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A generalized collapsing sandpile model

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Abstract. In this paper, we introduce a new model for the collapsing sandpile and we prove existence and uniqueness of a solution for the corresponding initial value problem. Moreover, we prove the convergence of the time-stepping approximation of the solution. We use subgradient flows for variational problems with time dependent gradient constraints. These gradient constraints are interpreted as the critical angles of the sandpile. In particular, our model produces an evolution in time of avalanches in a drying of a sandpile, rather than instantaneous collapse.

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1. Introduction. It is well known that granular materials like sand, gravel or broken stones have an angle limit, the so-called angle of repose. It corresponds to the steepest angle which the surface of a mass of particles in bulk make with the ground. It is determined by the friction, the cohesion and the shapes of the particles, and it is affected by the moisture of particles. In a wet state, granular materials have generally a larger angle of repose than in their dry state. Indeed, the surface tension of water ties the seeds together, and the angle of repose is increased to more vertical. So, in drying the angle of repose becomes less vertical, avalanches occur and carry sand from the top to the bottom of the pile. We say that the sandpile collapses. This is a typical example of self-organized critical phenomena exhibited by driven systems which reach a critical state by their intrinsic dynamics. In [8], to describe the instanteneous collapse of the sandpile, the authors used a model based on the limit $p \to \infty$ of the p-Laplacian evolutions with "unstable" initial data. Roughly speaking, in their model the initial data corresponds to the initial profile of the sandpile and

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the limit as $p \to \infty$ of the solution corresponds to the profile of the sandpile when it collapses. Our approach here is different, we introduce and study a general model for the description of the collapse of a sandpile. Notice that rescaling the problem considered in [8] produces a particular situation of the model we are giving here (cf. Remark 1 and [7]).

Using mainly the angle of repose, several models were built for the study of the evolution of a pile of granular materials [1,4,9,13]. In [13] (see also [1]), the author used subgradient flow for variational problems with gradient constraint to model the sandpile growth. The gradient constraint is interpreted as a critical angle. The flow is modeled as a thin boundary layer moving down the slope of the growing pile. The dynamic is described by using the continuity equation in fluid dynamics and a phenomenological equation combining the stability angle and the fact that the sand flow is directed towards the steepest descent. For the collapse of the sandpile, we suggest to use a time-dependent stability angle. Indeed, assuming that the moisture of the material is changing in time, we can assume that the angle of repose is a given time dependent function. Since we are dealing with problems without external source and the avalanches occur only when the sand is more vertical than the angle of stability, then the case of nondecreasing (in time) repose angle is more interesting. We assume that the tangent of the repose angle is a given nondecreasing function $c : t \in [0, T) \to c(t) \in \mathbb{R}^+$ (additional comments are given in Remark 1 at the end of the paper). So, if $u(t)$ represents the height function of the profile at time t, then the critical slope (stability condition) is given by

$$|\nabla u(t)| \leq c(t). \tag{1}$$

If there exists a time $\tau > t$ such that $u(t)$ satisfies the instability condition

$$\|\nabla u(\tau)\|_\infty > c(\tau),$$

then (1) forces $u(t)$ to rearrange itself to achieve the profile in a stable configuration. To describe this process, we use the continuity equation

$$\partial_t u + \nabla \cdot \Phi = 0, \tag{2}$$

where $\Phi = \Phi(\nabla u)$ is the horizontal projection of the flux of the material. Since the surface flow is directed towards the steepest descent, then

$$\Phi = -m\nabla u,$$

where $m \geq 0$ is an unknown scalar function. Now, taking into account the critical slope and assuming that no dynamic occurs for pile surfaces which are inclined less, we get

$$\begin{cases} |\nabla u(t,x)| \leq c(t) \\ \text{and} \\ |\nabla u(x,t)| < c(t) \Longrightarrow m = 0. \end{cases}$$

So, if the initial free surface is given by

$$u(x, 0) = u_0$$

then, the height u of the pile satisfies the following nonlinear PDE

$$(P) \quad \begin{cases} \left.\begin{array}{l} u_t - \nabla \cdot (m\, \nabla u) = 0, \ m \geq 0 \\ \\ |\nabla u| \leq c(t), \ m\, (|\nabla u| - c(t)) = 0 \end{array}\right) & \text{in } Q := (0,T) \times \Omega \\ \\ u = 0 & \text{on } \Sigma := (0,T) \times \Gamma \\ \\ u(0) = u_0, & \end{cases}$$

where $m : Q \to \mathbb{R}^+$ is also unknown.

In the case where $c \equiv 1$, (P) corresponds to the well known Prigozhin model for the sandpile. Indeed, taking a nonnegative function (instead of 0) in the second member of the first equation, the model describes the growth of a sandpile with respect to an external source of sand (cf. [1,13]). Existence, uniqueness and numerical analysis are well known for the problem in that case (cf. [1,2,6]).

The collapsing sandpile problem was studied in [8] and [7] by using the p-Laplacian equation and letting $p \to \infty$. This approach gives an instantaneous study of the collapse of a sandpile. The model (P) is a generalization of [8] and [7] (see Remark 1) and can describe the succession of avalanches in any given scaling of time.

Our aim is to prove existence and uniqueness of a solution for (P). Moreover, we prove the convergence of the approximation of the solution of (P) by Euler implicit time discretization. Recall that the main interest of the approximation by Euler implicit time discretization remains on its application to numerical analysis (cf. [7]). Indeed, it transforms the problem into projections on convex sets, so that one can use the numerical algorithms introduced in [6] for numerical simulation.

In the next section, we set and prove our main result of existence, uniqueness and convergence of Euler implicit time discretization.

2. Main results and proofs. We assume that

$$c \in W^{1,\infty}(0,T) \quad \text{and} \quad \min_{t \in (0,T)} c(t) =: \delta > 0. \tag{3}$$

For a given $[b,a]$ compact interval of $[0,\infty)$, we say that $(d_i)_{i=0}^n$ is an ε-discretization of $[b,a]$ provided

$$\left(\begin{array}{l} \varepsilon = \varepsilon(n), \ \lim_{n\to\infty} \varepsilon(n) = 0, \ |d_i - d_{i-1}| \leq \varepsilon, \quad \text{for any } i = 1, \ldots n \\ \\ \text{and } d_0 = b < d_1 < d_2 < \cdots < d_n = a. \end{array} \right.$$

For a given $\varepsilon > 0$, we say that $(t_i)_{i=0}^n$ is an ε-discretization of $[0,T)$, provided $\varepsilon = \varepsilon(n)$, $\lim_{n\to\infty} \varepsilon(n) = 0$, $t_0 = 0 < t_1 < t_2 < \cdots < t_n = T$ and $t_i - t_{i-1} = \varepsilon$, for any $i = 0, ...n$. For any $d > 0$, we denote by $K(d)$ the convex set given by

$$K(d) = \left\{ z \in W^{1,\infty}(\Omega) \cap W_0^{1,1}(\Omega);\ |\nabla z| \leq d \right\} \quad \text{a.e. in } \Omega.$$

We say that u_ε is an ε-approximate solution of (P) in $[0,T)$, if there exist $(t_i)_{i=0,\ldots n}$ and an ε-discretization of $[0,T)$, such that

$$u_\varepsilon(t) = u_i \quad \text{for } \ t \in [t_i, t_{i+1}[, \ \ i = 0,\ldots n-1 \tag{4}$$

and, for $i = 1,\ldots,n$, the u_i solve the Euler implicit time discretization of (P)

$$\begin{cases} \left.\begin{array}{l} u_i - \nabla\cdot(m_i\,\nabla u_i) = u_{i-1}, \ m_i \geq 0 \\ \\ |\nabla u_i| \leq c(t_i), \ m_i\,(|\nabla u_i| - c(t_i)) = 0 \end{array}\right| \quad \text{in } \Omega \\ \\ u_i = 0 \quad \text{ on } \partial\Omega. \end{cases} \tag{5}$$

So, the generic problem for the study of (P) is

$$\begin{cases} \left.\begin{array}{l} v - \nabla\cdot(m\,\nabla v) = g, \ \Phi = |\Phi|\,\nabla v \\ \\ |\nabla v| \leq d, \ |\Phi|\,(|\nabla u| - d) = 0 \end{array}\right| \quad \text{in } \Omega \\ \\ v = 0 \quad \text{ on } \partial\Omega, \end{cases} \tag{6}$$

where g (resp. $d > 0$) is a given function (resp. real parameter).

It is well known by now (see for instance [10]), that for any $g \in L^2(\Omega)$, (6) has a unique solution v given by

$$v = I\ \mathbb{P}_K(d),$$

where $\mathbb{P}_{K(d)}$ denotes the projection onto the convex $K(d)$, with respect to the $L^2(\Omega)$ norm. The connection between $v = \ \mathbb{P}_K(d)$ and the formulation in divergence form (6) is more or less well known by now; for more details in this direction we refer the reader to [10] and the references therein.

Recall also that, $\mathbb{P}_{K(d)} = (I + \partial\mathbb{I}_{K(d)})^{-1}$, where $\partial\mathbb{I}_{K(d)}$ is the sub-differential of the indicator function of $K(d)$ given in $L^2(\Omega)$ by

$$v \in \partial\mathbb{I}_K(g) \quad \text{if and only if} \quad \int_\Omega v(z-g) \leq 0 \quad \text{for any } z \in K.$$

In particular, this gives rise to the concept of solutions for problem (P).

Definition 1. For a given $u_0 \in K(c(0))$, we say that u is a solution of (P) provided $u \in W^{1,1}(0,T; L^2(\Omega))$, $u(0) = u_0$ and

$$-u_t(t) \in \partial\mathbb{I}_{K(c(t))}(u(t)) \quad \text{for any } t \in (0,T).$$

Our main result is

Theorem 1. *Let $u_0 \in K(c(0))$ and $0 < T < \infty$. Then (P) has a unique solution u and for any sequence $\varepsilon \to 0$, if u_ε is an ε-approximate solution, then*

$$u_\varepsilon \to u \quad \text{in } \mathcal{C}([0,T); L^2(\Omega)), \quad \text{as } \varepsilon \to 0.$$

Moreover, if for $i = 1,\ 2$ u_i is the solution corresponding to u_{0i}, then

$$\int_\Omega (u_1 - u_2)^+ \leq \int_\Omega (u_{01} - u_{02})^+ \quad \text{in } \mathcal{D}'(0,T)$$

In particular, if $u_0 \geq 0$, then $u \geq 0$ a.e. in Ω.

Proof. It is not difficult to see that u is a solution of (P) if and only if $v(t) = u(t)/c(t)$ is a solution of

$$\begin{cases} v_t(t) + \partial\mathbb{I}_{K(1)}\ (v(t)) \ni f(t) & \text{a.e. } t \in (0,T) \\ \\ v(0) = u_0/c(0), \end{cases} \tag{7}$$

with $f(t) = -\dfrac{c'(t)}{c(t)}\ v(t)$. Thanks to [5, Proposition 3.13], and (3), for any $v_0 \in K(1)$, (7) has a unique solution $v \in W^{1,\infty}(0,T;L^2(\Omega))$, $u(0) = u_0$ and

$$f(t) - v_t(t) \in \partial\mathbb{I}_{K(1)}(v(t)) \quad \text{ for any } t \in (0,T).$$

This ends up the proof of the existence of a solution of (P). To prove the convergence of the ε-approximate solution, let us consider, for $i = 1, \ldots n$,

$$u_i = \mathbb{P}_{K(c(t_i))} u_{i-1},$$

i.e.,

$$u_i + \partial\mathbb{I}_{K(c(t_i)}(u_i) \ni u_{i-1} \qquad \text{for } i = 1, \ldots, n.$$

Setting, for $i = 0, \ldots.n$,

$$z_i = u_i/c(t_i),$$

it is not difficult to see that

$$z_i = \mathbb{P}_{K(1)} \left(\frac{c(t_{i-1})}{c(t_i)} z_{i-1} \right) \quad \text{ for } \quad i = 1, \ldots, n. \tag{8}$$

Now, let us consider the Euler implicit discretization in time associated with (7)

$$v_i + \partial\mathbb{I}_{K(1)}(v_i) \ni v_{i-1} - \frac{c(t_i) - c(t_{i-1})}{c(t_i)}\ v(t_{i-1}), \quad i = 1, 2, \ldots, n, \tag{9}$$

where, we take as an approximation of f its discretization in $[0,t)$,

$$f_i = \frac{c(t_i) - c(t_{i-1})}{t_i - t_{i-1}}\ \frac{v(t_{i-1})}{c(t_i)}, \quad \text{ for } \quad i = 1, \ldots, n.$$

Defining v_ε by $v_\varepsilon(t) = v_i$ for $t \in [t_i, t_{i+1}[$ and for $i = 0, 1, \ldots n-1$, we know by standard theory of nonlinear semigroups (see for instance [3, Theorem 4.6],) that

$$v_\varepsilon \to v \quad \text{ in } \quad \mathcal{C}([0,T); L^2(\Omega)), \text{ as } \varepsilon \to 0, \tag{10}$$

where v is the solution of (7). It is not difficult to see that (9) is equivalent to

$$v_i = \mathbb{P}_{K(1)} \left(\frac{c(t_{i-1})}{c(t_i)}\ v_{i-1} - \frac{c(t_i) - c(t_{i-1})}{c(t_i)}\ (v(t_{i-1}) - v_{i-1}) \right),$$

so that, by using (8) and the L^2-contraction property of $\mathbb{P}_{K(1)}$, for $i = 1, \ldots n$, we get

$$\|v_i - z_i\|_2 \leq \frac{c(t_{i-1})}{c(t_i)}\ \|v_{i-1} - z_{i-1}\|_2 + \left| \frac{c(t_i) - c(t_{i-1})}{c(t_i)} \right|\ \|v(t_{i-1}) - v_{i-1}\|_2. \tag{11}$$

Since $v_0 = z_0 = u_0/c(0)$, iterating (11) for $i = k, \dots 1$, we obtain

$$\begin{aligned}\|v_k - z_k\|_2 &\le \sum_{i=1}^{k} \left| \frac{c(t_{k-i+1}) - c(t_{k-i})}{c(t_k)} \right| \, \|v(t_{k-i}) - v_{k-i}\|_2 \\ &\le \frac{1}{\delta} \sum_{i=1}^{k} |c(t_{k-i+1}) - c(t_{k-i})| \, \|v(t_{k-i}) - v_{k-i}\|_2 \\ &\le \frac{\|c'\|_\infty}{\delta} \sum_{i=0}^{k-1} (t_{k-i+1} - t_{k-i}) \, \|v(t_{k-i}) - v_{k-i}\|_2.\end{aligned}$$

Considering $\overline{v}_\varepsilon$ given by $\overline{v}_\varepsilon(t) = v(t_i)$ for $t \in [t_i, t_{i+1}[$ and $i = 0, 1, \dots, n-1$, we obtain

$$\begin{aligned}\|v_k - z_k\|_2 &\le \frac{\|c'\|_\infty}{\delta} \sum_{i=0}^{k-1} \int_{t_i}^{t_{i+1}} \|\overline{v}_\varepsilon(t) - v_\varepsilon(t)\|_2 \, dt \\ &\le \frac{\|c'\|_\infty}{\delta} \int_0^T \|\overline{v}_\varepsilon(t) - v_\varepsilon(t)\|_2 \, dt\end{aligned}$$

and

$$\|v_\varepsilon(t) - z_\varepsilon(t)\|_2 \le \frac{\|c'\|_\infty}{\delta} \int_0^T \|\overline{v}_\varepsilon(t) - v_\varepsilon(t)\|_2 \, dt, \qquad \text{for any } t \in [0, T).$$

Since, as $\varepsilon \to 0$, $\overline{v}_\varepsilon \to v$ in $\mathcal{C}([0,T); L^2(\Omega))$, then

$$\lim_{\varepsilon \to 0} \sup_{t \in [0,T)} \|v_\varepsilon(t) - z_\varepsilon(t)\|_2 = 0. \tag{12}$$

Now, let us consider c_ε defined by $c_\varepsilon(t) = c(t_i)$ for $t \in [t_i, t_{i+1}[$ and $i = 0, 1, \dots n-1$. It is clear that

$$\begin{aligned}\|u(t) - u_\varepsilon(t)\|_2 &= c_\varepsilon(t) \left\| \frac{u(t)}{c_\varepsilon(t)} - z_\varepsilon(t) \right\|_2 \\ &\le \|c\|_\infty \left(\left\| \frac{u(t)}{c_\varepsilon(t)} - v(t) \right\|_2 + \|v(t) - v_\varepsilon(t)\|_2 + \|v_\varepsilon(t) - z_\varepsilon(t)\|_2 \right).\end{aligned}$$

So, combining (12) with the fact that $v_\varepsilon \to v$ in $\mathcal{C}([0,T); L^2(\Omega))$ and $c_\varepsilon \to c$ in $\mathcal{C}([0,T))$, we deduce that

$$\lim_{\varepsilon \to 0} \sup_{t \in [0,T)} \|u(t) - u_\varepsilon(t)\|_2 = 0$$

and the second part the theorem follows. At last, the contraction property is a consequence of nonlinear semigroup theory and the fact that the operator $\partial \mathbb{I}_K$ is a preserving order maximal monotone graph in $L^2(\Omega)$. □

Remarks 1. 1. In this paper, we study the collapsing problem with an arbitrary time dependent angle of repose. The exact value of $c(t)$ is closely connected to the time stepping realization of the avalanches in the concret stituation. In [7], the authors rescale the model proposed by [8] and produce a particular situation of our model here. In this case c is given by (cf. [7])

$$c(t) = \frac{1}{t + \|\nabla u_0\|^{-1}_{L^\infty(\Omega)}}$$

and $T = 1 - \|\nabla u_0\|^{-1}_{L^\infty(\Omega)}$. Another more realistic value for $c(t)$ is given in [11] by using a stochastic description of the collapse of piles of cubes. In [11], the value of c is obtained by assuming that the times of realization of the avalanches are random variables independent and identically exponentially distributed with a given constant mean.

2. Let u_0 be the profile of an initial unstable sandpile. Assume for instance that the gradient constraint of stable sandpile is equal to 1. Then, the collapse of u_0 may be described by the initial value problem (P) with a given nonnegative $c \in W^{1,\infty}(0,T)$, such that $\|\nabla u_0\|_{L^\infty(\Omega)} \leq c(0)$ and $c(T) = 1$. It would be interesting to know if the final profile $u(1)$ (the stable profile associated with u_0) depends on the value of c in $(0,T)$ or not.
3. For numerical simulations, we see that the time-stepping approximations of the solution of (P) transform the problems into a sequence of projections on convex sets. More precisely, denoting by u the solution of (P) and using Theorem 1, it is not difficult to see that the characterization of the final profile $u(T)$ is given by

$$u(T) = \lim_{n\to\infty} \mathbb{P}_{K(c(T))}\mathbb{P}_{K(c(t_{n-1}))} \dots \mathbb{P}_{K(c(1))}\mathbb{P}_{K(c(0))}\, u_0, \tag{13}$$

where $\left(t_i\right)_{i=0}^{n}$ is an ε-discretization of $[0,T)$. For more details in this direction we refer the readers to [7].

References

[1] G. Aronson, L.C. Evans, and Y. Wu, Fast/Slow diffusion and growing sandpiles, J. Differential Equations, **131** (1996), 304–335.

[2] J. W. Barrett and L. Prigozhin, Dual formulation in Critical State Problems, Interfaces and Free Boundaries **8** (2009), 349–370.

[3] P. H. Benilan, M. G. Crandall, and A. Pazy, Evolution Equations Governed by Accretive Operators, Book, not published.

[4] J. P. Bouchaud et al., A model for the Dynamic of Sandpile Surfaces, J. Phys. I France **4** (1994), 1383–1410.

[5] H. Brézis, Opérateurs maximaux monotones et semigroups de contractions dans les espaces de Hilbert, Math. studies 5, North Holland (1973).

[6] S. Dumont and N. Igbida, Back on a Dual Formulation for the Growing Sandpile Problem, European J. Appl. Math. **20** (2009), 169–185.

[7] S. Dumont and N. Igbida, On the collapsing sandpile problem, submitted (2008).

[8] L. C. Evans, M. Feldman, and R. F. Gariepy, Fast/Slow diffusion and collapsing sandpiles, J. Differential Equations **137** (1997), 166–209.

[9] L. C. EVANS AND F. REZAKHANLOU, A stochastic model for sandpiles and its continuum limit, Comm. Math. Phys. **197** (1998), 325–345.

[10] N. IGBIDA, Evolution Monge-Kantorovich equation, submitted (2008).

[11] N. IGBIDA, Stochastic Model for Collapsing Sandpiles, Preprint (2009).

[12] N. IGBIDA, Equivalent Formulations for Monge-Kantorovich Equation, Nonlinear Anal. TMA **71** (2009), 3805–3813.

[13] L. PRIGOZHIN, Variational model of sandpile growth, Euro. J. Appl. Math. **7** (1996), 225–236.

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