_\epsilon \to z$ in $L^1(Q_T)$ and $w_\epsilon \to w$ in $L^1(S_T)$, having in mind (31), (32) and using monotonicity argument we obtain that $z \in \gamma(u)$ a.e. in Q_T, $w \in \beta(u)$ a.e. on S_T.

Since $\{Du_\epsilon\}$ is bounded in $L^p(Q_T)$, by (H$_2$) we have $\{|\mathbf{a}(x,Du_\epsilon)|\}$ is bounded in $L^{p'}(Q_T)$, then we can assume that

$$\mathbf{a}(x,Du_\epsilon) \rightharpoonup \Phi \quad \text{ weakly in } \ L^{p'}(Q_T) \ \text{ as } \ \epsilon \to 0^+. \tag{33}$$

From (24), we have

$$\begin{aligned} \int_\Omega \mathbf{a}(x,Du_\epsilon(t))\cdot Dv + \int_\Omega \frac{z_\epsilon(t)-z_\epsilon(t-\epsilon)}{\epsilon} v + \int_{\partial\Omega} \frac{w_\epsilon(t)-w_\epsilon(t-\epsilon)}{\epsilon} v \\ = \int_\Omega f_\epsilon(t) v + \int_{\partial\Omega} g_\epsilon(t) v \end{aligned} \tag{34}$$

for all $v \in W^{1,p}(\Omega)$. Then, given $\psi \in W^{1,1}(0,T;W^{1,1}(\Omega)\cap L^\infty(\Omega)) \cap L^p(0,T;W^{1,p}(\Omega))$, $\psi(0) = \psi(T) = 0$, from (34), we get

$$\begin{aligned} \int_0^T \int_\Omega \mathbf{a}(x,Du_\epsilon(t))\cdot D\psi + \int_\Omega \int_0^T \frac{z_\epsilon(t)-z_\epsilon(t-\epsilon)}{\epsilon}\psi(t) \\ + \int_{\partial\Omega}\int_0^T \frac{w_\epsilon(t)-w_\epsilon(t-\epsilon)}{\epsilon}\psi(t) = \int_0^T \int_\Omega f_\epsilon(t)\psi + \int_0^T \int_{\partial\Omega} g_\epsilon(t)\psi. \end{aligned} \tag{35}$$

Now,

$$\lim_{\epsilon\to 0}\int_\Omega\int_0^T \frac{z_\epsilon(t)-z_\epsilon(t-\epsilon)}{\epsilon}\psi(t)$$

$$= \lim_{\epsilon\to 0}\left(-\int_\Omega\int_0^{T-\epsilon} z_\epsilon(t)\frac{\psi(t+\epsilon)-\psi(t)}{\epsilon}+\int_\Omega\int_{T-\epsilon}^T \frac{z_\epsilon(t)\psi(t)}{\epsilon}-\int_\Omega\int_0^\epsilon \frac{z_0\psi(t)}{\epsilon}\right)$$

$$= -\int_0^T\int_\Omega z(t)\psi_t.$$

Similarly,

$$\lim_{\epsilon\to 0}\int_{\partial\Omega}\int_0^T \frac{w_\epsilon(t)-w_\epsilon(t-\epsilon)}{\epsilon}\psi(t) = -\int_0^T\int_{\partial\Omega} w(t)\psi_t.$$

Therefore, taking limit in (35) as $\epsilon\to 0^+$, we obtain that

$$\begin{aligned}\int_0^T\int_\Omega \Phi\cdot D\psi-\int_0^T\int_\Omega z(t)\psi_t-\int_0^T\int_{\partial\Omega} w(t)\psi_t\\ = \int_0^T\int_\Omega f(t)\psi+\int_0^T\int_{\partial\Omega} g(t)\psi.\end{aligned}\tag{36}$$

Thus, to finish the proof of the existence, we only need to show that $\Phi=\mathbf{a}(x,Du)$. To do that we apply the Minty-Browder's method.

It is enough to prove that

$$\limsup_{\epsilon\to 0}\int_{Q_T}\mathbf{a}(x,Du_\epsilon)\cdot Du_\epsilon \le \int_{Q_T}\Phi\cdot Du.\tag{37}$$

Indeed, for any $\rho\in L^p(0,T;W^{1,p}(\Omega))$.

$$\int_{Q_T}\mathbf{a}(x,D\rho)\cdot D(u_\epsilon-\rho)\le\int_{Q_T}\mathbf{a}(x,Du_\epsilon)\cdot D(u_\epsilon-\rho),$$

so that, passing to the limit and using (37), we get

$$\int_{Q_T}\mathbf{a}(x,D\rho)\cdot D(u-\rho)\le\int_{Q_T}\Phi\cdot D(u-\rho).$$

Then taking $\rho=u\pm\lambda\xi$, for $\lambda>0$ and $\xi\in L^p(0,T;W^{1,p}(\Omega))$, we get

$$\int_{Q_T}\mathbf{a}(x,D(u+\lambda\rho))\cdot D\xi=\int_{Q_T}\Phi\cdot D\xi,$$

and by letting $\lambda\to 0$, we obtain

$$\int_{Q_T}\mathbf{a}(x,D(u))\cdot D\xi=\int_{Q_T}\Phi\cdot D\xi,\quad \text{for any } \xi\in L^p(0,T;W^{1,p}(\Omega)),$$

which implies that

$$\mathbf{a}(x, D(u)) = \Phi \quad \text{a.e. in } Q.$$

Now, let us prove (37). Thanks to (26) and Fatou's Lemma, we have

$$\begin{aligned} \limsup_{\epsilon\to 0} \int_0^T \int_\Omega \mathbf{a}(x, Du_\epsilon(t)) \cdot Du_\epsilon(t)\, dt \leq -\int_\Omega \big(j_\gamma^*(z(T)) - j_\gamma^*(z_0)\big) \\ - \int_{\partial\Omega} \big(j_\beta^*(w(T)) - j_\beta^*(w_0)\big) + \int_0^T \int_\Omega fu + \int_0^T \int_{\partial\Omega} gu. \end{aligned} \tag{38}$$

On the other hand, (36) can be rewritten as follows

$$\int_0^T \int_\Omega z(t)\psi_t + \int_0^T \int_{\partial\Omega} w(t)\psi_t = \int_0^T (F(t), \psi(t))dt,$$

where F is given by

$$(F(t), \psi(t)) = \int_\Omega \Phi(t) \cdot D\psi(t) - \int_\Omega f(t)\psi(t) - \int_{\partial\Omega} g(t)\psi(t).$$

Then, by Lemma 4.1 applied to this F, $H(x, r) = r$ and $\psi(t, x) = \xi(x)\phi(t)$, $\xi \in \mathcal{D}(\mathbb{R}^N)$, $\xi = 1$ in Ω, $\phi \in \mathcal{D}(]0, T[)$, we get

$$\frac{d}{dt} \int_\Omega j_\gamma^*(z) + \frac{d}{dt} \int_{\partial\Omega} j_\beta^*(w) = (F, u) \quad \text{in } \mathcal{D}'(]0, T[), \tag{39}$$

Therefore,

$$\int_\Omega j_\gamma^*(z) + \int_{\partial\Omega} j_\beta^*(w) \in W^{1,1}(]0, T[).$$

So, integrating on $]0, T[$ in (39) we obtain

$$\begin{aligned} \int_0^T \int_\Omega \Phi \cdot Du = -\int_\Omega \big(j_\gamma^*(z(T)) - j_\gamma^*(z_0)\big) - \int_{\partial\Omega} \big(j_\beta^*(w(T)) - j_\beta^*(w_0)\big) \\ + \int_0^T \int_\Omega fu + \int_0^T \int_{\partial\Omega} gu. \end{aligned}$$

From here and (38) we obtain (37). □

5 Contraction principle

Our main tool to prove the contraction principle is the concept of *integral solution* due to Ph. Bénilan (see [9], [13]).

Definition 5.1 A function $V = (z, w) \in C([0,T] : X)$ is an *integral solution* of (10) in $[0,T]$ if for every $(\hat{f}, \hat{g}) \in \mathcal{B}^{\gamma,\beta}(\hat{z}, \hat{w})$, we have

$$\frac{d}{dt}\int_{\Omega} |z(t) - \hat{z}| + \frac{d}{dt}\int_{\partial\Omega} |w(t) - \hat{w}|$$

$$\leq \int_{\Omega} (f(t) - \hat{f})\mathrm{sign}_0(z(t) - \hat{z}) + \int_{\{x\in\Omega: z(t)=\hat{z}\}} |f(t) - \hat{f}|$$

$$+ \int_{\partial\Omega} (g(t) - \hat{g})\mathrm{sign}_0(w(t) - \hat{w}) + \int_{\{x\in\partial\Omega: w(t)=\hat{w}\}} |g(t) - \hat{g}|$$

in $\mathcal{D}'(]0,T[)$, and $V(0) = (z_0, w_0)$.

Since $\mathcal{B}^{\gamma,\beta}$ is accretive in X, it is well known (see, [9], [13]) that mild solutions and integral solutions of problem (10) coincide, and a contraction principle holds. We shall prove in Theorem 5.3 that a weak solution of $P_{\gamma,\beta}(f, g, z_0, w_0)$ in $[0,T]$ is an integral solution of (10). Consequently, since, in fact, $\mathcal{B}^{\gamma,\beta}$ is T-accretive in X, the contraction principle (3) given in Theorem 2.4 follows.

To prove Theorem 5.3, the main difficulties are due to the nonlinear and non-homogeneous boundary conditions and to the jumps of γ and β. In [17], to obtain the L^1-contraction principle for a similar problem in the case $\beta = \{0\} \times \mathbb{R}$, and for γ having a set of jumps without density points, the authors give an improvement of the "hole filling" argument of [21] and use the doubling variable in time technique. This technique can be adapted to our problem. Now, by the Nonlinear Semigroup Theory, we are able to simplify the proof without using the doubling variable in time technique and without imposing any condition on the jumps of γ and β.

Lemma 5.2 *Let (z, w) be a weak solution of problem $P_{\gamma,\beta}(f, g, z_0, w_0)$ in $[0,T]$. Let $u \in L^p(0,T; W^{1,p}(\Omega))$ such that $z \in \gamma(u)$ a.e. in Q_T, $w \in \beta(u)$ a.e. on S_T as in Definition 2.2. Let $\hat{z}, \hat{f} \in L^1(\Omega)$ and $\hat{u} \in W^{1,p}(\Omega)$, $\hat{z} \in \gamma(\hat{u})$ a.e. in Ω, such that*

$$\int_{\Omega} \mathbf{a}(x, D\hat{u}) \cdot D\psi = \int_{\Omega} \hat{f}\psi, \quad \forall \psi \in W_0^{1,p}(\Omega) \cap L^{\infty}(\Omega). \tag{40}$$

Then, for any $\psi \in \mathcal{D}(\Omega)$, $\psi \geq 0$,

$$\frac{d}{dt}\int_{\Omega} |z(t) - \hat{z}|\psi + \int_{\Omega} sign_0(u(t) - \hat{u})(\mathbf{a}(x, Du(t)) - \mathbf{a}(x, D\hat{u})) \cdot D\psi$$

$$\leq \int_{\Omega} (f(t) - \hat{f}) sign_0(z(t) - \hat{z})\psi + \int_{\{x\in\Omega: z(t)=\hat{z}\}} |f(t) - \hat{f}|\psi$$

in $\mathcal{D}'(]0,T[)$.

Proof. Let us take in Lemma 4.1 the function F given by

$$(F(t), \psi(t)) = \int_{\Omega} \mathbf{a}(x, Du(t)) \cdot D\psi(t) - \int_{\Omega} f(t)\psi(t) - \int_{\partial\Omega} g(t)\psi(t)$$

for all $\psi \in W^{1,1}(0,T;W^{1,1}(\Omega)\cap L^\infty(\Omega))\cap L^p(0,T;W^{1,p}(\Omega))$, $\psi(0)=\psi(T)=0$, and

$$H(x,r)=\frac{1}{k}T_k(r-\hat{u}(x)+k\rho(x)),$$

where $\rho \in W^{1,p}(\Omega)$, $-1\le\rho\le 1$. Then, for any $\psi\in\mathcal{D}(\Omega)$, $\psi\ge 0$, having in mind (40), we have

$$\begin{gathered}
\frac{d}{dt}\int_\Omega\left(\int_{\hat{z}}^{z(t)}\frac{1}{k}T_k((\gamma^{-1})^0(\tau)-\hat{u}+k\rho))d\tau\right)\psi \\
+\int_\Omega(\mathbf{a}(x,Du(t))-\mathbf{a}(x,D\hat{u}))\cdot D\left(\frac{1}{k}T_k(u(t)-\hat{u}+k\rho)\psi\right) \\
=\int_\Omega(f(t)-\hat{f})\frac{1}{k}T_k(u(t)-\hat{u}+k\rho)\psi,
\end{gathered}\tag{41}$$

in $\mathcal{D}'(]0,T[)$.

Now, it is easy to see that

$$\begin{aligned}
&\lim_{k\to 0}\int_{\hat{z}}^{z(t)}\frac{1}{k}T_k((\gamma^{-1})^0(\tau)-\hat{u}+k\rho))d\tau \\
&=\int_{\hat{z}}^{z(t)}\left[\mathrm{sign}_0((\gamma^{-1})^0(\tau)-\hat{u})+\rho\chi_{\{\tau:(\gamma^{-1})^0(\tau)=\hat{u}\}}\right]d\tau \\
&=\int_{\hat{z}}^{z(t)}\left[\mathrm{sign}_0(\tau-\hat{z})+(\rho-\mathrm{sign}_0(\tau-\hat{z}))\chi_{\{\tau:(\gamma^{-1})^0(\tau)=\hat{u}\}}+\mathrm{sign}_0((\gamma^{-1})^0(\tau)-\hat{u})\chi_{\{\tau=\hat{z}\}}\right]d\tau \\
&=\int_{\hat{z}}^{z(t)}\left[\mathrm{sign}_0(\tau-\hat{z})+(\rho-\mathrm{sign}_0(\tau-\hat{z}))\chi_{\{\tau:(\gamma^{-1})^0(\tau)=\hat{u}\}}\right]d\tau \\
&=|z(t)-\hat{z}|+\int_{\hat{z}}^{z(t)}(\rho-\mathrm{sign}_0(\tau-\hat{z}))\chi_{\{\tau:(\gamma^{-1})^0(\tau)=\hat{u}\}}
\end{aligned}$$

Hence, taking limits in (41) as k goes to 0, we get

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega\left(|z(t)-\hat{z}|+\int_{\hat{z}}^{z(t)}(\rho-\mathrm{sign}_0(\tau-\hat{z}))\chi_{\{\tau:(\gamma^{-1})^0(\tau)=\hat{u}\}}\right)\psi \\
&\quad+\int_\Omega\mathrm{sign}_0(u(t)-\hat{u})(\mathbf{a}(x,Du(t))-\mathbf{a}(x,D\hat{u}))\cdot D\psi \\
&\le\int_\Omega(f(t)-\hat{f})\left(\mathrm{sign}_0(z(t)-\hat{z})+\mathrm{sign}_0(u(t)-\hat{u})\chi_{\{x\in\Omega:z(t)=\hat{z}\}}\right)\psi \\
&\quad+\int_\Omega(f(t)-\hat{f})\left(\rho-\mathrm{sign}_0(z(t)-\hat{z})\right)\chi_{\{x\in\Omega:u(t)=\hat{u}\}})\psi,
\end{aligned}$$

and integrating between $\hat{t}, t \in]0,T[$, we get

$$\int_\Omega |z(t)-\hat{z}|\psi - \int_\Omega |z(\hat{t})-\hat{z}|\psi$$

$$+\int_\Omega \int_{z(\hat{t})}^{z(t)} (\rho - \mathrm{sign}_0(\tau-\hat{z}))\chi_{\{\tau:(\gamma^{-1})^0(\tau)=\hat{u}\}}\psi$$

$$+\int_{\hat{t}}^t \int_\Omega \mathrm{sign}_0(u(\tau)-\hat{u})(\mathbf{a}(x,Du(\tau))-\mathbf{a}(x,D\hat{u}))\cdot D\psi$$

$$\leq \int_{\hat{t}}^t \int_\Omega (f(\tau)-\hat{f})\left(\mathrm{sign}_0(z(\tau)-\hat{z})+\mathrm{sign}_0(u(\tau)-\hat{u})\chi_{\{x\in\Omega:z(\tau)=\hat{z}\}}\right)\psi$$

$$+\int_{\hat{t}}^t \int_\Omega (f(\tau)-\hat{f})\left(\rho-\mathrm{sign}_0(z(\tau)-\hat{z})\right)\chi_{\{x\in\Omega:u(\tau)=\hat{u}\}})\psi.$$

Since in the last expression there are no space derivatives of ρ, we can take, for each t fixed, $\rho = \mathrm{sign}_0(z(t)-\hat{z})$. Then the second term in the above expression is positive and we have, for any $\hat{t}, t \in]0,T[$,

$$\begin{aligned}
&\int_\Omega |z(t)-\hat{z}|\psi - \int_\Omega |z(\hat{t})-\hat{z}|\psi \\
&+\int_{\hat{t}}^t \int_\Omega \mathrm{sign}_0(u(\tau)-\hat{u})(\mathbf{a}(x,Du(\tau))-\mathbf{a}(x,D\hat{u}))\cdot D\psi \\
&\leq \int_{\hat{t}}^t \int_\Omega (f(\tau)-\hat{f})\left(\mathrm{sign}_0(z(\tau)-\hat{z})+\mathrm{sign}_0(u(\tau)-\hat{u})\chi_{\{x\in\Omega:z(\tau)=\hat{z}\}}\right)\psi \\
&+\int_{\hat{t}}^t \int_\Omega (f(\tau)-\hat{f})\left(\mathrm{sign}_0(z(t)-\hat{z})-\mathrm{sign}_0(z(\tau)-\hat{z})\right)\chi_{\{x\in\Omega:u(\tau)=\hat{u}\}})\psi.
\end{aligned} \tag{42}$$

Let

$$\varphi_1(t) := \int_\Omega |z(t)-\hat{z}|\psi,$$

$$\varphi_2(\tau) := -\int_\Omega \mathrm{sign}_0(u(\tau)-\hat{u})(\mathbf{a}(x,Du(\tau))-\mathbf{a}(x,D\hat{u}))D\psi$$

$$+\int_\Omega (f(\tau)-\hat{f})(\mathrm{sign}_0(z(\tau)-\hat{z})+\mathrm{sign}_0(u(\tau)-\hat{u}))\chi_{\{x\in\Omega:z(\tau)(x)=\hat{z}(x)\}}\psi$$

and

$$\varphi_3(t,\tau) := \int_\Omega (f(\tau)-\hat{f})(\mathrm{sign}_0(z(t)-\hat{z})-\mathrm{sign}_0(z(\tau)-\hat{z}))\chi_{\{x\in\Omega:u(\tau)(x)=\hat{u}(x)\}}\psi.$$

Then, taking in (42) $\hat{t} = t - h$, $h > 0$, dividing by h and letting h go to 0, we get for any $\eta \in \mathcal{D}(]0,T[)$, $\eta \geq 0$,

$$
\begin{aligned}
-\int_0^T \varphi_1(t)\eta_t(t)dt &= -\lim_{h\to 0^+}\int_0^T \varphi_1(t)\frac{\eta(t+h)-\eta(t)}{h}dt \\
&= \lim_{h\to 0^+}\int_0^T \frac{\varphi_1(t)-\varphi_1(t-h)}{h}\eta(t)dt \qquad (43)\\
\leq \lim_{h\to 0^+}\bigg(\int_0^T \frac{1}{h}\left(\int_{t-h}^t \varphi_2(\tau)d\tau\right)\eta(t)dt &+ \int_0^T \frac{1}{h}\left(\int_{t-h}^t \varphi_3(t,\tau)d\tau\right)\eta(t)dt\bigg).
\end{aligned}
$$

Now, by the Dominate Convergence Theorem,

$$
\begin{aligned}
\lim_{h\to 0^+}\int_0^T \frac{1}{h}\left(\int_{t-h}^t \varphi_2(\tau)d\tau\right)\eta(t)dt &= -\lim_{h\to 0^+}\int_0^T \left(\int_0^t \varphi_2(\tau)d\tau\right)\frac{\eta(t+h)-\eta(t)}{h}dt \\
&= -\int_0^T\left(\int_0^t \varphi_2(\tau)d\tau\right)\eta_t(t)dt = \int_0^T \varphi_2(t)\eta(t)dt.
\end{aligned}
$$

On the other hand, for h small enough,

$$
\int_0^T \frac{1}{h}\left(\int_{t-h}^t \varphi_3(t,\tau)d\tau\right)\eta(t)dt = \int_0^T \frac{1}{h}\left(\int_\tau^{\tau+h}\varphi_3(t,\tau)\eta(t)dt\right)d\tau.
$$

Now,

$$
\begin{aligned}
&\left|\int_0^T \frac{1}{h}\left(\int_\tau^{\tau+h}\varphi_3(t,\tau)\eta(t)dt\right)d\tau\right| \\
&\leq \int_0^T \frac{1}{h}\left(\int_\tau^{\tau+h}\int_\Omega |f(\tau)-\hat{f}||\mathrm{sign}_0(z(t)-\hat{z})-\mathrm{sign}_0(z(\tau)-\hat{z})|\eta(t)\psi(x)dxdt\right)d\tau \\
&\leq \|\psi\|_{L^\infty(\Omega)}\|\eta\|_{L^\infty(0,T)}\int_0^T\bigg[\int_\Omega |f(\tau)-\hat{f}|dx \\
&\qquad\qquad \times\frac{1}{h}\int_\tau^{\tau+h}\|\mathrm{sign}_0(z(t)-\hat{z})-\mathrm{sign}_0(z(\tau)-\hat{z})\|_{L^\infty(\Omega)}dt\bigg]d\tau.
\end{aligned}
$$

Moreover, for all Lebesgue's point of the $L^1(0,T;L^\infty(\Omega))$-function $\mathrm{sign}_0(z(.)-\hat{z})$, and so, for *a.e.* $\tau \in]0,T[$, we have

$$
\lim_{h\to 0^+}\frac{1}{h}\int_\tau^{\tau+h}\|\mathrm{sign}_0(z(t)-\hat{z})-\mathrm{sign}_0(z(\tau)-\hat{z})\|_{L^\infty(\Omega)}dt = 0.
$$

Consequently, since

$$
\left(\int_\Omega |f(\tau)-\hat{f}|dx\right)\frac{1}{h}\int_\tau^{\tau+h}\|\mathrm{sign}_0(z(t)-\hat{z})-\mathrm{sign}_0(z(\tau)-\hat{z})\|_{L^\infty(\Omega)}dt \leq 2\int_\Omega |f(\tau)-\hat{f}|dx,
$$

which is a function of $L^1(0,T)$, by the Dominate Convergence Theorem, we get

$$\lim_{h\to 0^+}\int_0^T \frac{1}{h}\left(\int_{t-h}^{t}\varphi_3(t,\tau)d\tau\right)\eta(t)dt = \lim_{h\to 0^+}\int_0^T \frac{1}{h}\left(\int_{\tau}^{\tau+h}\varphi_3(t,\tau)\eta(t)dt\right)d\tau = 0.$$

Therefore, from (43) we obtain that

$$\frac{d}{dt}\int_\Omega |z(t)-\hat{z}|\psi + \int_\Omega \mathrm{sign}_0(u(t)-\hat{u})(\mathbf{a}(x,Du(t)-\mathbf{a}(x,D\hat{u}))\cdot D\psi$$

$$\leq \int_\Omega (f(t)-\hat{f})\left(\mathrm{sign}_0(z(t)-\hat{z})+\mathrm{sign}_0(u(t)-\hat{u})\chi_{\{x\in\Omega: z(t)=\hat{z}\}}\right)\psi$$

in $\mathcal{D}'(]0,T[)$. □

Theorem 5.3 *Let (z,w) be a weak solution of $P_{\gamma,\beta}(f,g,z_0,w_0)$ in $[0,T]$. Let $(\hat{f},\hat{g})\in\mathcal{B}^{\gamma,\beta}(\hat{z},\hat{w})$. Then,*

$$\frac{d}{dt}\int_\Omega |z(t)-\hat{z}| + \frac{d}{dt}\int_{\partial\Omega}|w(t)-\hat{w}|$$

$$\leq \int_\Omega (f(t)-\hat{f})sign_0(z(t)-\hat{z}) + \int_{\{x\in\Omega: z(t)=\hat{z}\}}|f(t)-\hat{f}|$$

$$+\int_{\partial\Omega}(g(t)-\hat{g})sign_0(w(t)-\hat{w}) + \int_{\{x\in\partial\Omega: w(t)=\hat{w}\}}|g(t)-\hat{g}|$$

in $\mathcal{D}'(]0,T[)$, that is, since $(z(0),w(0))=(z_0,w_0)$, (z,w) is an integral solution of (10) in $[0,T]$.

Proof. Let $u\in L^p(0,T;W^{1,p}(\Omega))$ such that $z\in\gamma(u)$ a.e. in Q_T, $w\in\beta(u)$ a.e. on S_T as in Definition 2.2, and let $\hat{u}\in W^{1,p}(\Omega)$ such that $\hat{z}\in\gamma(\hat{u})$ *a.e.* in Ω and $\hat{w}\in\gamma(\hat{u})$ *a.e.* in $\partial\Omega$ as in the definition of $\mathcal{B}^{\gamma,\beta}$.

Thanks to Lemma 5.2, we have that, for any $\psi\in\mathcal{D}(\Omega)$, $0\leq\psi\leq 1$,

$$\frac{d}{dt}\int_\Omega |z(t)-\hat{z}|\psi + \int_\Omega \mathrm{sign}_0(u(t)-\hat{u})(\mathbf{a}(x,Du(t)-\mathbf{a}(x,D\hat{u}))\cdot D\psi$$

$$\leq \int_\Omega (f(t)-\hat{f})\mathrm{sign}_0(z(t)-\hat{z})\psi + \int_{\{x\in\Omega: z(t)=\hat{z}\}}|f(t)-\hat{f}|\psi \tag{44}$$

in $\mathcal{D}'(]0,T[)$. Now, for the second term in the above expression we have that

$$\int_\Omega \mathrm{sign}_0(u(t)-\hat{u})(\mathbf{a}(x,Du(t)-\mathbf{a}(x,D\hat{u}))\cdot D\psi$$

$$= \int_\Omega \mathrm{sign}_0(u(t)-\hat{u})(\mathbf{a}(x,Du(t)-\mathbf{a}(x,D\hat{u}))\cdot D(\psi-1) \tag{45}$$

$$\geq \lim_{k\to 0}\int_\Omega (\mathbf{a}(x,Du(t))-\mathbf{a}(x,D\hat{u}))\cdot D\left(\frac{1}{k}T_k(u(t)-\hat{u}+k\rho)(\psi-1)\right),$$

where $\rho \in W^{1,p}(\Omega)$, $-1 \leq \rho \leq 1$. Using again Lemma 4.1 we get

$$\begin{aligned}
&\int_\Omega (\mathbf{a}(x, Du(t) - \mathbf{a}(x, D\hat{u})) \cdot D \left(\frac{1}{k} T_k(u(t) - \hat{u} + k\rho)(\psi - 1) \right) \\
&= -\frac{d}{dt} \int_\Omega \left(\int_{\hat{z}}^{z(t)} \frac{1}{k} T_k((\gamma^{-1})^0(s) - \hat{u} + k\rho)) ds \right) (\psi - 1) \\
&\quad + \frac{d}{dt} \int_{\partial\Omega} \left(\int_{\hat{w}}^{w(t)} \frac{1}{k} T_k((\beta^{-1})^0(s) - \hat{u} + k\rho)) ds \right) \\
&\quad + \int_\Omega (f(t) - \hat{f}) \frac{1}{k} T_k(u(t) - \hat{u} + k\rho)(\psi - 1) \\
&\quad - \int_{\partial\Omega} (g(t) - \hat{g}) \frac{1}{k} T_k(u(t) - \hat{u} + k\rho),
\end{aligned} \tag{46}$$

which converges as k goes to 0 to

$$\begin{aligned}
&-\frac{d}{dt} \int_\Omega \left(|z(t) - \hat{z}| - \int_{\hat{z}}^{z(t)} (\rho - \mathrm{sign}_0(s - \hat{z})) \chi_{\{s:(\gamma^{-1})^0(s) = \hat{u}\}} \right) (\psi - 1) \\
&+ \frac{d}{dt} \int_{\partial\Omega} \left(|w(t) - \hat{w}| + \int_{\hat{w}}^{w(t)} (\rho - \mathrm{sign}_0(s - \hat{w})) \chi_{\{s:(\beta^{-1})^0(s) = \hat{u}\}} \right) \\
&+ \int_\Omega (f(t) - \hat{f}) \left(\mathrm{sign}_0(z(t) - \hat{z}) + \mathrm{sign}_0(u(t) - \hat{u}) \chi_{\{x \in \Omega : z(t) = \hat{z}\}} \right) (\psi - 1) \\
&+ \int_\Omega (f(t) - \hat{f}) \left(\rho - \mathrm{sign}_0(z(t) - \hat{z}) \right) \chi_{\{x \in \Omega : u(t) = \hat{u}\}})(\psi - 1) \\
&- \int_{\partial\Omega} (g(t) - \hat{g}) \left(\mathrm{sign}_0(w(t) - \hat{w}) + \mathrm{sign}_0(u(t) - \hat{u}) \chi_{\{x \in \partial\Omega : w(t) = \hat{w}\}} \right) \\
&- \int_{\partial\Omega} (g(t) - \hat{g}) \left(\rho - \mathrm{sign}_0(w(t) - \hat{w}) \right) \chi_{\{x \in \partial\Omega : u(t) = \hat{u}\}}).
\end{aligned}$$

Therefore, taking into account (45) and (46) in (44), replacing ψ by ψ_n such that $L^1(\Omega)$- $\lim_n \psi_n = 1$, and taking limits as n goes to $+\infty$ we obtain

$$\begin{aligned}
&\frac{d}{dt} \int_\Omega |z(t) - \hat{z}| + \frac{d}{dt} \int_{\partial\Omega} \left(|w(t) - \hat{w}| + \int_{\hat{w}}^{w(t)} (\rho - \mathrm{sign}_0(s - \hat{w})) \chi_{\{s:(\beta^{-1})^0(s) = \hat{u}\}} \right) \\
&\leq \int_\Omega (f(t) - \hat{f}) \mathrm{sign}_0(z(t) - \hat{z}) + \int_{\{x \in \Omega : z(t) = \hat{z}\}} |f(t) - \hat{f}| \\
&+ \int_{\partial\Omega} (g(t) - \hat{g}) \left(\mathrm{sign}_0(w(t) - \hat{w}) + \mathrm{sign}_0(u(t) - \hat{u}) \chi_{\{x \in \partial\Omega : w(t) = \hat{w}\}} \right)
\end{aligned}$$

$$+\int_{\partial\Omega}(g(t)-\hat{g})\,(\rho-\mathrm{sign}_0(w(t)-\hat{w}))\,\chi_{\{x\in\partial\Omega: u(t)=\hat{u}\}}).$$

Finally, by a similar argument to the one used in Lemma 5.2, we finish the proof. □

Remark 5.4 It is easy to see that Theorem 2.5 also holds for data $(z_0,w_0)\in V^{1,p}(\Omega)\times V^{1,p}(\partial\Omega)$ and $(f,g)\in L^{p'}(0,T;V^{1,p}(\Omega))\times L^{p'}(0,T;V^{1,p}(\partial\Omega))$ satisfying conditions (4), (5) and (6). In particular, if $p>N$, for data $(z_0,w_0)\in L^1(\Omega)\times L^1(\partial\Omega)$ and $(f,g)\in L^1(0,T;L^1(\Omega))\times L^1(0,T;L^1(\partial\Omega))$ satisfying conditions (4), (5) and (6).

6 Appendix

Let us give here the proof of Theorem 3.5. For this we need to prove some previous lemmas.

Lemma 6.1 *Assume $\gamma,\beta:\mathbb{R}\to\mathbb{R}$ are strictly increasing functions with Ran(γ) = Ran(β) = $\mathbb{R}$. Let $\phi\in L^\infty(\Omega)$ and $\psi\in L^\infty(\partial\Omega)$. Then, if $[u,z,w]$ is a weak solution of problem $(S^{\gamma,\beta}_{\phi,\psi})$, we have*

$$\inf\left\{\gamma^{-1}(\inf\phi),\beta^{-1}(\inf\psi)\right\}\le u\le\max\left\{\gamma^{-1}(\sup\phi),\beta^{-1}(\sup\psi)\right\}.$$

Proof. By (ii) of Theorem 3.3, if

$$a:=\inf\left\{\gamma^{-1}(\inf\phi),\beta^{-1}(\inf\psi)\right\}\quad\text{and}\quad b:=\max\left\{\gamma^{-1}(\sup\phi),\beta^{-1}(\sup\psi)\right\},$$

we have

$$\int_\Omega(\gamma(a)-z)^+ +\int_{\partial\Omega}(\beta(a)-w)^+\le\int_\Omega(\gamma(a)-\phi)^+ +\int_{\partial\Omega}(\beta(a)-\psi)^+$$

and

$$\int_\Omega(z-\gamma(b))^+ +\int_{\partial\Omega}(w-\beta(b))^+\le\int_\Omega(\phi-\gamma(b))^+ +\int_{\partial\Omega}(\psi-\beta(b))^+,$$

and from here the result follows. □

Let now, for $m,n,r,l\in\mathbb{N}$, $\gamma_l^{m,n}(s)=\gamma_l(s)+\frac{1}{l}|s|^{p-2}s+\frac{1}{m}s^+-\frac{1}{n}s^-$ and $\beta_r^{m,n}(s)=\beta_r(s)+\frac{1}{m}s^+-\frac{1}{n}s^-$, where γ_l and β_r are the Yosida approximation of γ and β respectively. Then, by the above lemma, if $[u^{m,n}_{r,l},z^{m,n}_{r,l},w^{m,n}_{r,l}]$ is the weak solution of $(S^{\gamma_l^{m,n},\beta_r^{m,n}}_{\phi,\psi})$, for $\phi\in L^\infty(\Omega)$ and $\psi\in L^\infty(\partial\Omega)$, then

$$\inf\left\{(\gamma^{m,n}_{r,l})^{-1}(\inf\phi),(\beta^{m,n}_{r,l})^{-1}(\inf\psi)\right\}\le u^{m,n}_{r,l}$$
$$\le\sup\left\{(\gamma^{m,n}_{l})^{-1}(\sup\phi),(\beta^{m,n}_{r})^{-1}(\sup\psi)\right\}.$$

Since

$$\gamma^{m,n}(s):=(\liminf_{l\to+\infty}\gamma_l^{m,n})(s)=\gamma(s)+\frac{1}{m}s^+-\frac{1}{n}s^-$$

and

$$\beta^{m,n}(s) := (\liminf_{r\to+\infty} \beta_r^{m,n})(s) = \beta(s) + \frac{1}{m}s^+ - \frac{1}{n}s^-,$$

it follows the next lemma.

Lemma 6.2 *Assume* $\lim_l \lim_r u_{r,l}^{m,n} = u^{m,n}$ *a.e. in* Ω *or* $\lim_r u_{r,r}^{m,n} = u^{m,n}$ *a.e. in* Ω. *Let* $\phi \in L^\infty(\Omega)$ *and* $\psi \in L^\infty(\partial\Omega)$. *Then*

$$\inf\left\{\inf(\gamma^{m,n})^{-1}(\inf\phi), \inf(\beta^{m,n})^{-1}(\inf\psi)\right\} \le u^{m,n}$$

$$\le \sup\left\{\sup(\gamma^{m,n})^{-1}(\sup\phi), \sup(\beta^{m,n})^{-1}(\sup\psi)\right\}.$$

Let $n(m)$ be a subsequence in $\mathbb{N}$. Since

$$\liminf_{m\to\infty} \gamma^{m,n(m)} = \gamma \quad \text{and} \quad \liminf_{m\to\infty} \beta^{m,n(m)} = \beta,$$

the following result holds.

Lemma 6.3 *Assume* $\lim_{m\to\infty} u^{m,n(m)} = u$ *a.e. in* Ω. *If* $a_0 \le \phi \le a_1$ *and* $b_0 \le \psi \le b_1$, *where*

- $\gamma_- < a_0 < 0$ *if* $\gamma_- < 0$ *and* $0 \le a_0$ *if* $\gamma_- = 0$,
- $0 < a_1 < \gamma_+$ *if* $\gamma_+ > 0$ *and* $a_1 \le 0$ *if* $\gamma_+ = 0$,
- $\beta_- < b_0 < 0$ *if* $\beta_- < 0$ *and* $0 \le b_0$ *if* $\beta_- = 0$,
  and
- $0 < b_1 < \beta_+$ *if* $\beta_+ > 0$ *and* $b_1 \le 0$ *if* $\beta_+ = 0$,

then

$$\inf\{A_0, B_0\} \le u \le \sup\{A_1, B_1\},$$

where $A_0 = \inf\gamma^{-1}(a_0)$ *if* $\gamma_- < 0$, $A_0 = 0$ *if* $\gamma_- = 0$, $B_0 = \inf\beta^{-1}(b_0)$ *if* $\beta_- < 0$, $B_0 = 0$ *if* $\beta_- = 0$, $A_1 = \sup\gamma^{-1}(a_1)$ *if* $\gamma_+ > 0$, $A_1 = 0$ *if* $\gamma_+ = 0$, $B_1 = \sup\beta^{-1}(b_1)$ *if* $\beta_+ > 0$ *and* $B_1 = 0$ *if* $\beta_+ = 0$.

Proof of Theorem 3.5. It is obvious that

$$\overline{D(\mathcal{B}^{\gamma,\beta})}^{L^1(\Omega)\times L^1(\partial\Omega)} \subset \left\{(z,w) \in L^1(\Omega)\times L^1(\partial\Omega) : \ \gamma_- \le z \le \gamma_+, \ \beta_- \le w \le \beta_+\right\}.$$

To obtain the another inclusion, it is enough to take $(z,w) \in L^\infty(\Omega)\times L^\infty(\partial\Omega)$, with $a_0 \le z \le a_1$ and $b_0 \le w \le b_1$, where the constants a_i, b_i, $i = 0,1$, satisfy

- $\gamma_- < a_0 < 0$ if $\gamma_- < 0$ and $a_0 = 0$ if $\gamma_- = 0$,
- $0 < a_1 < \gamma_+$ if $\gamma_+ > 0$ and $a_1 = 0$ if $\gamma_+ = 0$,

- $\beta_- < b_0 < 0$ if $\beta_- < 0$ and $b_0 = 0$ if $\beta_- = 0$,

  and

- $0 < b_1 < \beta_+$ if $\beta_+ > 0$ and $b_1 = 0$ if $\beta_+ = 0$,

and to prove that $(z, w) \in \overline{D(\mathcal{B}^{\gamma,\beta})}^X$.

Given $(z, w) \in L^\infty(\Omega) \times L^\infty(\partial\Omega)$ with $a_0 \leq z \leq a_1$ and $b_0 \leq w \leq b_1$, we set

$$(z_n, w_n) = \left(I + \frac{1}{n}\mathcal{B}^{\gamma,\beta}\right)^{-1} (z, w), \quad n \in \mathbb{N}.$$

Let us see that there exists a subsequence, denoted equal, such that

$$(z_n, w_n) \to (z, w) \quad \text{in } L^1(\Omega) \times L^1(\partial\Omega),$$

which implies that $(z, w) \in \overline{D(\mathcal{B}^{\gamma,\beta})}^X$.

Since $((z_n, w_n), n(z - z_n, w - w_n)) \in \mathcal{B}^{\gamma,\beta}$, there exist $u_n \in W^{1,p}(\Omega)$, such that $[u_n, z_n, w_n]$ is a weak solution of problem $(S^{\gamma,\beta}_{z_n+n(z-z_n),w_n+n(w-w_n)})$. Hence, $z_n(x) \in \gamma(u_n(x))$ *a.e.* in Ω, $w_n(x) \in \beta(u_n(x))$ *a.e.* in $\partial\Omega$ and

$$\frac{1}{n}\int_\Omega \mathbf{a}(x, Du_n) \cdot D\phi + \int_\Omega z_n\phi + \int_{\partial\Omega} w_n\phi = \int_\Omega z\phi + \int_{\partial\Omega} w\phi, \tag{47}$$

for all $\phi \in W^{1,p}(\Omega)$.

Note that if $\mathbf{a}_n(x, \xi) := \frac{1}{n}\mathbf{a}(x, \xi)$, then $[u_n, z_n, w_n]$ is a weak solution of the problem

$$({}_{\mathbf{a}_n}S^{\gamma,\beta}_{z,w}) \begin{cases} -\text{div } \mathbf{a}_n(x, Du) + \gamma(u) \ni z & \text{in } \Omega \\ \\ \mathbf{a}_n(x, Du) \cdot \eta + \beta(u) \ni w & \text{on } \partial\Omega, \end{cases}$$

and by uniqueness, we can consider that $[u_n, z_n, w_n]$ is the weak solution of problem $({}_{\mathbf{a}_n}S^{\gamma,\beta}_{z,w})$ given in Theorem 3.3. This construction is done as follows (see [6]). Firstly, we find a weak solution $[(u_n)^{m,k}_r, (z_n)^{m,k}_r, (w_n)^{m,k}_r]$ of $({}_{\mathbf{a}_n}S^{\gamma^{m,k}_r,\beta^{m,k}_r}_{z,w})$ in the case $\mathrm{Dom}(\beta) = \mathbb{R}$, and $[(u_n)^{m,k}_{r,l}, (z_n)^{m,k}_{r,l}, (w_n)^{m,k}_{r,l}]$ of $({}_{\mathbf{a}_n}S^{\gamma^{m,k}_l,\beta^{m,k}_r}_{z,w})$ in the case $\mathbf{a}$ smooth. In the case $\mathrm{Dom}(\beta) = \mathbb{R}$, taking limits as r goes to $+\infty$, we have

$$\lim_r (u_n)^{m,k}_r = (u_n)^{m,k} \text{ in } L^1(\Omega),$$

$$\lim_r (z_n)^{m,k}_r = (z_n)^{m,k} \text{ weakly in } L^1(\Omega),$$

$$\lim_r (w_n)^{m,k}_r = (w_n)^{m,k} \text{ weakly in } L^1(\partial\Omega),$$

$[(u_n)^{m,k}, (z_n)^{m,k}, (w_n)^{m,k}]$ being a weak solution of $(S^{\gamma^{m,k},\beta^{m,k}}_{z,w})$; in the case $\mathbf{a}$ smooth, taking limits as l goes to $+\infty$ we get

$$\lim_l (u_n)^{m,k}_{r,l} = (u_n)^{m,k}_r \text{ in } L^1(\Omega),$$

$$\lim_{l}(z_n)^{m,k}_{r,l} = (z_n)^{m,k}_r \ \text{ weakly in } L^1(\Omega),$$

$$\lim_{l}(w_n)^{m,k}_{r,l} = (w_n)^{m,k}_r \ \text{ weakly in } L^1(\partial\Omega),$$

$[(u_n)^{m,k}_r, (z_n)^{m,k}_r, (w_n)^{m,k}_r]$ being a weak solution of $(S^{\gamma^{m,k},\beta^{m,k}_r}_{z,w})$, and taking limits as r goes to $+\infty$, we obtain

$$\lim_{r}(u_n)^{m,k}_r = (u_n)^{m,k} \ \text{ in } L^1(\Omega),$$

$$\lim_{r}(z_n)^{m,k}_r = (z_n)^{m,k} \ \text{ weakly in } L^1(\Omega),$$

$$\lim_{r}(w_n)^{m,k}_r = (w_n)^{m,k} \ \text{ weakly in } L^1(\partial\Omega),$$

$[(u_n)^{m,k}, (z_n)^{m,k}, (w_n)^{m,k}]$ being a weak solution of $(S^{\gamma^{m,k},\beta^{m,k}}_{z,w})$. Moreover, in the case $\mathbf{a}$ smooth,

$$(w_n)^{m,k} << w - \mathbf{a}_n(x, D(\hat{u}_n)^{m,k}) \cdot \eta,$$

being $[(\hat{u}_n)^{m,k}, (\hat{z}_n)^{m,k}]$ the weak solution of

$$\begin{cases} -\text{div } \mathbf{a}_n(x, D(\hat{u}_n)^{m,k}) + \gamma((\hat{u}_n)^{m,k}) + \dfrac{1}{m}((\hat{u}_n)^{m,k})^+ - \dfrac{1}{k}((\hat{u}_n)^{m,k})^- \ni z & \text{in } \Omega \\ (\hat{u}_n)^{m,k} = 0 & \text{on } \partial\Omega. \end{cases}$$

Finally, passing to the limit in m for an adequate subsequence $\{k(m)\}$ in $\mathbb{N}$, we have

$$\begin{aligned} &\lim_{m\to\infty}(u_n)^{m,k(m)} = u_n \ \text{ in } L^1(\Omega), \\ &\lim_{m\to\infty}(z_n)^{m,k(m)} = z_n \ \text{ in } L^1(\Omega), \\ &\lim_{m\to\infty}(w_n)^{m,k(m)} = w_n \ \text{ in } L^1(\partial\Omega). \end{aligned} \tag{48}$$

Under the assumption $\mathbf{a}$ smooth,

$$\lim_{m\to\infty}(\hat{u}_n)^{m,k(m)} = \hat{u}_n \ \text{ in } L^1(\Omega),$$

$$\lim_{m\to\infty}(\hat{z}_n)^{m,k(m)} = \hat{z}_n \ \text{ in } L^1(\Omega),$$

$$\lim_{m\to\infty}\mathbf{a}_n(x, D(\hat{u}_n)^{m,k(m)}) \cdot \eta = \mathbf{a}_n(x, D\hat{u}_n) \cdot \eta \ \text{ in } L^1(\partial\Omega),$$

$[\hat{u}_n, \hat{z}_n]$ being the weak solution of

$$\begin{cases} -\text{div } \mathbf{a}_n(x, D\hat{u}_n) + \gamma(\hat{u}_n) \ni z & \text{in } \Omega \\ \hat{u}_n = 0 & \text{on } \partial\Omega. \end{cases}$$

Moreover (see [6]),

$$\hat{z}_n << z, \tag{49}$$

$$w_n << w - \mathbf{a}_n(x, D\hat{u}_n) \cdot \eta, \tag{50}$$

and

$$\int_{\partial\Omega} |\mathbf{a}_n(x, D\hat{u}_n) \cdot \eta| \leq \int_\Omega |z - \hat{z}_n|. \tag{51}$$

Observe that, by Lemmas 6.1, 6.2 and 6.3, $\{u_n\}$ is uniformly bounded in $L^\infty(\Omega)$; similarly, $\{\hat{u}_n\}$ is uniformly bounded in $L^\infty(\Omega)$. Therefore, since $\mathrm{Dom}(\gamma) = \mathbb{R}$, $\{z_n\}$ and $\{\hat{z}_n\}$ are uniformly bounded in $L^\infty(\Omega)$, so there exists a subsequence, denoted equal, such that z_n and $\hat{z}_n$ are weakly convergent in $L^1(\Omega)$. Also, in the case $\mathrm{Dom}(\beta) = \mathbb{R}$, there exists a subsequence, denoted equal, such that w_n is weakly convergent in $L^1(\partial\Omega)$.

We claim now that

$$\lim_{n\to\infty} \int_\Omega z_n \phi = \int_\Omega z\phi \quad \text{for every} \quad \phi \in \mathcal{D}(\Omega). \tag{52}$$

Taking $\phi = u_n$ in (47), since $z_n(x) \in \gamma(u_n(x))$ *a.e.* in Ω, $w_n(x) \in \beta(u_n(x))$ *a.e.* in $\partial\Omega$, and $\{u_n\}$ is bounded in $L^\infty(\Omega)$, we get

$$\int_\Omega \mathbf{a}(x, Du_n) \cdot Du_n \leq n \left(\int_\Omega z u_n + \int_{\partial\Omega} w u_n \right) \leq nC$$

Now, using (H_1) and (H_2), we have

$$\left(\int_\Omega |\mathbf{a}(x, Du_n)|^{p'} \right)^{1/p'} \leq \sigma \left(\int_\Omega \left(\varrho(x) + |Du_n|^{p-1} \right)^{p'} \right)^{1/p'} \leq$$

$$\leq \sigma \left(\left(\int_\Omega \varrho(x)^{p'} \right)^{1/p'} + \left(\int_\Omega |Du_n|^p \right)^{1/p'} \right) \leq$$

$$\leq \sigma \left(\left(\int_\Omega \varrho(x)^{p'} \right)^{1/p'} + \left(\frac{1}{\lambda} \int_\Omega \mathbf{a}(x, Du_n) \cdot Du_n \right)^{1/p'} \right)$$

$$\leq \sigma \|\varrho\|_{L^{p'}(\Omega)} + \sigma \left(\frac{C}{\lambda} n \right)^{1/p'}.$$

Consequently,

$$\left(\int_\Omega \left| \frac{1}{n} \, \mathbf{a}(x, Du_n) \right|^{p'} \right)^{1/p'} \leq \frac{\sigma \|\varrho\|_{L^{p'}(\Omega)}}{n} + \sigma \left(\frac{C/\lambda}{n^{p'-1}} \right)^{1/p'}. \tag{53}$$

On the other hand, taking $\phi \in \mathcal{D}(\Omega)$ in (47) we have that

$$\frac{1}{n} \int_\Omega \mathbf{a}(x, Du_n) \cdot D\phi + \int_\Omega z_n \phi = \int_\Omega z\phi.$$

By (53), we get (52). Consequently

$$z_n \rightharpoonup z \quad \text{weakly in } L^1(\Omega). \tag{54}$$

Having in mind (54) and (53), it follows, from (47), that

$$\int_{\partial\Omega} w_n \phi \to \int_{\partial\Omega} w\phi \quad \text{for any } \phi \in W^{1,p}(\Omega) \cap L^\infty(\Omega). \tag{55}$$

Therefore, in the case $\mathrm{Dom}(\beta) = \mathbb{R}$, by (55) we get that

$$w_n \rightharpoonup w \quad \text{weakly in } L^1(\partial\Omega).$$

Similarly, we get $\hat{z}_n \rightharpoonup z$ weakly in $L^1(\Omega)$, hence by (49), $\hat{z}_n \to z$ in $L^1(\Omega)$. Therefore, in the case $\mathbf{a}$ smooth, from (50), (51) and a similar argument to the above one, we get that

$$w_n \rightharpoonup w \quad \text{weakly in } L^1(\partial\Omega).$$

Observe that for any $b \geq 0$ and $c \geq 0$, we also have

$$(z_n - b)^+ \rightharpoonup z_b \geq (z - b)^+,$$

$$(w_n - c)^+ \rightharpoonup w_c \geq (w - c)^+.$$

Now, if $c \notin \mathrm{Ran}(\beta)$,

$$\int_{\partial\Omega} (w_n - c)^+ \leq 0,$$

therefore

$$\int_{\partial\Omega} (w - c)^+ \leq \int_{\partial\Omega} w_c \leq 0,$$

and

$$w_c = (w - c)^+.$$

On the other hand, if $c \in \mathrm{Ran}(\beta)$, there exists $a \geq 0$ such that $c \in \beta(a)$, taking $b \in \gamma(a)$, since $[a, b, c]$ is an entropy solution of the problem $({}_{\mathbf{a}_n} S_{b,c}^{\gamma,\beta})$, we have

$$\int_\Omega (z_n - b)^+ + \int_{\partial\Omega} (w_n - c)^+ \leq \int_\Omega (z - b)^+ + \int_{\partial\Omega} (w - c)^+. \tag{56}$$

Taking limits in (56), we get

$$\int_\Omega (z - b)^+ + \int_{\partial\Omega} (w - c)^+ \leq \int_\Omega z_b + \int_{\partial\Omega} w_c \leq \int_\Omega (z - b)^+ + \int_{\partial\Omega} (w - c)^+,$$

hence

$$w_c = (w - c)^+.$$

Consequently, we obtain, for any $c \geq 0$,

$$(w_n - c)^+ \rightharpoonup (w - c)^+ \quad \text{weakly in } L^1(\partial\Omega). \tag{57}$$

Working similarly, we also get

$$(w_n + c)^- \rightharpoonup (w + c)^- \quad \text{weakly} \ \text{ in } L^1(\partial\Omega). \tag{58}$$

By (57) and (58), working as in the proof of [12, Proposition 2.11], we obtain that

$$w_n \to w \quad \text{in } L^1(\partial\Omega).$$

For $b \geq 0$, we have that

$$(z_n - b)^+ \rightharpoonup z_b \geq (z - b)^+.$$

Now, if $b \notin \mathrm{Ran}(\gamma)$,

$$\int_\Omega (z - b)^+ \leq \int_\Omega z_b \leq 0,$$

hence

$$z_b = (z - b)^+.$$

On the other hand, if $b \in \mathrm{Ran}(\gamma)$, there exists $a \geq 0$ such that $b \in \gamma(a)$. In the case $a \in \mathrm{Dom}(\beta)$, taking $c \in \beta(a)$, we obtain that

$$\int_\Omega (z_n - b)^+ + \int_{\partial\Omega} (w_n - c)^+ \leq \int_\Omega (z - b)^+ + \int_{\partial\Omega} (w - c)^+. \tag{59}$$

And in the case, $a \notin \mathrm{Dom}(\beta)$ (therefore we are assuming **a** smooth), we take $b^m = b + \frac{1}{m} a$, which belongs to $\gamma^{m,k(m)}(a)$ and satisfies

$$\lim_{m\to\infty} b^m = b.$$

Now, since $[(u_n)_r^{m,k}, (z_n)_r^{m,k}, (w_n)_r^{m,k}]$ is the weak solution of $(S_{z,w}^{\gamma^{m,k},\beta_r^{m,k}})$, we have that

$$\int_\Omega \left((z_n)_r^{m,k} - b^m\right)^+ + \int_{\partial\Omega} \left((w_n)_r^{m,k(m)} - \beta_r^{m,k}(a)\right)^+$$

$$\leq \int_\Omega (z - b^m)^+ + \int_{\partial\Omega} \left(w - \beta_r^{m,k}(a)\right)^+ .$$

Then, letting r go to $+\infty$ and having in mind that $\lim_r \beta_r^{m,k}(a) = +\infty$, we get

$$\int_\Omega ((z_n)^{m,k} - b^m)^+ \leq \int_\Omega (z - b^m)^+. \tag{60}$$

Let us take the subsequence $k(m)$ used in (48). Then, taking limits when m goes to $+\infty$ in (60) with $k = k(m)$,

$$\int_\Omega (z_n - b)^+ \leq \int_\Omega (z - b)^+. \tag{61}$$

Now, letting n go to $+\infty$ in (59) and (61), we have that

$$\int_\Omega z_b \leq \int_\Omega (z-b)^+,$$

and therefore $z_b = (z-b)^+$. Hence, for any $b \geq 0$,

$$(z_n - b)^+ \rightharpoonup (z-b)^+ \quad \text{weakly in } L^1(\Omega).$$

Similarly, we can get

$$(z_n + b)^- \rightharpoonup (z+b)^- \quad \text{weakly in } L^1(\Omega).$$

From these convergences we obtain that

$$z_n \to z \quad \text{in } L^1(\Omega),$$

and the proof concludes. $\square$

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