Evolution Monge-Kantorovich Equation

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Abstract

The paper deals with Monge-Kantorovich equation (MK for short) in an open bounded domain Ω with Dirichlet boundary condition. We study existence and uniqueness of a solution to the associated evolution problem (EMK for short) and we prove the convergence to a solution of MK, when time goes to ∞. A solution is a couple (u, Φ), where u is the potential and Φ is the transportation flux. We study the problem for a given Radon measure source term and we show how to use the numerical method of [25] to provide numerical approximation of the solution of MK.

1 Introduction

The original optimal mass transport problem (which goes back to Monge in 1781 in [36]) consists in minimizing

$$\int_\Omega |x - t(x)|\, d\mu^+(x) \tag{1}$$

among admissible transports t, which are measurable functions $t\ :\ \mathrm{spt}(\mu^+) \to \mathrm{spt}(\mu^-)$ such that $t_\# \mu^+ = \mu^-$; i.e. $\mu^-(B) = \mu^+(t^{-1}(B))$, for any measurable set B of $\mathbb{R}^N$ with $N \geq 1$. Here $|.|$ denotes the Euclidean norm of $\mathbb{R}^N$, μ^+ and μ^- are respectively the positive and negative part of a Radon measure μ (satisfying $\mu(\mathbb{R}^N) = 0$ in the original Monge problem). The problem was reformulated by Kantorovich in 1942 into a relaxed variational formulation (the so called Monge-Kantorovich problem : see for instance [27] and [2] for a complete survey on the problem) and got a great variety of applications. It was generalized in many different ways.

Existence of an optimal transport t is a very delicate problem that was solved in the last decade (see [28], [15], [3], [18], [19] and the references therein). An interesting tool to fashion an optimal transport map t is the so called Monge-Kantorovich equation (the MK equation as called by Bouchitté, Buttazzo and Seppecher in [10]) :

$$\begin{cases} -\nabla \cdot \Phi = \mu & \text{in } \mathcal{D}'(\mathbb{R}^N) \\ \\ |\nabla u| \leq 1 \ \text{ and } \ \Phi = |\Phi|\, \nabla_{|\Phi|} u. \end{cases} \tag{2}$$

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Here $|\Phi|$ denotes the total variation measure of the $\mathbb{R}^N-$valued measure Φ and $\nabla_{|\Phi|}u$ denotes the tangential gradient with respect to $|\Phi|$ (cf. [10]). Indeed, this strongly nonlinear pde contains all the information concerning Monge optimal mass transportation problem. Actually, it is known (cf. [28], [2], [27] and the references therein) that a function u solving (2) together with Φ are meaningful in the context of transport problem (1). Indeed, the vector $-\Phi$ provides the direction of the optimal transportation and the quantity $|\Phi|$ gives the density of the transport. Recall that we are using the tangential gradient of u because Φ is a Radon measure and u is only Lipschitz in general. We refer the reader to the papers [22], [23] and [24] for more details concerning the regularity of the solutions of (2).

In addition to the interesting formulation of Monge problem in terms of nonlinear PDE, the equation (2) is closely connected to *Beckmann's minimal flow problem* (cf. [7]), that Beckman himself called in the '50s "a continuous model of transportation". Roughly speaking, in an urban area where μ^+ and μ^- represent the distributions of residents and services in the city respectively, we can model the consumers traffic by a traffic flow field Φ. In this situation, the equation

$$-\nabla\cdot\Phi = \mu \qquad \text{in } \mathcal{D}'(\mathbb{R}^N) \tag{3}$$

gives the relationship between the excess demand and the traffic flow. Beckmann's problem aims to find Φ with minimal total variation among those satisfying the equation (3). It is well known by now that such vector field is given by Φ satisfying (2) (see [20] and [31] and the references therein for the equivalence between both formulations). Recall that, while classical Monge problem is stated in $\mathbb{R}^N$, classical Beckmann's problem is stated in a bounded domain with appropriate boundary condition on Φ. In this paper, we'll consider the equation (2) in a bounded domain large enough such that Dirichlet boundary condition would be enough to describe many concrete situations.

Let $\Omega\subset\mathbb{R}^N$ be a bounded open domain. Our main interest in this paper is to study the evolution problem associated with (2) in Ω with Dirichlet boundary condition. That is

$$(P_\mu)\qquad \begin{cases} \partial_t u(t) - \nabla\cdot(\Phi(t)) = \mu(t) & in\quad \Omega,\ \text{for } t\in(0,T)\\[2mm] |\nabla u(t)|\le 1 \ \text{ and } \ \Phi(t) = |\Phi(t)|\,\nabla_{|\Phi(t)|}u(t) & in\quad \Omega,\ \text{for } t\in(0,T)\\[2mm] u=0 & on\quad \Sigma := (0,T)\times\Gamma\\[2mm] u(0)=u_0 & in\quad \Omega, \end{cases}$$

where $\partial_t u$ denotes the partial derivative with respect to t, $u_0\in W^{1,\infty}(\Omega)\cap H^1_0(\Omega)$, $\|\nabla u_0\|_\infty\le 1$, and μ is a bounded Radon measure such that $t\in(0,T)\to\mu(t)\in\mathcal{M}_b(\Omega)$ is an L^1 weakly measurable map into $\mathcal{M}_b(\Omega)$ (the set of bounded Radon measures concentrated in Ω) ; i.e. $\mu\in L^1(0,T;w^*-\mathcal{M}_b(\Omega)))$. In addition to its interest for the study of the sandpile (cf. [37], [27] and [29]) and in the study of mass optimization problem (cf. [9] and [11]), a particular interest of the evolution equation (P_μ) remains in the numerical approximation of the flux Φ of (2). Indeed, the numerical approximation of the transport flux Φ of the problem (2) is difficult because the numerical instabilities. A possible approach is to approximate it by the optimal evolutionary flux, for which stable numerical method are develop recently (see [5], [6] and [25]). We prove the existence of a solution (u, Φ) and the uniqueness of the potential u. Moreover, we prove that as $t \to \infty$ the solution $(u(t), \Phi(t))$ converges to a solution of the stationary problem (2). At last, we show how to use the numerical method of [25] to provide a numerical approximation of a solution (u, Φ) of (2).

Like for the stationary problem (2), the main difficulty in the study of (P_μ) is connected with the regularity of Φ. To to handle this difficulty, we use essentially a weak formulation based on a characterization of the total variation of the flux Φ. The heuristic is the following. One could first look for a couple (u, Φ), with Φ in the space of vector valued Radon measures, which solves

$$\begin{cases} \partial_t u(t) - \nabla \cdot (\Phi(t)) = \mu(t) & in \quad \Omega, \text{ for } t \in (0,T) \\ u = 0 & on \quad \Sigma \\ u(0) = u_0 & in \quad \Omega. \end{cases} \tag{4}$$

By the bound on the norm of ∇u and the first equation of (4) we have

$$|\Phi|(Q) \geq \iint_Q \nabla u \cdot \Phi = \iint_Q u d\mu - \frac{1}{2}\frac{d}{dt}\iint_Q u^2\, dxdt.$$

And, then proving the opposite inequality

$$|\Phi|(Q) \leq \iint_Q u d\mu - \frac{1}{2}\frac{d}{dt}\iint_Q u^2\, dxdt,$$

would permits to write $\Phi = \nabla u\, |\Phi|$. Of course, the gradient of u in the last equation needs to be handle in a right way.

In the following section, we begin by giving some preliminaries and notations that will be used throughout the paper. Then, we summarize our main results and we show formally how to use the numerical method of [25] to provide a numerical approximation of the solution $(u(t), \Phi(t))$ of (P_μ). Taking t large enough in (P_μ), we give some numerical approximation of the transport flux Φ solution of (2). Section 3, is devoted to the proofs. For the uniqueness of the potential we use doubling and dedoubling variables technics. As for the existence, we consider the $p-$Laplacien evolution equation and we let $p \to \infty$.

2 Preliminaries and main results

2.1 Preliminaries and notations

Let us begin with some preliminaries concerning $\mathbb{R}^N$-valued Radon measure and notations that we use in this paper (for more details one can see for instance [40]). Throughout the paper $\Omega \subset \mathbb{R}^N$ is a bounded domain with Lipschitz boundary. We denote by $\mathcal{L}^N$ the $N-$dimensional Lebesgue measure of $\mathbb{R}^N$. For $1 \leq p < +\infty$, $L^p(\Omega)$, $W^{1,p}(\Omega)$ and $W_0^{1,p}(\Omega)$ denote respectively, with respect to $\mathcal{L}^N$, the standard Lebesgue space, Sobolev space and the closure of $\mathcal{D}(\Omega)$ in $W^{1,p}(\Omega)$. We denote by $\mathcal{M}(\Omega)$ (resp. $\mathcal{M}_b(\Omega)$) the space of all Radon measures in Ω (resp. with bounded total variation $|\mu|(\Omega)$). For any $\Phi \in \mathcal{M}_b(\Omega)^N$ the total variation measure associated with Φ, denoted again by $|\Phi|$, is defined by

$$|\Phi|(B) = \sup\left\{\sum_{i=1}^{\infty} |\Phi(B_i)| \ ; \ B = \cup_{i=1}^{\infty} B_i, \ B_i \text{ a Borelean set}\right\}$$

and belongs to $\mathcal{M}^+(\Omega)$ (the set of nonnegative Radon measure). For any $\Phi \in \mathcal{M}_b(\Omega)^N$, we use polar factorization $\dfrac{\Phi}{|\Phi|}$ of Φ, to denote the density of Φ with respect to $|\Phi|$. So, every $\Phi \in \mathcal{M}_b(\Omega)^N$ can be identified with linear application $\eta \in \mathcal{C}_c(\Omega)^N \to \displaystyle\int_\Omega \frac{\Phi}{|\Phi|} \cdot \eta \, d|\Phi|$. To simplify the presentation we'll use the notation

$$\int_\Omega \eta \, d\Phi := \int_\Omega \frac{\Phi}{|\Phi|} \cdot \eta \ d|\Phi|, \text{ for any } \eta \in \mathcal{C}_c(\Omega)^N.$$

For any $\Phi \in \mathcal{M}_b(\Omega)^N$ and $\nu \in \mathcal{M}_b(\Omega)$, we say that $-\nabla \cdot \Phi = \nu$ in $\mathcal{D}'(\Omega)$ provided

$$\int_\Omega \nabla \xi \, d\Phi = \int_\Omega \xi \ d\nu \quad \text{ for any } \xi \in \mathcal{D}(\Omega) \ .$$

Since $\mathcal{M}_b(\Omega) = (\mathcal{C}_c(\Omega))^*$ and $\mathcal{C}_c(\Omega)$ is separable, then, for a given $T > 0$, any weak*-measurable function $\psi \,:\, (0,T) \to \mathcal{M}_b(\Omega)$ is such that $t \in (0,T) \to |\psi(t)|(\Omega)$ is measurable (see [21]). So, for any $1 \leq q \leq \infty$, we define

$$L^q(0,T; w^* - \mathcal{M}_b(\Omega)) = \Big\{\psi \ : \ (0,T) \to \mathcal{M}_b(\Omega) \ \text{ weak}^*\text{-measurable} \ ; \ \int_0^T |\psi(t)|^q(\Omega) \, dt < \infty\Big\}.$$

Recall that the space $L^q(0,T; w^* - \mathcal{M}_b(\Omega))$ equipped with the norm

$$\|\psi\|_{L^q(0,T;w^*-\ \mathcal{M}_b(\Omega))} = \begin{cases} \left(\displaystyle\int_0^T (|\psi(t)|(\Omega))^q \, dt\right)^{\frac{1}{q}} & \text{if } q < \infty \\ \\ \text{ess-sup}_{t\in(0,T)} |\psi(t)|(\Omega) & \text{if } q = \infty \end{cases}$$

is a Banach space. If $q > 1$, then (cf. [17]) $L^q(0,T; w^* - \mathcal{M}_b(\Omega))$ can be identified with $\Big(L^{q'}(0,T; \mathcal{C}_0(\Omega))\Big)^*$ the dual space of $L^{q'}(0,T; \mathcal{C}_0(\Omega))$, where $q' = \dfrac{q}{q-1}$. The identification is given by the application

$$\mathcal{I} \ : \ L^q(0,T; w^* - \mathcal{M}_b(\Omega)) \to \Big(L^{q'}(0,T; \mathcal{C}_0(\Omega))\Big)^* \quad \text{ with } \quad \mathcal{I}(\mu)(\xi) = \int_0^T \int_\Omega \xi(t) \, d\mu(t).$$

The set $BV(0,T; w^* - \mathcal{M}_b(\Omega))$ is the subspace of $L^1(0,T; w^* - \mathcal{M}_b(\Omega))$ defined by $\mu \in BV(0,T; w^* - \mathcal{M}_b(\Omega))$ if and only if $\mu \in L^1(0,T; w^* - \mathcal{M}_b(\Omega))$ and

$$V(\mu, T) := \limsup_{h\to 0} \frac{1}{h} \int_0^{T-h} |\mu(\tau + h) - \mu(\tau)|(\Omega) \, d\tau \ < \ \infty\Big\}.$$

If $\mu \in BV(0,T; w^* - \mathcal{M}_b(\Omega))$, then it is essentially bounded and has an essential limit from the right, denoted by $\mu(t+)$, for every $t \in [0,T)$. We also use the notation

$$V(\mu, t+) = \limsup_{h \to 0} \frac{1}{h} \int_0^t |\mu(\tau+h) - \mu(\tau)|(\Omega)\, d\tau \quad \text{ for } 0 \leq t < T.$$

To end up these preliminaries, we recall the following result that follows from [10] (a detailled proof can be found in [31]).

Lemma 2.1 *Let $v \in K$ and $\Phi \in \mathcal{M}_b(\Omega)^N$ such that $-\nabla \cdot \Phi =: \nu \in \mathcal{M}_b(\Omega)$. Then,*

$$\Phi = |\Phi|\, \nabla_{|\Phi|} v \quad \textit{if and only if} \quad |\Phi|(\Omega) \leq \int_\Omega v\, d\nu.$$

To simplify the notation, our integrals are with respect to dt, dx or $dtdx$ over $(0,T)$, Ω or $Q := (0,T) \times \Omega$ respectively, when omitted, unless otherwise indicated. Moreover, for any $\mu \in L^q(0,T; w^* - \mathcal{M}_b(\Omega))$, we denote by

$$\iint_Q \xi\, d\mu := \int_0^T \int_\Omega \xi(t)\, d\mu(t), \quad \text{ for any } \xi \in L^{q'}(0,T; \mathcal{C}_0(\Omega)).$$

Throughout this section, $\Omega \subset \mathbb{R}^N$ is a bounded domain with $\mathcal{C}^1$ boundary Γ, $0 < T < \infty$, $Q = (0,T) \times \Omega$, $\Sigma = (0,T) \times \Gamma$. We denote by K the convex set given by

$$K = \Big\{ z \in W^{1,\infty}(\Omega) \cap H_0^1(\Omega) \ ; \ |\nabla z| \leq 1 \ \ \mathcal{L}^N - \text{ a.e. in } \Omega \Big\}$$

and

$$K_T = \Big\{ z \in \mathcal{C}([0,T); L^2(\Omega)) \cap L^2(0,T; H_0^1(\Omega)) \ ; \ z(t) \in K \text{ for any } t \in [0,T) \Big\}.$$

It is not difficult to see that

$$K_T \subset \cap_{q \geq 1} L^\infty(0,T; W_0^{1,q}(\Omega)) \quad \text{ and } \quad K_T \subset L^\infty(0,T; \mathcal{C}_0(\Omega)).$$

So, for any $u \in K_T$ and $\mu \in L^1(0,T; w^* - \mathcal{M}_b(\Omega))$ the quantity $\iint_Q u\, d\mu$ is well defined.

Recall that (see for instance [31] and the references therein) a solution u of (2) maybe given by

$$\mu \in \partial I\!\!I_K(u),$$

where K is the set of $1-$Lipchitz continuous function u null on the boundary of Ω. Here, $\partial I\!\!I_K$ is the usual sub-differential operator of the indicator function of K, defined in $L^2(\Omega)$, by

$$I\!\!I_K(u) = \begin{cases} 0 & \text{if } u \in K \\ +\infty & \text{otherwise.} \end{cases}$$

So, one expect the equivalence between a solution u of (P_μ) and the solution of

$$(5) \qquad \begin{cases} \partial_t u + \partial I\!\!I_K(u) \ni \mu & \text{in } (0,T) \\ u(0) = u_0. \end{cases}$$

2.2 Existence, uniqueness and large time behavior

This formal transformation of (P_μ) into (5) gives rise rather the notion of variational solution for (P_μ). Existence and uniqueness of u solving (5) (variational solution) is more or less well known by now for regular data. Indeed, (5) is an evolution problem governed by a sub-differential operator. So, by using classical results from nonlinear semigroup theory, existence and uniqueness of a solution u of (5) for any L^p source term μ, with $1 \leq p \leq \infty$, holds to be true (see [14] for $p = 2$ and [13] for $p \neq 2$). For general Radon measure source μ we have the following result

Theorem 2.2 *For any $u_0 \in K$ and $\mu \in L^1(0,T; w^* - \mathcal{M}_b(\Omega))$, (5) has a unique variational solution u ; i.e. $u \in K_T$, $u(0) = u_0$ and for any $\xi \in K$*

$$\text{(6)} \qquad \frac{1}{2}\frac{d}{dt}\int_\Omega |u(t) - \xi|^2 \leq \int_\Omega (u(t) - \xi)\, d\mu(t) \quad \text{ in } \mathcal{D}'(0,T).$$

Moreover, if u_i is a variational solution of (P_{μ_i}), for $i \in \{1,\ 2\}$, then

$$\text{(7)} \qquad \frac{d}{dt}\int_\Omega |u_1(t) - u_2(t)| \leq |\mu_1(t) - \mu_2(t)|(\Omega) \quad \text{ in } \mathcal{D}'(0,T)$$

and

$$\text{(8)} \qquad \frac{1}{2}\frac{d}{dt}\int_\Omega |u_1(t) - u_2(t)|^2 \leq \int_\Omega (u_1(t) - u_2(t))\, d(\mu_1(t) - \mu_2(t)) \quad \text{ in } \mathcal{D}'(0,T).$$

Recall that, particular situation $\mu = \sum_{k=1}^{m} f_k(t)\delta_{d_k}$, where δ_{d_k} denotes the Dirac mass at the point d_k and the source function f_k is nonnegative ($k = 1,..,m$), was studied in [4] (see also [38]). The authors show existence and uniqueness of a variational solution by letting $p \to \infty$ in the p-Laplacian equation :

$$\text{(9)} \qquad \begin{cases} \partial_t u - \Delta_p u_p = \mu & \text{in} \quad (0,\infty) \times \mathbb{R}^N \\ \\ u(0) = u_0. \end{cases}$$

Now, having in mind the roles of the flux Φ for concrete situations like Monge problem, mass optimization and sandpile, we focus our attention on the formulation of the solution u of (P_μ) with a flux Φ for any given source term $\mu \in L^1(0,T, w^* - \mathcal{M}_b(\Omega))$. The following theorem provide the right weak formulations in divergence form of the solution of (P_μ).

Theorem 2.3 *Let $u_0 \in K$ and $\mu \in BV(0,T; w^* - \mathcal{M}_b(\Omega))$. Then, u is the variational solution of (5) if and only if $u \in \mathcal{C}([0,T); L^2(\Omega)) \cap K_T$, $u(0) = u_0$, $\partial_t u \in L^\infty(0,T; w^* - \mathcal{M}_b(\Omega))$ and there exists $\Phi \in L^\infty(0,T; w^* - \mathcal{M}_b(\Omega))$, such that $\partial_t u - \nabla \cdot \Phi = \mu$ in $\mathcal{D}'(Q)$ and Φ satisfies one of* *the following equivalent formulations :*

$$\Phi(t) = |\Phi(t)| \, \nabla_{|\Phi(t)|} u(t) \quad \mathcal{L}^1 - a.e. \ t \in (0,T) \tag{10}$$

$$|\Phi(t)|(\Omega) \leq \int_\Omega u(t)\, d\mu(t) - \frac{1}{2}\frac{d}{dt}\int_\Omega u^2(t) \quad \mathcal{L}^1 - a.e. \ t \in (0,T) \tag{11}$$

$$|\Phi|(Q) \leq \iint_Q u\, d\mu - \frac{1}{2}\int_\Omega u(T)^2 + \frac{1}{2}\int_\Omega u_0^2. \tag{12}$$

Using the fact that a solution u of (P_μ) is such that $|\nabla u(t)| \leq 1$ $\mathcal{L}^N$-a.e. in Ω for $\mathcal{L}^1$-a.e. $t \in (0,T)$, one can prove that (11) (resp. (12)) is equivalent to

$$|\Phi(t)|(\Omega) = \int_\Omega u(t)\, d\mu(t) - \frac{1}{2}\frac{d}{dt}\int_\Omega u^2(t) \quad \mathcal{L}^1 - \text{a.e. } t \in (0,T)$$

$$\Big(\text{resp. } |\Phi|(Q) = \iint_Q u\, d\mu - \frac{1}{2}\int_\Omega u(T)^2 + \frac{1}{2}\int_\Omega u_0^2\Big).$$

Notice that, the regularity of Φ as well as its uniqueness hold not to be true in general. For more details on these questions, we refer the readers to the papers [16], [2], [22], [23], [24] and [28].

Theorem 2.3 treats the case with μ is regular with respect to t. For general μ ; i.e. $\mu \in L^1(0,T; w^* - \mathcal{M}_b(\Omega))$, we can prove that if u is a variational solution then, there exists $\Phi \in \mathcal{M}_b(Q)^N$ such that $\partial_t u - \nabla \cdot \Phi = \mu \quad$ in $\mathcal{D}'(Q)$ and (12) is fulfilled. But, in connection with the decomposition of Φ with respect to t and x, the converse implication as well as uniqueness are not clear in general. So, in order to give a complete description of u with flux $\Phi \in \mathcal{M}_b(Q)^N$, we replace (11) and (12) with the weak formulation bellow

$$\sup_{\eta \in \mathcal{C}_0(\Omega)^N, \ \|\eta\|_\infty \leq 1} \iint_Q \sigma\, \eta\, d\Phi \leq \frac{1}{2}\int_0^T\!\!\int_\Omega u^2\, \sigma_t + \iint_Q \sigma\, u\, d\mu, \tag{13}$$

for any $\sigma \in \mathcal{D}(0,T)$ such that $\sigma \geq 0$.

To simplify the presentation we introduce the following definition.

Definition 2.4 *Let $\mu \in L^1(0,T; w^* - \mathcal{M}_b(\Omega))$ and $u_0 \in K$. A function (u,Φ) is said to be a weak solution of (P_μ) provided $u \in \mathcal{C}([0,T); L^2(\Omega)) \cap K_T$, $u(0) = u_0$, $\Phi \in \mathcal{M}_b(Q)^N$, $\partial_t u - \nabla \cdot \Phi = \mu \quad$ in $\mathcal{D}'(Q)$ and (13) is fulfilled.*

It is clear that both (10), (11) and (12) imply (13), but the converse part is not true in general. The connection between the variational solution and solution with a flux satisfying (13) is summarized in the following theorem.

Theorem 2.5 *Let $u_0 \in K$ and $\mu \in L^1(0,T; w^* - \mathcal{M}_b(\Omega))$. Then, u is the variational solution of (5) if and only if $u \in \mathcal{C}([0,T); L^2(\Omega)) \cap K_T$, $u(0) = u_0$ and there exists $\Phi \in \mathcal{M}_b(Q)^N$ such that $\partial_t u - \nabla \cdot \Phi = \mu \quad$ in $\mathcal{D}'(Q)$ and (13) is fulfilled. Moreover, (13) is equivalent to*

$$\sup_{\eta \in \mathcal{C}_0(\Omega)^N, \ \|\eta\|_\infty \leq 1} \iint_Q \sigma\, \eta\, d\Phi = \frac{1}{2}\int_0^T\!\!\int_\Omega u^2\, \partial_t\sigma + \iint_Q \sigma\, u\, d\mu \tag{14}$$

for any $\sigma \in \mathcal{D}(0,T)$ such that $\sigma \geq 0$.

As we said in the introduction, a main feature of the weak solution of (P_μ) is the description of the equilibrium solutions which are the solutions of MK. The connection between the weak solution of (P_μ) and the weak solution (2) is given in the following theorem.

Theorem 2.6 *Let $u_0 \in K$, $\mu \in \mathcal{M}_b(\Omega)$ and (u, Φ) be a weak solution of (P_μ). As $t \to \infty$,*

$$u(t) \to u^* \quad \text{uniformly in } \Omega,$$

and the there exists a subsequence that we denote again by t, such that

$$\Phi(t) \to \Phi^* \quad \text{in } \mathcal{M}_b(\Omega) - \text{weak}^*.$$

Moreover, (u^*, Φ^*) is a weak solution of (2) ; that is u^* is a Kantorovich potential and Φ^* is the flux transport.

Remark 2.7 *Here, let us see that a solution of MK in $\mathbb{R}^N$ may be describe by a solution in a bounded domain with Dirichlet boundary condition whenever the source term μ is compactly supported. Indeed, it is enough to prove that if $f \in L^2(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ is compactly supported, then $u := \mathbb{P}_{\tilde{K}} f$ is compactly supported, where*

$$\tilde{K} = \Big\{ z \,:\, \mathbb{R}^N \to \mathbb{R}\,;\ |u(x) - v(y)| \leq |x - y| \ \ \mathcal{L}^{2N} - \ \text{a.e. } (x, y) \in \mathbb{R}^N \times \mathbb{R}^N \Big\}.$$

To see that, we assume without loose of generality that support$(f) \subseteq B(0, R)$, for a given $R > 0$ and we prove that there exists $R' > 0$, such that

$$\text{support}(u) \subseteq B(0, R'). \tag{15}$$

It is not difficult to see that

$$|f(x)| \leq d_R(x) := (\|f\|_{L^\infty(\mathbb{R}^2)} + (R - |x|))^+, \qquad \text{a.e. } x \in \mathbb{R}^2. \tag{16}$$

Moreover, we see that $d_R \in \tilde{K}$, so that

$$\mathbb{P}_{\tilde{K}} f \leq \mathbb{P}_{\tilde{K}} d_R = d_R, \qquad \mathcal{L}^2 - \ \text{a.e. in } \mathbb{R}^2$$

In the same way, we can prove that

$$\mathbb{P}_{\tilde{K}} f \geq -d_R, \qquad \mathcal{L}^2 - \ \text{a.e. in } \mathbb{R}^2.$$

This implies that

$$\text{Support}(u) \subseteq B(0, R') \text{ where } R' = R + \|f\|_{L^\infty(\mathbb{R}^2)},$$

and the proof is complete.

2.3 Numerical computation

2.3.1 A numerical algorithm (P_μ) (cf. [25])

In [25] (a joint paper with S. Dumont), we study the numerical analysis of the problem (5) to provide numerical approximation of the solution u of (5). However, by using duality arguments, the numerical method of [25] provide moreover a numerical approximation of the flux Φ. Thus, combining the numerical method of [25] and Theorem 2.6, we can obtain a numerical approximation of the solution (u,ϕ) of (P_μ). For completeness let us give formally the main ideas of the method.

For $\varepsilon > 0$, we consider $\Big(t_i,\mu_i\Big)_{i=1,\dots n}$ an $\varepsilon-$ discretization, i.e. $t_0 = 0 < t_1 < \dots < t_{n-1} < T = t_n$ with $t_i - t_{i-1} = \varepsilon$, $\mu_1, \dots\mu_n \in L^2(\Omega)$, such that

$$\mu_\varepsilon := \sum_{i=1}^n \mu_i \chi_{[t_{i-1},t_i)} \to \mu \quad \text{ in } L^\infty(0,T;w^* - \mathcal{M}_b(\Omega)).$$

The Euler implicit time discretization of (P_μ) is given by

$$\text{(17)} \qquad \begin{cases} u_i - \nabla\cdot(\Phi_i) = \varepsilon\,\mu_i + u_{i-1} \quad \text{ in } \Omega \\ \\ |\nabla u_i| \leq 1 \;\text{ and }\; \Phi_i = |\Phi_i|\,\nabla_{|\Phi_i|} u_i \quad \text{ in } \Omega \\ \\ u_i = 0 \quad \text{ on } \partial\Omega, \end{cases}$$

for $i = 1,..n$. Then, a numerical approximation of the solution (u,Φ) of (P_μ) is given by a numerical computation of the $\varepsilon-$approximate solution $(u_\varepsilon, \Phi_\varepsilon)$ given by

$$\text{(18)} \qquad u_\varepsilon(t) = \begin{cases} u_0 \quad \text{ for any } t \in]0,t_1], \\ \\ u_i \quad \text{ for any } t \in]t_{i-1},t_i],\; i = 1,\dots n \end{cases}$$

and

$$\text{(19)} \qquad \Phi_\varepsilon(t) = \Phi_i \quad \text{ for any } t \in]t_{i-1},t_i],\; i = 1,\dots n.$$

Thanks to [25], we know that setting

$$g_i := \varepsilon\,\mu_i - u_{i-1} \quad i = 1,\dots n,$$

a numerical approximation of u_i can be given by the following optimization problem

$$\text{(20)} \qquad \sup_{\sigma \in H_{div}(\Omega)} -G(\sigma)$$

where

$$G(\sigma) = \frac{1}{2}\int_\Omega |div\sigma|^2 + \int_\Omega g_i\, div\sigma + |\sigma|(\Omega)$$

and

$$H_{div}(\Omega) = \Big\{ w \in \Big(L^2(\Omega) \Big)^N \ ; \ \ \mathrm{div}(w) \in L^2(\Omega) \Big\}.$$

Indeed, u_i is characterized by

$$J(u_i) = \inf\{J(z), \ z \in W^{1,\infty}(\Omega)\}, \tag{21}$$

where

$$J(z) = \int_\Omega |z - g_i|^2 + I\!\!I_K,$$

and (20) is the dual problem of (21). The maximization problem (20) has a solution Φ which is not in $H_{div}(\Omega)$ in general but is a vector valued Radon measure Φ such that $div(\Phi) \in L^2(\Omega)$ and, we have

$$u_i = g_i + div(\Phi). \tag{22}$$

Moreover, thanks to [31], taking $\Phi_i := \Phi$, the couple (u_i, Φ_i) is also a weak solution of (17). In other words, the maximization problem (20) and (22) provides a couple (u_i, Φ_i) which is a solution of (17).

So, to give a numerical approximation of the solution (u_i, Φ_i) of (17), for $i = 1, ...n$, we can follow the same algorithm of [25]. That is

- Solve the maximization problem (20) by using Raviart Thomas finite element of the lowest order [39]. Denoting h the average length of the elements and V_h the space of finite elements, we compute σ_h the solution of the problem

$$G(\sigma_h) = \inf_{q_h \in V_h} G(q_h), \tag{23}$$

- Compute

$$u_h = g_i + div(\sigma_h).$$

- The couple (u_h, w_h) is the numerical approximation of (u_i, Φ_i).

2.3.2 Numerical results

Here we use the numerical codes of [25] to give some numerical examples in $\mathbb{R}^2$ where the flux of the Monge-Kantorovich problem is computed by using the evolution problem (P_μ) with large t. Recall that the standard Monge optimal transportation is posed in all $\mathbb{R}^2$ and the measure $\mu = \mu^+ - \mu^-$ is such that μ^+ and μ^- are nonnegative, has disjoint support and satisfies the balance mass condition

$$\mu^+(\mathbb{R}^2) = \mu^-(\mathbb{R}^2).$$

Our results here as well as the result of [25] are set in a bounded domain with Dirichlet boundary condition on the potential u. However, thanks to Remark 2.7, by taking a large domain with respect to the support of μ, the numerical method introduced in [25] still works here, since in this case the support of Φ will be included in the interior of Ω. This means that all the transportation takes place between μ^+ and μ^-, there no exchanges with the boundary.

In the following examples, we take $\Omega = [0, 10] \times [0, 10]$, and as in [25] the minimization of G on V_h is implemented using a relaxation procedure.

Example 1 In this example we take

$$\mu^+ = \delta_{\{4\}\times[4,5]} \quad \text{and} \quad \mu^- = \delta_{[5,6]\times\{6\}}$$

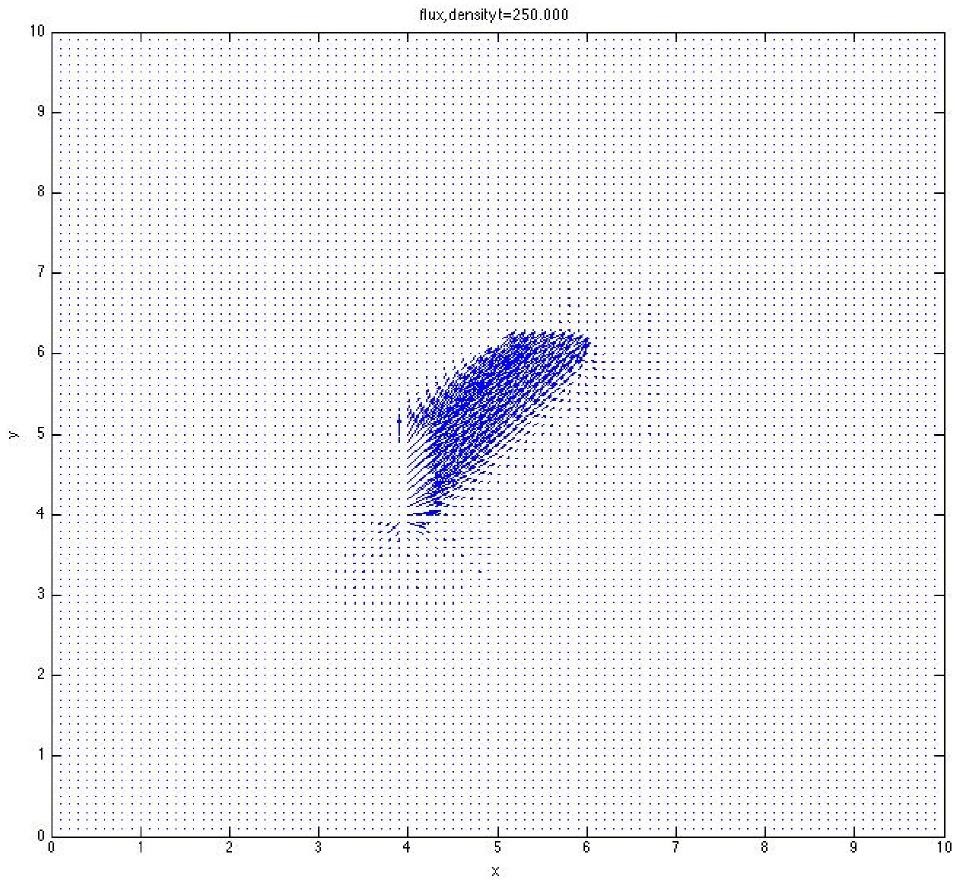

Figure 1: Representation of the flux of MK equation (N=2)

Example 2 In this example we take

$$\mu^+ = \delta_{(5,3)} + \delta_{(3,5)} + \delta_{(6,7)} + \delta_{(7,4)} \quad \text{and} \quad \mu^- = 2\delta_{(5,5)} + 2\delta_{(5,6)}$$

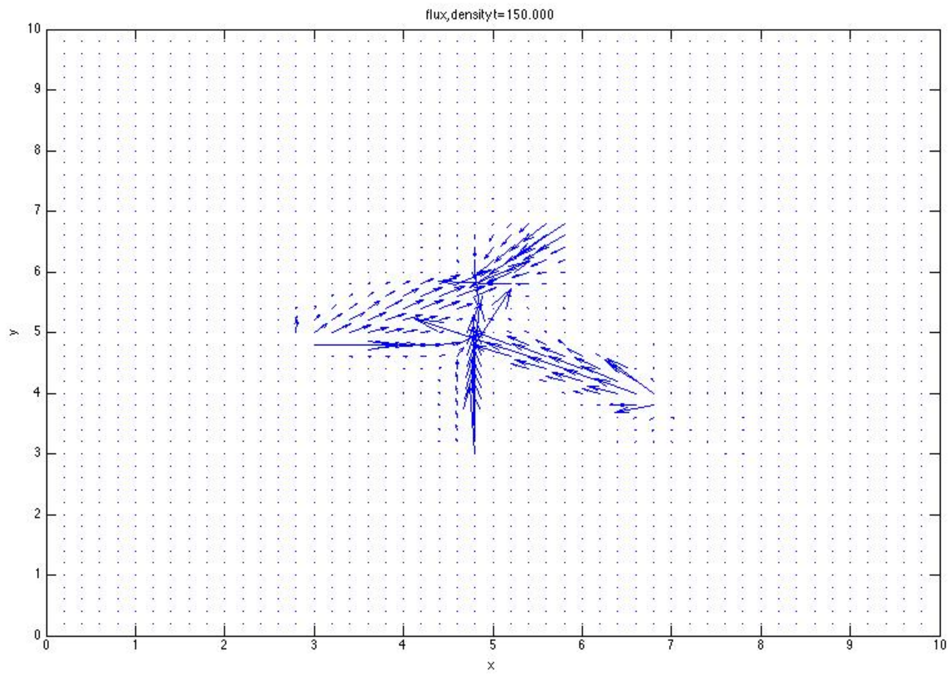

Figure 2: Representation of the flux of MK equation (N=2)

Example 3 In this example we take

$$\mu^+ = \delta_{(5,3)} + \delta_{(3,5)} + \delta_{(6,7)} + \delta_{(7,4)} \quad \text{and} \quad \mu^- = 3\delta_{(5,5)} + \delta_{(5,6)}$$

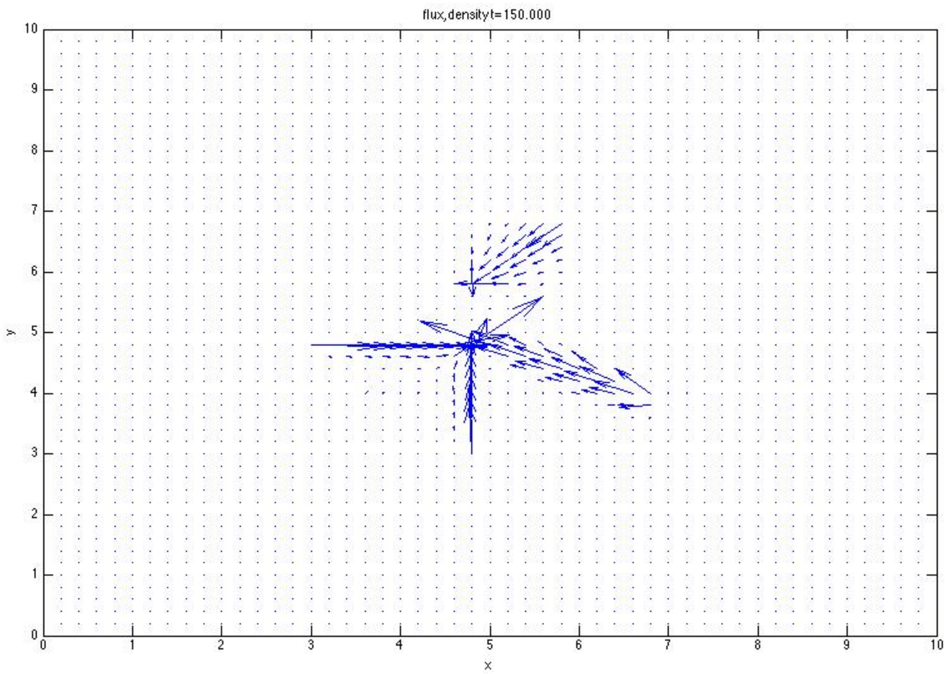

Figure 3: Representation of the flux of MK equation (N=2)

Example 3 In this example we take

$$\mu^+ = \delta_{(5,3)} + \delta_{(3,5)} + \delta_{(6,7)} + \delta_{(7,4)} \quad \text{and} \quad \mu^- = \delta_{(5,5)} + 3\delta_{(5,6)}$$

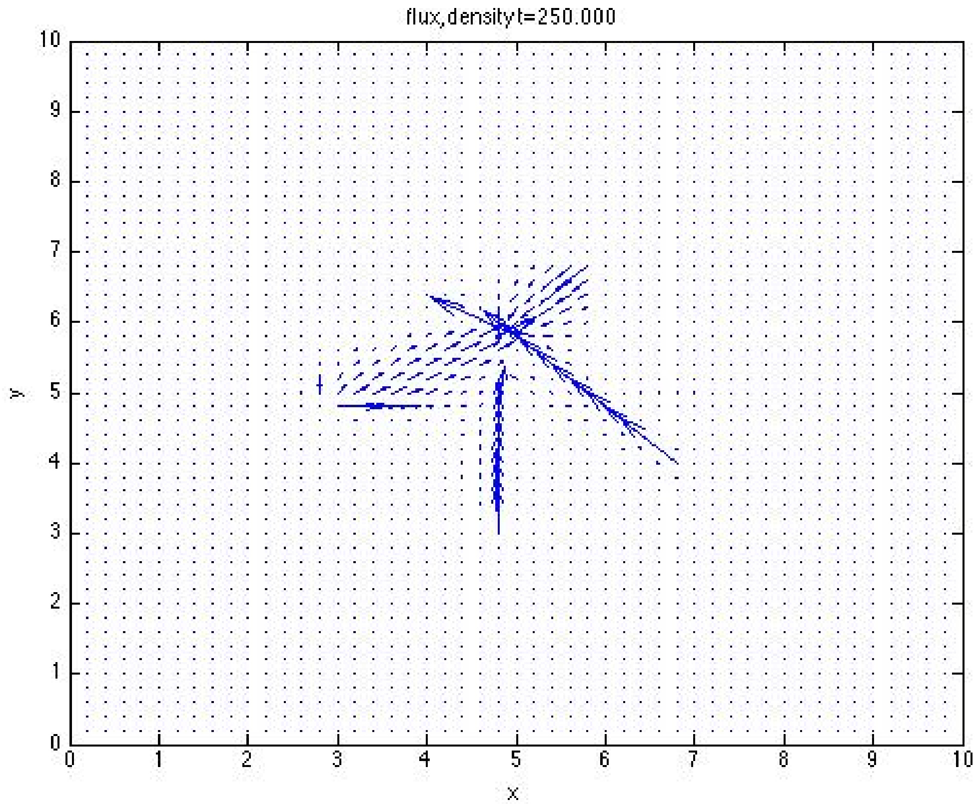

Figure 4: Representation of the flux of MK equation (N=2)

3 Proofs

3.1 Uniqueness of variational solution

Proposition 3.1 *For any $\mu \in L^1(0,T,w^* - \mathcal{M}_b(\Omega))$ and $u_0 \in K$, the problem (5) has at most one variational solution.*

Proof : We use doubling and dedoubling variables techniques. So, let $\sigma = \sigma(t_1,t_2) \in \mathcal{D}((0,T)^2)$, and set $u_1 = u_1(t_1)$ and $u_2 = u_2(t_2)$. Then, by definition

$$(24) \quad -\frac{1}{2}\int_0^T\int_\Omega \sigma_{t_1}(t_1,t_2)\ |u_1(t_1) - u_2(t_2)|^2\ dt_1\, dx \le \int_0^T\int_\Omega (u_1(t_1) - u_2(t_2))\, \sigma(t_1,t_2)\, d\mu(t_1)\, dt_1,$$

and

$$(25)\quad -\frac{1}{2}\int_0^T\int_\Omega \sigma_{t_2}(t_1,t_2)\,|u_1(t_1)-u_2(t_2)|^2\,dt_2\,dx \le -\int_0^T\int_\Omega (u_1(t_1)-u_2(t_2))\,\sigma(t_1,t_2)\,d\mu(t_2)\,dt_2.$$

Integrating (24) (resp. (25)) with respect to t_2 (resp. t_1) and adding the resulting equations, we get

$$\begin{aligned}
-\frac{1}{2}\int_0^T\int_0^T\int_\Omega |u_1(t_1)-u_2(t_2)|^2\,(\sigma_{t_1}(t_1,t_2)+\sigma_{t_2}(t_1,t_2))\,dt_1\,dt_2\,dx \\
\le \int_0^T\int_0^T\int_\Omega (u_1(t_1)-u_2(t_2))\,\sigma(t_1,t_2)\,d(\mu(t_1)-\mu(t_2))\,dt_1\,dt_2 \\
\le 2C\int_0^T\int_0^T \sigma(t_1,t_2)\,|\mu(t_1)-\mu(t_2)|(\Omega)\,dt_1\,dt_2,
\end{aligned}$$

where we used the fact that, for $i=1,\ 2$, $u_i\in K$ and C is a constant depending only on N and Ω such that $\|z\|_\infty\le C$ for any $z\in K$. Now, to dedouble variables, let $\xi\in\mathcal{D}(0,T)$, $\xi\ge 0$ and ρ_ε be the usual mollifiers in $\mathbb{R}$. Set

$$\sigma(t_1,t_2)=\rho_\varepsilon\Big(\frac{t_1-t_2}{2}\Big)\,\xi\Big(\frac{t_1+t_2}{2}\Big),$$

we have

$$\sigma_{t_1}(t_1,t_2)+\sigma_{t_2}(t_1,t_2)=\rho_\varepsilon(\frac{t_1-t_2}{2})\,\xi'(\frac{t_1+t_2}{2}).$$

Then

$$\begin{aligned}
-\frac{1}{2}\int_0^T\int_0^T\int_\Omega \rho_\varepsilon(\frac{t_1-t_2}{2})\,\xi'(\frac{t_1+t_2}{2})\,|u_1(t_1)-u_2(t_2)|^2\,dt_1\,dt_2\,dx \\
\le 2C\int\int \rho_\varepsilon\Big(\frac{t_1-t_2}{2}\Big)\,\xi\Big(\frac{t_1+t_2}{2}\Big)\,|\mu(t_1)-\mu(t_2)|(\Omega)dt_1\,dt_2.
\end{aligned}$$

Setting,

$$(\tau_1,\tau_2)=\tau(t_1,t_2)=\Big(\frac{t_1+t_2}{2},\frac{t_1-t_2}{2}\Big),$$

we get

$$\begin{aligned}
-\frac{1}{2}\int_0^T\int_0^T\int_\Omega \rho_\varepsilon(\tau_2)\,\xi'(\tau_1)\,|u_1(\frac{\tau_1+\tau_2}{2})-u_2(\frac{\tau_1-\tau_2}{2})|\,d\tau_1\,d\tau_2\,dx \\
\le C\int_0^T\int_0^T \rho_\varepsilon(\tau_2)\,\xi(\tau_1)\,|\mu(\frac{\tau_1+\tau_2}{2})-\mu(\frac{\tau_1-\tau_2}{2})|(\Omega)\,d\tau_1 d\tau_2.
\end{aligned}$$

Since, $t\in(0,T)\to|\mu(t)|(\Omega)$ is in $L^1(0,T)$, then by letting $\varepsilon\to 0$, we get

$$-\frac{1}{2}\int_0^T\int_\Omega |u_1(t)-u_2(t)|^2\,\xi_t\le 0,$$

for any $\xi \in \mathcal{D}(0,T)$, which implies that

$$\frac{d}{dt}\int_\Omega |u_1(t)-u_2(t)| \leq 0 \quad \text{ in } \mathcal{D}'(0,T).$$

At last, since, $u \in \mathcal{C}([0,T);L^2(\Omega))$ and $u_1(0)=u_2(0)$, then $u_1=u_2$, $\mathcal{L}^{N+1}$-a.e. in Q. ∎

3.2 A weak solution is a variational solution

To prove that weak solutions are variational solution, we begin first by to prove the following Lemma.

Lemma 3.2 *For any $z \in K_T$, there exists $(z_\varepsilon)_{\varepsilon>0}$ a sequence in $L^\infty(0,T;\mathcal{C}_0^1(\Omega)) \cap K_T$ such that, as $\varepsilon \to 0$, for any $q \geq 1$,*

$$z_\varepsilon \to z \quad \text{ in } L^q(0,T;W_0^{1,q}(\Omega)) - weak$$

and

$$z_\varepsilon \to z \quad \text{ uniformly in } \quad Q.$$

Proof : For a given $\epsilon >$, we consider the application $I_\epsilon \, : \, \mathbb{R} \to \mathbb{R}$, defined by

$$I_\epsilon(r) = \begin{cases} 0 & \text{if } |r| \leq \epsilon \\ r - sign(r)\,\epsilon & \text{if } |r| > \epsilon. \end{cases}$$

Then, we choose

$$\tilde{z}_\epsilon(t,x) = I_\epsilon(z(t)), \quad \text{ for any } t \in [0,T).$$

One sees that $\tilde{z}_\epsilon(t)$ is compactly supported in $\omega \subset\subset \Omega$ and $\tilde{z}_\epsilon \in K_T$. So that, there exists $0<\alpha<1$, such that

$$z_\epsilon(t) = \tilde{z}_\epsilon(t) * \rho_{\alpha\epsilon} \quad \in K \cap \mathcal{D}(\Omega), \quad \text{ for any } \epsilon > 0.$$

Moreover, for any $q \geq 1$, z_ϵ is bounded in $L^q(0,T;W_0^{1,q}(\Omega))$, and the results of the lemma follows. ∎

Proposition 3.3 *Let $\mu \in L^1(0,T;w^*-\mathcal{M}_b(\Omega))$ and $u_0 \in K$. If (u,Φ) is a weak solution of (P_μ), then*

$$\sup_{\eta \in \mathcal{C}_0(\Omega)^N,\ \|\eta\|_\infty \leq 1} \iint_Q \sigma\,\eta\,d\Phi \geq \frac{1}{2}\int_0^T\int_\Omega u^2\,\partial_t\sigma + \iint_Q \sigma\,u\,d\mu \,, \tag{26}$$

for any $\sigma \in \mathcal{D}(0,T)$ such that $\sigma \geq 0$. In addition, u is a variational solution of (5).

Proof : For any $h > 0$ and $\varepsilon > 0$, let us consider

$$u_\varepsilon^h(t,x) = \frac{1}{2h}\int_{t-h}^{t+h} u_\varepsilon(s,x)ds \quad \text{for any } (t,x) \in Q,$$

where $(u_\varepsilon)_{\varepsilon>0}$ a sequence of $L^\infty(0,T;\mathcal{C}_0^1(\Omega))$ given by Lemma 3.2 extended by 0 outside $(0,T)$; i.e. $\|\nabla u_\varepsilon\|_\infty \leq 1$ and, as $\varepsilon \to 0$,

$$u_\varepsilon \to u \quad \text{in } L^q(0,T;W_0^{1,q}(\Omega)) \quad \text{and in } L^q(\Omega),$$

for any $q \geq 1$. It is clear that $u_\varepsilon^h \in W^{1,q}(0,T;W_0^{1,q}(\Omega))$ and, for any $t \in (0,T)$,

$$u_\varepsilon^h(t) \longrightarrow \frac{1}{2h}\int_{t-h}^{t+h} u(\tau)\,d\tau =: u_h(t), \quad \text{in } L^q(\Omega), \quad \text{as } \varepsilon \to 0,$$

for any $q \geq 1$. Taking $\sigma\, u_\varepsilon^h$ as a test function in $\partial_t u - \nabla\cdot\Phi = \mu$, we get

$$\int\!\!\int_Q \sigma\,\nabla u_\varepsilon^h\,d\Phi = \int_0^T\!\!\int_\Omega \sigma_t\,u\,u_\varepsilon^h + \int_0^T\!\!\int_\Omega \sigma(t)\,u(t)\,\frac{u_\varepsilon(t+h)-u_\varepsilon(t-h)}{2\,h} + \int\!\!\int_Q \sigma\,u\,d\mu.$$

Letting $\varepsilon \to 0$, then $h \to 0$, and using Lebesgue dominated convergence Theorem, we obtain

$$\lim_{h\to 0}\lim_{\varepsilon\to 0}\int\!\!\int_Q \sigma\,\nabla u_\varepsilon^h\,d\Phi = \int_0^T\!\!\int_\Omega \sigma_t\,u^2 + \lim_{h\to 0}\int_0^T\!\!\int_\Omega \sigma(t)\,u(t)\,\frac{u(t+h)-u(t-h)}{2h} + \int\!\!\int_Q \sigma\,u\,d\mu. \tag{27}$$

Since, $u \in \mathcal{C}([0,T);L^2(\Omega))$, then

$$\begin{aligned}\lim_{h\to 0}\int_0^T\!\!\int_\Omega \sigma(t)\,u(t)\,\frac{u(t+h)-u(t-h)}{2h} &= -\lim_{h\to 0}\int_0^T\!\!\int_\Omega u(t-h)\,u(t)\,\frac{\sigma(t+h)-\sigma(t-h)}{2h}\\ &= -\frac{1}{2}\int_0^T\!\!\int_\Omega u^2\,\sigma_t.\end{aligned}$$

So, (27) implies that

$$\begin{aligned}\frac{1}{2}\int_0^T\!\!\int_\Omega \sigma_t\,u^2 + \int\!\!\int_Q \sigma\,u\,d\mu &= \lim_{h\to 0}\lim_{\varepsilon\to 0}\int\!\!\int_Q \sigma\,\nabla u_\varepsilon^h\,d\Phi\\ &\leq \sup_{\eta\in\mathcal{C}_0(\Omega)^N,\ \|\eta\|_\infty\leq 1}\int\!\!\int_Q \sigma\,\eta\,d\Phi.\end{aligned}$$

and the proof of (26) is complete. Now, let $\xi \in K$ and $\sigma \in \mathcal{D}(0,T)$ be such that $\sigma \geq 0$. We consider $\xi_\varepsilon \in K \cap \mathcal{D}(\Omega)$ as given by Lemma 3.2. Notice that in this case, we can assume that $z_\varepsilon \in K \cap \mathcal{C}_0^1(\Omega)$ and $z_\varepsilon \to z$ in $\mathcal{C}_0(\Omega)$. Then taking $\sigma\, \xi_\epsilon$ as a test function in $\partial_t u - \nabla \cdot \Phi = \mu$ and using (13), we get

$$\begin{aligned}\int_0^T\int_\Omega \sigma_t\, u\, \xi + \int_0^T\int_\Omega \sigma\, \xi\, d\mu &= \lim_{\epsilon\to 0} \iint_Q \sigma\, \nabla\xi_\epsilon\, d\Phi \\ &\leq \sup_{\eta\in\mathcal{C}_0(\Omega)^N,\ \|\eta\|_\infty\leq 1} \iint_Q \sigma\, \eta\, d\Phi \\ &\leq \frac{1}{2}\int_0^T\int_\Omega \sigma_t\, u^2 + \iint_Q \sigma\, u\, d\mu.\end{aligned}$$

This implies that

$$\int_0^T\int_\Omega \sigma_t\, (\frac{1}{2}u^2 - u\,\xi) + \iint_Q \sigma\,(u-\xi)\, d\mu \geq 0,$$

which is equivalent to (6), for any $\xi \in K$. At last, one sees easily that a solution in the sense of Theorem 2.5 is a weak solution. Thus, both weak solution and a solution in the sense of Theorem 2.5 are variational solution, and their uniqueness follows by Proposition 3.1. ∎

3.3 Existence of a weak solution

Now, in order to prove the existence parts of Theorem 2.2, Theorem 2.3 and Theorem 2.5, we consider the p-Laplacian evolution equation

$$(P_\mu^p)\qquad \begin{cases} \partial_t u - \nabla\cdot(|\nabla u|^{p-2}\nabla u) = \mu & \text{in } Q \\ u = 0 \quad \text{on } \Sigma \\ u(0) = u_{0p}, \end{cases}$$

with $p \geq N+1$, $\mu \in L^\infty(0,T; w^* - \mathcal{M}_b(\Omega))$ and $u_{0p} \in W_0^{1,p}(\Omega)$. Since

$$\mathcal{M}_b(\Omega) \subset W^{-1,(N+1)'}(\Omega) \subset W^{-1,p'}(\Omega),$$

with continuous injection, then

$$L^\infty(0,T; w^* - \mathcal{M}_b(\Omega)) \subset L^{p'}(0,T; W^{-1,p'}(\Omega)).$$

So, thanks to [35] (see also [1]), there exists a unique solution u_p of the problem (P_μ^p), in the sense that $u \in L^p(0,T; W_0^{1,p}(\Omega))$, $\partial_t u \in L^{p'}(0,T; W^{-1,p'}(\Omega))$, $u(0) = u_0$ and $\partial_t u - \Delta_p u = \mu$ in $\mathcal{D}'(Q)$. Moreover, for any $p \geq q \geq N+1$, we have

$$\begin{aligned}\int_\Omega |\nabla u_p(t)|^p &= \int_\Omega u_p(t)\, d\mu(t) - \Big\langle \partial_t u_p(t), u_p(t)\Big\rangle_{W^{-1,q'}(\Omega), W_0^{1,q}(\Omega)} \quad \mathcal{L}^1 - \text{a.e. } t \in (0,T) \\ &= \int_\Omega u_p(t)\, d\mu(t) - \frac{1}{2}\frac{d}{dt}\int_\Omega u_p^2(t) \quad \mathcal{L}^1 - \text{a.e. } t\in(0,T).\end{aligned} \tag{28}$$

Integrating (28) with respect to t and using Young inequality, we get

$$\iint_Q |\nabla u_p|^p \le C_q \, |Q|^{\frac{p-q}{(p-1)q}} \|\mu\|^{\frac{p}{p-1}}_{L^\infty(0,T;w^*-\,\mathcal{M}_b(\Omega))} + \frac{1}{2}\int_\Omega u_{0p}^2$$

and

$$\left(\iint_Q |\nabla u_p|^q\right)^{\frac{1}{q}} \le |Q|^{\frac{1}{p}} \left(C_q \, |Q|^{\frac{p-q}{(p-1)q}} \|\mu\|^{\frac{p}{p-1}}_{L^\infty(0,T;w^*-\,\mathcal{M}_b(\Omega))} + \frac{1}{2}\int_\Omega u_{0p}^2\right)^{\frac{1}{p}} \tag{29}$$

where $C_q = C(q,\Omega,N)$ denotes a constant independent of p. Recall also, that if u_i is the solution of $(P^p_{\mu_i})$, for $i=1,\ 2$, then

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |u_1-u_2|^2 \le \int_\Omega (u_1-u_2)\, d(\mu_1-\mu_2) \quad \text{ in } \mathcal{D}'(0,T) \tag{30}$$

and

$$\frac{d}{dt}\int_\Omega |u_1-u_2| \le |\mu_1-\mu_2|(\Omega) \quad \text{ in } \mathcal{D}'(0,T). \tag{31}$$

Lemma 3.4 *If, moreover $\mu \in BV(0,T;w^*-\mathcal{M}_b(\Omega))$ and $\Delta_p u_{0p} \in L^1(\Omega)$, then*

1. *$\partial_t u_p \in L^\infty(0,T;w^*-\mathcal{M}_b(\Omega))$ and*

$$\|\partial_t u_p\|_{L^\infty(0,T;w^*-\mathcal{M}_b(\Omega))} \le |\mu(0^+)|(\Omega) + \|\Delta_p u_{0p}\|_{L^1(\Omega)} + V(\mu,T). \tag{32}$$

2. *For any $p \ge q \ge N+1$, $u_p \in L^\infty(0,T;W^{1,q}_0(\Omega))$ and*

$$\|u_p\|_{L^\infty(0,T;W^{1,q}_0(\Omega))} \le C_q^{\frac{1}{p-1}} \, |\Omega|^{\frac{p-q}{(p-1)\,q}} \|\mu+\partial_t u_p\|^{\frac{1}{p-1}}_{L^\infty(0,T;w^*-\mathcal{M}_b(\Omega))}. \tag{33}$$

Proof :

1. We see that $w_p(t) = u_p(t+h)$ is a solution of

$$\begin{cases} \partial_t w - \Delta_p w = \mu(.+h) \quad \text{ in } Q_h := (0,T-h)\times\Omega \\ w = 0 \quad \text{ on } \Sigma \\ w(0) = u_p(h), \end{cases}$$

so that by (31), we have

$$\int_\Omega |u_p(t)-u_p(t+h)| \le \int_\Omega |u_{0p}-u_p(h)| + \int_0^t |\mu(s)-\mu(s+h)|(\Omega)\, ds. \tag{34}$$

Since, $u_{0p} \in W_0^{1,p}(\Omega)$ and $\Delta_p u_{0p} \in L^1(\Omega)$, then u_{0p} is a solution of $(P^p_{(\Delta_p u_{0p})})$, and by applying again (31) we have

$$\begin{aligned}\int_\Omega |u_{0p} - u_p(h)| &\leq \int_0^h |\mu(t) - \Delta_p u_{0p}\, \mathcal{L}^N|(\Omega) \\ &\leq \int_0^h |\mu(\tau)|(\Omega)\, d\tau + h \int_\Omega |\Delta_p u_{0p}|.\end{aligned}$$

So,

$$\begin{aligned}\int_\Omega |u_p(t+h) - u_p(t)| \leq \int_0^h |\mu(\tau)|(\Omega) + h \int_\Omega |\Delta_p u_{0p}| \\ + \int_0^T |\mu(\tau+h) - \mu(\tau)|(\Omega)\, d\tau.\end{aligned} \tag{35}$$

Dividing by h and letting $h \to 0$, we obtain that $\partial_t u_p \in L^\infty(0,T; w^* - \mathcal{M}_b(\Omega))$ and satisfies (32).

2. Thanks to (28), we have

$$\begin{aligned}\int_\Omega |\nabla u_p|^p &\leq \|u_p\|_{W^{1,q}(\Omega)}\, \|\mu + \partial_t u_p\|_{W^{-1,q'}(\Omega)} \\ &\leq C_q \left(\int_\Omega |\nabla u_p|^q\right)^{\frac{1}{q}} \|\mu + \partial_t u_p\|_{L^\infty(0,T;w^* - \mathcal{M}_b(\Omega))}.\end{aligned}$$

By using Holder inequality, we deduce that

$$\left(\int_\Omega |\nabla u_p|^q\right)^{\frac{1}{q}} \leq |\Omega|^{\frac{p-q}{pq}} \left(C_q \left(\int_\Omega |\nabla u_p|^q\right)^{\frac{1}{q}} \|\mu + \partial_t u_p\|_{L^\infty(0,T;w^* - \mathcal{M}_b(\Omega))}\right)^{\frac{1}{p}},$$

and (33) follows.

Proposition 3.5 *Let $\mu \in BV(0,T; w^* - \mathcal{M}_b(\Omega))$ and $u_{0p} \in W_0^{1,p}(\Omega)$. If $\Delta_p u_{0p}$ is bounded in $L^1(\Omega)$ and, as $p \to \infty$, $u_{0p} \to u_0$ in $L^2(\Omega)$, then there exists $(u, \Phi) \in L^\infty(0,T; W_0^{1,q}(\Omega)) \times L^\infty(0,T; w^* - \mathcal{M}_b(\Omega)^N)$, for any $N+1 \leq q < \infty$, such that*

1. *As $p \to \infty$,*
$$u_p \to u \quad in \quad L^q(0,T; W_0^{1,q}(\Omega)) - weak, \quad and\ in\ L^q(Q)$$
for any $q \geq N+1$.

2. *There exists $p_k \to \infty$, such that*
$$|\nabla u_{p_k}|^{p_k - 2} \nabla u_{p_k} \to \Phi \quad in \quad L^\infty(0,T; w^* - \mathcal{M}_b(\Omega)^N) - weak^*$$
and Φ satisfies (12).

Proof : Thanks to Lemma 3.4, for fixed $q \geq N+1$, $(u_p)_{p\geq q}$ and $(\partial_t u_p)_{p\geq q}$ are bounded, respectively, in $L^\infty(0,T;W^{1,q}(\Omega)$ and $L^\infty(0,T;w^*-\mathcal{M}_b(\Omega))$. So, there exists $u \in L^\infty(0,T;W_0^{1,q}(\Omega))$ and a subsequence that we denote again by p, such that

$$u_p \to u \quad \text{in } L^q(0,T;W_0^{1,q}(\Omega)) - \text{weak}, \tag{36}$$

and

$$\partial_t u_p \to \partial_t u \quad \text{in } L^{q'}(0,T;w^*-\mathcal{M}_b(\Omega)) - \text{weak}^*. \tag{37}$$

Since $L^{q'}(0,T;w^*-\mathcal{M}_b(\Omega)) \subset L^{q'}(0,T;W^{-1,q'}(\Omega))$, then $u_p \to u$ in $\mathcal{C}([0,T],L^2(\Omega))$ and $u(0) = u_0$. Moreover, since $W_0^{1,q}(\Omega) \subset \mathcal{C}_0(\Omega)$, with compact injection, then we can set that

$$u_p \to u \quad \text{in } L^q(0,T;\mathcal{C}_0(\Omega)) - \text{weak} ; \tag{38}$$

in the sense that, for any $\nu \in L^{q'}(0,T;w^*-\mathcal{M}_b(\Omega))$, $\displaystyle\iint_Q u_p \, d\nu \to \iint_Q u \, d\nu$. Thanks again to Lemma 3.4 and Holder inequality, we see that $|\nabla u_p|^{p-2}\nabla u_p$ is bounded in $L^\infty(0,T;L^1(\Omega))$. Thus, there exists $\Phi \in L^\infty(0,T;w^*-\mathcal{M}_b(\Omega)^N)$, such that

$$|\nabla u_p|^{p-2}\nabla u_p \to \Phi \quad \text{in } L^\infty(0,T;w^*-\mathcal{M}_b(\Omega)^N) - \text{weak}^*. \tag{39}$$

Passing to the limit in $\partial_t u_p - \nabla\cdot(|\nabla u_p|^{p-2}\nabla u_p) = \mu$, we obtain $\partial_t u - \nabla\cdot\Phi = \mu$ in $\mathcal{D}'(Q)$. Using (28) and Holder inequality, we get

$$\int_\Omega |\nabla u_p(t)|^{p-1} \leq |\Omega|^{\frac{1}{p}} \left(\int_\Omega u_p(t)\, d\mu(t) - \frac{1}{2}\frac{d}{dt}\int_\Omega u_p^2(t) \right)^{\frac{p-1}{p}} \qquad \mathcal{L}^1 - \text{a.e. } t \in (0,T),$$

so that

$$\int_0^T\int_\Omega |\nabla u_p|^{p-1} \leq |\Omega|^{\frac{1}{p}}\, |T|^{\frac{1}{p}} \left(\int_0^T\int_\Omega u_p\, d\mu - \frac{1}{2}\int_\Omega u_p(T)^2 + \frac{1}{2}\int_\Omega u_0^2 \right)^{\frac{p-1}{p}}. \tag{40}$$

Thanks to (38) and (39), we deduce (12), by letting $p \to \infty$ in (40). By uniqueness of u, the convergence of u_p holds to be true for the sequence $(u_p)_{p\geq N}$, and for any $N+1 \leq q < \infty$. ∎

3.4 Proofs of the main theorems

First, let us prove the equivalence between (10), (11) and (12).

Lemma 3.6 *Let* $\mu \in L^1(0,T;w^*-\mathcal{M}_b(\Omega))$, $u_0 \in K$, $u \in \mathcal{C}([0,T);L^2(\Omega)) \cap K_T$, $u(0) = u_0$ *and* $\Phi \in L^1(0,T;w^*-\mathcal{M}_b(\Omega))$ *is such that* $\partial_t u - \nabla\cdot\Phi = \mu$ *in* $\mathcal{D}'(Q)$. *Then (10), (11) and (12) are equivalent, and implies (13).*

Proof : The equivalence between (10) and (11) is a simple consequence of Lemma 3.2. In addition, it is not difficult to see that (11) implies (12). To complete the proof, let us show that (12) implies (11). Since $\nu := \nabla \cdot \Phi \in L^1(0,T; w^* - \mathcal{M}_b(\Omega))$, then by using the approximation

$$u^h(t,x) = \frac{1}{2h}\int_{t-h}^{t+h} u(s,x)ds \quad \text{for any } (t,x) \in Q,$$

one proves exactly in the same way as for Proposition 3.3, that

$$\int_0^T \sigma \int_\Omega u(t)\, d\nu(t) = \frac{1}{2}\int_0^T\int_\Omega u^2\, \sigma_t(t) + \iint_Q \sigma\, u\, d\mu, \tag{41}$$

for any $\sigma \in \mathcal{D}(0,T)$. In addition, since $\Phi \in L^1(0,T; w^* - \mathcal{M}_b(\Omega)^N)$, then by using Proposition 3.3, we deduce that

$$\int_0^T \sigma(t)\, |\Phi(t)|(\Omega)\, dt \geq \frac{1}{2}\int_0^T\int_\Omega u^2\, \sigma_t + \iint_Q \sigma\, u\, d\mu \; = \int_0^T \sigma(t)\left(\int_\Omega u(t)\, d\nu(t)\right) dt. \tag{42}$$

This implies that

$$\int_\Omega u(t)\, d\nu(t) \leq |\Phi(t)|(\Omega) \quad \mathcal{L}^1- \text{ a.e. } t \in (0,T).$$

Thanks to (12), we have

$$\int_{[0,T]} |\Phi(t)|(\Omega)\, dt \leq \int_0^T\int_\Omega u(t)\, d\nu(t)\, dt$$

we deduce that

$$\int_\Omega u(t)\, d\nu(t) = |\Phi(t)|(\Omega) \quad \mathcal{L}^1- \text{ a.e. } t \in (0,T)$$

and (11) follows. At last, since

$$\sup_{\eta \in \mathcal{C}_0(\Omega)^N,\ \|\eta\|_\infty \leq 1} \iint_Q \sigma\, \eta\, d\Phi \leq \int_0^T \sigma(t)\, |\Phi(t)|(\Omega)\, dt,$$

for any $\sigma \in \mathcal{D}(0,T)$ with $\sigma \geq 0$, then (11) implies (13). And the proof is finished.
∎

Proof of Theorem 2.3 : Thanks to Proposition 3.3 and Lemma 3.6, it is enough to prove existence of (u,Φ) such that uK_T, $u(0) = u_0$, $\partial_t u \in L^\infty(0,T; w^* - \mathcal{M}_b(\Omega))$, $\Phi \in L^\infty(0,T; w^* - \mathcal{M}_b(\Omega))$, $\partial_t u - \nabla\cdot\Phi = \mu \quad$ in $\mathcal{D}'(Q)$ and Φ satisfies (12). To this aim, we consider the elliptic problem

$$\begin{cases} z - \Delta_p z = u_0 & \text{in } \Omega \\ \\ z \in W_0^{1,p}(\Omega). \end{cases}$$

It is well know by now that this problem has a unique solution. Let us denote this solution by u_{0p}. Since, $u_0 \in K$, then it is not difficult to see that, letting $p \to \infty$, $\Delta_p u_{0p}$ is bounded in $L^1(\Omega)$

and $u_{0p} \to u_0$ in $\mathcal{C}_0(\Omega)$. So, by considering u_p the solution of (P^p_μ), and letting $p \to \infty$, the result follows by using Proposition 3.5. ∎

Proof of Theorem 2.5 : By using again Proposition 3.3 and Lemma 3.6, it is enough to prove existence of weak solution. Since $\mathcal{M}_b(\Omega) \subset W^{-1,\frac{N+1}{N}}(\Omega)$ with continuous injection, then $L^1(0,T;w^* - \mathcal{M}_b(\Omega)) \subset L^1(0,T;w^* - W^{-1,\frac{N+1}{N}}(\Omega))$ with continuous injection. So, for any $\mu \in L^1(0,T;w^* - \mathcal{M}_b(\Omega))$, there exists $F \in L^1(Q)^N$ such that $\mu = \nabla \cdot F$. Let us denote by F_n the regularization by convolution of F and set $\mu_n = \nabla \cdot F_n$. Thanks to Theorem 2.3, the regularized problem (P_{μ_n}) has a weak solution $(u_n, \Phi_n) \in \mathcal{C}([0,T);L^2(\Omega)) \times L^\infty(0,T;w^* - \mathcal{M}_b(\Omega)^N)$. By letting $p \to \infty$ in (30) and using Lemma 3.5, we see that, for any $n \geq q$,

$$\begin{aligned} \frac{1}{2}\int_\Omega (u_n(t) - u_q(t))^2 &\leq \int_0^t\int_\Omega (\mu_n - \mu_q)\,(u_n - u_q) \\ &\leq -\int_0^t\int_\Omega (F_n - F_q)\cdot\nabla(u_n - u_q) \\ &\leq 2\int_0^T\int_\Omega |F_n - F_q|, \end{aligned}$$

so that,

$$\lim_{n,\ q\ \to\infty} \int_\Omega (u_n(t) - u_q(t))^2 = 0\ .$$

This implies that u_n is a Cauchy sequence in $\mathcal{C}([0,T);L^2(\Omega))$, and there exists $u \in \mathcal{C}([0,T);L^2(\Omega))$, such that

$$u_n \to u \qquad \text{in } \mathcal{C}([0,T);L^2(\Omega)). \tag{43}$$

Moreover, since $u_n \in K_T$ for any $t \in [0,T)$, then for any $q \geq 1$, $(u_n)_{n\geq 1}$ is bounded in $L^\infty(0,T;W^{1,q}_0(\Omega))$ and

$$u_n \to u \qquad \text{in } L^q(0,T;W^{1,q}_0(\Omega)) - \text{weak}. \tag{44}$$

On the other hand, thanks to (12), we have

$$|\Phi_n|(Q) \leq \frac{1}{2}\int_\Omega u_n(t)^2 - \frac{1}{2}\int_\Omega u_0^2 + \iint_Q u_n\, d\mu_n,$$

which implies that $(\Phi_n)_{n\geq 1}$ is bounded in $\mathcal{M}_b(Q)^N$. So, there exists $\Phi \in \mathcal{M}_b(Q)^N$, and a subsequence that we denote again by n, such that

$$\Phi_n \to \Phi \quad \text{in} \quad \mathcal{M}_b(Q)^N - \text{weak}^*.$$

Passing to the limit in the equation, we deduce that $\partial_t u - \nabla\cdot\Phi = \mu$ in $\mathcal{D}'(Q)$. To prove that (13) is fulfilled, let us consider $\sigma \in \mathcal{D}(0,T)$ and $\eta \in \mathcal{C}_0(\Omega)$ such that $\|\eta\|_\infty \leq 1$. In addition, using (43) and (44), we get

$$\begin{aligned}
\iint_Q \sigma\, \eta\, d\Phi &= \lim_{n\to\infty} \iint_Q \sigma\, \eta\, d\Phi_n \\
&\leq \lim_{n\to\infty} \left\{ \iint_Q u_n\, \sigma\, d\mu_n + \frac{1}{2} \iint_Q u_n^2\, \sigma_t \right\} \\
&\leq \lim_{n\to\infty} \left\{ - \iint_Q \sigma\, \nabla u_n \cdot F_n + \frac{1}{2} \iint_Q u_n^2\, \sigma_t \right\} \\
&\leq - \iint_Q \sigma\, \nabla u \cdot F + \frac{1}{2} \iint_Q u^2\, \sigma_t \\
&\leq \iint_Q u\, \sigma\, d\mu + \frac{1}{2} \iint_Q u^2\, \sigma_t
\end{aligned}$$

which implies that u is a weak solution. ∎

Proof of Theorem 2.2 : The uniqueness of variational solution follows by Proposition 3.1. As to the existence, this is a consequence of Proposition 3.3 and Theorem 2.5. At last, since variational solution are unique and are obtained as a limit as $p \to \infty$ of the solution of (P_μ^p), then the contraction properties (7) and (8) follows by passing to the limit in (31) and (30), respectively. ∎

Lemma 3.7 *Under the assumptions of Theorem 2.6, as $t \to \infty$,*

(45) $$\left(u(t), \frac{d}{dt} u(t) \right) \to (u^*, 0) \qquad \text{in } L^2(\Omega) \times L^2(\Omega),$$

and u^* is a variational solution (2).

Proof : Thanks to [31], we know that (2) has a variational solution v. That is $v \in K$ and

$$\int_\Omega \geq 0 \qquad \text{for any } \xi \in K.$$

Now, let us consider the functional $\tilde{\Phi} : L^2(\Omega) \to [0, \infty]$ defined by

$$\tilde{\Phi}(z) := I\!\!I_K(z) + \int_\Omega z\, d\mu - \min_{\eta \in K} \int_\Omega z\, d\mu.$$

Then, it is clear that the variational solution u of (P_μ) is the unique solution of the evolution problem

$$\begin{cases} \dfrac{d}{dt} u(t) + \partial \tilde{\Phi}(u(t)) \ni 0 \quad \mathcal{L}^1 - \text{ a.e. } t \in (0, \infty) \\ \\ u(0) = 0. \end{cases}$$

Since, K is compact in $L^2(\Omega)$, the result of the lemma follows from Theorems 3.10 and 3.11 of [14]. ∎

Proof of Theorem 2.6 : It remains to prove the convergence of the flux $\Phi(t)$. Recall that

$$|\Phi(t)|(\Omega) \leq \int_\Omega u(t)\, d\mu - \frac{1}{2}\frac{d}{dt}\int_\Omega u^2(t) \quad \mathcal{L}^1 - \text{a.e. } t \in (0,T). \tag{46}$$

Using Lemma 3.7, we deduce that $\Big(\Phi(t)\Big)_{t>0}$ is bounded in $\mathcal{M}_b(\Omega)^N$. So, there exists $\Phi^* \in \mathcal{M}_b(\Omega)^N$ and a subsequence that we denote again by t, such that

$$\Phi(t) \to \Phi^* \quad \text{ in } \mathcal{M}_b(\Omega)^N - \text{weak}^*.$$

Letting $t \to \infty$ in (46) and using Lemma 3.7 again, we deduce that

$$|\Phi^*|(\Omega) \leq \int_\Omega u^*\, d\mu,$$

where u^* is a variational solution (2). This ends up the proof of the theorem. ∎

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