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THE MESA-LIMIT OF THE POROUS-MEDIUM EQUATION AND THE HELE–SHAW PROBLEM

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Abstract. We are interested in the limit, as $m \to \infty$, of the solution u_m of the porous-medium equation $u_t = \Delta u^m$ in a bounded domain Ω with Neumann boundary condition, $\frac{\partial u^m}{\partial n} = g$ on $\partial\Omega$, and initial datum $u(0) = u_0 \geq 0$. It is well known by now that this kind of limit turns out to be singular. In the case $g \equiv 0$, it was proved that there exists an initial boundary layer $\underline{u}_0$, the so-called mesa, and $u_m(t) \to \underline{u}_0$ in $L^1(\Omega)$, for any $t > 0$, as $m \to \infty$. In this work, we generalize this result to the case of arbitrary $g \in L^2(\partial\Omega)$, we prove that the initial boundary layer is still $\underline{u}_0$ and in general (even in the regular case) the limit function is not a solution of a Hele–Shaw problem. There exists a time interval I where the limit of u_m, as $m \to \infty$, is the unique solution of a Hele–Shaw problem and elsewhere, u_m conveges to the constant function $\frac{1}{|\Omega|}(\int_\Omega u_0 + t\int_{\partial\Omega} g)$.

1. Introduction

Let Ω be a bounded domain of $\mathbb{R}^N$ with smooth boundary Γ. For $m \geq 1$, we consider the porous-medium equation with nonhomogeneous Neumann boundary condition

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u^m & \text{on } Q = (0,\infty) \times \Omega \\ \dfrac{\partial u^m}{\partial n} = g & \text{on } \Sigma = (0,\infty) \times \partial\Omega \\ u(0) = u_0 \end{cases} \tag{1.1}$$

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where $g \in L^2(\Gamma)$ and $u_0 \in L^2(\Omega)$. For any $m \geq 0$ (cf. Proposition 3), there exists a unique weak solution u of (1.1) in the following sense:

$$\left\{\begin{array}{l} u \in L^2(Q),\ w := |u|^{m-1}u \in L^2_{loc}\left(0,\infty; H^1(\Omega)\right) \\ \displaystyle\int_0^\infty\int_\Omega \xi_t u + \int_\Omega \xi(0,\cdot)u_0 = \int_0^\infty\int_\Omega Dw\cdot D\xi + \int_0^\infty\int_\Gamma \xi g \\ \forall \xi \in \mathcal{C}^1([0,\infty)\times\overline{\Omega}) \text{ compactly supported.}\end{array}\right. \tag{1.2}$$

We denote by u_m this solution. We are interested in the behavior of u_m, as $m \to \infty$.

Formally, we see that as $m \to \infty$ the equation

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u^m & \text{on } Q \\ \dfrac{\partial u^m}{\partial n} = g & \text{on } \Sigma \end{cases}$$

converges to

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta w = 0 & \text{in } Q \\ u \in \operatorname{sign}(w) & \text{in } Q \\ \dfrac{\partial w}{\partial \eta} = g & \text{on } \Sigma, \end{cases} \tag{1.3}$$

which is a weak formulation of a Hele–Shaw-type problem. In fact, the Hele–Shaw problem is a one-phase free-boundary problem modeling the evolution of a slow, incompressible, viscous fluid moving between slightly separated plates, so that the pressure $w = w(x,t) \geq 0$ is such that there exists a phase function u and (u,w) satisfies (1.3) (cf. [19, 17] and [11, 10] for physical and mathematical formulation, respectively). A sign condition on g corresponds to the injection through Γ if $g \geq 0$ and to the suction if $g \leq 0$. This case turns out to be an ill-posed problem under general conditions on g (see [10]). In this work, although we consider time-independent data, since no restriction on the sign of g is assumed we will consider the (mathematical model) generalized free-boundary problem associated to (1.3) that we call the generalized Hele–Shaw problem.

Since the range of a solution of (1.3) remains in $[-1,+1]$, u_0 is an inconsistent initial datum for (1.3) if $\|u_0\|_\infty > 1$. This implies that the limit of u_m may be singular and an initial boundary layer appears in general, when one passes to the limit. On the other hand, we see that a solution of (1.3) satisfies

$$\frac{d}{dt}\int_\Omega u = \int_\Gamma g,$$

so that, if $\int_\Gamma g \neq 0$, then a solution of (1.3) is not defined for large t. This implies that the limit of u_m is not a solution of a Hele–Shaw problem in all of $(0,\infty)$. This formal analysis shows that, even in the regular case $\|u_0\|_\infty \leq 1$, the problem is completely different from the similar one with the Dirichlet boundary condition (see [16, 18, 20]). Indeed, it was proved in [12] that if the Dirichlet boundary condition is prescribed on the boundary Γ, then the limit is a solution of the Hele–Shaw problem.

Recall that the case $g \equiv 0$ and $u_0 \geq 0$ is completely solved. It was proved in [2] that $u_m(t) \to \underline{u}_0$ in $L^1(\Omega)$ for $t > 0$, where

$$\underline{u}_0 = \begin{cases} ⨍ u_0 := \dfrac{1}{|\Omega|}\displaystyle\int_\Omega u_0 & \text{if} \quad ⨍ u_0 \geq 1 \\ u_0\chi_{[w=0]} + \chi_{[w>0]} & \text{if} \quad ⨍ u_0 < 1 \end{cases} \tag{1.4}$$

with $w \in H^1(\Omega)$ the unique solution of the so-called "mesa problem"

$$w \in H^2(\Omega),\ w \geq 0,\ 0 \leq \Delta w + u_0 \leq 1,$$
$$w(\Delta w + u_0 - 1) = 0 \text{ a.e. } \Omega \ \text{ and } \frac{\partial w}{\partial n} = 0 \text{ on } \Sigma.$$

However, to our knowledge, the case $g \not\equiv 0$ was an open problem and there was no result concerning the limit as $m \to \infty$ of u_m even in the regular case. The aim of this paper is to characterize this limit, for any $u_0 \in L^2(\Omega)$, $u_0 \geq 0$ and any $g \in L^2(\Gamma)$. We show that, as $m \to \infty$,

$$u_m \to u \quad \text{in } \mathcal{C}\left((0,\infty); L^1(\Omega)\right), \tag{1.5}$$

where, setting

$$\mu(t) = ⨍ u_0 + \frac{t}{|\Omega|}\int_\Gamma g \quad \text{for } t \geq 0,$$

and defining $I = \{t \geq 0 : |\mu(t)| \leq 1\} := [a,b]$ with $a = b = +\infty$ if $I = \emptyset$, u is the unique solution of

$$\begin{cases} i)\ \ u \in \mathcal{C}\left([0,\infty); L^1(\Omega)\right),\ u(0) = \underline{u}_0, \\ ii)\ u(t) \equiv \mu(t) \text{ a.e. on } \Omega, \text{ for any } t \in (0,a] \cup [b,\infty) \\ iii)\ \exists\ w \in L^2_{loc}(a,b; H^1(\Omega)) \text{ such that } u \in sign(w) \ \text{ a.e. in } \Omega \\ \quad \text{and } \displaystyle\int_a^b\int_\Omega (\xi_t u - Dw \cdot D\xi) = \int_a^b\int_\Gamma \xi g,\ \forall \xi \in \mathcal{C}^1((a,b)\times\overline{\Omega}), \\ \quad \text{compactly supported;} \end{cases} \tag{1.6}$$

here $\underline{u}_0$ is given by (1.4). So, the limit function u is a solution of a Hele–Shaw problem for $t \in I$ and u is a constant function in Ω, for $t \in \mathbb{R}^+ \setminus I$. On the other hand, we see that I may be empty; for instance, when $\int_\Gamma g \geq 0$ and $⨍ u_0 \geq 1$, then $u(t) = ⨍ u_0 + \frac{t}{|\Omega|} \int_\Gamma g \geq 1$, for all $t \geq 0$.

The existence and uniqueness of a weak solution of (1.3) and (1.1) were extensively studied in the case where the Dirichlet boundary condition is prescribed at some part of the lateral boundary, but there are few works with the Neumann boundary condition; we cite for instance [11, 13]. Briefly, the main difficulty in considering the Neumann boundary condition remains in the control of the H^1-norm of w in Ω; the L^2-norm of Dw in Ω is insufficient, and we must control the average of w; this is the aim of Lemma 3 and Lemma 4.

Finally, we notice that using the results of [14], all the arguments of this paper remain true for the study of the limit of the solution of the Stefan problem with nonhomogeneous Neumann boundary condition, as the specific heat c goes to 0. In other words, the limit, as $c \to 0$, of the solution of the Stefan problem with nonhomogeneous Neumann boundary condition is the unique solution of (1.6) with the same initial data $\underline{u}_0$ given by (1.4).

To prove these results, we will use abstract arguments of nonlinear semigroup theory. So, we will be interested in the limit, as $m \to \infty$, of the solution to the stationary problem

$$v = \Delta v^m + f \text{ on } \Omega, \quad \frac{\partial v^m}{\partial n} = g \text{ on } \partial\Omega$$

for any $f \in L^1(\Omega)$ and $g \in L^1(\Gamma)$. This is the aim of Section 2. We recall that this problem was completely solved when $g \equiv 0$ (see [5] for $\left|⨍ f\right| \leq 1$ and [6] for any $f \in L^1(\Omega)$). In Section 3, we give a new proof of existence and uniqueness of a weak solution to the generalized Hele–Shaw problem (1.3) under natural conditions on initial data $\chi_0 \in L^2(\Omega)$ and $g \in L^2(\Gamma)$. In Section 4, we prove existence and uniqueness of a solution to (1.1) and (1.6) and we prove the convergence result (1.5).

2. The elliptic problem

We consider, first, the elliptic problem

$$v = \Delta v^m + f \text{ on } \Omega, \quad \frac{\partial v^m}{\partial n} = g \text{ on } \partial\Omega \tag{2.1}$$

with $f \in L^1(\Omega)$ and $g \in L^1(\Gamma)$. Applying Theorem 22 in [8], for any $m > 0$, there exists a unique solution v of (2.1) in the sense that

$$\begin{cases} v \in L^1(\Omega), \ w := |v|^{m-1} v \in W^{1,1}(\Omega), \ \text{a.e. } \Omega \\ \displaystyle\int_\Omega Dw \cdot D\xi = \int_\Omega (f - v)\xi + \int_\Gamma g\xi, \ \forall \xi \in \mathcal{C}^1(\overline{\Omega}). \end{cases} \tag{2.2}$$

Moreover, if v and $\hat{v}$ are two solutions corresponding to $f, \hat{f} \in L^1(\Omega)$ and $g, \hat{g} \in L^1(\Gamma)$ then (cf. Proposition E in [5])

$$\int_\Omega (v - \hat{v})^+ \le \int_\Omega (f - \hat{f})^+ + \int_\Gamma (g - \hat{g})^+ \tag{2.3}$$

and

$$\int_\Omega |v - \hat{v}| \le \int_\Omega |f - \hat{f}| + \int_\Gamma |g - \hat{g}|. \tag{2.4}$$

As $m \to \infty$, one has

Proposition 1. *Let* $f \in L^1(\Omega)$, $g \in L^1(\Gamma)$ *and for* $m > 0$, *let* v_m *be the unique solution of* (2.1).

1) *If*

$$\Big| {\textstyle -\!\!\!\!\!\!\int} f + \frac{1}{|\Omega|} \int_\Gamma g \Big| < 1,$$

there exists a unique solution (v, w) *of*

$$\left\{ \begin{array}{l} v \in L^1(\Omega), \ w \in W^{1,1}(\Omega), \ v \in sign(w) \ a.e. \ on \ \Omega \\ \displaystyle\int_\Omega Dw \cdot D\xi = \int_\Omega (f - v)\xi + \int_\Gamma g\xi, \ \forall \ \xi \in \mathcal{C}^1(\overline{\Omega}), \end{array} \right. \tag{2.5}$$

$v_m \to v$ *in* $L^1(\Omega)$ *and* $|v_m|^{m-1} v_m \rightharpoonup w$ *in* $W^{1,1}(\Omega)$, *as* $m \to \infty$.

2) *If*

$$\Big| {\textstyle -\!\!\!\!\!\!\int} f + \frac{1}{|\Omega|} \int_\Gamma g \Big| \ge 1,$$

then $v_m \to {\textstyle -\!\!\!\!\!\!\int} f + \frac{1}{|\Omega|} \int_\Gamma g$ *in* $L^1(\Omega)$, *as* $m \to \infty$.

First, we prove the following lemma.

Lemma 1. *Let* $f \in L^1(\Omega)$, $g \in L^1(\Gamma)$ *and* v_m *be the solution of* (2.1). *Then,* v_m *is precompact in* $L^1(\Omega)$.

Proof. According to [5] (step 3 of the proof of Theorem B′), for all $\omega \subset\subset \Omega$, v_m is precompact in $L^1(\omega)$, and since

$$\|v_m\|_{L^1(\Omega)} \le \|f\|_{L^1(\Omega)} + \|g\|_{L^1(\Gamma)},$$

there exists $m_k \to \infty$ and $v \in L^1(\Omega)$ such that

$$v_{m_k} \to v \qquad \text{a.e. } \Omega. \tag{2.6}$$

First, we assume that $f \in L^2(\Omega)$ and $g \in L^2(\Gamma)$; then we have

$$\|v_m\|_{L^2(\Omega)} \le C \left(\|f\|_{L^2(\Omega)} + \|g\|_{L^2(\Gamma)} \right),$$

where C depends only on Ω. This implies that v_m is weakly precompact in $L^2(\Omega)$ and in $L^1(\Omega)$. Then, using (2.6) we deduce that v_m is precompact in $L^1(\Omega)$.

Now, let $f \in L^1(\Omega)$ and $g \in L^1(\Gamma)$. We consider $f_\varepsilon \in L^2(\Omega)$ and $g_\varepsilon \in L^2(\Gamma)$ such that $f_\varepsilon \to f$ in $L^1(\Omega)$ and $g_\varepsilon \to g$ in $L^1(\Gamma)$, as $\varepsilon \to 0$. Using the first step of the proof, we denote by $v_{m\varepsilon}$ the corresponding solution, which is compact in $L^1(\Omega)$. Using (2.4) for $m \geq n \geq 1$, we have

$$\begin{aligned} \|v_n - v_m\|_1 &\leq \|v_n - v_{n\varepsilon}\|_1 + \|v_m - v_{m\varepsilon}\|_1 + \|v_{n\varepsilon} - v_{m\varepsilon}\|_1 \\ &\leq 2\{\|f - f_\varepsilon\|_1 + \|g - g_\varepsilon\|_1\} + \|v_{n\varepsilon} - v_{m\varepsilon}\|_1. \end{aligned}$$

So,

$$\limsup_{n\to\infty} \|v_n - v_m\|_1 \leq 2\{\|f - f_\varepsilon\|_1 + \|g - g_\varepsilon\|_1\} \quad \to 0, \text{ as } \varepsilon \to 0;$$

then v_m is precompact in $L^1(\Omega)$. □

Proof of Proposition 1. If

$$|\fint f + \frac{1}{|\Omega|}\int_\Gamma g| < 1,$$

then using Lemma 1, part 1) of the proposition follows exactly in the same way as in Theorem B in [5].

Let us prove part 2). Due to (2.4), it is enough to prove it for $f \in L^2(\Omega)$, $g \in L^2(\Gamma)$ and $|\fint f + \frac{1}{|\Omega|}\int_\Gamma g| > 1$. We may assume without lost of generality that $\fint f + \frac{1}{|\Omega|}\int_\Gamma g > 1$.

According to [5], we have

$$\{(v_m)^m - C_m\}_{m\geq 1} \text{ is bounded in } W^{1,1}(\Omega) \tag{2.7}$$

where $C_m = \fint (v_m)^m$. Using Lemma 1, there exists $m_k \to \infty$ such that $v_k := v_{m_k} \to v$ in $L^1(\Omega)$, and using (2.7) we have $\tilde{w}_k := (v_{m_k})^{m_k} - C_{m_k} \rightharpoonup \tilde{w}_\infty$ in $W^{1,1}(\Omega)$ and almost everywhere on Ω. Then, using Jensen's inequality and the fact that $\fint v_k = \fint f + \frac{1}{|\Omega|}\int_\Gamma g > 1$, we have

$$\fint (v_k^+)^{m_k} \geq \left(\fint v_k^+\right)^{m_k} \geq \left(\fint f + \frac{1}{|\Omega|}\int_\Gamma g\right)^{m_k} \to \infty \ ;$$

since

$$C_{m_k}\frac{|\{v_k > 0\}|}{|\Omega|} \geq \fint (v_k^+)^{m_k} - \fint |\tilde{w}_k|$$

we deduce $C_{m_k} \to \infty$. Then $\frac{\tilde{w}_k}{C_{m_k}} \to 0$ almost everywhere and

$$\left(\frac{v_k}{C_{m_k}}\right)^{\frac{1}{m_k}} = \left(1 + \frac{\tilde{w}_k}{C_{m_k}}\right)^{\frac{1}{m_k}} \to 1 \text{ a.e.},$$

so that $v = \lim_{m_k\to\infty}(C_{m_k})^{\frac{1}{m_k}}$ is constant almost everywhere on Ω and equal to $⨍ v = ⨍ f + \frac{1}{|\Omega|}\int_\Gamma g$. □

These results may be stated in terms of operators in $L^1(\Omega)$. For $m \geq 1$ and $g \in L^1(\Gamma)$, let A_m^g be the operator defined in $L^1(\Omega)$, by

$$A_m^g v = -\Delta |v|^{m-1} v \tag{2.8}$$

with

$$\begin{aligned}\mathcal{D}(A_m) \quad = \quad & \{v \in L^1(\Omega)\ ;\ w := |v|^{m-1}v \in W^{1,1}(\Omega),\ \Delta w \in L^1(\Omega) \\ & \qquad \text{and } \int_\Omega (Dw \cdot D\xi + \Delta w \xi) = \int_\Gamma g\xi, \quad \forall \xi \in \mathcal{C}^1(\overline{\Omega})\}.\end{aligned}$$

Then A_m^g is m-accretive in $L^1(\Omega)$ and $A_m^g \to A^g$ in the graph sense, where A^g is the multivalued m-accretive operator in $L^1(\Omega)$ defined by

$$z \in A^g v \Leftrightarrow \begin{cases} v,\ z \in L^1(\Omega),\ ⨍ z = \frac{1}{|\Omega|}\int_\Gamma g \text{ and} \\ \text{either } v = \mu \text{ a.e. on } \Omega \text{ with } \mu \in \mathbb{R},\ |\mu| \geq 1 \\ \ \text{or there exists } w \in W^{1,1}(\Omega) \text{ such that} \\ v \in sign(w) \text{ a.e. on } \Omega \text{ and} \\ \int_\Omega Dw \cdot D\xi = \int_\Omega z\xi + \int_\Gamma g\xi \quad \forall \xi \in \mathcal{C}^1(\overline{\Omega}). \end{cases} \tag{2.9}$$

Indeed, A^g being defined as above, for $f \in L^1(\Omega)$, we have

$$v + A^g v \ni f \Leftrightarrow \begin{cases} v \in L^1(\Omega) \ \ ⨍ v = ⨍ f + \frac{1}{|\Omega|}\int_\Gamma g \text{ and} \\ \text{either } v = \mu \text{ a.e. on } \Omega \text{ with } \mu \in \mathbb{R},\ |\mu| \geq 1 \text{ or} \\ \text{there exists } w \text{ such that } (v, w) \text{ is the solution of (2.9),} \end{cases}$$

so that according to Proposition 1, there exists a unique solution v of $v + A^g v \ni f$ and

$$v = \lim_{m\to\infty} (I + A_m^g)^{-1} f.$$

Corollary 1. *Let $f \in L^1(\Omega)$, $g \in L^1(\Gamma)$ and consider $f_m \in L^1(\Omega)$, $g_m \in L^1(\Gamma)$ such that, as $m \to \infty$,*

$$g_m \to g \quad \text{ in } L^1(\Gamma) \text{ and } f_m \to f \quad \text{ in } L^1(\Omega).$$

Then

$$(I + A_m^{g_m})^{-1} f_m \to (I + A^g)^{-1} f \quad \text{ in } L^1(\Omega), \text{ as } m \to \infty.$$

Proof. Using (2.4), we have

$$\begin{aligned}&\|(I + A_m^{g_m})^{-1} f_m - (I + A^g)^{-1} f\|_1 \\ &\leq \|(I + A_m^{g_m})^{-1} f_m - (I + A_m^g)^{-1} f\|_1 + \|(I + A_m^g)^{-1} f - (I + A^g)^{-1} f\|_1\end{aligned}$$

$$\leq \int_\Gamma |g_m - g| + \int_\Omega |f_m - f| + \|(I + A^g_m)^{-1}u_0 - (I + A^g)^{-1}u_0\|_1.$$

Then, using the Proposition 1, the result of the corollary follows. □

Proposition 2. *For any $g \in L^1(\Gamma)$, $\overline{\mathcal{D}(A^g)} = D_1 \cup D_2 =: D$, where $D_1 = \{u \in L^\infty(\Omega) : |u| \leq 1\}$, $D_2 = \{u \equiv \mu : \mu \in \mathbb{R},\ |\mu| \geq 1\}$ and $\overline{\mathcal{D}(A^g)}$ denote the closure in $L^1(\Omega)$ of the domain of A^g.*

Proof. By the definition of A^g it is clear that $\overline{\mathcal{D}(A^g)} \subseteq D$ and $D_2 \subset \overline{\mathcal{D}(A^g)}$. Now we prove that $D_1 \subseteq \overline{\mathcal{D}(A^g)}$. For this aim let $u \in D_1$ and consider u_ε a sequence of D_1, such that $|⨍ u_\varepsilon + \frac{\varepsilon}{|\Omega|}\int_\Gamma g| \leq 1$ for all $\varepsilon > 0$ and $u_\varepsilon \to u$ in $L^1(\Omega)$, as $\varepsilon \to 0$. Using Proposition 1, $u_\varepsilon \in R(I + \varepsilon A^g)$ and $(I + \varepsilon A^g)^{-1}u_\varepsilon \in \mathcal{D}(A^g)$, for all $\varepsilon > 0$.

Next, we show that

$$(I + \varepsilon A^g)^{-1}u_\varepsilon \to u \quad \text{in } L^1(\Omega) \text{ as } \varepsilon \to 0, \tag{2.10}$$

which concludes the proof. Since $\text{sign}(\varepsilon r) = \text{sign}(r)$ for all $\varepsilon > 0$ and $r \in \mathbb{R}$, $(I + \varepsilon A^g)^{-1}u_\varepsilon = (I + A^{\varepsilon g})^{-1}u_\varepsilon$, and using Corollary 1, we have

$$(I + \varepsilon A^g)^{-1}u_\varepsilon \to (I + A^0)^{-1}u \quad \text{in } L^1(\Omega) \text{ as } \varepsilon \to 0.$$

As $\|u\|_\infty \leq 1$ then $(I + A^0)^{-1}u = u$ so that (2.10) follows. □

Now if $u_0 \in L^1(\Omega)$ and $g \in L^1(\Gamma)$ are given, by the general theory of evolution equations (see [1], [4], [9]), for any $m \geq 1$ there exists a unique mild solution $u_m \in \mathcal{C}\left([0,\infty); L^1(\Omega)\right)$ of

$$\frac{du_m}{dt} + A^g_m u_m \ni 0 \text{ on } (0,\infty) \quad u_m(0) = u_0 \tag{2.11}$$

which is given by the exponential formula

$$u_m(t) = e^{-tA^g_m}u_0 = \lim_{n\to\infty}\left(I + \frac{t}{n}A^g_m\right)^{-n}u_0.$$

Using the Brezis–Pazy theorem, for regular perturbations of a nonlinear semigroup, and the Proposition 1, we have

Corollary 2. *Assume that $u_0 \in D$. If for $m \geq 1$, $g_m \in L^1(\Gamma)$ and $u_{0m} \in L^1(\Omega)$ are such that $g_m \to g$ in $L^1(\Gamma)$ and $u_{0m} \to u_0$ in $L^1(\Omega)$, as $m \to \infty$, then*

$$e^{-tA^g_m}u_{0m} \to e^{-tA^{gm}}u_0 \quad \text{in } \mathcal{C}\left([0,\infty); L^1(\Omega)\right), \text{ as } m \to \infty.$$

3. The generalized Hele–Shaw problem

The aim of this section is the study, using nonlinear semigroup theory, of the existence and uniqueness of a weak solution to the two-phase Hele–Shaw problem (1.3) with a natural initial datum $\chi_0 \in D_1$.

Theorem 1. *Let $\chi_0 \in D_1$, $g \in L^2(\Gamma)$,*

$$\mu(t) = \fint \chi_0 + \frac{t}{|\Omega|}\int_\Gamma g, \qquad \textit{for any } t \geq 0 \tag{3.1}$$

and

$$T = \max\{\ t > 0 : |\mu(t)| \leq 1\ \}. \tag{3.2}$$

If $T > 0$, then there exists a unique solution u of the generalized Hele–Shaw problem in the following sense:

$$\begin{cases} u \in L^\infty(Q_T),\ \exists\ w \in L^2\left(0,T;H^1(\Omega)\right),\ u \in sign(w)\ a.e.\ Q_T \\ and \iint \xi_t u + \int \xi(0,\cdot)\chi_0 = \iint Dw\cdot D\xi + \int\int_\Gamma g\xi \\ \forall \xi \in H^1(Q_T),\ \xi(T,\cdot) \equiv 0 \end{cases} \tag{3.3}$$

where $Q_T = (0,T)\times\Omega$. Moreover, $u(t) = e^{-tA^g}\chi_0$, for any $t \in [0,T)$.

Remark 1. This theorem gives the existence and uniqueness of a weak solution u to the generalized Hele–Shaw problem with initial data $\chi_0 \in D_1$. Actually, if we assume that $g \geq 0$ and $\chi_0 \geq 0$, then using (2.3) we have $u \geq 0$ so that u is the unique weak solution of the one-phase Hele–Shaw problem. Note that there exist particular choices of negative g and nonnegative χ_0 such that the one-phase Hele–Shaw problem still has a solution (cf. [10]).

Remark 2. In the one-phase Hele–Shaw problem, T as given above is the time when the physical model breaks down (cf. [11]).

In order to prove the theorem, we need the following lemmas:

Lemma 2. *Let $f \in L^2(\Omega)$, $g \in L^2(\Gamma)$ and $w \in H^1(\Omega)$ such that*

$$\int_\Omega Dw\cdot D\xi = \int_\Omega f\xi + \int_\Gamma g\xi, \quad \forall\ \xi \in \mathcal{C}^1(\overline{\Omega}).$$

Then for all $\xi \in W^{2,1}(\Omega)\cap L^\infty(\Omega)$, $\xi \geq 0$ and $\frac{\partial\xi}{\partial\eta} = 0$ on Γ we have

$$\int_\Omega w^+(-\Delta\xi) \leq \int_{[w>0]} f\xi + \int_{\Gamma\cap[w>0]} g\xi, \tag{3.4}$$

and

$$\int_\Omega w^-(-\Delta\xi) \leq \int_{[w<0]} (-f)\xi + \int_{\Gamma\cap[w<0]} (-g)\xi. \tag{3.5}$$

Proof. Since $z = -w$ is the solution of

$$\int_\Omega Dz \cdot D\xi = \int_\Omega (-f)\xi + \int_\Gamma (-g)\xi, \quad \forall\ \xi \in \mathcal{C}^1(\overline{\Omega}),$$

(3.5) is a consequence of (3.4).

Let us prove (3.4). We consider the sequence j_q defined by

$$j_q(r) = (r^+)^{\frac{1}{q}+1}, \quad \forall r \in \mathbb{R}$$

where $q \in \mathbb{N}$ and $q \geq 1$. Then, $j_q(w) \in H^1(\Omega)$ and for all ξ as in the lemma we have

$$\begin{aligned}
&\int_\Omega j_q(w)(-\Delta\xi) = \int Dj_q(w) \cdot D\xi = \int_\Omega j_q'(w) Dw \cdot D\xi \\
&= \int_\Omega Dw \cdot D(\xi j_q'(w)) - \int_\Omega \xi j_q''(w)|Dw|^2 \\
&= \int_\Omega f\xi j_q'(w) + \int_\Gamma g\xi j_q'(w) - \int \xi j_q''(w)|Dw|^2 \leq \int_\Omega f\xi j_q'(w) + \int_\Gamma g\xi j_q'(w).
\end{aligned}$$

This implies that, for all $q \geq 1$, we have

$$\int_\Omega (w^+)^{\frac{1}{q}+1}(-\Delta\xi) \leq (\frac{1}{q} + 1 \left\{ \int_\Omega f\xi(w^+)^{\frac{1}{q}} + \int_\Gamma g\xi(w^+)^{\frac{1}{q}} \right\}.$$

As $q \to \infty$, we obtain (3.4). $\square$

Lemma 3. *Let $\varepsilon > 0$, $u, \hat{u} \in L^\infty(\Omega)$, $g \in L^2(\Gamma)$ and $w \in H^1(\Omega)$ such that $u \in sign(w)$ almost everywhere in Ω, $|\hat{u}| \leq 1$ and*

$$\int_\Omega Dw \cdot D\xi = \int_\Omega \frac{u - \hat{u}}{\varepsilon} + \int_\Gamma g\xi, \quad \forall \xi \in \mathcal{C}^1(\overline{\Omega}),\ \forall\ \varepsilon > 0.$$

If $|⨍ u| < 1$, then

$$\|w\|_{L^1(\Omega)} \leq \frac{C}{1 - |⨍ u|} \|g\|_{L^1(\Gamma)}$$

where C is a constant depending only on Ω.

Proof. First, applying Lemma 2, for any $\xi \in W^{2,1}(\Omega) \cap L^\infty(\Omega)$ with $\xi \geq 0$, $\frac{\partial \xi}{\partial n} = 0$ on $\partial\Omega$, we have

$$\int_\Omega w^+(-\Delta\xi) \leq \int_{\Gamma \cap [w>0]} \xi g - \int_{[w>0]} \frac{u - \hat{u}}{\varepsilon} \xi \leq \int_{\Gamma \cap [w>0]} \xi g \leq \|\xi\|_{L^\infty(\Omega)} \|g\|_{L^1(\Gamma)}.$$

Let ξ_0 be the solution of

$$\begin{cases} -\Delta\xi_0 = u - ⨍ u & in \quad \Omega \\ \frac{\partial\xi_0}{\partial n} = 0 & on \quad \partial\Omega \\ ⨍\xi_0 = 0; & \end{cases}$$

we have $\xi_0 \in W^{2,p}(\Omega)$, for any $1 < p < \infty$, and

$$\|\xi_0\|_{L^\infty} \leq C\Big\|u - ⨍ u\Big\|_{L^\infty} \leq C,$$

where C is a constant depending only on Ω. Set $\xi = \xi_0 + C$; we have $\xi \geq 0$ and

$$\int_\Omega w^+\Big(u - ⨍ u\Big) = \int_\Omega |w|(-\Delta\xi) \leq 2C\|g\|_{L^1(\Gamma)},$$

and since $w^+u = w^+$ almost everywhere in Ω, we have

$$\Big(1 - ⨍ u\Big)\int_\Omega w^+ \leq 2C\|g\|_{L^1(\Gamma)}. \tag{3.6}$$

Now, using (3.5), with ξ_1 the solution of

$$\begin{cases} -\Delta\xi_1 = -u + ⨍ u & in \quad \Omega \\ \frac{\partial\xi_1}{\partial n} = 0 & on \quad \partial\Omega \\ ⨍\xi_1 = 0, & \end{cases}$$

and since $w^-u = -w^-$, we have

$$\Big(1 + ⨍ u\Big)\int_\Omega w^- \leq 2C\|g\|_{L^1(\Gamma)}. \tag{3.7}$$

From (3.6) and (3.7) we deduce that

$$\Big(1 - \Big|⨍ u\Big|\Big)\int_\Omega |w| \leq 2C\|g\|_{L^1(\Gamma)},$$

which completes the proof. □

Proof of Theorem 1. First we show that the mild solution is a solution of (3.3). By definition of a mild solution, $u(t) = L^1 - \lim u_\varepsilon(t)$ uniformly for $t \in [0,T]$, where for $\varepsilon > 0$, u_ε is an ε-approximate solution corresponding to a subdivision $t_0 = 0 < t_1 < \cdots < t_{n-1} < T = t_n$ with $t_i - t_{i-1} < \varepsilon$, defined by $u_\varepsilon(0) = \chi_0$, $u_\varepsilon(t) = u_i$ for $t \in (t_{i-1}, t_i]$ where $u_i \in L^1(\Omega)$ satisfies

$$\frac{u_i - u_{i-1}}{t_i - t_{i-1}} + A^g u_i \ni 0;$$

that is, there exists w_ε defined by $w_\varepsilon(t) = w_i$ on (t_{i-1}, t_i) where for all $i = 1, \dots, n$,

$$\begin{cases} w_i \in H^1(\Omega), \ u_i \in \mathrm{sign}(w_i) \text{ a.e. } \Omega, \\ \displaystyle\int_\Omega Dw_i \cdot D\xi = \int_\Gamma g\xi - \int \frac{u_i - u_{i-1}}{t_i - t_{i-1}}\xi, \ \forall\, \xi \in \mathcal{C}^1(\overline{\Omega}). \end{cases} \tag{3.8}$$

Since $T > 0$ and $u_\varepsilon(t) \to u(t)$ in $L^1(\Omega)$ as $\varepsilon \to 0$ uniformly for $t \in [0, T]$, for $\varepsilon > 0$ small enough, one has $\left|⨍ u_i\right| \leq \theta$ for $i = 1, \dots, n$ with $\theta < 1$ independent of ε. Using Lemma 3,

$$\left|⨍ w_i\right| \leq C_1 \|g\|_{L^1(\Gamma)} \qquad \text{for } i = 1, \dots, n, \tag{3.9}$$

with C_1 independent of ε.

By density we can replace in (3.8) ξ by w_i; we get

$$\int_\Omega |Dw_i|^2 = \int_\Gamma g w_i - \int_\Omega \frac{|w_i| - w_i u_{i-1}}{t_i - t_{i-1}} \leq \|w_i\|_{L^2(\Gamma)} \|g\|_{L^2(\Gamma)},$$

and then, by the Poincaré inequality, using (3.9),

$$\|Dw_i\|_{L^2(\Omega)} \leq C_2 \|g\|_{L^2(\Gamma)} \tag{3.10}$$

with C_2 independent of ε.

It follows from (3.9) and (3.10) that w_ε is bounded in $L^\infty(0, T; H^1(\Omega))$ as $\varepsilon \to 0$. Let $\varepsilon_k \to 0$ such that $w_{\varepsilon_k} \rightharpoonup w$ in $L^2(0, T; H^1(\Omega))$. On the other hand, since $u_\varepsilon \to u$ in $L^1(Q)$ and $u_\varepsilon \in sign(w_\varepsilon)$ almost everywhere on Q, at the limit $u \in sign(w)$ almost everywhere on Q.

Finally, let $\tilde{u}_\varepsilon$ be the function from $[0, T]$ into $L^1(\Omega)$ defined by $\tilde{u}_\varepsilon(t_i) = u_i$ and suppose $\tilde{u}_\varepsilon$ is linear in $[t_{i-1}, t_i]$; for $\xi \in H^1(\overline{Q})$ with $\xi(T, \cdot) \equiv 0$

$$\int_0^T \int_\Omega \tilde{u}_\varepsilon \xi_t + \int_\Omega \chi_0 \xi(0, \cdot) = \int_0^T \int_\Omega Dw_\varepsilon \cdot D\xi + \int_0^T \int_\Gamma w_\varepsilon g.$$

Passing to the limit we conclude that u is a solution of (3.3).

To complete the proof, we have to show the uniqueness of the solution to (3.3). If (u_1, w_1) and (u_2, w_2) satisfy (3.3), then

$$\int_0^T \int_\Omega (u_1 - u_2)\xi_t + D(w_1 - w_2) \cdot D\xi = 0$$

for all $\xi \in \mathcal{C}^1(\overline{Q})$ with $\xi(T, \cdot) \equiv 0$ with $u_1 \in \mathrm{sign}(w_1)$ and $u_1 \in \mathrm{sign}(w_1)$ almost everywhere on Q. So, applying Lemma A in the appendix of [6] with $H = L^2(\Omega)$, $V = H^1(\Omega)$, $a(u, v) = \int DuDv$, $u = u_1 - u_2$ and $v = w_1 - w_2$, the uniqueness follows. $\square$

Remark 3. In the proof of Theorem 1, we see that the main difficulty in considering the Neumann boundary condition remains in the control of the H^1-norm of w_ϵ with the L^2-norm of Dw_ϵ in Ω. This is obvious if one prescribed Dirichlet boundary condition in some part of Γ, by using the Poincaré inequality; otherwise, we need to control the average of $of w_\varepsilon$ in Ω; this is the aim of Lemma 3.

4. The limit as $m \to \infty$

Now, let $g \in L^2(\Omega,)\ u_0 \in L^2(\Omega)$ and consider the porous-medium equation (1.1). First, we state the following existence and uniqueness result of a weak solution:

Proposition 3. *For any $m \geq 1$, there exists a unique u_m, a solution of* (1.1) *in the sense of*

$$\begin{cases} u_m \in L^2(Q),\ w_m := |u_m|^{m-1}u_m \in L^2_{loc}\left(0,\infty; H^1(\Omega)\right) \\ \displaystyle\int_0^\infty\int_\Omega \xi_t u_m + \int_\Omega \xi(0,\cdot)u_0 = \int_0^\infty\int_\Omega Dw_m.D\xi + \int_0^\infty\int_\Gamma \xi g \\ \forall \xi \in \mathcal{C}^1([0,\infty)\times\overline{\Omega}) \text{ compactly supported.} \end{cases} \tag{4.1}$$

Moreover, $u_m(t) = e^{-tA_m^g}u_0$, *for any* $t \geq 0$.

And, as $m \to \infty$, we have

Theorem 2. *Set*

$$\mu(t) = ⨍ u_0 + \frac{t}{|\Omega|}\int_\Gamma g \quad \text{for } t \geq 0, \quad I = \{t \geq 0 : |\mu(t)| \leq 1\} := [a,b]$$

with $a = b = +\infty$ *if* $I = \emptyset$, *and let* u_m *be the solution of* (4.1).

There exists $u \in \mathcal{C}([0,\infty); L^1(\Omega))$ *such that*

$$u_m \to u \quad \text{in } \mathcal{C}((0,\infty); L^1(\Omega)),\ \text{as } m \to \infty. \tag{4.2}$$

If $u_0 \geq 0$, *then* u *is the unique solution of the following problem:*

$$\begin{cases} i)\ \ u \in \mathcal{C}\left([0,\infty); L^1(\Omega)\right),\ u(0) = \underline{u}_0, \\ ii)\ u(t) \equiv \mu(t) \text{ a.e. on } \Omega \text{ for any } t \in (0,a] \cup [b,\infty) \\ iii)\ \exists\ w \in L^2_{loc}(a,b; H^1(\Omega)) \text{ such that } u \in sign(w) \text{ a.e. on } \Omega \\ \quad \text{and } \displaystyle\int_a^b\int_\Omega \xi_t u - Dw\cdot D\xi = \int_a^b\int_\Gamma \xi g,\ \forall \xi \in \mathcal{C}^1((a,b)\times\overline{\Omega}), \\ \qquad \text{compactly supported} \end{cases} \tag{4.3}$$

where $\underline{u}_0$ *is given by* (1.4).

In order to prove Proposition 3, we need the following result:

Lemma 4. *For any $m > 0$, there exists a constant C depending on m, Ω and N, such that*

$$\|u^m\|_{L^2(\Omega)} \leq C\left\{\|u\|^m_{L^1(\Omega)} + \|Du^m\|_{L^2(\Omega)}\right\}$$

for any $u \in L^1(\Omega)$ such that $u^m := |u|^{m-1}u \in H^1(\Omega)$.

Proof. Let $u \in L^1(\Omega)$ such that $u^m := |u|^{m-1}u \in H^1(\Omega)$ for $m > 0$ fixed. Using Lemma A.16 of [3], we have

$$\|u^m\|_{L^2(\Omega)} \leq \lambda^m|\Omega|^{\frac{1}{2}} + K\Big\{\Big(\frac{|\Omega|}{|[|u| < \lambda]|}\Big)^{\frac{1}{2}} + 1\Big\}\|Du^m\|_{L^2(\Omega)}$$

for all $\lambda > 0$. On the other hand, we see that

$$|[|u| < \lambda]| = |\Omega| - |[|u| \geq \lambda]| \geq |\Omega| - \frac{1}{\lambda}\|u\|_{L^1(\Omega)},$$

so that

$$\|w\|_{L^2(\Omega)} \leq \lambda^m|\Omega|^{\frac{1}{2}} + K\Big\{\Big(\frac{\lambda|\Omega|}{\lambda|\Omega| - \|u\|_{L^1(\Omega)}}\Big)^{\frac{1}{2}} + 1\Big\}\|Du^m\|_{L^2(\Omega)}$$

for all $\lambda > \frac{1}{|\Omega|}\|u\|_{L^1(\Omega)}$. Choosing, for instance, $\lambda = \frac{2}{|\Omega|}\|u\|_{L^1(\Omega)}$, the result follows. $\square$

Proof of Proposition 3. To show the uniqueness of a solution u of (4.1), we apply Lemma A in the appendix of [6] in the same way as in the proof of Proposition 2.

To prove that the mild solution $u = u_m$ satisfies (4.1), we consider, as in the proof of Proposition 2, an ε-approximate solution u_ε corresponding to a subdivision $t_0 < t_1 < \cdots < t_{n-1} < T \leq t_n$. We have $u_\varepsilon(t) = u_i$ on $(t_{i-1}, t_i]$ with $u_i \in L^2(\Omega)$, $w_i := |u_i|^{m-1}u_i \in H^1(\Omega)$ and

$$\int_\Omega Dw_i \cdot D\xi = \int_\Gamma g\xi - \int \frac{u_i - u_{i-1}}{t_i - t_{i-1}}\xi,\ \forall\, \xi \in \mathcal{C}^1(\overline{\Omega}). \tag{4.4}$$

It follows that

$$\|u_i\|_{L^1(\Omega)} \leq \|u_0\|_{L^1(\Omega)} + i\varepsilon\int_\Gamma |g|,$$

so that

$$\|u_\varepsilon(t)\|_{L^1(\Omega)} \leq M_1 := \|u_0\|_1 + T\int_\Gamma |g|, \quad \forall t \in [0, T], \tag{4.5}$$

and, using Lemma 4 and (4.5), we have

$$\|w_i\|_{H^1(\Omega)} \leq C\left(1 + \|Dw_i\|_{L^2(\Omega)}\right) \tag{4.6}$$

with C independent of ε. Now, replacing ξ by w_i in (4.4), we get

$$\begin{aligned}\frac{1}{m+1}\int_\Omega |u_i|^{m+1} + \varepsilon\int_\Omega |Dw_i|^2 &\le \varepsilon\int_\Gamma g w_i + \frac{1}{m+1}\int_\Omega |u_{i-1}|^{m-1}\\ &\le \varepsilon\|g\|_{L^2(\Gamma)}\|w_i\|_{H^1(\Omega)} + \frac{1}{m+1}\int_\Omega |u_{i-1}|^{m+1}.\end{aligned}$$

This implies that

$$\begin{aligned}&\frac{1}{m+1}\int_\Omega |u_i|^{m+1} + \varepsilon\int_\Omega |Dw_i|^2\\ &\le \frac{1}{m+1}\int_\Omega |u_{i-1}|^{m+1} + \varepsilon C\|g\|_{L^2(\Gamma)}\left(1+\|Dw_i\|_{L^2(\Omega)}\right)\end{aligned}$$

so that $w_\varepsilon := |u_\varepsilon|^{m-1}u_\varepsilon$ satisfies

$$\begin{aligned}&\frac{1}{m+1}\int |u_\varepsilon|^{m+1} + \int_0^T\int_\Omega |Dw_\varepsilon|^2\\ &\le \frac{1}{m+1}\int |u_0|^{m+1} + TC\|g\|_{L^2(\Gamma)} + \left(\int_0^T\int_\Omega |Dw_\varepsilon|^2\right)^{1/2}.\end{aligned}$$

This implies that Dw_ε is bounded in $L^2(Q)$; then (4.6) implies that w_ε is bounded in $L^2(0,T;H^1(\Omega))$, and there exists a subsequence that we denote again by ε such that, as $\varepsilon \to 0$,

$$w_\varepsilon \to |u|^{m-1}u \quad \text{weakly in } L^2(0,T;H^1(\Omega)).$$

At last, let $\tilde{u}_\varepsilon$ be the function from $[0,t_n]$ into $L^1(\Omega)$ defined by $\tilde{u}_\varepsilon(t_i) = u_i$, where $\tilde{u}_\varepsilon$ is linear in $[t_{i-1},t_i]$; for $\xi \in W^{1,1}(0,T;L^1(\Omega)) \cap L^2(0,T;H^1(\Omega))$ with $\xi(T,\cdot) \equiv 0$

$$\int_0^T\int_\Omega \tilde{u}_\varepsilon\xi_t + \int_\Omega u_0\xi(0,\cdot) = \int_0^T\int_\Omega Dw_\varepsilon\cdot D\xi + \int_0^T\int_\Gamma g\xi. \tag{4.7}$$

Passing to the limit in (4.7), we get that u is a solution of (4.1), which ends the proof of the proposition. □

Proof of Theorem 2. First, we prove the uniqueness of a solution u of (4.3). By definition, a solution $u(t)$ of (4.3) is perfectly defined on $[0,a] \cup [b,\infty)$. On the other hand, applying Theorem 1, for $a < \alpha < \beta < b$, we find $u = u_\alpha$ on $(\alpha,\beta)\times\Omega$, where u_α is the mild solution of

$$\begin{cases} \dfrac{du_\alpha}{dt} + A^g u_\alpha \ni 0 \ \text{ on } \ (\alpha,\beta)\\ u_\alpha(\alpha) = u(\alpha). \end{cases}$$

Then, if u_1 and u_2 are two solutions of (4.3), by the contraction property of mild solutions, we obtain

$$\|u_1(t) - u_2(t)\|_{L^1} \leq \|u_1(\alpha) - u_2(\alpha)\|_{L^1}, \quad \forall\ a < \alpha \leq t < b.$$

Since $u_1(\alpha) - u_2(\alpha) \to 0$ in $L^1(\Omega)$ as $\alpha \to a$, we conclude $u_1 = u_2$ on $(a,b) \times \Omega$.

For the existence of a solution to (4.3), let $u(t) = e^{-tA^g}\underline{u}_0$, for $t \geq 0$. By assumption, u satisfies (4.3–i). Being a mild solution it is clear that $u(t) \in D$, for any $t \geq 0$ and $⨍ u(t) = \mu(t)$; then u satisfies (4.3–ii). At last, by Theorem 1, u satisfies (4.3–iii).

Now, as the solution of (4.1) (respectively (4.3)) is given by $u_m(t) = e^{-tA^g_m}u_0$ (respectively $u(t) = e^{-tA^g}\underline{u}_0$), the convergence result (4.2) follows from the following lemma, which is based on an idea of [7] (see also [15]), and this ends the proof of the theorem.

Lemma 5. *Let $u_0 \in L^1(\Omega)$, $u_0 \geq 0$ and $g \in L^1(\Gamma)$. As $m \to \infty$, we have*

$$e^{-tA^g_m}u_0 \to e^{-tA^g}\underline{u}_0 \text{ in } \mathcal{C}\left((0,\infty); L^1(\Omega)\right) \tag{4.8}$$

where $\underline{u}_0$ is given by (1.4).

Proof. Let $0 < \delta \leq t_1 < t_2 < \infty$. For all $t \in [t_1, t_2]$, we have

$$\begin{aligned}\|e^{-tA^g_m}u_0 - e^{-tA^g}\underline{u}_0\|_1 &\leq \|e^{-tA^g_m}u_0 - e^{-(t-\delta)A^g_m}e^{-\delta A^0_m}u_0\|_1 \\ &\quad + \|e^{-(t-\delta)A^g_m}e^{-\delta A^0_m}u_0 - e^{-(t-\delta)A^g}\underline{u}_0\|_1 + \|e^{-(t-\delta)A^g}\underline{u}_0 - e^{-tA^g}\underline{u}_0\|_1.\end{aligned}$$

Using the L^1 contraction property of the operators A^g_m and A^g, we have

$$\begin{aligned}\|e^{-tA^g_m}u_0 - e^{-tA^g}\underline{u}_0\|_1 &\leq \|e^{-\delta A^g_m}u_0 - e^{-\delta A^0_m}u_0\|_1 \\ &\quad + \|e^{-(t-\delta)A^g_m}e^{-\delta A^0_m}u_0 - e^{-(t-\delta)A^g}\underline{u}_0\|_1 + \|\underline{u}_0 - e^{-\delta A^g}\underline{u}_0\|_1.\end{aligned}$$

And, since

$$\|(I + \lambda A^g_m)^{-1}u_0 - (I + \lambda A^0_m)^{-1}u_0\|_1 \leq \lambda \int_\Gamma |g|, \quad \forall\ \lambda > 0,$$

$$\|e^{-\delta A^g_m}u_0 - e^{-\delta A^0_m}u_0\|_1 \leq \delta \int_\Gamma |g|$$

and

$$\begin{aligned}\|e^{-tA^g_m}u_0 - e^{-tA^g}\underline{u}_0\|_1 \quad &\leq \quad \delta \int_\Gamma g + \|e^{-(t-\delta)A^g_m}e^{-\delta A^0_m}u_0 - e^{-(t-\delta)A^g_\infty}\underline{u}_0\|_1 \\ &\qquad + \|\underline{u}_0 - e^{-\delta A^g_\infty}\underline{u}_0\|_1.\end{aligned} \tag{4.9}$$

Recall that, as $m \to \infty$ (cf. [2]),

$$e^{-\delta A_m^0} u_0 \to \underline{u}_0 \quad \text{ in } L^1(\Omega)$$

and $\underline{u}_0 \in D$; then, using the Corollary 2, we have

$$\sup_{t\in[t_1,t_2]} \|e^{-(t-\delta)A_m^g} e^{-\delta A_m^0} u_0 - e^{-(t-\delta)A_\infty^g} \underline{u}_0\|_1 \to 0, \tag{4.10}$$

and (4.9) implies

$$\lim_{m\to\infty} \sup_{t\in[t_1,t_2]} \|e^{-tA_m^g} u_0 - e^{-tA_\infty^g} \underline{u}_0\|_1 \le \delta \int_\Gamma g + \|\underline{u}_0 - e^{-\delta A_\infty^g} \underline{u}_0\|_1 \quad \forall\ \delta > 0.$$

At last, let $\delta \to 0$; then the result follows. □

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