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Equivalent formulations for Monge–Kantorovich equation

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ABSTRACT

In this paper, we study some equivalent formulations in divergence form for the optimization problem $\max\left\{\int_\Omega \xi \,\mathrm{d}\mu\,;\ \xi \in W_0^{1,1}(\Omega) \text{ s.t. } |\nabla\xi(x)| \le k(x) \text{ a.e. } x \in \Omega\right\}$ where $k \in \mathcal{C}(\overline{\Omega})$ and $k > 0$ in Ω. This is the so called dual equation of Monge–Kantorovich problem.

1. Introduction and main result

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded open domain of $\mathbb{R}^N$ with $\mathcal{C}^1$ smooth boundary Γ, and μ a bounded Radon measure concentrated in Ω. We are interested in the study of the optimization problem

$$\max\left\{\int_\Omega \xi \,\mathrm{d}\mu;\ \xi \in K\right\}, \tag{1}$$

where

$$K = \left\{z \in W_0^{1,1}(\Omega);\ |\nabla z(x)| \le k(x) \text{ a.e. } x \in \Omega\right\}$$

and $k \in \mathcal{C}(\overline{\Omega})$ is a positive continuous function. In the case where $k \equiv 1$, this is the so called dual equation of Monge–Kantorovich problem. It is of wide interest for Monge optimal mass transport problem (cf. [1,2] and the references therein). In one hand, it was used by Kantorovich for the study of existence of a solution for his relaxed formulation of the original Monge problem. On the other hand, it appears in numerous papers, that the PDE in divergence form behind (1) contains all the information concerning the original Monge problem (cf. [24,23,3,2,1]). The case where k is an x-dependent function appears in the study of optimal mass transport problem in inhomogeneous domain to treat problems with, say, subregions through which mass transportation is forbidden or, on the contrary, where it is free of charge (cf. [4]). It appears also in the study of mass optimization problem (cf. [5]). Existence of a solution of (1) is well known by now for any bounded Radon measure μ. Our aim in this paper, is to show the equivalence between (1) and formulations in divergence form. Since their interest for the Monge–Kantorovich problem and related problems, this kind of question was already studied in previous papers. So, a part of our results are well known by now and may exist in the literature in a more general setting and proved by using sophisticated arguments (see for instance [6,7] and the references therein). Our aim here is to give simple and direct proofs for our (more or less) simple situation.

Perhaps the main difficulty in the study of the PDE associated with (1) is the non-regularity of the flux. Roughly speaking, in general the PDE associated with (1) is a divergence of non-regular flux (a measure). To close the problem, the flux needs

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to be expressed depending on the gradient. Since in general the gradient is no more than L^∞, then the gradient should be taken in an unusual sense. Otherwise, different expressions have been used to close the problem depending on the studied issues : optimal mass transport problem, mass optimization, evolution Monge–Kantorovich equation (sandpile equation).

Here, we focus our attention on the three following divergence formulations :

$$\begin{cases} -\nabla\cdot\Phi=\mu & \text{in } \mathcal{D}'(\Omega)\\ k\Phi=|\Phi|\nabla_{|\Phi|}u, \end{cases} \tag{2}$$

$$\begin{cases} -\nabla\cdot\Phi=\mu & \text{in } \mathcal{D}'(\Omega)\\ \displaystyle\int_\Omega k\mathrm{d}|\Phi|\leq\int_\Omega u\mathrm{d}\mu, \end{cases} \tag{3}$$

and

$$\begin{cases} \displaystyle\int_\Omega k\mathrm{d}|\Phi|=\min\left\{\int_\Omega k\mathrm{d}|\nu|\,;\ -\nabla\cdot\nu=\mu \text{ in } \mathcal{D}'(\Omega)\right\}\\ \displaystyle\quad=\int_\Omega u\mathrm{d}\mu, \end{cases} \tag{4}$$

where $|\Phi|$ (resp. $|\nu|$) denotes the total variation measure of Φ (resp. ν) and $\nabla_{|\Phi|}$ denotes the tangential gradient with respect to $|\Phi|$ (see the following section for preliminaries and references). We prove the following:

Theorem 1. *Let $\mu\in\mathcal{M}_b(\Omega)$, where $\mathcal{M}_b(\Omega)$ the set of bounded Radon measure, and $u\in K$. Then u is a solution of (1), i.e.*

$$\int_\Omega (u-\xi)\mathrm{d}\mu\geq 0 \quad \textit{for any } \xi\in K, \tag{5}$$

if and only if there exists $\Phi\in\mathcal{M}_b(\Omega)^N$ such that (u,Φ) satisfies (3). Moreover, we have

1. (2) $\Longleftrightarrow$ (3) $\Longleftrightarrow$ (4).
2. *If $\Phi\in\mathcal{M}_b(\Omega)^N$ is such that $-\nabla\cdot\Phi=\mu$ in $\mathcal{D}'(\Omega)$ and*

$$\int_\Omega k\mathrm{d}|\Phi|=\min\left\{\int_\Omega k\mathrm{d}|\nu|\,;\ -\nabla\cdot\nu=\mu \textit{ in } \mathcal{D}'(\Omega)\right\},$$

then, there exists $v\in K$ such that

$$\int_\Omega k\mathrm{d}|\Phi|=\int_\Omega v\mathrm{d}\mu.$$

The main interest in the formulation (1)–(4) is their connection with the Monge optimal mass transport problem (cf. [3], [2,1] and the references therein) as well as mass optimization (cf. [5,8]) and sandpile (cf. [9,10,2] and [11]). The formulation (2) is the so called Monge–Kantorovich equation (the MK equation as called by Bouchitté, Buttazzo and Seppecher in [12]). It is very connected to Monge–Kantorovich problem for optimal mass transportation. Perhaps Kantorovich (in 1940) was the first who introduced this connection. In addition, Evans and Gangbo (cf. [3]) give rigorous proof for this connection in the case where μ is regular enough (see also [2]). Indeed, in the case where $k\equiv 1$, under additional assumptions on μ, Evans and Gangbo prove in [3] that a related PDE to (1) in divergence form is given by

$$\begin{cases} \left.\begin{aligned} &-\nabla\cdot\Phi=\mu,\ \Phi:=m\nabla u\\ &m\geq 0,\ |\nabla u|\leq 1,\ m(|\nabla u|-1)=0 \end{aligned}\right\} & \text{in } \Omega\\ u=0 & \text{on } \Gamma. \end{cases} \tag{6}$$

The unknown function m is in $L^\infty(\Omega)$ in their case and contains all the information concerning the optimal mass transportation. For the general case, $m\in\mathcal{M}_b^+(\Omega)$ and (6) may be written in the form

$$\begin{cases} -\nabla\cdot\Phi=\mu,\ \ \Phi:=m\nabla_m u & \text{in } \Omega,\\ |\nabla u|\leq k \quad \text{in } \Omega, \quad |\nabla_m u|=k \quad m\text{-a.e. in } \Omega, & \\ u=0 & \text{on } \Gamma. \end{cases} \tag{7}$$

The connection between (7) and optimal transportation has been proved in [1] by using the equivalent formulation (2), where the tangential gradient introduced in [12] (see also [5,8] and the references therein) plays a fundamental role. For the regularity of m with respect to additional assumptions on μ, we refer the reader to the papers [13,14], [15,16] and the references therein. Notice that for the particular case where μ is a nonnegative regular function, explicit formulation for m is given in [17] (see also [18]).

The formulation (4) is the natural dual formulation of (1). Its connection with (2) appeared in [12] for the study of mass optimization problem. In the context of mass optimal transportation, (4) is in connection with minimizing Monge work (for more details in this direction see [12,6]). As to the formulation (3), it appears in [19] (see also [4]) and seems to be very useful for the study of the evolution problem associated with the Monge–Kantorovich equation and the sandpile problem. Notice that the equivalence between (2), (3) and (7) is related to the following obvious fact : assuming $\Phi : \Omega \to \mathbb{R}^N$ and $u : \Omega \to \mathbb{R}$ are regular and $|\nabla u(x)| \leq k(x)$, the following assertions are equivalent

- $m(|\nabla u| - k) = 0$ and $\Phi = m\nabla u$ in Ω
- $mk = |\Phi|$ and $k|\Phi| = \Phi \cdot \nabla u$ in Ω
- $mk = |\Phi|$ and $\int_\Omega k|\Phi| \leq \int_\Omega \Phi \cdot \nabla u$

Assuming $u = 0$ on the boundary and integrating by parts the last equation, we deduce the equivalence between the preceding assertions and the fact that

$$mk = |\Phi| \quad \text{and} \quad \int_\Omega k|\Phi| \leq -\int_\Omega u\nabla \cdot \Phi.$$

In the following section, we begin by giving some preliminaries that will be used throughout the paper. Then, we prove some more or less well known results concerning 1-Lipchitz continuous functions that will be used in this paper. Section 3 is devoted to the proofs of the main theorem.

2. Preliminaries

Let $\Omega \subset \mathbb{R}^N$ be a bounded open domain. We denote by $\mathcal{L}^N$ the N-dimensional Lebesgue measure of $\mathbb{R}^N$. For $1 \leq p < +\infty$, $L^p(\Omega)$, $W^{1,p}(\Omega)$ and $W_0^{1,p}(\Omega)$ denote respectively, with respect to $\mathcal{L}^N$, the standard Lebesgue space, Sobolev space and the closure of $\mathcal{D}(\Omega)$ in $W^{1,p}(\Omega)$. Otherwise, we denote by $L^p_\mu(\Omega)$, the standard L^p space with respect to the measure μ.

We denote by $\mathcal{M}(\Omega)$ the space of all Radon measures in Ω. We recall that $\mathcal{M}(\Omega)$ can be identified with the dual space of the set of continuous functions with compact support in Ω. In other words, $\mathcal{M}(\Omega) = \left(\mathcal{C}_c(\Omega)\right)^*$, in the sense that, every $\mu \in \mathcal{M}(\Omega)$ is identified to the linear application $\xi \in \mathcal{C}_c(\Omega) \to \int_\Omega \xi \mathrm{d}\mu$. The set $\mathcal{M}(\Omega)$ can be identified also with the dual space of the set of continuous functions $\mathcal{M}(\Omega) = \left(\mathcal{C}(\overline{\Omega})\right)^*$, in the sense that, every $\mu \in \mathcal{M}(\Omega)$ is equal to $\tilde{\mu} \in \left(\mathcal{C}(\overline{\Omega})\right)^*$ with $\tilde{\mu}(\partial\Omega) = 0$. So, for any $\mu \in \mathcal{M}(\Omega)$ and $\xi \in \mathcal{C}(\overline{\Omega})$, we use the notation $\int_\Omega \xi \mathrm{d}\mu$ for the quantity $\left\langle \tilde{\mu}, \xi \right\rangle$.

For $\mu \in \mathcal{M}(\Omega)$, we denote by μ^+, μ^- and $|\mu|$ the positive part, negative part and the total variation measure associated with μ, respectively. Then we denote, $\mathcal{M}_b(\Omega)$ the space of Radon measures with bounded total variation $|\mu|(\Omega)$. Recall that $\mathcal{M}_b(\Omega)$ equipped with the norm $|\mu|(\Omega)$ is a Banach space.

We denote by $\mathcal{M}(\Omega)^N$ the space of $\mathbb{R}^N$-valued Radon measures of Ω ; i.e. $\mu \in \mathcal{M}(\Omega)^N$ if and only if $\mu = (\mu_1, \ldots, \mu_n)$ with $\mu_i \in \mathcal{M}(\Omega)$. We recall that the total variation measure associated with $\mu \in \mathcal{M}(\Omega)^N$, denoted again by $|\mu|$, is defined by

$$|\mu|(B) = \sup\left\{\sum_{i=1}^{\infty} |\mu(B_i)| \,;\; B = \bigcup_{i=1}^{\infty} B_i,\; B_i \text{ a Borelean set}\right\}$$

and belongs to $\mathcal{M}^+(\Omega)$, the set of nonnegative Radon measure. The subspace $\mathcal{M}_b(\Omega)^N$ equipped with the norm $\|\mu\| = |\mu|(\Omega)$ is a Banach space. It is clear that $\mathcal{M}(\Omega)^N$ endowed with the norm $\| \ \|$ is isometric to the dual of $\mathcal{C}_c(\Omega)^N$. The duality is given by

$$\left\langle \mu, \xi \right\rangle = \sum_{i=1}^{N} \int_\Omega \xi_i \mathrm{d}\mu_i,$$

for any $\mu = (\mu_1, \ldots, \mu_N) \in \mathcal{M}(\Omega)^N$ and $\xi = (\xi_1, \ldots, \xi_N) \in \mathcal{C}_c(\Omega)^N$.

For any $\mu \in \mathcal{M}_b(\Omega)^N$ and $\nu \in \mathcal{M}_b(\Omega)^+$, μ is absolutely continuous with respect to ν; denoted by $\mu \ll \nu$, provided $\nu(A) = 0$ implies $|\mu|(A) = 0$, for any $A \subset \Omega$. Thanks to Radon–Nicodym Decomposition Theorem, we know that for any $\mu \in \mathcal{M}_b(\Omega)^N$ and $\nu \in \mathcal{M}_b(\Omega)$ such that $\mu \ll \nu$, there exists unique bounded $\mathbb{R}^N$-valued Radon measure denoted by $D_\nu\mu$, such that

$$\mu(A) = \int_A D_\nu\mu \mathrm{d}\nu \quad \text{for any } A \subseteq \Omega;$$

$D_\nu\mu \in \mathcal{M}_b(\Omega)^N$ is the density of μ with respect to ν, that can be computed by differentiating. In particular, it is not difficult to see that, for any $\mu \in \mathcal{M}(\Omega)^N$, we have $\mu \ll |\mu|$, $D_{|\mu|}\mu \in L^1_{|\mu|}(\Omega)^N$ and $|D_{|\mu|}\mu| = 1$, $|\mu|$-a.e. in Ω (see for instance [20]). In connection with the polar factorization, in general, $D_{|\mu|}\mu$ is denoted by $\frac{\mu}{|\mu|}$. So, for any $\mu \in \mathcal{M}_b(\Omega)^N$, we have

$$\mu(A) = \int_A \frac{\mu}{|\mu|} \mathrm{d}|\mu|, \quad \text{for any Borel set } A \subseteq \Omega.$$

So, every $\mu \in \mathcal{M}_b(\Omega)^N$ can be identified with the linear application

$$\xi \in \mathcal{C}_c(\Omega)^N \to \int_\Omega \frac{\mu}{|\mu|} \cdot \xi \mathrm{d}|\mu|.$$

For any $\Phi \in \mathcal{M}_b(\Omega)^N$ and $\nu \in \mathcal{M}_b(\Omega)$, we say that $-\nabla \cdot \Phi = \nu$ in $\mathcal{D}'(\Omega)$ provided

$$\int_\Omega \frac{\Phi}{|\Phi|} \cdot \nabla\xi \mathrm{d}|\Phi| = \int_\Omega \xi \mathrm{d}\nu \quad \text{for any } \xi \in \mathcal{C}_0^1(\Omega),$$

where $\mathcal{C}_0^1(\Omega)$ is the set of $\mathcal{C}^1$ function in Ω, such that ξ and $\nabla\xi$ are null on the boundary of Ω. In particular, $-\nabla \cdot \Phi = \nu$ in $\mathcal{D}'(\Omega)$ is equivalent to $-\nabla \cdot \left(\frac{\Phi}{|\Phi|}|\Phi|\right) = \nu$ in $\mathcal{D}'(\Omega)$.

Let $\nu \in \mathcal{M}_b(\Omega)^+$ be given. To define the tangential gradient with respect to ν (see [12]), recall the sets

$$\mathcal{N}_\nu := \left\{\xi \in L^\infty_\nu(\Omega)^N; \exists u_n \in \mathcal{C}^\infty(\Omega),\ u_n \to 0 \text{ in } \mathcal{C}(\Omega) \text{ and } Du_n \to \xi \text{ in } \sigma(L^\infty_\nu(\Omega)^N, L^1_\nu(\Omega)^N)\right\}$$

and

$$\mathcal{N}_\nu^\perp := \left\{\eta \in L^1_\nu(\Omega)^N\ ;\ \int_\Omega \eta \cdot \xi \mathrm{d}\nu = 0,\ \forall \xi \in \mathcal{N}_\nu\right\}$$

For ν-a.e. $x \in \Omega$, we define the tangent space $T_\nu(x)$ to the measure ν, as the subspace of $\mathbb{R}^N$:

$$T_\nu(x) = \left\{A \in \mathbb{R}^N\ ;\ \exists\, \xi \in \mathcal{N}_\nu^\perp,\ A = \xi(x)\right\}.$$

Then (cf. Proposition 3.2 of [21]) the operator $\nabla_\nu : Lip(\Omega) \to L^\infty_\nu(\Omega)^N$ is the continuous linear operator such that for any $u \in \mathcal{C}^1(\Omega)$,

$$\nabla_\nu u(x) = P_{T_\nu(x)} \nabla u(x) \quad \nu\text{-a.e. } x \in \Omega,$$

where $P_{T_\nu(x)}$ is the orthogonal projector on $T_\nu(x)$, $Lip(\Omega)$ is the set of Lipchitz continuous function equipped with the uniform convergence and $L^\infty_\nu(\Omega)^N$ is equipped with the weak star topology. A $\mathbb{R}^N$-valued Radon measure Φ is said to be a tangential measure on Ω provided there exists $\nu \in \mathcal{M}_b(\Omega)^+$ and $\sigma \in L^1_\nu(\Omega)^N$, such that $\sigma(x) \in T_\nu(x)$, ν-a.e. $x \in \Omega$ and $\Phi = \sigma\nu$. At last, thanks to Proposition 3.5 of [21], we know that for any tangential measure $\Phi = \sigma\nu$ on Ω, such that $-\nabla \cdot \Phi = \mu \in \mathcal{M}_b(\Omega)$, we have the following integration by parts

$$\int_\Omega u \mathrm{d}\mu = \int_\Omega \sigma \cdot \nabla_\nu u \mathrm{d}\nu, \tag{8}$$

for any $u \in Lip(\Omega)$ null on the boundary of Ω.

To prove Theorem 1, we use in Section 3. Below are the following two lemmas :

Lemma 1. *For any $z \in K$, there exists $(z_\varepsilon)_{\varepsilon>0}$ a sequence in $\mathcal{D}(\Omega) \cap K$ such that*

$$z_\varepsilon \to z \quad \text{in } \mathcal{C}_0(\Omega) \text{ and in } W^{1,\infty}(\Omega)\text{-weak}^*.$$

Proof. Let d be the solution of the maximization problem

$$\int_\Omega d(x)\mathrm{d}x = \max_{\xi\in K} \int_\Omega \xi(x)\mathrm{d}x.$$

Then, for any $u \in K$, we have $u \leq d$ in Ω. Indeed, for any $u \in K$, taking $\tilde{u}(x) = \max(u(x), d(x))$ for any $x \in \Omega$, we have $\tilde{u} \in K$, $\tilde{u} \geq d$ in Ω and $\int_\Omega \tilde{u}(x)\mathrm{d}x \leq \int_\Omega d(x)\mathrm{d}x$, so that $\tilde{u} = d$ in Ω and, then $u \leq d$ in Ω. Now, let us consider

$$d_\varepsilon = (d - \varepsilon)^+ \quad \text{in } \Omega.$$

We see that $d_\varepsilon \in K$ and, for any $\varepsilon > 0$, there exists $\Omega_\varepsilon \subset\subset \Omega$, such that d_ε is compactly supported in Ω_ε. Let us denote by

$$\tilde{z}_\varepsilon(x) = \begin{cases} (d_\varepsilon(x) - z^-(x))^+ - (d_\varepsilon(x) - z^+(x))^+ & \text{if } x \in \Omega_\epsilon \\ 0 & \text{if } x \in \mathbb{R}^N \setminus \Omega_\epsilon. \end{cases}$$

It is not difficult to verify that $\tilde{z}_\epsilon \in K$ and that $\tilde{z}_\epsilon$ is supported in Ω_ϵ. Let $(\rho_\lambda)_{\lambda>0}$ be the standard sequence of mollifiers. We denote by $m := \min\left\{k(x)\ ;\ x \in \Omega\right\}$ and, for any $\varepsilon > 0$, we denote by $\omega(\varepsilon)$ the modulus of continuity of z ; i.e. $\omega(\varepsilon) = \sup_{|x-y|\leq\varepsilon} |z(x) - z(y)|$. Now, take $\alpha > 0$ small enough so that, for any $0 < \varepsilon < 1$, we have

$$z_\epsilon := \frac{m}{m + \omega(\varepsilon)} \tilde{z}_\epsilon * \rho_{\alpha\epsilon} \quad \in \mathcal{D}(\Omega)$$

and

$$|\nabla z_\varepsilon(x)| \le \frac{m}{m+\omega(\varepsilon)}\Big(k(x)+\omega(\varepsilon)\Big) \le k(x) \quad \text{a. e. } x \in \Omega.$$

Since, as $\epsilon \to 0$, d_ε converges to d in $\mathcal{C}_0(\Omega)$, $\omega(\varepsilon) \to 0$ and z_ϵ is bounded in $W^{1,\infty}(\Omega)$, then the result of the lemma follows. ■

Lemma 2. *For any $u \in K$ and $\nu \in \mathcal{M}_b(\Omega)^+$, we have*

$$|\nabla_\nu u| \le k \quad \nu\text{-a.e. in } \Omega.$$

Proof. If $u \in \mathcal{D}(\Omega)$, then $\nabla_\nu u(x)$ coincides with $P_{T_\nu(x)}\nabla u(x)$, ν-a.e. $x \in \Omega$, and then the property is true. For $u \in K$, let $(u_\varepsilon)_{\varepsilon>0}$ be a sequence of regularization of u as given by Lemma 1. Thanks to the first part of the proof, we get $|\nabla u_\varepsilon| \le k$, ν-a.e. in Ω. At last, thanks to the continuity of the operator ∇_ν (cf. Proposition 3.2 of [21]), $\nabla_\nu u_\epsilon \to \nabla_\nu u$ in $L^\infty(\Omega, d\nu)$-weak*, which implies that $|\nabla_\nu u| \le k\nu$-a.e. in Ω. ■

3. Proof of Theorem 1

The proof of Theorem 1, follows as a consequence of the sequence of the lemmas below.

Lemma 3. *Let $\mu \in \mathcal{M}_b(\Omega)$, $u \in K$ and $\Phi \in \mathcal{M}_b(\Omega)^N$. If (u, Φ) satisfies (3), then*

$$\int_\Omega k d|\Phi| = \int_\Omega u d\mu. \tag{9}$$

and u satisfies (5).

Proof. Let $u_\varepsilon \in \mathcal{D}(\Omega) \cap K$ be the approximation of u as given by Lemma 1. Then the Eq. (9) is a simple consequence of (3) and the fact that

$$\begin{aligned}\int_\Omega u d\mu &= \lim_{\varepsilon\to 0}\int_\Omega u_\varepsilon d\mu \\ &= \lim_{\varepsilon\to 0}\int_\Omega \nabla u_\varepsilon \cdot \frac{\Phi}{|\Phi|} d|\Phi| \\ &\le \int_\Omega k d|\Phi|.\end{aligned}$$

As to (5), it follows again from Lemma 1. Indeed, for any $\xi \in K$, we have

$$\begin{aligned}\int_\Omega \xi d\mu &= \lim_{\varepsilon\to 0}\int_\Omega \frac{\Phi}{|\Phi|}\cdot\nabla\xi_\varepsilon d|\Phi| \\ &\le \int_\Omega k d|\Phi| = \int_\Omega u d\mu,\end{aligned}$$

where we used $\xi_\varepsilon \in \mathcal{D}(\Omega) \cap K$ the approximation of ξ as given by Lemma 1. ■

Lemma 4. *Let $\mu \in \mathcal{M}_b(\Omega)$, $u \in K$ and $\Phi \in \mathcal{M}_b(\Omega)^N$ be given such that $-\nabla\cdot\Phi = \mu$ in $\mathcal{D}'(\Omega)$. Then, the following assumptions are equivalent :*

1. $\int_\Omega k d|\Phi| \le \int_\Omega u d\mu$.
2. $\int_\Omega k d|\Phi| = \lim_{\varepsilon\to 0}\int_\Omega \frac{\Phi}{|\Phi|}\cdot\nabla u_\varepsilon d|\Phi|$, *for any $u_\varepsilon \in K \cap \mathcal{D}(\Omega)$, such that $u_\varepsilon \to u$ in $W^{1,\infty}(\Omega)$-weak*.*
3. $k\Phi = |\Phi|\nabla_{|\Phi|}u$.

Proof. Step 1: 1 ⇔ 2. First, notice that since $-\nabla\cdot\Phi = \mu$ in $\mathcal{D}'(\Omega)$, then

$$\int_\Omega u d\mu = \lim_{\varepsilon\to 0}\int_\Omega u_\varepsilon d\mu = \lim_{\varepsilon\to 0}\int_\Omega \nabla u_\varepsilon \cdot \frac{\Phi}{|\Phi|} d|\Phi|,$$

for any $u_\varepsilon \in \mathcal{D}(\Omega)$, such that $u_\epsilon \to u$ in $\mathcal{C}_0(\Omega)$. So, the equivalence between 1 and 2 is an obvious consequence of Lemma 1 and the first part of Lemma 3.

Step 2: 1 ⇒ 3. Now assume that (u, Φ) satisfies 1. Thanks to the continuity of the operator $\nabla_{|\Phi|}$, we have

$$\int_\Omega k d|\Phi| \le \int_\Omega u d\mu = \lim_{\epsilon\to 0}\int_\Omega \nabla u_\epsilon \cdot \frac{\Phi}{|\Phi|} d|\Phi| = \int_\Omega \nabla_{|\Phi|}u \cdot \frac{\Phi}{|\Phi|} d|\Phi|, \tag{10}$$

In addition, since by Lemma 2, we have $|\nabla_{|\Phi|}u| \leq k|\Phi|$-a.e. in Ω, then (10) implies that

$$\nabla_{|\Phi|}u \cdot \frac{\Phi}{|\Phi|} = k \quad |\Phi|\text{-a.e. in } \Omega,$$

which implies 3.

Step 3: $3 \Rightarrow 1$. Assume that (u, Φ) satisfies 3., then $\nabla_{|\Phi|}u \cdot \frac{\Phi}{|\Phi|} = k$, $|\Phi|$-a.e. in Ω. By using (8), we get

$$\int_\Omega k \mathrm{d}|\Phi| = \int_\Omega \frac{\Phi}{|\Phi|} \cdot \nabla_{|\Phi|} u \mathrm{d}|\Phi| = \int_\Omega u \mathrm{d}\mu,$$

which implies 2. ∎

Lemma 5. *Let $\nu \in \mathcal{M}_b(\Omega)$ and $u \in K$. If (u, Φ) is a weak solution of (3), then*

$$\int_\Omega k \mathrm{d}|\Phi| = \min \left\{ \int_\Omega k \mathrm{d}|\nu| \, ; \; -\nabla \cdot \nu = \mu \text{ in } \mathcal{D}'(\Omega) \right\}.$$

Proof. Let $\nu \in \mathcal{M}_b(\Omega)$ be such that $-\nabla \cdot \nu = \mu$ in $\mathcal{D}'(\Omega)$. Since, (u, Φ) is a weak solution then, by using $u_\epsilon \in \mathcal{D}(\Omega) \cap K$ the approximation of u as given by Lemma 1, we have

$$\begin{aligned}\int_\Omega k \mathrm{d}|\Phi| &\leq \int_\Omega u \, \mathrm{d}\mu = \lim_{\epsilon\to 0} \int_\Omega u_\epsilon \, \mathrm{d}\mu = \lim_{\epsilon \to 0} \int_\Omega \nabla u_\epsilon \cdot \frac{\nu}{|\nu|} \, \mathrm{d}|\nu| \\ &\leq \int_\Omega k \mathrm{d}|\nu|,\end{aligned}$$

and the proof is complete. ∎

As a consequence of Lemmas 4 and 5, we deduce that (3) implies (2) and (5) is equivalent to (3) and (3) implies (4). Now, let us prove both that (4) implies (3) and, that (5) implies (3). To this end, we consider the elliptic equation

$$\begin{cases} \lambda \, u_\varepsilon - \nabla \cdot w_\varepsilon = \nu_\varepsilon \\ w_\varepsilon = \phi_\epsilon(x, \nabla u_\varepsilon) \\ u_\varepsilon = 0 \end{cases} \begin{matrix} \text{in } \Omega \\ \\ \text{on } \partial\Omega, \end{matrix} \tag{S_ϵ}$$

where ν_ε is a given measure in $\mathcal{M}_b(\Omega)$, $\lambda \geq 0$ is fixed and, for any $\varepsilon > 0$ and $x \in \Omega$, $\phi_\varepsilon(x, .) \; : \; \mathbb{R}^N \to \mathbb{R}^N$ is given by

$$\phi_\varepsilon(x, r) = \frac{1}{\varepsilon}\Big((|r| - k(x))^+ \Big)^{(p-1)} \frac{r}{|r|}, \qquad \text{for any } r \in \mathbb{R}^N,$$

with $p > N$ fixed. It is not difficult to see that ϕ_ε satisfies the following properties

(i) for any $r_1, \; r_2 \; \in \mathbb{R}^N$ and $x \in \Omega$, $\Big(\phi_\varepsilon(x, r_1) - \phi_\varepsilon(x, r_2)\Big) \cdot (r_1 - r_2) \geq 0$.
(ii) there exists $\varepsilon_0 > 0$ and $A > 1$ such that $\phi_\varepsilon(x, r) \cdot r \geq |r|^p$ for any $x \in \Omega$, $|r| \geq A$ and $\varepsilon < \varepsilon_0$.
(iii) for any $\varepsilon > 0$, $r \in \mathbb{R}^N$ and $x \in \Omega$, $k|\phi_\varepsilon(x, r)| \leq \phi_\varepsilon(x, r) \cdot r$.

So (see for instance [22]), for any $\nu_\varepsilon \in W^{-1,p'}(\Omega)$, (S_ε) has a unique weak solution u_ε. In particular, since for any $p > N$, $\mathcal{M}_b(\Omega)$ may be injected continuously into $W^{-1,p'}(\Omega)$, then for any $\nu_\varepsilon \in \mathcal{M}_b(\Omega)$, (S_ε) has a unique weak solution u_ε.

Lemma 6. *Let $(\nu_\varepsilon)_{0<\varepsilon<\varepsilon_0}$ a bounded sequence in $W^{-1,p'}(\Omega)$ and $(u_\varepsilon)_{0<\varepsilon<\varepsilon_0}$ the sequence of solutions of (S_ε). Then,*

1. *$(u_\varepsilon)_{0<\varepsilon<\varepsilon_0}$ is bounded in $W_0^{1,p}(\Omega)$.*
2. *$\Big(\Phi_\varepsilon(., \nabla u_\varepsilon)\Big)_{0<\varepsilon<\varepsilon_0}$ is bounded $L^1(\Omega)^N$.*
3. *For any Borel set $B \subseteq \Omega$,*

$$\liminf_{\varepsilon \to 0} \int_B k|\nabla u_\varepsilon|^{p-1} \leq \int_B k^p.$$

Proof. 1. Taking u_ε as a test function in (S_ε), we get

$$\begin{aligned}\frac{1}{\varepsilon}\int_\Omega (|\nabla u_\varepsilon| - k)^{+(p-1)} |\nabla u_\varepsilon| &= \int_\Omega \Phi_\varepsilon(., \nabla u_\varepsilon) \cdot \nabla u_\varepsilon \\ &= \int_\Omega u_\varepsilon \mathrm{d}\nu_\varepsilon - \lambda \int_\Omega u_\varepsilon^2 \\ &\leq C \|\nu_\varepsilon\|_{W^{-1,p'}(\Omega)} \|\nabla u_\varepsilon\|_{L^p(\Omega)},\end{aligned} \tag{11}$$

where C is the constant of Poincaré inequality. Using (11) and property (ii) of ϕ_ε, for any $0<\varepsilon<\varepsilon_0$, we get

$$\begin{aligned}\int_\Omega |\nabla u_\varepsilon|^p &= \int_{[|\nabla u_\varepsilon|\le A]} |\nabla u_\varepsilon|^p + \int_{[|\nabla u_\varepsilon|>A]} |\nabla u_\varepsilon|^p \\ &\le \int_{[|\nabla u_\varepsilon|\le A]} |\nabla u_\varepsilon|^p + \frac{1}{\varepsilon}\int_\Omega (|\nabla u_\varepsilon|-k)^{+(p-1)}|\nabla u_\varepsilon| \\ &\le \int_{[|\nabla u_\varepsilon|\le A]} |\nabla u_\varepsilon|^p + C\|\nu_\varepsilon\|_{W^{-1,p'}(\Omega)}\|\nabla u_\varepsilon\|_{L^p(\Omega)} \\ &\le |A|^p|\Omega| + C\|\nu_\varepsilon\|_{W^{-1,p'}(\Omega)}\|\nabla u_\varepsilon\|_{L^p(\Omega)}.\end{aligned}$$

Thus, by Young inequality, u_ε is bounded in $W_0^{1,p}(\Omega)$.

2. Since $\min\left\{k(x)\ ;\ x\in\Omega\right\}>0$, then the second part of the lemma follows by the assumption (iii) on Φ_ε and the first part.
3. Now, let $B\subseteq\Omega$ be a fixed Borel set. We have,

$$\begin{aligned}\left(\int_B |k\nabla u_\varepsilon|^{p-1}\right)^{\frac{1}{p-1}} &\le \left(\int_B k(|\nabla u_\varepsilon|-k)^{+(p-1)}\right)^{\frac{1}{p-1}} + \left(\int_B k^p\right)^{\frac{1}{p-1}} \\ &\le \left(\int_B (|\nabla u_\varepsilon|-k)^{+(p-1)}|\nabla u_\varepsilon|\right)^{\frac{1}{p-1}} + \left(\int_B k^p\right)^{\frac{1}{p-1}} \\ &\le \left(C\varepsilon\|\nu_\varepsilon\|_{W^{-1,p'}(\Omega)}\|\nabla u_\varepsilon\|_{L^p(\Omega)}\right)^{\frac{1}{p-1}} + \left(\int_B k^p\right)^{\frac{1}{p-1}}.\end{aligned}$$

Letting $\varepsilon\to 0$, and using the fact that u_ε and ν_ε are bounded respectively in $W_0^{1,p}(\Omega)$ and $W_0^{-1,p'}(\Omega)$, we obtain

$$\liminf_{\varepsilon\to 0}\int_B k|\nabla u_\varepsilon|^{p-1} \le \int_B k^p. \quad \blacksquare$$

Lemma 7. *Under the assumptions of Lemma 6, suppose that $\nu\in\mathcal{M}_b(\Omega)$ is such that*

$$\nu_\varepsilon \to \nu \quad \textit{weakly in } \mathcal{M}_b(\Omega).$$

Then, there exists a subsequence that we denote again by ε, such that, as $\varepsilon\to 0$,

$$u_\varepsilon \to u \quad \textit{in } \mathcal{C}_0(\Omega) \textit{ and in } W^{1,\infty}(\Omega)\textit{-weak}^*, \tag{12}$$

$$\Phi_\varepsilon(.,\nabla u_\varepsilon) \to \Phi \quad \textit{in } \mathcal{M}_b(\Omega)^N\textit{-weak}^*, \tag{13}$$

and

$$\int_\Omega k|\Phi_\varepsilon(.,\nabla u_\varepsilon)| \to \int_\Omega k\,\mathrm{d}|\Phi|. \tag{14}$$

Moreover, $u\in K$ and (u,Φ) satisfies

$$\begin{cases}\lambda u - \nabla\cdot\Phi = \nu \quad \textit{in } \mathcal{D}'(\Omega) \\ \displaystyle\int k\,\mathrm{d}|\Phi| \le \int u\,\mathrm{d}\nu - \lambda\int u^2.\end{cases} \tag{15}$$

Proof. Thanks to Lemma 6, there exists $u\in W_0^{1,p}(\Omega)$, $\Phi\in\mathcal{M}_b(\Omega)^N$ and a subsequence that we denote again by ε, such that (12) and (13) are fulfilled and $\lambda u-\nabla\cdot\Phi=\nu$ in $\mathcal{D}'(\Omega)$. Thanks to the third part of Lemma 6 and (12), we have

$$\int_B k|\nabla u|^{p-1} \le \liminf_{\varepsilon\to 0}\int_B k|\nabla u_\varepsilon|^{p-1} \le \int_B k^p,$$

for any Borel set $B\subseteq\Omega$. Thus $|\nabla u|\le k$, a.e. in Ω, and then $u\in K$. To prove (14), we see that using property (iii) of Φ and (12), we have

$$\begin{aligned}\limsup_{\varepsilon\to 0}\int_\Omega k|\phi_\varepsilon(.,\nabla u_\varepsilon)| &\le \limsup_{\varepsilon\to 0}\int_\Omega \phi_\varepsilon(.,\nabla u_\varepsilon)\cdot\nabla u_\varepsilon \\ &\le \limsup_{\varepsilon\to 0}\left(\int_\Omega u_\varepsilon\,\mathrm{d}\nu_\varepsilon - \lambda\int_\Omega u_\varepsilon^2\right) \\ &\le \int_\Omega u\,\mathrm{d}\nu - \lambda\int_\Omega u^2.\end{aligned} \tag{16}$$

In addition, we have

$$\int_\Omega u\mathrm{d}\nu = \lim_{\varepsilon\to 0}\int_\Omega u\mathrm{d}\nu_\varepsilon = \lim_{\varepsilon\to 0}\Big(\int_\Omega \phi_\varepsilon(.,\nabla u_\varepsilon)\cdot\nabla u + \lambda\int_\Omega u_\varepsilon u\Big)$$
$$\leq \liminf_{\varepsilon\to 0}\int_\Omega k|\phi_\varepsilon(.,\nabla u_\varepsilon)| + \lambda\int_\Omega u^2. \tag{17}$$

So, (16) and (17) implies that

$$\lim_{\varepsilon\to 0}\int_\Omega k|\phi_\varepsilon(.,\nabla u_\varepsilon)| = \int_\Omega u\mathrm{d}\nu - \lambda\int_\Omega u^2. \tag{18}$$

and, using (13), we get

$$\int k\mathrm{d}|\Phi| \leq \lim_{\varepsilon\to 0}\int_\Omega k|\phi_\varepsilon(.,\nabla u_\varepsilon)| = \int_\Omega u\mathrm{d}\nu - \lambda\int_\Omega u^2. \tag{19}$$

This ends up as the proof of (15). At last, using u_ε the approximation as given by Lemma 1, we see that

$$\begin{aligned}\int_\Omega u\mathrm{d}\nu - \lambda\int_\Omega u^2 &= \lim_{\varepsilon\to 0}\int_\Omega u_\varepsilon\mathrm{d}\nu - \lambda\int_\Omega u_\varepsilon u\\ &= \lim_{\varepsilon\to 0}\int_\Omega \nabla u_\varepsilon\cdot\frac{\Phi}{|\Phi|}\mathrm{d}|\Phi|\\ &\leq \int_\Omega k\mathrm{d}|\Phi|.\end{aligned}$$

Combining this with (15), we obtain

$$\int_\Omega k\mathrm{d}|\Phi| = \int_\Omega u\mathrm{d}\nu - \lambda\int_\Omega u^2,$$

so that, by using (14) and (18) follows. ■

Lemma 8. *Let $\nu\in\mathcal{M}_b(\Omega)$. If $\lambda\neq 0$, then the following assumptions are equivalent :*

1. *$v\in K$ and $\int_\Omega (v-z)\mathrm{d}\nu \geq \int_\Omega \lambda v(v-z)\mathrm{d}x$ for any $z\in K$.*
2. *For any $\nu_\varepsilon\in\mathcal{M}_b(\Omega)$ such that*

$$\nu_\varepsilon\to\nu \quad \textit{weakly in } \mathcal{M}_b(\Omega),$$

we have $v = \mathcal{C}_0(\Omega) - \lim_{\varepsilon\to 0} u_\varepsilon$, where u_ε is the solution of (S_ε).

Proof. Since $\lambda\neq 0$, then the proof is a simple consequence of Lemmas 3 and 7 and the uniqueness of v given by 1. ■

Lemma 9. *If u is a solution of (1), then there exists $\Phi\in\mathcal{M}_b(\Omega)$, such that (u,Φ) satisfies (3).*

Proof. Assume that u is a solution of (1); i.e. $u\in K$ and $\int_\Omega u\mathrm{d}\mu = \max_{\xi\in K}\int_\Omega u\mathrm{d}\mu$. Taking in Lemma 8 $\lambda = 1$, $\nu = \mu + u$ and $v = u$, we deduce that $u = \mathcal{C}_0(\Omega) - \lim_{\varepsilon\to 0} u_\varepsilon$, where u_ε is the solution of

$$\begin{cases} u_\varepsilon - \nabla\cdot\Phi_\varepsilon(x,\nabla u_\varepsilon) = \mu + u & \text{in } \mathcal{D}'(\Omega)\\ u_\varepsilon\in W_0^{1,p}(\Omega).\end{cases}$$

Then, using Lemma 7, the result of the lemma follows. ■

Proof of Theorem 1. Thanks to Lemmas 3 and 9, the first part of the theorem follows ; i.e. u is a solution of (5) if and only if there exists $\Phi\in\mathcal{M}_b(\Omega)^N$ such that (u,Φ) satisfies (3). The equivalence between (2)and (3) follows by Lemma 4. As a consequence of Lemma 5, we have (3) implies (4) and the fact that (4) implies (3) follows by definition of Φ in (4). To prove the last part of the theorem, let $\Phi\in\mathcal{M}_b(\Omega)^N$ be such that $-\nabla\cdot\Phi = \mu$ in $\mathcal{D}'(\Omega)$ and

$$\int_\Omega k\mathrm{d}|\Phi| = \min\left\{\int_\Omega k\mathrm{d}|\nu|\,;\; -\nabla\cdot\nu = \mu \text{ in } \mathcal{D}'(\Omega)\right\}.$$

Let us prove that, there exists $v\in K$ such that

$$\int_\Omega k\mathrm{d}|\Phi| = \int_\Omega v\mathrm{d}\mu.$$

To this end, let $\mu_\varepsilon\in W^{-1,p'}(\Omega)$ be such that

$$\mu_\varepsilon\to\mu \quad \text{weakly in } \mathcal{M}_b(\Omega)$$

and let us consider u_ε a solution of (S_ε) with $\lambda = 0$. Then, by definition of Φ in (4), we have

$$\int_\Omega k\mathrm{d}|\Phi| \le \int_\Omega k(x)|\Phi_\varepsilon(x, \nabla u_\varepsilon(x))|\mathrm{d}x.$$

So, using (14) and (15), we deduce that

$$\int_\Omega k\mathrm{d}|\Phi| \le \int_\Omega u\mathrm{d}\nu,$$

and the proof is complete. ∎

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