LARGE TIME BEHAVIOR OF SOLUTIONS TO SOME DEGENERATE PARABOLIC EQUATIONS

N. Igbida

C.M.A.F. Universidade de Lisboa

1649-003 Lisboa, Portugal

Article paru dans Comm. Partial Differential Equations 26 (2001), no. 7-8, 1385–1408

Abstract : The purpose of this paper is to study the limit in $L^1(\Omega)$, as $t \to \infty$, of solutions of initial-boundary-value problems of the form $u_t - \Delta w = 0$ and $u \in \beta(w)$ in a bounded domain Ω with general boundary conditions $\frac{\partial w}{\partial \eta} + \gamma(w) \ni 0$. We prove that a solution stabilizes by converging as $t \to \infty$ to a solution of the associated stationary problem. On the other hand, since in general these solutions are not unique, we characterize the true value of the limit and comment the results on the related concrete situations like the Stefan problem and the filtration equation.
Key words : Degenerate parabolic equation, Stefan problem, Filtration equation, Asymptotic behavior, Nonlinear semigroup.

1 Introduction

This paper deals with the large time behavior of the solution of the initial-boundary-value problems of the form :

$$(P) \qquad \begin{cases} u_t - \Delta w = 0, \quad u \in \beta(w) & \text{in } Q = (0,\infty)\times\Omega \\ -\dfrac{\partial w}{\partial \eta} \in \gamma(w) & \text{on } \Sigma = (0,\infty)\times\partial\Omega \\ u(0) = u_0 & \text{in } \Omega \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with a smooth boundary Γ, $\dfrac{\partial w}{\partial \eta}$ is the normal derivative of w and the nonlinearities β and γ are maximal monotone graphs in $\mathbb{R}$ (see [9]) such that $\mathcal{D}(\gamma)\cap\mathcal{D}(\beta) \neq \emptyset$. In particular γ may be multi-valued and this allows the boundary conditions to include the Dirichlet (taking $\mathcal{D}(\gamma) = \{0\}$) and the Neumann condition (taking $\gamma \equiv 0$) as well as many other possibilities. Also, β may be multivalued, so that (P) is a mathematical model of various phenomena with changes of states where discontinuous points of β are values at which phase transitions take place. On the other hand, β may be a continuous function in $\mathbb{R}$, then (P) is the filtration equation which includes the flow in Porous Media, the heat flow in material with temperature dependent conductivity and biological models.
Due to general conditions on β, γ and u_0 it is known that it is necessary to introduce a suitable class of generalized solutions. In [7], the authors treat (P) in the context of nonlinear semigroups theory and prove that problem of type (P) has a unique generalized solution u. Moreover, $u(t) = S(t)u_0$ where $S(t)$ is a continuous nonlinear semigroup of order preserving contraction operators in $L^1(\Omega)$. We are interested to the limit of $u(t)$, as $t \to \infty$.

Setting $\varphi := \beta^{-1}$, the case φ strictly increasing continuous function is probably the most covered in the literature. N. D. Alikakos and R. Rostamian proved for $\varphi(r) = |r|^{m-1}r$ and Neumann boundary condition that $u(t) \to ⨍ u_0$, in $L^1(\Omega)$, as $t \to \infty$. Diaz and Vrabie treated the case $\varphi \in \mathcal{C}(\mathbb{R}) \times \mathcal{C}^1(\mathbb{R}\setminus\{0\})$, $\varphi(0) = 0$, $\varphi'(r) \geq C|r|^{\alpha-1}$ ($C > 0$ and $\alpha > 1$) with Dirichlet boundary condition and obtained $u(t) \to 0$, in $L^1(\Omega)$, as $t \to \infty$; also A. Pazy (cf. [29]) obtained the more precise estimate $|u(t)| \leq M(1/t)^{\frac{1}{\alpha-1}}$, for any $t > 0$, and also some results about the regularising effect in the case $\alpha > \frac{N-2}{N}$ (see also [5] and [34] in relation with the regularising effect from $L^1(\Omega)$ to $L^\infty(\Omega)$).

More generally, Mazon and Toledo (cf. [25]) considered the case of general γ and $\varphi \in \mathcal{C}(\mathbb{R}) \times \mathcal{C}^1(\mathbb{R} \setminus \{a_1, ..., a_n\})$, $\varphi(0) = 0$ with $\varphi'(r) > 0$ for $r \neq a_i$, $i = 1, .., n$. They proved that $u(t)$ stabilizes by converging in $L^1(\Omega)$, as $t \to \infty$, to a constant function k with $k \in \gamma^{-1}(0)$.

It this paper we generalize these results, to a more general case where β is any maximal monotone graph everywhere defined. Moreover, since the limiting (as $t \to \infty$) process exhibits some non-uniqueness properties, we also purpose to recognize the true value of the limit of the solution. Actually, we prove that $u(t)$ stabilizes, as $t \to \infty$, to a function $\underline{u} \in \mathcal{K}$, where $\mathcal{K}$ is a set of equilibrium points of (P), in other words, the set of solutions of the stationary problem associated to (P) given by

$$\mathcal{K} = \Big\{ \ z \in L^1(\Omega) \ ; \ \exists \ c \in \gamma^{-1}(0) \text{ s.t. } \ z(x) \in \beta(c) \text{ a.e. } x \in \Omega \ \Big\}.$$

If β is singlevalued on $\gamma^{-1}(0)$, then $\mathcal{K} = \Big\{ \ z \equiv \beta(c) \ ; \quad c \in \gamma^{-1}(0) \ \Big\}$, so that, if in addition $\gamma^{-1}(0) = \{0\}$, then $\mathcal{K}$ is a singleton and we deduce that $u(t) \to 0$, in $L^1(\Omega)$ as $t \to \infty$. But, in general $\mathcal{K}$ is not a singleton and it will be interesting to recognize $\underline{u}$ among $\mathcal{K}$ (one could see [9] where the problem of the characterization of the limit of a solution for general evolution problem governed by a maximal monotone graph in Hilbert spaces is posed on page 167 Pb. 14). In the case of Neumann boundary condition, the conservation of the average is an ingredient that we use to identify this limit for a large class of initial datum $u_0 \in L^1(\Omega)$. Otherwise, we restrict ourselves to linear boundary conditions and prove that the limit of $u(t)$, as $t \to \infty$ is the unique solution of some elliptic problem which depend on u_0.

For instance, assume that $u_0 \geq 0$, $\gamma \equiv 0$ and φ is the continuous function given by

$$\varphi(r) = \begin{cases} (r-1)^+ & \text{if } r \geq 0 \\ \\ cr & \text{if } r < 0 \end{cases} \tag{1.1}$$

with $c \geq 0$. Then, (P) reads the following initial Neumann boundary value problem

$$
(1.2) \qquad \begin{cases} u_t - \Delta\varphi(u) = 0, & \text{in } Q \\ \dfrac{\partial}{\partial\eta}\varphi(u) = 0 & \text{on } \Sigma \\ u(0) = u_0 & \text{in } \Omega \end{cases}
$$

which arises in connection with the Stefan problem (see for instance [13], [31] and the references therein). It is not difficult to prove that, for any $u_0 \in L^\infty(\Omega)$, the generalized solution u is the unique weak solution of (1.2). The limit of $u(t)$ as $t \to \infty$, is closely connected to a problem that attracted considerable interest ; it concerns the nature and the evolution of the so called "Mushy region", which is the interior of the set in which $\varphi(u) = 0$, i.e. $M = [0 < u < 1]$. But the problem seems not to be well understood in general. So far, despite the nonincreasing property of $M(\tau) = M \cap \{t = \tau\}$ for the case of Dirichlet boundary condition (cf. [33] and [21]), to our knowledge there is no results on the characterization of the limiting mushy region, i.e. the part of $M(0)$ that will never be reduced by the diffusion. Notice that in this direction, for the case of nonhomogeneous Dirichlet boundary condition, interesting results about the disappearance of $M(t)$ after a finite time (resp. existence for each time), if some energical criterion is fulfilled (resp. if it is not fulfilled) was obtained first by Meirmanov [27] and generalized by Gotz and Zaltzman [22] (one can see also [26] and [30] where examples of the appearance of the mushy region were constructed). Notice also, that other related asymptotic behavior concerning the limit of w, as $t \to \infty$, may be found in [16, 23, 32, 8] and the references therein. Actually, our results (cf. Theorem 2., Corollary 4. and Proposition 3.) applied to (1.2) implies that $M(t)$ is nonincreasing and

$$
(1.3) \qquad u(t) \to \underline{u}_0 \text{ in } L^1(\Omega), \text{ as } t \uparrow \infty
$$

and

$$
(1.4) \qquad \underline{u}_0 = \begin{cases} ⨍ u_0 & \text{if } ⨍ u_0 \notin (0,1) \\ u_0 \chi_{[\underline{w}=0]} + \chi_{[\underline{w}>0]} & \text{if } ⨍ u_0 \in (0,1) \text{ and } u_0 \geq 0 \end{cases}
$$

where $\underline{w}$ is the unique solution of

$$
(1.5)\qquad \begin{cases} \underline{w} \in H^2(\Omega),\ \underline{w} \geq 0,\ 0 \leq \Delta \underline{w} + u_0 \leq 1, \\[2ex] \underline{w}(\Delta \underline{w} + u_0 - 1) = 0 \text{ a.e } \Omega, \\[2ex] \dfrac{\partial \underline{w}}{\partial \eta} = 0 \text{ on } \Gamma. \end{cases}
$$

So that, as $t \uparrow \infty$, $M(t)$ disappears if $⨍ u_0 \notin (0,1)$; otherwise it may not be empty and the set $[\underline{w} = 0]$ (resp. $[\underline{w} > 0]$) represents the mushy region that will never be reduced by the diffusion (resp. the mushy region that will be transformed by the diffusion).

It is interesting to observe that (1.5) is the so called "mesa problem" and $\underline{u}_0$ given by (1.4) is a projection of u_0 on the mesa of height 1. This kind of projections appears as an initial boundary layer in the study of the singular limit of solutions of some degenerate evolution problems with inconsistent initial data (see [18, 12, 20, 6, 24]). For instance, for the porous medium equation $u_t = \Delta u^m$ with Neumann boundary condition and nonnegative initial data u_0, it was proved in [6] that the solution u_m converges, as $m \to \infty$, to $\underline{u}_0$. In fact, there is a close connection between the limit of u_m as $m \to \infty$ and the limit of a solution of (1.2) as $t \to \infty$. Through the transformation $v_k(x,t) = u(x,kt)$, one sees formally that

$$
(1.6)\qquad \lim_{k\to\infty} v_k(x,t) = \lim_{t\to\infty} u(x,t),
$$

v_k is a solution of

$$
\begin{cases} \dfrac{\partial v_k}{\partial t} = \Delta w_k,\ w_k \in \varphi_k(v_k) & \text{in } (0,\infty) \times \Omega \\[2ex] \dfrac{\partial w_k}{\partial \eta} = 0 & \text{on } (0,\infty) \times \partial\Omega \\[2ex] v_k(0) = u_0 & \text{in } \Omega \end{cases}
$$

with $\varphi_k(r) = k\varphi(r)$, for any $r \in \mathbb{R}$ and φ_k converges to the graph

$$\varphi_\infty(r) = \begin{cases} 0 & \text{if } 0 \le r \le 1 \\ \\ [0,\infty) & \text{if } r = 1 \\ \\ (-\infty, 0] & \text{if } r = 0, \end{cases}$$

which is also the limit of the application $r \to (r^+)^m$, as $m \to \infty$. So that, one expects that the limit (1.6) is also $\underline{u}_0$.

Let us notice that problems of type (P) may be studied in the framework of nonlinear evolution equations governed by sub-differential operators in a Hilbert space H, where H is a quotient space of the dual space of $H^1(\Omega)$. So that, applying general stability results of Brezis (cf. [9]) and Bruck (cf. [11]), one deduces that a solution $u(t)$, converges in H to a solution of the stationary problem, as $t \to \infty$. This approach was used by Damlamian and Kenmochi (cf. [16]) for mixed time dependent boundary conditions and by Haraux and Kenmochi (cf. [23]) for time dependent Neumann boundary conditions, but the characterization of the limit of $u(t)$, was left open. Our approach is completely different and is inspired by the works of [7, 25] and independently by the works of [18, 12, 20, 6] concerning singular limits (mesa problem) of porous medium equation.

To give a brief description of our approach, notice that in order to prove the stabilisation results, one need the orbits of the semigroup $S(t)$, i.e. $\{S(t)u_0 \ ; \ t \ge 0\}$, to be relatively compact. For this aim, we will use regularity results of [7] for the elliptic problem associated to (P) to show that the resolvent are relatively compact from $L^\infty(\Omega)$ into $L^1(\Omega)$. Then, we deduce that the orbits are relatively compact in $L^1(\Omega)$ by using the same arguments of [25]. On the other hand, to recognize the true value of the limit of $u(t)$, we will integrate the equation of (P) with respect to t and pass to the limit in the integrated equation. We also use the decreasing property of the "Mushy regions" of (P) (cf. Proposition 3.) to prove that the limit is a projection on a mesa.

In Section 2, we will give some preliminaries and state our main results. In Section 3, we recall some results of [7] concerning the elliptic problem associated to (P), for general graphs φ and γ, and prove the L^1-convergence result for a solution $u(t)$, as $t \to \infty$. We also give the definition of "Mushy regions" for the problem (P) and prove their decreasing property. In Section 4, we prove characterization results of the true value of the limit among solutions of the stationary problem.

2 Preliminaries and main results

Throughout this section, Ω is a bounded domain of $\mathbb{R}^N$ with smooth boundary Γ, φ and γ are maximal monotone graphs in $\mathbb{R}$ such that

$$(H_1) \qquad \mathcal{D}(\varphi) = \mathbb{R},$$

$$(H_2) \qquad \text{either } \mathcal{D}(\gamma) = \mathbb{R} \text{ or } \mathcal{D}(\gamma) = \{0\}$$

and

$$(H_3) \qquad 0 \in \varphi(0) \cap \gamma(0)$$

and, we consider the following evolution problem

$$Pe(u_0, \varphi, \gamma) \qquad \begin{cases} u_t - \Delta w = 0, \quad w \in \varphi(u) & \text{in } Q = (0,\infty) \times \Omega \\ -\dfrac{\partial w}{\partial \eta} \in \gamma(w) & \text{on } \Sigma = (0,\infty) \times \partial\Omega \\ u(0) = u_0 & \text{in } \Omega, \end{cases}$$

with $u_0 \in L^1(\Omega)$. Due to general conditions on φ, γ and u_0, it is necessary to introduce a suitable class of generalized solutions. In [7], the authors treat $Pe(u_0, \varphi, \gamma)$ in the contest of nonlinear semigroups theory. We work here from this point of view, a solution of $Pe(u_0, \varphi, \gamma)$ will be a generalized solution $u \in \mathcal{C}([0,\infty), L^1(\Omega))$ with $u(0) = u_0$. By definition, u is the limit in $\mathcal{C}([0,\infty), L^1(\Omega))$ of sequence of classical solutions u_k of approximating problems in which u_0, φ and γ are replaced by smooth functions. Actually, it is well known that, under the hypothesis $(H_1) - (H_3)$, $Pe(u_0, \varphi, \gamma)$ has a unique generalized solution u, moreover

$$u(t) = S(t)u_0$$

where $S(t)$ is a continuous nonlinear semigroup of order preserving contraction operators in $L^1(\Omega)$.

We denote by φ_0 the principal part of φ, which is the function

$$\varphi_0(r) = \min\{s\ ;\ s \in \varphi(r)\} \qquad \text{for any } r \in \mathbb{R}$$

and by φ^{-1} the inverse of φ, which is the maximal monotone graph given by $r \in \varphi^{-1}(s)$ if and only if $s \in \varphi(r)$. We also introduce the set

$$\mathcal{E} = \Big\{ r \in \mathbb{R}\ ;\ \varphi_0^{-1} \text{ is discontinuous on } r \Big\},$$

where φ_0^{-1} is the principal part of φ^{-1}.
Obviously, if φ is a nondecreasing continuous function, then $\varphi_0 = \varphi$ and φ^{-1} may be multivalued. In particular, $\varphi^{-1}(r)$ is a subinterval of $\mathbb{R}$, for any $r \in \mathcal{E}$.
At last, let us introduce the set $\mathcal{K}_{\varphi\gamma}$, defined by

$$\mathcal{K}_{\varphi\gamma} = \Big\{\ z \in L^1(\Omega)\ ;\ \exists\, c \in \gamma^{-1}(0) \text{ s.t. } z(x) \in \varphi^{-1}(c) \text{ a.e. } x \in \Omega\ \Big\}.$$

It is not difficult to see that $\mathcal{K}_{\varphi\gamma}$ is a closed subset of $L^1(\Omega)$ and is contained in the set of stationary solutions of $Pe(u_0, \varphi, \gamma)$, so that, for any $z \in \mathcal{K}_{\varphi\gamma}$, $S(t)z = z$, for any $t \geq 0$.
To prove the stabilization result, one needs the orbits $\{S(t)u_0\ ;\ t \geq 0\}$ to be relatively compact. Now, it is not possible to obtain this result from the compactness of the semigroup because it is known that if $\varphi(r) = |r|^{m-1}r$ and γ corresponds to Dirichlet boundary condition, then $S(t)\ :\ L^1(\Omega) \to L^1(\Omega)$ is compact if $m > \frac{N-2}{N}$ $(N \geq 3)$ (see [3]) but for $0 < m \leq \frac{N-2}{N}$, even the resolvent are not compact (see [10]). However, for general boundary condition and φ any increasing (strictly) continuous function everywhere defined, Mazon and Toledo proved in their work [25] (see also [2]) that $S(t)u_0$ is relatively compact in $L^1(\Omega)$, for any $u_0 \in L^1(\Omega)$ (one can see also [17] and [1] for Dirichlet and Neumann boundary conditions respectively). The following Theorem is a generalization of those results.

Theorem 1. *For any $u_0 \in L^1(\Omega)$, there exists $\underline{u} \in \mathcal{K}_{\varphi\gamma}$, such that*

$$S(t)u_0 \to \underline{u} \quad \text{in } L^1(\Omega),\ \text{as } t \to \infty.$$

Using this Theorem, let us define the operator $\mathcal{L}_{\varphi\gamma}$ in $L^1(\Omega)$, by

$$\mathcal{L}_{\varphi\gamma}(u_0) = \lim_{t\to\infty} S(t)u_0, \qquad \text{for any } u_0 \in L^1(\Omega). \tag{2.1}$$

Corollary 1. *For any φ and γ such that $(H_1)-(H_3)$ are fulfilled, the operator $\mathcal{L}_{\varphi\gamma}$ is well defined from $L^1(\Omega)$ to $\mathcal{K}_{\varphi\gamma}$ and is an order preserving contraction in $L^1(\Omega)$.*

In the following, we will be interested in recognizing the true value of $\mathcal{L}_{\varphi\gamma}(u_0)$ among the elements of $\mathcal{K}_{\varphi\gamma}$. First, we remark that the structure of $\mathcal{K}_{\varphi\gamma}$ is closely connected to $\gamma^{-1}(0)$ and values of φ^{-1} on $\gamma^{-1}(0)$. So that, as an immediate consequence of Theorem 1., we have :

Corollary 2. *If φ_0^{-1} is continuous in $\gamma^{-1}(0)$, then, for any $u_0 \in L^1(\Omega)$, there exists $k \in \gamma^{-1}(0)$, such that*

$$\mathcal{L}_{\varphi\gamma}(u_0) \equiv \varphi^{-1}(k).$$

In particular, if $\varphi^{-1}(0) = \gamma^{-1}(0) = \{0\}$, then

$$\mathcal{L}_{\varphi\gamma}(u_0) = 0, \qquad \textit{for any } u_0 \in L^1(\Omega).$$

Now, assume that $\gamma \equiv 0$, it is not difficult to see that $⨍ S(t)u_0 = ⨍ u_0$, for any $u_0 \in L^1(\Omega)$. This extra property of the solution gives further information on $\mathcal{L}_{\varphi\gamma}(u_0)$, which enable us to characterize it's value under additional conditions on u_0. This is the aim of the following Theorem.

Theorem 2. *If $\gamma \equiv 0$, then for any $u_0 \in L^1(\Omega)$, $⨍ \mathcal{L}_{\varphi\gamma}(u_0) = ⨍ u_0$ and*

$$\exists\ c \in \varphi(⨍ u_0) \ \textit{such that}\ \mathcal{L}_{\varphi\gamma}(u_0) \in \varphi^{-1}(c) \ \textit{a.e. in}\ \Omega.$$

If, in addition, $\varphi_0(⨍ u_0) \notin \mathcal{E}$, then $\mathcal{L}_{\varphi\gamma}(u_0) = ⨍ u_0$.

Remark 1. *Corollary 2. and the second part of Theorem 2. generalize the results of [25] stated in the case* φ *a strictly increasing function (see also [1] and [17]). Notice also, that the second case of Corollary 2. is the unique one where* $\mathcal{K}_{\varphi\gamma}$ *has a single element.*

The second part of our main results deals with the the characterization of $\mathcal{L}_{\varphi\gamma}(u_0)$ in the case of linear boundary condition and φ is any maximal monotone graph everywhere defined, with $0 \in \varphi(0)$. So, we will assume that γ is such that

$$(H_4) \qquad \gamma(r) = \alpha r \text{ for any } r \in \mathbb{R}$$

with $\alpha \in [0, \infty]$, where the case $\alpha = \infty$ corresponds to Dirichlet boundary condition. We will also assume that the initial data $u_0 \in L^1(\Omega)$ satisfies

$$\varphi_0(\textstyle{⨍} u_0) := m_0 \in \mathcal{E} \quad \text{if } \alpha = 0. \tag{2.2}$$

Theorem 3. *Assume that* (H_1), (H_3) *and* (H_4) *are fulfilled and let* $u_0 \in L^\infty(\Omega)$ *satisfying (2.2). Setting*

$$[l, L] = \begin{cases} \varphi^{-1}(0) & \text{if } \alpha > 0 \\ \\ \varphi^{-1}(m_0) & \text{if } \alpha = 0, \end{cases} \tag{2.3}$$

we have $l \leq \mathcal{L}_{\varphi\gamma}(u_0) \leq L$ *and*

$$\mathcal{L}_{\varphi\gamma}(u_0) = u_0 + \Delta \underline{w} \text{ a.e. in } \Omega \tag{2.4}$$

where $\underline{w}$ *satisfies*

$$\begin{aligned} &\underline{w} \in H^2(\Omega), \ \frac{\partial \underline{w}}{\partial \eta} + \gamma(\underline{w}) \ni 0 \text{ a.e. on } \Gamma \\ &\underline{w} = 0 \text{ a.e. in } \{ \, x \in \Omega \ ; \ l < \mathcal{L}_{\varphi\gamma}(u_0)(x) < L \} \, . \end{aligned} \tag{2.5}$$

Corollary 3. *Under the assumptions of Theorem 3., there exists* $A \subseteq [l \leq u_0 \leq L]$, A_1 *and* A_2 *disjoint subsets of* Ω *such that*

$$\mathcal{L}_{\varphi\gamma}(u_0) = u_0.\chi_A + l.\chi_{A_1} + L.\chi_{A_2}. \tag{2.6}$$

Now, for the sequel, let us introduce the following stationary problem

$$\begin{cases} v - \Delta w = u_0, \;\; w \in \partial I_C(v) & \text{in } \Omega \\ \\ \dfrac{\partial w}{\partial \eta} + \gamma(w) \ni 0 & \text{on } \Gamma \end{cases} \tag{2.7}$$

where $C = [l, L]$ is given by (2.3) and ∂I_C is the sub-differential of I_C, the indicatrice function of C ($I_C(r) = 0$, if $r \in C$ and $I_C(r) = +\infty$, elsewhere). More precisely,

$$\partial I_C(r) = \begin{cases} (-\infty, 0] & \text{if } r = l \\ \\ 0 & \text{if } l < r < L \\ \\ [0, \infty) & \text{if } r = L. \end{cases}$$

Using the results of [7], for any $u_0 \in L^1(\Omega)$ satisfying (2.2), the problem (2.7) has a unique solution that we denote by $\underline{u}_0$, in the following sense

$$\begin{cases} \underline{u}_0 \in L^1(\Omega), \;\; \exists \; \underline{w} \in W^{1,1}(\Omega), \;\; \underline{w} \in \partial I_C(\underline{u}_0) \text{ a.e. in } \Omega, \\ \\ \exists \; \underline{z} \in L^1(\Gamma), \;\; \underline{z} \in \gamma(\underline{w}) \text{ a.e. on } \Gamma \text{ and} \\ \\ \displaystyle\int_\Omega D\underline{w}.D\xi + \int_\Gamma \underline{z}\xi = \int_\Omega (u_0 - \underline{u}_0)\xi, \;\; \forall \xi \in W^{1,\infty}(\Omega). \end{cases} \tag{2.8}$$

Theorem 4. *Assume that* (H_1), (H_3) *and* (H_4) *are fulfilled and let* $u_0 \in L^1(\Omega)$ *satisfying (2.2). If* $u_0 \geq l$, *then* $\mathcal{L}_{\varphi\gamma}(u_0) \geq l$ *and* $\mathcal{L}_{\varphi\gamma}(u_0) = \underline{u}_0$, *where* $\underline{u}_0$ *is given by (2.8).*

Corollary 4. *If, in addition of the hypothesis of Theorem 4., $u_0 \in L^\infty(\Omega)$, then*

$$\mathcal{L}_{\varphi\gamma}(u_0) = u_0 \chi_{[\underline{w}=0]} + L\chi_{[\underline{w}>0]},$$

where $\underline{w}$ is the unique solution of

$$\text{(2.9)} \qquad \begin{cases} \underline{w} \in H^2(\Omega),\ \underline{w} \geq 0,\ l \leq \Delta\underline{w} + u_0 \leq L, \\ \\ \underline{w}(\Delta\underline{w} + u_0 - L) = 0 \ a.e\ \Omega, \\ \\ \dfrac{\partial \underline{w}}{\partial \eta} + \gamma(\underline{w}) \ni 0 \ on\ \Gamma. \end{cases}$$

Remark 2. *In order to give a brief description of our results, let us come back to the example φ given by (1.1), for which the Problem $Pe(u_0, \varphi, \gamma)$ models a free boundary problem involving a solid-liquid phase change of Stefan type. Then, the functions u and $w = \varphi(u)$ represent respectively the Enthalpy and the Temperature of a material assumed to be of H_2O-based system and for which $w = 0$ is the only temperature at which transition of phase takes place. The function u is sometimes called a phase function, since it characterizes the regions occupied either by the liquid or the solid or the mushy region. Indeed, $Q_l(t) = [u(t) \geq 1]$, $Q_s(t) = [u(t) \leq 0]$ and $M(t) = [0 < u(t) < 1]$ represent respectively liquid, solid and mushy region. So, the characterization of the limit of $u(t)$ in $L^1(\Omega)$, as $t \to \infty$, describes the limits of $Q_l(t)$, $Q_s(t)$ and $M(t)$.*

i - *Dirichlet boundary condition : $\varphi(u) = a$ on Γ, with $a \in I\!R$.*
If $a \neq 0$ and $a = \varphi(\alpha)$, then Corollary 2. implies that $u(t) \to \alpha$. Indeed, it is sufficient to take $\psi(r) = \varphi(r + \alpha) - a$ and apply the second part of Corollary 2. with ψ and $u_0 - \alpha$ instead of φ and u_0, respectively. This means that, if we prescribed a liquid (resp. solid) temperature a on the boundary, then all the material in the domain Ω will be in it's liquid (resp. solid) phase at the temperature a. All the mushy region is transformed by the diffusion in this case.
If $a = 0$, the situation is different, Theorem 4. implies that $u(t) \to \underline{u}_0$, which depends on the initial data u_0 and is given by (2.6). This means

that if we prescribe the temperature of phase transition on the boundary, then all the phases of the material may remains as $t \to \infty$, *we may have liquid at the temperature* 1, *the solid at the temperature* 0 *and a part of the initial mushy region may also holds. Corollary 4. characterizes each region in the case where the diffusion in the liquid or the solid is neglected (i.e. one phase Stefan problem).*

ii - *Neumann boundary condition :* $\dfrac{\partial \varphi(u)}{\partial n} = 0$ *on* Γ, *i.e. there is no exchange of Temperature with the exterior of* Ω *through* Γ.
If $⨍ u_0 \notin (0,1)$, *then Theorem 2. implies that* $u(t) \to ⨍ u_0$, *as* $t \to \infty$. *This traduces the fact that the Enthalpy average is either big (i.e.* $⨍ u_0 \geq 1$*) or small (i.e.* $⨍ u_0 \leq 0$*) enough to transform all the material to it's either a liquid (i.e.* $u \geq 1$*) or a solid phase (i.e.* $u \leq 0$*), at the temperature* $\varphi(⨍ u_0)$.
Conversely, if $⨍ u_0 \in (0, L)$, *then Enthalpy average is favorable to get the material, as* $t \to \infty$, *in it's three phases and Corollary 4. characterizes each region, as* $t \to \infty$, *in function of those at time* $t = 0$, *in the case where* $u_0 \geq 0$. *If we abort sign condition on* u_0 *the problem is still open in general, however, under additional condition on* u_0, *Theorem 3. can give a characterization of each region, this will be done in forthcoming works.*

3 Stabilization results

First, let us recall some results of [7] concerning the elliptic problem

$$Ps(f, \phi, \gamma) \qquad \begin{cases} v = \Delta w + f, \ w \in \phi(v) & \text{in } \Omega \\ \\ \dfrac{\partial w}{\partial \eta} + z = 0, \ z \in \gamma(w) & \text{on } \Gamma \end{cases}$$

where ϕ and γ are maximal monotone graphs in $\mathbb{R}$. According to [7], for any $f \in L^1(\Omega)$, ϕ and γ such that $0 \in \phi(0) \cap \gamma(0)$ and

$$B_- < \int_\Omega f < B_+$$

where $B_- = |\Omega| \inf \phi^{-1} + |\Gamma| \inf \gamma$ and $B_+ = |\Omega| \sup \phi^{-1} + |\Gamma| \sup \gamma$, there exists a unique (u, w, z) solution of $Ps(f, \phi, \gamma)$ in the sense

$$\left\{ \begin{array}{l} v \in L^1(\Omega),\ \ w \in W^{1,1}(\Omega),\ \ w \in \phi(v) \text{ a.e. in } \Omega, \\ \\ z \in L^1(\Gamma),\ \ z \in \gamma(w) \text{ a.e. on } \Gamma \text{ and} \\ \\ \displaystyle\int_\Omega Dw.D\xi + \int_\Gamma z\xi = \int_\Omega (f - v)\xi,\ \ \forall \xi \in W^{1,\infty}(\Omega) \end{array} \right. \tag{3.1}$$

and, for any $f_1,\ f_2 \in L^1(\Omega)$, if (v_i, w_i, z_i) is the solution of $Ps(f_i, \phi, \gamma)$ for $i = 1,\ 2$, then

$$\int_\Omega (v_1 - v_2)^+ + \int_\Gamma (z_1 - z_2)^+ \leq \int_\Omega (f_1 - f_2)^+$$

and

$$\int_\Omega |v_1 - v_2| + \int_\Gamma |z_1 - z_2| \leq \int_\Omega |f_1 - f_2|\,.$$

Moreover, if $f \in L^\infty(\Omega)$ then the solution $(v, w, z) \in L^\infty(\Omega) \times H^2(\Omega) \times L^2(\Gamma)$ and one has the following estimates

$$\|v\|_\infty \leq \|f\|_\infty \tag{3.2}$$

and

$$\|w\|_{H^1(\Omega)} \leq C\, \|f\|_\infty\,. \tag{3.3}$$

where C is a constant which depends only on Ω.

Now, assuming that φ and γ are maximal monotone graphs such that (H_1), (H_2) and (H_3) are full field, we define the operator (possibly multivalued) $A_{\varphi\gamma}$, in $L^1(\Omega)$ by

$$\begin{aligned} A_{\varphi\gamma}v &= \Big\{ f \in L^1(\Omega) \ ; \ \exists\, w \in W^{1,1}(\Omega),\ w \in \varphi(v) \text{ a.e. in } \Omega \\ &\quad \exists\, z \in L^1(\Gamma) \text{ s.t. } z \in \gamma(w) \text{ a.e. on } \Gamma \text{ and} \\ &\quad \int_\Omega Dw.D\xi + \int_\Gamma z\xi = \int_\Omega f\xi \text{ for any } \xi \in W^{1,\infty}(\Omega) \Big\}. \end{aligned}$$

Using the preceding arguments it is clear that, for any $\lambda > 0$, the resolvent of $A_{\varphi\gamma}$ defined by $\mathcal{J}_\lambda = (I + \lambda A_{\varphi\gamma})^{-1}$ is an everywhere defined order preserving contraction in $L^1(\Omega)$; so that $A_{\varphi\gamma}$ is m-T-accretive in $L^1(\Omega)$. Moreover, we have (cf. [7])

$$\overline{\mathcal{D}(A_{\varphi\gamma})}^{L^1} = L^1(\Omega).$$

So, using the general theory of nonlinear semigroup of evolution equation, $A_{\varphi\gamma}$ generates a continuous nonlinear semigroup of order preserving contraction $S(t)$, in $L^1(\Omega)$. Moreover, for any $u_0 \in L^1(\Omega)$, $S(t)u_0$ is the unique generalized solution of $Pe(u_0, \varphi, \gamma)$ (cf. Theorem I. of [7]).
By definition of $S(t)$,

$$S(t)u_0 = L^1 - \lim_{\varepsilon \to 0} u_\varepsilon(t) \tag{3.4}$$

uniformly for $t \in [0, \tau]$, where for $\varepsilon > 0$, u_ε is an $\varepsilon-$ approximate solution corresponding to a subdivision $t_0 = 0 < t_1 < ... < t_{n-1} < \tau \leq t_n$, with $t_i - t_{i-1} = \varepsilon$ and defined by $u_\varepsilon(0) = u_0$, $u_\varepsilon(t) = u_i$ for $t \in]t_{i-1}, t_i]$ where $u_i \in L^1(\Omega)$ satisfies

$$\frac{u_i - u_{i-1}}{\varepsilon} + A_{\varphi\gamma}u_i \ni 0. \tag{3.5}$$

In other words, the generalized solution u of $Pe(u_0, \varphi, \gamma)$ is given by the exponential formula

$$\begin{aligned} S(t)u_0 &= e^{-tA_{\varphi\gamma}}u_0 \\ &= \lim_{n\to\infty} \mathcal{J}^n_{t/n}u_0. \end{aligned} \tag{3.6}$$

Proposition 1. *If $u_0 \in L^\infty(\Omega)$, then, the generalized solution u of $Pe(u_0, \varphi, \gamma)$ satisfies*

$$
(3.7)\begin{cases} u \in L^\infty(Q) \cap \mathcal{C}([0,\infty); L^1(\Omega)),\ \exists\ w \in L^2_{loc}\left(0,\infty; H^1(\Omega)\right), \\ w \in \varphi(u)\ \ a.e.\ \ in\ Q,\ \exists\ z \in L^2_{loc}\left(0,\infty; L^2(\Gamma)\right),\ z \in \gamma(w)\ \ a.e.\ \ in\ \Sigma \\ \int_0^\tau \int_\Omega \xi_t u + \int_\Omega \xi(0) u_0 = \int_0^\tau \int_\Omega Dw.D\xi \\ \qquad + \int_0^\tau \int_\Gamma \xi z + \int_\Omega \xi(\tau) u(\tau),\quad \forall \xi \in \mathcal{C}^1([0,\tau]\times\overline{\Omega})\ \ and\ \tau > 0. \end{cases}
$$

Moreover, for any $\tau \geq 0$,

$$
(3.8) \qquad \|u(\tau)\|_\infty \leq \|u_0\|_\infty
$$

and

$$
(3.9) \qquad \int_\Omega j(u(\tau)) + \int_0^\tau \int_\Omega |Dw|^2 + \int_0^\tau \int_\Gamma zw \leq \int_\Omega j(u_0)
$$

where $j\ :\ I\!R \to [0,\infty]$ *is a proper convex s.c.i. function such that* $\varphi = \partial j$.

Proof : This is a quite standard result (cf. [4]). For completeness let give the arguments. Using (3.4) and (3.5), let u_ε the $\varepsilon-$approximate solution with $\varepsilon = \dfrac{\tau}{n}$ and, for $i = 1,\ ....,n$, let $(w_i, z_i) \in H^2(\Omega) \times L^2(\Gamma)$ such that

$$
(3.10) \qquad \begin{cases} u_i - \varepsilon\Delta w_i = u_{i-1},\ w_i \in \varphi(u_i) & \text{in } \Omega \\ \dfrac{\partial w_i}{\partial \eta} + z_i = 0,\ z_i \in \gamma(w_i) & \text{on } \Gamma. \end{cases}
$$

It follows that $u_i \in L^\infty(\Omega)$ and $\|u_i\|_\infty \leq \|u_0\|_\infty$, so that

$$
(3.11) \qquad \|u_\varepsilon\|_\infty \leq \|u_0\|_\infty.
$$

On the other hand, multiplying (3.10) by w_i and using the fact that

$$
\int_\Omega (u_{i-1} - u_i) w_i \leq \int_\Omega j(u_{i-1}) - \int_\Omega j(u_i)
$$

we have

$$
(3.12) \qquad \int_\Omega j(u_i) + \varepsilon \int_\Omega |Dw_i|^2 + \varepsilon \int_\Gamma z_i w_i \leq \int_\Omega j(u_{i-1}).
$$

Adding (3.12) from $i = 1$, to n, we get

$$\int_\Omega j(u_\varepsilon(\tau)) + \int_0^\tau \int_\Omega |Dw_\varepsilon|^2 + \int_0^\tau \int_\Gamma w_\varepsilon z_\varepsilon \leq \int_\Omega j(u_0) \tag{3.13}$$

where $w_\varepsilon : [0,\tau] \to H^1(\Omega)$ (resp. $z_\varepsilon : [0,\tau] \to L^2(\Gamma)$) and $w_\varepsilon(t) = w_i$ (resp. $z_\varepsilon(t) = z_i$), for any $t \in]t_{i-1}, t_i]$, $i = 1, ...n$. Thanks to (H_1) and (3.11), w_ε is bounded in $L^\infty((0,\tau)\times\Omega)$, then using the fact that $j \geq 0$ and $z_\varepsilon w_\varepsilon \geq 0$, a.e. in $[0,\tau]\times\Gamma$, we deduce from (3.13) that w_ε is bounded in $L^2(0,\tau; H^1(\Omega))$, then thanks to (H_2), z_ε is bounded in $L^\infty((0,\tau)\times\Gamma)$.
Let $w \in L^2(0,\tau; H^1(\Omega))$, $z \in L^2((0,\tau)\times\Gamma)$ and $\varepsilon_k \to 0$, such that $z_{\varepsilon_k} \to z$ weakly in $L^2((0,\tau)\times\Gamma)$, $w_{\varepsilon_k} \to w$ weakly in $L^2(0,\tau; H^1(\Omega))$ and strongly in $L^2((0,\tau)\times\Omega)$ and in $L^2((0,\tau)\times\Gamma)$. Since, for any $t > 0$, $w_{\varepsilon_k}(t) \in \varphi(u_{\varepsilon_k}(t))$ a.e. in Ω (resp. $z_{\varepsilon_k}(t) \in \gamma(w_{\varepsilon_k}(t))$ a.e. on Γ), then using the strong convergence of $u_{\varepsilon_k}(t)$ in $L^1(\Omega)$ (resp. $w_{\varepsilon_k}(t)$ in $L^2(\Gamma)$) and the weak convergence of $w_{\varepsilon_k}(t)$ in $L^2(\Omega)$ (resp. $z_{\varepsilon_k}(t)$ in $L^2(\Gamma)$), we obtain $w(t) \in \varphi(u(t))$ a.e. in Ω (resp. $z(t) \in \gamma(w(t))$ a.e. on Γ).
At last, let $\tilde{u}_\varepsilon$ be the function from $[0,\tau]$ into $L^1(\Omega)$, defined by $\tilde{u}_\varepsilon(t_i) = u_i$, $\tilde{u}_\varepsilon$ is linear in $[t_{i-1}, t_i]$, then (3.10) implies that

$$\begin{aligned} \int_0^\tau \int_\Omega \tilde{u}_\varepsilon \xi_t + \int_\Omega \xi(0) u_0 &= \int_0^\tau \int_\Omega Dw_\varepsilon . D\xi \\ &\quad + \int_0^\infty \int_\Gamma \xi z_\varepsilon + \int_\Omega \xi(\tau) u_\varepsilon(\tau) \end{aligned} \tag{3.14}$$

for any $\xi \in \mathcal{C}^1([0,\tau]\times\overline{\Omega})$. Letting $\varepsilon \to 0$ in (3.11), (3.13) and (3.14), the results of the Proposition follows. ∎

Remark 3. *In general we do not know if weak solutions of $Pe(u_0, \varphi, \gamma)$, i.e. functions $u \in \mathcal{C}\left([0,\infty), L^1(\Omega)\right)$ satisfying (3.7) with $\xi(\tau) = 0$, are unique. However, this is true in the case of linear boundary conditions and also in the case γ and φ locally Lipschitz continuous functions (see for instance [28]).*

Proposition 2. *Under the assumptions $(H_1) - (H_3)$, for any $u_0 \in L^1(\Omega)$, $S(t)u_0$ is relatively compact in $L^1(\Omega)$.*

In order to prove this Proposition, let us prove, first, the following Lemma.

Lemma 1. *Let $f \in L^\infty(\Omega)$ and $\lambda > 0$. For any $y \in I\!R^N$ and $\xi \in \mathcal{C}^2(\Omega)$ supported in $\{x \in \Omega \ ; \ distance(x,\Gamma) < |y|\}$, we have*

$$\int_\Omega \xi(x)\, |\mathcal{J}_\lambda f(x+y) - \mathcal{J}_\lambda f(x)|\, dx \ \leq \ C\, |y|\, \|\Delta\xi\|_\infty\, \|f\|_\infty + \int_\Omega \xi(x)\, |f(x+y) - f(x)|\, dx$$

where C is a constant depending only on Ω.

Proof : Set $v = \mathcal{J}_\lambda(f)$ and let $(w,z) \in H^1(\Omega) \times L^2(\Gamma)$, such that (v,w,z) is the solution of $Ps(f, \lambda\varphi, \gamma)$.
Using the results of [7], for any $y \in \mathbb{R}^N$ and $\xi \in \mathcal{C}^2(\Omega)$ supported in $\{x \in \Omega \ ; \ \mathrm{distance}(x,\Gamma) < |y|\}$, we have

$$\begin{aligned} \int_\Omega \xi(x)\, |v(x+y) - v(x)|\, dx \ &\leq \ \int_\Omega |\Delta\xi|\, |w(x+y) - w(x)|\, dx \\ &\quad + \int_\Omega \xi(x)\, |f(x+y) - f(x)|\, dx \\ &\leq \ |y|\, \|\Delta\xi\|_\infty\, |\Omega|^{\frac{1}{2}}\, \|\nabla w\|_2 \\ &\quad + \int_\Omega \xi(x)\, |f(x+y) - f(x)|\, dx \end{aligned}$$

then, using (3.3), the result follows. ∎

Proof of Proposition 2. : First, using Lemma 1., we see that for any $\lambda > 0$ fixed and B a bounded subset of $L^\infty(\Omega)$, $\mathcal{J}_\lambda B$ is a relatively compact subset of $L^1(\Omega)$. Indeed, for any $\{f_n\} \subseteq B$, with an appropriate choice of ξ, we have

$$\lim_{|y|\to 0} \sup_n \int_{\Omega'} |\mathcal{J}_\lambda f_n(x+y) - \mathcal{J}_\lambda f_n(x)| = 0$$

for any $\Omega' \subset\subset \Omega$ which implies, with (3.2), that $\{\mathcal{J}_\lambda f_n\}$ is relatively compact in $L^1(\Omega)$. Then, the proof of the relative compactness of $S(t)u_0$, in $L^1(\Omega)$, follows exactly in the same way of the proof of Theorem 2.2 in [25] (see also [14] Theorem 3). In fact, one proves, firstly, that $S(t)u_0$ is relatively compact for any $u_0 \in L^\infty(\Omega) \cap \mathcal{D}(A_{\varphi\gamma})$ by using the inequality

$$\|S(t)u_0 - \mathcal{J}_\lambda S(t)u_0\|_1 \leq \lambda \inf\{\|v\|_1 \ ; \ v \in A_{\varphi\gamma}u_0\} \ ;$$

then, the compactness of a subsequence of $S(t)u_0$, for $u_0 \in L^1(\Omega)$, follows by approximation of u_0 and the fact that

$$\sup_{t\geq 0}\inf_{s\geq 0} \|S(t)u_0 - S(s)z\|_1 \leq \|u_0 - z\|_1, \quad \text{for any } z \in L^1(\Omega). \tag{3.15}$$

∎

Now, we use terminology and notation from topological dynamics : for any $u_0 \in L^1(\Omega)$, we define the $\omega-$limit set of $Pe(u_0, \varphi, \gamma)$ by

$$\omega_{\varphi\gamma}(u_0) = \left\{ \underline{u} \in L^1(\Omega) \ ; \ \underline{u} = L^1 - \lim_{t_n\to\infty} S(t_n)u_0 \text{ for some sequence } t_n \right\}.$$

This set is possibly empty. Now, it is well known (see [15]) that if $S(t)u_0$ is relatively compact, then $\omega_{\varphi\gamma}(u_0)$ is a non empty compact and connected subset of $L^1(\Omega)$. Furthermore $\omega_{\varphi\gamma}(u_0)$ is invariant under $S(t)$, i.e., $S(t)\omega_{\varphi\gamma}(u_0) \subseteq \omega_{\varphi\gamma}(u_0)$ for any $t \geq 0$. An equilibrium or stationary point is any $z \in L^1(\Omega)$ such that $\omega_{\varphi\gamma}(z) = \{S(t)z\} = \{z\}$.

Corollary 5. *For any $u_0 \in L^1(\Omega)$, $\omega_{\varphi\gamma}(u_0) \neq \emptyset$.*

Proof of Theorem 1. : Using the fact that $\mathcal{K}_{\varphi\gamma}$ is a closed subset of $L^1(\Omega)$ and the inequality (3.15), one see that it is sufficient to prove the Theorem for any $u_0 \in L^\infty(\Omega)$. So, assume that $u_0 \in L^\infty(\Omega)$ and consider $(w, z) \in L^2_{loc}(0, \infty; H^1(\Omega)) \times L^2_{loc}(0, \infty; L^2(\Gamma))$, such that (u, w, z) satisfy (3.7) with $u(t) = S(t)u_0$. Thanks to (3.9) and since $j \geq 0$ and for any $t \geq 0$ $w(t)z(t) \geq 0$ a.e. in Ω, there exists $t_n \to \infty$,

$$\lim_{t_n\to\infty} \int_\Omega |Dw(t_n)|^2 + \int_\Gamma w(t_n)z(t_n) = 0, \tag{3.16}$$

then using (H_1), (3.8) and Poincaré inequality, we deduce that $w(t_n)$ is bounded in $H^1(\Omega)$, as $t_n \to \infty$. On the other hand, thanks to (H_1), (H_2) and (3.8), $z(t_n)$ is bounded in $L^\infty(\Gamma)$. Thanks to Proposition 2., let $\underline{u} \in L^1(\Omega)$ and $t_{nk} \to \infty$, such that $u(t_{nk}) \to \underline{u}$ in $L^1(\Omega)$ and, let $\underline{z} \in L^2(\Gamma)$, $\underline{w} \in H^1(\Omega)$ such that $z(t_{nk}) \to \underline{z}$ weakly in $L^2(\Gamma)$ and $w(t_{nk}) \to \underline{w}$ weakly in $H^1(\Omega)$ and strongly in $L^2(\Omega)$ and in $L^2(\Gamma)$. Then, as in the proof of Proposition 1., by using standard compactness and monotony arguments, we obtain

(3.17) $$\underline{w} \in \varphi(\underline{u}) \text{ a.e. } \Omega \text{ and } \underline{z} \in \gamma(\underline{w}) \text{ a.e. } \Gamma.$$

Passing to the limit in (3.16), through the subsequence t_{nk}, we get

$$\int_\Omega |D\underline{w}|^2 + \int_\Gamma \underline{w}\ \underline{z} \leq 0.$$

So, since $\underline{w}\ \underline{z} \geq 0$ a.e. on Γ, then $D\underline{w} \equiv 0$ a.e. in Ω and $\underline{w}\ \underline{z} = 0$ a.e. on Γ ; which implies that there exists $c \in \mathbb{R}$ such that

(3.18) $$\underline{w} \equiv c \text{ a.e. in } \Omega \quad \text{ and } \underline{z}c = 0 \text{ a.e. on } \Gamma.$$

From this we deduce that $\underline{u} \in \mathcal{K}_{\varphi\gamma}$, which implies that $S(t)\underline{u} = \underline{u}$. Then, the Theorem is an obvious consequence of the contraction property of $S(t)$. ∎

Before to end up this section, we introduce, the so called "Mushy regions" (in terms of Stefan Problem),

$$M_r = \Big\{(x,t) \in [0,\infty) \times \Omega\ ;\ S(t)u_0(x) \in \mathrm{int}(\varphi^{-1}(r))\Big\},$$

for any $r \in \mathcal{E}$, and

$$M_r(t_0) = \{\ x \in \Omega\ ;\ (t_0, x) \in M_r\ \}.$$

The following Proposition is a generalization of results of [33] and [21], and will be useful in the following section.

Proposition 3. *Under the hypothesis $(H_1) - (H_3)$, for any $r \in \mathcal{E}$ and $u_0 \in L^1(\Omega)$,*

(3.19) $$M_r(t_2) \subseteq M_r(t_1) \quad \textit{for any } t_2 > t_1,$$

in the sense of $mes(M_r(t_2) \setminus M_r(t_1)) = 0$.

Proof : Let $r \in \mathcal{E}$ fixed and set $(a,b) = \mathrm{int}(\varphi^{-1}(r))$. Since, $S(t)u_0$ depends continuously in u_0, it is sufficient to prove (3.19), for $u_0 \in L^\infty(\Omega)$. So, assume that $u_0 \in L^\infty(\Omega)$ and consider u_ε, the $\varepsilon-$approximate solution in $[0,\tau]$ given by (3.4) with $\varepsilon = \dfrac{\tau}{n}$. We have

(3.20) $$\{x \in \Omega\ ;\ u_\varepsilon(\tau, x) \in (a,b)\} \subseteq \{x \in \Omega\ ; u_0(x) \in (a,b)\}.$$

Indeed, by definition of u_ε, for $(w_i, z_i) \in H^1(\Omega) \times L^2(\Gamma)$ given by (3.5), for $i = 0, ...n = \tau/\varepsilon$, we have $\Delta w_i = 0$ and $u_i = u_{i-1}$ a.e. in $\{x \in \Omega\ ; u_i(x) \in (a,b)\}$, so that

$$\begin{aligned}
\{x \in \Omega\ ; u_\varepsilon(\tau, x) \in (a,b)\} &= \{x \in \Omega\ ; u_n(x) \in (a,b)\} \\
&\subseteq \{x \in \Omega\ ; u_{n-1}(x) \in (a,b)\} \\
&\subseteq \{x \in \Omega\ ;\ u_i(x) \in (a,b)\},
\end{aligned}$$

for any $0 \leq i \leq n-1$ and $i = 0$, (3.20) follows. Using (3.4) and (3.20), we get

$$\{x \in \Omega\ ;\ S(\tau)u_0(x) \in (a,b)\} \subseteq \{x \in \Omega\ ;\ u_0(x) \in (a,b)\} \tag{3.21}$$

for any $\tau \geq 0$.
At last, since for $0 \leq t_1 < t_2 < \infty$, $S(t_2)u_0 = S(t_2 - t_1)S(t_1)u_0$, then the result follows by replacing u_0 by $S(t_1)u_0$, in (3.21). ∎

4 Characterization of the limit

In this section, we assume that γ satisfies (H_4), $u_0 \in L^1(\Omega)$ satisfies (2.2) and we introduce the set

$$\begin{aligned}
S(u_0) = &\{x \in \Omega\ ;\ x \text{ is a Lebesgue point of } \mathcal{L}_{\varphi\gamma}(u_0) \\
&\text{and } \exists\ t_n \to \infty,\ S(t_n)u_0(x) \to \mathcal{L}_{\varphi\gamma}(u_0)(x)\}.
\end{aligned} \tag{4.1}$$

Using Theorem 1., we have mes $\{\Omega \setminus S(u_0)\} = 0$, for any $u_0 \in L^1(\Omega)$. We begin this section by proving a particular case of Theorem 3..

Proposition 4. *Assume φ and γ satisfy (H_1), (H_3) and (H_4), with $\varphi^{-1}(0) = [l, L]$. If $u_0 \in L^\infty(\Omega)$, with $⨍ u_0 \in [l, L]$ if $\alpha = 0$, then*

$$l \leq \mathcal{L}_{\varphi\gamma}(u_0) \leq L \text{ a.e. in } \Omega \tag{4.2}$$

and

$$\mathcal{L}_{\varphi\gamma}(u_0) = u_0 + \Delta \underline{w} \ \textit{a.e. in } \Omega \tag{4.3}$$

where

$$\underline{w} \in H^2(\Omega), \ \frac{\partial \underline{w}}{\partial \eta} + \gamma(\underline{w}) \ni 0 \ \textit{a.e. on } \Gamma \tag{4.4}$$

and

$$\underline{w} = 0 \ \textit{a.e. in } \{ \ x \in \Omega \ ; \ l < \mathcal{L}_{\varphi\gamma}(u_0)(x) < L \} \,. \tag{4.5}$$

If, in addition, $u_0 \geq 0$, then $\mathcal{L}_{\varphi\gamma}(u_0) \geq 0$ and there exists $\underline{w} \geq 0$ such that (4.3) and (4.4) holds and

$$\underline{w} = 0 \ \textit{a.e. in } \{ \ x \in \Omega \ ; \ 0 \leq \mathcal{L}_{\varphi\gamma}(u_0)(x) < L \} \,. \tag{4.6}$$

In order to prove this proposition let prove the following Lemma :

Lemma 2. *Let f_n be a sequence of $L^1(\Omega)$, $f \in L^1(\Omega)$, such that $f_n \to f$ in $L^1(\Omega)$. If $x_0 \in \Omega$ is a Lebesgue point of f such that $\theta_1 < f(x_0) < \theta_2$, for $\theta_1, \ \theta_2 \ \in I\!R$, then, for any $\delta > 0$,*

$$mes\{x \in B(x_0, \delta) \ ; \ \theta_1 < f(x) < \theta_2\} > 0 \tag{4.7}$$

and, there exist $n_0 = n_0(\theta_1, \theta_2, \delta) > 0$, such that

$$mes\{x \in B(x_0, \delta) \ ; \ \theta_1 < f_n(x) < \theta_2\} > 0 \qquad \textit{for any } n \geq n_0. \tag{4.8}$$

Proof : Let $\delta > 0$ fixed. If (4.7) is not true, then

$$f(x) \in (-\infty, \theta_1] \cup [\theta_2, \infty), \ \text{a.e. } x \in B(x_0, \delta). \tag{4.9}$$

On the other hand, since x_0 is a Lebesgue point of f, then (cf. [19])

$$f(x_0) = \sup\left\{ t \in \mathbb{R} \ ; \ \lim_{r \to 0} \frac{|B(x_0, r) \cap [f < t]|}{|B(x_0, r)|} = 0 \right\},$$

so that, using the fact that $\theta_1 < f(x_0) < \theta_2$ and (4.9), we get

$$\begin{aligned} f(x_0) &= \sup\left\{ t \in [\theta_1, \theta_2] \; ; \; \lim_{r \to 0} \frac{|B(x_0, r) \cap [f < t]|}{|B(x_0, r)|} = 0 \right\} \\ &= \theta_2, \end{aligned}$$

which contradicts the hypothesis of the Lemma.
Now assume, that (4.8) is not true, then one can construct a sequence $n_k \to \infty$, such that

$$\text{mes}\,\{x \in B(x_0, \delta) \; ; \; \theta_1 < f_{n_k}(x) < \theta_2\} = 0 \quad \text{for any } n_k,$$

which implies that

$$f_{n_k}(x) \in (-\infty, \theta_1] \cup [\theta_2, \infty) \quad \text{a.e. } x \in B(x_0, \delta), \text{ for any } n_k. \tag{4.10}$$

Since, $f_n \to f$, in $L^1(\Omega)$, then there exists a subsequence of n_k, that we denote again by n_k, such that $f_{n_k} \to f$ a.e. in Ω and, then (4.10) implies $f(x) \in (-\infty, \theta_1] \cup [\theta_2, \infty)$ a.e. $x \in B(x_0, \delta)$, which contradicts (4.7). This ends up the proof of the Lemma. ∎

Proof of Proposition 4. : Firstly, we assume that $\alpha > 0$. Obviously (4.2) is a direct consequence of Theorem 1.. In order to prove (4.3) and (4.4), we consider $u(t) = S(t)u_0$ and $w \in L^2_{loc}(0, \infty; H^1(\Omega))$ given by Proposition 1., such that $w \in \varphi(u)$ a.e. in Q and

$$\int_0^\tau \int_\Omega \xi_t u + \int_\Omega \xi(0) u_0 = \int_0^\tau \int_\Omega Dw.D\xi + \alpha \int_0^\tau \int_\Gamma \xi w + \int_\Omega \xi(\tau) u(\tau) \tag{4.11}$$

for any $\xi \in \mathcal{C}^1([0, \tau] \times \overline{\Omega})$ and $\tau > 0$. It is clear that, for any $t \geq 0$,

$$W(t) = \int_0^t w(s) ds \; \in H^1(\Omega) \tag{4.12}$$

and, by appropriate choice of ξ in (4.11), $W(t)$ is a weak solution of

$$\begin{cases} -\Delta W(t) = u_0 - u(t) & \text{in } \Omega \\ \\ \dfrac{\partial W(t)}{\partial \eta} + \alpha W(t) = 0 & \text{on } \Gamma. \end{cases} \tag{4.13}$$

Since $u_0,\ u(t)\ \in L^\infty(\Omega)$, then $W(t) \in H^2(\Omega) \cap \mathcal{C}(\Omega)$ and (4.13) is satisfied a.e. in $\overline{\Omega}$. Applying Theorem 1., we have $u(t) \to \underline{u} := \mathcal{L}_{\varphi\gamma}(u_0)$ and, thanks to (4.13), there exists $\underline{w} \in H^2(\Omega)$ and a sequence $t_k \to \infty$, such that

(4.14)$W(t_k) \to \underline{w}$ weakly in $H^1(\Omega)$ and strongly in $L^2(\Omega)$ and in $L^2(\Gamma)$,

as $t_k \to \infty$, and $\underline{w}$ satisfies

$$\begin{cases} -\Delta \underline{w} = u_0 - \underline{u} & \text{a.e. in } \Omega \\ \\ \dfrac{\partial \underline{w}}{\partial \eta} + \alpha \underline{w} = 0 & \text{a.e. on } \Gamma \end{cases}$$

which ends up the proof of (4.3) and (4.4).
Now, let us prove (4.5). For this, we we consider $x_0 \in S(u_0)$ fixed such that $l < \underline{u}(x_0) < L$ and we claim that

(4.15) $$\underline{w}(x_0) = 0.$$

Using Lemma 2., for any $\delta > 0$, there exists $t_0 = t_0(l, L, \delta) \geq 0$, such that

$$\text{mes}\,\{x \in B(x_0, \delta)\ ;\ l < u(t,x) < L\} > 0 \quad \text{for any } t \geq t_0$$

then, thanks to Proposition 3.,

$$\text{mes}\,\{x \in B(x_0, \delta)\ ;\ l < u(t,x) < L\} > 0 \quad \text{for any } t \geq 0,$$

which implies, that

(4.16) $$\text{mes}\,\{x \in B(x_0, \delta)\ ;\ W(t,x) = 0\} > 0 \text{ for any } t \geq 0.$$

Since $W(t) \in \mathcal{C}(\Omega)$ and satisfies (4.16) for any $\delta > 0$, then $W(t, x_0) = 0$, for any $t \geq 0$, which implies (4.15).
In the case $\alpha = 0$, we see that, assuming $l \leq ⨍ u_0 \leq L$, implies with Theorem 2., that $l \leq \mathcal{L}_{\varphi\gamma}(u_0) \leq L$, then all the preceding arguments apply exactly in the same way of the case $\alpha > 0$.
To end up the proof, we see that if $u_0 \geq 0$, then $u(t) = S(t)u_0 \geq 0$ a.e. in Ω and thanks to (H_3), $w \in L^2_{loc}(0, \infty; H^1(\Omega))$ given by Proposition 1. is also

nonnegative. So, using (4.12) and (4.14) we deduce that $\underline{w} \geq 0$. ∎

Proof of Theorem 3. : It is clear that the case where either $\alpha > 0$ or $\alpha = 0$ and $m_0 = 0$ follows directly by Proposition 4.. In the case $\alpha = 0$ and $m_0 \neq 0$, we consider the graph $\psi(r) = \varphi(r + \fint u_0) - \varphi_0(\fint u_0)$ and $v_0 = u_0 - \fint u_0$. It is not difficult to see that $\mathcal{L}_{\varphi\gamma}(u_0) = \mathcal{L}_{\psi\gamma}(v_0) + \fint u_0$ and since $\psi(\fint v_0) = 0$, then the results of Theorem follows by applying Proposition 4. with φ and u_0 replaced by ψ and v_0. ∎

Proof of Corollary 3. : This Corollary is an immediate consequence of the fact that $l \leq \mathcal{L}_{\varphi\gamma}(u_0) \leq L$ and $\Delta\underline{w} = 0$ a.e. in $[\underline{w} = 0]$ which includes , by (2.5), $[l < \mathcal{L}_{\varphi\gamma}(u_0) < L]$. ∎

Proof of Theorem 4. : Using the contraction property of $Ps(u_0, \partial I_C, \gamma)$ and $Pe(u_0, \partial I_C, \gamma)$, it is enough to prove the Theorem for $u_0 \in L^\infty(\Omega)$.
In the case $\alpha > 0$ and $\alpha = 0$ with $m_0 = 0$, we can assume without loose of generality that $l = 0$, so that the second part of Proposition 4. implies that $\mathcal{L}_{\varphi\gamma}(u_0) =: \underline{u} \geq 0$ and there exists $\underline{w} \geq 0$ such that (4.3), (4.4) and (4.6) are full filed. On the other hand, it is clear that (4.6) with $\underline{u} \geq 0$ and $\underline{w} \geq 0$, implies that $\underline{w} \in \partial I_C(\underline{u})$, with $C = [l, L]$, so that $\underline{u}$ is a weak solution of $Ps(u_0, \partial I_C, \gamma)$, which is unique.
In the case $\alpha = 0$ and $m_0 \neq 0$, we consider $\psi(r) = \varphi(r+l) - m_0$ and $v_0 = u_0 - l$, so that applying the first part of the proof with φ and γ replaced respectively by ψ and v_0, we deduce that $\mathcal{L}_{\psi\gamma}(v_0)$ is the unique solution of $Ps(v_0, \partial I_{[0,L-l]}, \gamma)$. At last, we use the fact that $\partial I_{[0,L-l]}(r) = \partial I_{[l,L]}(r + l)$ for any $r \in [0, L - l]$ and $\mathcal{L}_{\varphi\gamma}(u_0) = \mathcal{L}_{\psi\gamma}(v_0) + c$ to conclude that $\mathcal{L}_{\varphi\gamma}(u_0)$ is the unique solution of $Ps(v_0, \partial I_{[l,L]}, \gamma)$. ∎

Proof of Corollary 4. : The Corollary is an obvious consequence of the regularity of the unique solution $(\underline{u}_0, \underline{w})$ of $Ps(u_0, \partial I_C, \gamma)$. In fact, if $u_0 \in L^\infty(\Omega)$, then $\underline{u}_0 \in L^\infty(\Omega)$ and $\underline{w} \in H^2(\Omega)$, so that $\Delta\underline{w} = 0$ a.e. in $[\underline{w} = 0]$ and then the Corollary follows. ∎

At last, let us prove the Theorem 2..

Proof of Theorem 2. : Assuming $\gamma \equiv 0$, we have $\fint \mathcal{J}_\lambda f = \fint f$, for any $\lambda > 0$ and $f \in L^1(\Omega)$. Then, for any $u_0 \in L^1(\Omega)$, (3.6) and Theorem 1. imply

$$\fint \mathcal{L}_{\varphi\gamma}(u_0) = \fint S(t)u_0 \tag{4.17}$$

$$= \fint u_0 \tag{4.18}$$

and, there exists $c \in \mathbb{R}$ such that $\mathcal{L}_{\varphi\gamma}(u_0) \in \varphi^{-1}(c)$. Using the fact that, $\varphi^{-1}(c)$ is a subinterval of $\mathbb{R}$, we get $\fint \mathcal{L}_{\varphi\gamma}(u_0) \in \varphi^{-1}(c)$, which implies, with (4.18), that $c \in \varphi(\fint u_0)$. This ends up the proof of the first part of the Theorem.

If, in addition, we assume that $\varphi_0(\fint u_0) \notin \mathcal{E}$, then $\varphi(\fint u_0) = \varphi_0(\fint u_0)$ and $\varphi^{-1}(\varphi_0(\fint u_0)) = \fint u_0$, so that $\mathcal{L}_{\varphi\gamma}(u_0) = \fint u_0$. ∎

ACKNOWLEDGEMENTS

This work was realized while the author was a postdoctoral student at the C.M.A.F. / Universidade de Lisboa and supported by CMAF/FCT. The paper was finished while he visited the LAMFA / Université de Picardie Jule-Verne that he wants to thank for the hospitality.

References

[1] N. D. Alikakos and R. Rostamian. Large time behavior of solutions of neumann boundary value problem for the porous medium equation. *Indiana Univ. Math. J.*, 30:749–785, 1981.

[2] F. Andrew, J.M. Mazon, and J. Toledo. Stabilization of solutions of the filtration equation with absorption and non-linear flux. *NODEA*, 2:267–289, 1995.

[3] P. BARAS. Compacité de l'operateur $f \to u$ solution d'une équation non linéaire $(du/dt) + Au \ni f$. *C. R. Acad. Sci. Paris Sér. A*, 286:1113–1116, 1978.

[4] Ph. BÉNILAN. Équation d'évolution dans un espace de Banach quelconque et applications. Thése, Orsay, 1972.

[5] Ph. BÉNILAN. Sur un problème d'évolution non monotone dans $L^2(\Omega)$. *Pub. Math. Fac. Sc. Besançon.*, 2, 1976.

[6] Ph. BÉNILAN, L. BOCCARDO, and M. HERRERO. On the limit of solution of $u_t = \Delta u^m$ as $m \to \infty$. In M. Bertch et al., editor, *Proceedings Int. Conf.*, Torino, 1989. Some Topics in Nonlinaer PDE's.

[7] Ph. BÉNILAN, M.G. CRANDALL, and P. SACKS. Some L^1 existence and dependence results for semilinear elliptic equations under nonlinear boundary conditions. *Appl. Math. Optim.*, 17:203–224, 1988.

[8] H. BREZIS. Monotonicity methods in Hilbert space and some applications to nonlinear partial differential equations. In E. Zarantonello, editor, *Contribution to Nonlinear Functionnal Analysis.* Academic Press, 1971.

[9] H. BREZIS. *Opérateurs Maximaux Monotones et Semi-Groupes de Contractions dans les espaces de Hilbert.* North-Holland, Amsterdam-London, 1973.

[10] H. BREZIS and A. FRIEDMAN. Nonlinear parabolic equations involving measures as initial conditions. *J. Math. Pures Appl.*, 62:73–97, 1983.

[11] R. E. BRUCK. Asymptotic convergence of nonlinear contraction semigroups in Hilbert space. *J. Func. Anal.*, 18:15–26, 1975.

[12] L. A. CAFFARELLI and A. FRIEDMAN. Asymptotic behavior of solution of $u_t = \Delta u^m$ as $m \to \infty$. *Indiana Univ. Math. J.*, pages 711–728, 1987.

[13] J. CRANK. *Free and Moving Boundary Problems.* Oxford Univ. Press, Oxford, 1984.

[14] C. M. DAFERMOS and M. SLERMOND. Asymptotic behavior for nonlinear contractions semigroup. *J. Func. Anal.*, 13:97–106, 1973.

[15] C.M DAFERMOS. Asymptotic behavior of solutions of evolution equations. In *Nonlinear Evolution Equations (M. G. Crandall ed).* Academic Press, 1978.

[16] A. DAMLAMIAN and N. KENMOCHI. Asymptotic behavior of solutions to a multi-phase Stefan problem. *Jap. J. Appl. Math.*, 3:15–36, 1986.

[17] T. I. DIAZ and I. I. VRABIE. Propriétés de compacité de l'opérateur de Green généralisé pour l'équation des milieux poreux. *C. R. Acad. Sci. Paris Série I*, 309:221–223, 1989.

[18] C.M. ELLIOT, M.A. HERRERO, J.R. KING, and J.R.OCKENDON. The mesa patterns for $u_t = \nabla(u^m \nabla u)$ as $m \to \infty$. *IMA J. Appl. Math.*, 37:147–154, 1986.

[19] L.C. EVANS and R.F. GARIEPY. *Mesure Theory and Fine Propetries of Functions*. Studies in Advanced Mathematics, CRC Press London, 1992.

[20] A. FRIEDMAN and S. HUANG. Asymptotic behavior of solutions of $u_t = \Delta\phi_m(u)$ as $m \to \infty$ with inconsistent initial values. In *Analyse Mathématique et applications*. Gauthier-Villars, 1988.

[21] I. G. GOTZ and B. B. ZALTZMAN. Nonincrease of mushy region in nonhomogeneous Stefan problem. *Quart. Appl. Mat.*, 49:741–746, 1991.

[22] I. G. GOTZ and B. B. ZALTZMAN. On the behavior of mushy region in a Stefan problem. In *Intenational Series of Numerical Mathematics*. Springer, 1991.

[23] A. HARAUX and N. KENMOCHI. Asymptotic behavior of solutions to some degenerate parabolic equations. *Func. Ekvacioj.*, 34:19–38, 1991.

[24] N. IGBIDA. Limite singulière de problèmes d'évolution non linéaires. Thèse de doctorat, Université de Franche-Comté, 1997.

[25] J. M. MAZON and J. TOLEDO. Asymptotic behavior of solutions of the filtration equation in bounded domains. *Dynam. Systems Appl.*, 3:275–295, 1994.

[26] A. M. MEIRMANOV. An example of nonexistence of a classical solution of the Stefan problem. *Dokl. Akad. Nauk SSSR 258*, 3:547–549, 1981. English trans., Soviet Math. Dokl. 23, No 3, 564-566 (1981).

[27] A. M. MEIRMANOV. On the desappearance of the mushy region in the Stefan problem with spherical symmetry. *Dinamika Sploshnoisredy lavrentyev Institute of Hydrodynamics*, 91:89–99, 1989.

[28] Ma. NIEZGODKA and I. PAWLOW. A generalized Stefan problem in several space variables. *Appl. Math. Optim.*, 9:193–223, 1983.

[29] A. PAZY. The Lyapunov method for semigroups of nonlinear contractions in Banach spaces. *J. Analyse Math.*, 40:239–262, 1981.

[30] M. Primicerio. Mushy region in phase-change problem. In Lang, editor, *Applied Nonlinear Functional Analysis*. Frankfurt/Main, 1982.

[31] J.F. Rodrigues. The stefan problem revisited. In J. F. Rodrigues, editor, *Mathematical Models for Phase Change Problems, International Series of Numerical Mathematics, vol. 88*. Birkhauser Verlag Basel, 1989.

[32] J.F. Rodrigues. Variational methods in the Stefan problem. In A. Visintin, editor, *Modeling and Analysis of Phase Transition and Hysteresis*. Springer, 1994.

[33] J. C.W. Rogers and A. E. Berger. Some properties of the nonlinear semi-group for the problem $u_t - \Delta f(u) = 0$. *Nonlin. Anal. TMA*, 8:909–939, 1984.

[34] L. Véron. Effets régularisants de semi-groupes non linéaires dans des espaces de Banach. *Ann. Fac. Sci. Toulouse Math.*, 5(2):171–200, 1979.