LARGE TIME BEHAVIOR OF SOLUTIONS TO SOME DEGENERATE PARABOLIC EQUATIONS

N. IGBIDA

C.M.A.F. Universidade de Lisboa
1649-003 Lisboa, Portugal


Abstract: The purpose of this paper is to study the limit in $L^1(\Omega)$, as $t \to \infty$, of solutions of initial-boundary-value problems of the form $u_t - \Delta w = 0$ and $u \in \beta(w)$ in a bounded domain $\Omega$ with general boundary conditions $\frac{\partial w}{\partial \eta} + \gamma(w) \ni 0$. We prove that a solution stabilizes by converging as $t \to \infty$ to a solution of the associated stationary problem. On the other hand, since in general these solutions are not unique, we characterize the true value of the limit and comment the results on the related concrete situations like the Stefan problem and the filtration equation.

Key words: Degenerate parabolic equation, Stefan problem, Filtration equation, Asymptotic behavior, Nonlinear semigroup.
1 Introduction

This paper deals with the large time behavior of the solution of the initial-boundary-value problems of the form:

\[
(P) \begin{cases}
    u_t - \Delta w = 0, & u \in \beta(w) \quad \text{in } Q = (0, \infty) \times \Omega \\
    -\frac{\partial w}{\partial \eta} \in \gamma(w) & \text{on } \Sigma = (0, \infty) \times \partial \Omega \\
    u(0) = u_0 & \text{in } \Omega
\end{cases}
\]

where \(\Omega\) is a bounded domain in \(\mathbb{R}^N\) with a smooth boundary \(\Gamma\), \(\frac{\partial w}{\partial \eta}\) is the normal derivative of \(w\) and the nonlinearities \(\beta\) and \(\gamma\) are maximal monotone graphs in \(\mathbb{R}\) (see [9]) such that \(D(\gamma) \cap D(\beta) \neq \emptyset\). In particular \(\gamma\) may be multivalued and this allows the boundary conditions to include the Dirichlet (taking \(D(\gamma) = \{0\}\)) and the Neumann condition (taking \(\gamma \equiv 0\)) as well as many other possibilities. Also, \(\beta\) may be multivalued, so that \((P)\) is a mathematical model of various phenomena with changes of states where discontinuous points of \(\beta\) are values at which phase transitions take place. On the other hand, \(\beta\) may be a continuous function in \(\mathbb{R}\), then \((P)\) is the filtration equation which includes the flow in Porous Media, the heat flow in material with temperature dependent conductivity and biological models.

Due to general conditions on \(\beta, \gamma\) and \(u_0\) it is known that it is necessary to introduce a suitable class of generalized solutions. In [7], the authors treat \((P)\) in the context of nonlinear semigroups theory and prove that problem of type \((P)\) has a unique generalized solution \(u\). Moreover, \(u(t) = S(t)u_0\) where \(S(t)\) is a continuous nonlinear semigroup of order preserving contraction operators in \(L^1(\Omega)\). We are interested to the limit of \(u(t)\), as \(t \to \infty\).

Setting \(\varphi := \beta^{-1}\), the case \(\varphi\) strictly increasing continuous function is probably the most covered in the literature. N. D. Alikakos and R. Rostamian proved for \(\varphi(r) = |r|^{m-1}r\) and Neumann boundary condition that \(u(t) \to 0\), in \(L^1(\Omega)\), as \(t \to \infty\). Diaz and Vrabie treated the case \(\varphi \in C(\mathbb{R}) \times C^1(\mathbb{R} \setminus \{0\})\), \(\varphi(0) = 0\), \(\varphi'(r) \geq C|r|^\alpha - 1\) (\(C > 0\) and \(\alpha > 1\)) with Dirichlet boundary condition and obtained \(u(t) \to 0\), in \(L^1(\Omega)\), as \(t \to \infty\); also A. Pazy (cf. [29]) obtained the more precise estimate \(|u(t)| \leq M(1/t)^{\alpha - 1}\), for any \(t > 0\), and also some results about the regularising effect in the case \(\alpha > \frac{N-2}{N}\) (see also [5] and [34] in relation with the regularising effect from \(L^1(\Omega)\) to \(L^\infty(\Omega)\)).
More generally, Mazon and Toledo (cf. [25]) considered the case of general \( \gamma \) and \( \varphi \in C(\mathbb{R}) \times C^1(\mathbb{R} \setminus \{a_1, ..., a_n\}) \), \( \varphi(0) = 0 \) with \( \varphi'(r) > 0 \) for \( r \neq a_i \), \( i = 1, ..., n \). They proved that \( u(t) \) stabilizes by converging in \( L^1(\Omega) \), as \( t \to \infty \), to a constant function \( k \) with \( k \in \gamma^{-1}(0) \).

It this paper we generalize these results, to a more general case where \( \beta \) is any maximal monotone graph everywhere defined. Moreover, since the limiting (as \( t \to \infty \)) process exhibits some non-uniqueness properties, we also purpose to recognize the true value of the limit of the solution. Actually, we prove that \( u(t) \) stabilizes, as \( t \to \infty \), to a function \( \underline{u} \in \mathcal{K} \), where \( \mathcal{K} \) is a set of equilibrium points of (P), in other words, the set of solutions of the stationary problem associated to (P) given by

\[
\mathcal{K} = \left\{ z \in L^1(\Omega) ; \exists c \in \gamma^{-1}(0) \text{ s.t. } z(x) \in \beta(c) \text{ a.e. } x \in \Omega \right\}.
\]

If \( \beta \) is singlevalued on \( \gamma^{-1}(0) \), then \( \mathcal{K} = \left\{ z = \beta(c) ; c \in \gamma^{-1}(0) \right\} \), so that, if in addition \( \gamma^{-1}(0) = \{0\} \), then \( \mathcal{K} \) is a singleton and we deduce that \( u(t) \to 0 \), in \( L^1(\Omega) \) as \( t \to \infty \). But, in general \( \mathcal{K} \) is not a singleton and it will be interesting to recognize \( \underline{u} \) among \( \mathcal{K} \) (one could see [9] where the problem of the characterization of the limit of a solution for general evolution problem governed by a maximal monotone graph in Hilbert spaces is posed on page 167 Pb. 14). In the case of Neumann boundary condition, the conservation of the average is an ingredient that we use to identify this limit for a large class of initial datum \( u_0 \in L^1(\Omega) \). Otherwise, we restrict ourselves to linear boundary conditions and prove that the limit of \( u(t) \), as \( t \to \infty \) is the unique solution of some elliptic problem which depend on \( u_0 \).

For instance, assume that \( u_0 \geq 0 \), \( \gamma \equiv 0 \) and \( \varphi \) is the continuous function given by

\[
\varphi(r) = \begin{cases} 
(r - 1)^+ & \text{if } r \geq 0 \\
0 & \text{if } r < 0
\end{cases}
\]

with \( c \geq 0 \). Then, (P) reads the following initial Neumann boundary value problem

\[
(1.1) 
\]
which arises in connection with the Stefan problem (see for instance [13], [31] and the references therein). It is not difficult to prove that, for any $u_0 \in L^\infty(\Omega)$, the generalized solution $u$ is the unique weak solution of (1.2). The limit of $u(t)$ as $t \to \infty$, is closely connected to a problem that attracted considerable interest; it concerns the nature and the evolution of the so called “Mushy region”, which is the interior of the set in which $\varphi(u) = 0$, i.e. $M = [0 < u < 1]$. But the problem seems not to be well understood in general. So far, despite the nonincreasing property of $M(\tau) = M \cap \{t = \tau\}$ for the case of Dirichlet boundary condition (cf. [33] and [21]), to our knowledge there is no results on the characterization of the limiting mushy region, i.e. the part of $M(0)$ that will never be reduced by the diffusion. Notice that in this direction, for the case of nonhomogeneous Dirichlet boundary condition, interesting results about the disappearance of $M(t)$ after a finite time (resp. existence for each time), if some energetical criterion is fulfilled (resp. if it is not fulfilled) was obtained first by Meirmanov [27] and generalized by Gotz and Zaltzman [22] (one can see also [26] and [30] where examples of the appearance of the mushy region were constructed). Notice also, that other related asymptotic behavior concerning the limit of $w$, as $t \to \infty$, may be found in [16, 23, 32, 8] and the references therein. Actually, our results (cf. Theorem 2., Corollary 4. and Proposition 3.) applied to (1.2) implies that $M(t)$ is nonincreasing and

\begin{equation}
(1.3) \quad u(t) \to u_0 \text{ in } L^1(\Omega), \text{ as } t \uparrow \infty
\end{equation}

and

\begin{equation}
(1.4) \quad u_0 = \begin{cases} 
\int u_0 & \text{if } \int u_0 \not\in (0, 1) \\
 u_0 \chi_{[w=0]} + \chi_{[w>0]} & \text{if } \int u_0 \in (0, 1) \text{ and } u_0 \ge 0
\end{cases}
\end{equation}

where $w$ is the unique solution of
\[ w \in H^2(\Omega), \ w \geq 0, \ 0 \leq \Delta w + u_0 \leq 1, \]
\[ w(\Delta w + u_0 - 1) = 0 \text{ a.e } \Omega, \]
\[ \frac{\partial w}{\partial \eta} = 0 \text{ on } \Gamma. \]

(1.5)

So that, as \( t \uparrow \infty \), \( M(t) \) disappears if \( \int u_0 \not\in (0,1) \); otherwise it may not be empty and the set \([w = 0]\) (resp. \([w > 0]\)) represents the mushy region that will never be reduced by the diffusion (resp. the mushy region that will be transformed by the diffusion).

It is interesting to observe that (1.5) is the so called “mesa problem” and \( u_0 \) given by (1.4) is a projection of \( u_0 \) on the mesa of height 1. This kind of projections appears as an initial boundary layer in the study of the singular limit of solutions of some degenerate evolution problems with inconsistent initial data (see [18, 12, 20, 6, 24]). For instance, for the porous medium equation \( u_t = \Delta u^m \) with Neumann boundary condition and nonnegative initial data \( u_0 \), it was proved in [6] that the solution \( u_m \) converges, as \( m \to \infty \), to \( u_0 \). In fact, there is a close connection between the limit of \( u_m \) as \( m \to \infty \) and the limit of a solution of (1.2) as \( t \to \infty \). Through the transformation \( v_k(x, t) = u(x, kt) \), one sees formally that

\[ \lim_{k \to \infty} v_k(x, t) = \lim_{t \to \infty} u(x, t), \]

(1.6)

\( v_k \) is a solution of

\[
\begin{aligned}
\frac{\partial v_k}{\partial t} &= \Delta w_k, \ w_k \in \varphi_k(v_k) \quad \text{in } (0, \infty) \times \Omega \\
\frac{\partial w_k}{\partial \eta} &= 0 \quad \text{on } (0, \infty) \times \partial \Omega \\
v_k(0) &= u_0 \quad \text{in } \Omega
\end{aligned}
\]

with \( \varphi_k(r) = k \varphi(r) \), for any \( r \in \mathbb{R} \) and \( \varphi_k \) converges to the graph.
\( \varphi_\infty(r) = \begin{cases} 
0 & \text{if } 0 \leq r \leq 1 \\
[0, \infty) & \text{if } r = 1 \\
(-\infty, 0] & \text{if } r = 0, 
\end{cases} \)

which is also the limit of the application \( r \to (r^+)^m \), as \( m \to \infty \). So that, one expects that the limit (1.6) is also \( \underline{u}_0 \).

Let us notice that problems of type (P) may be studied in the framework of nonlinear evolution equations governed by sub-differential operators in a Hilbert space \( H \), where \( H \) is a quotient space of the dual space of \( H^1(\Omega) \). So that, applying general stability results of Brezis (cf. [9]) and Bruck (cf. [11]), one deduces that a solution \( u(t) \), converges in \( H \) to a solution of the stationary problem, as \( t \to \infty \). This approach was used by Damlamian and Kenmochi (cf. [16]) for mixed time dependent boundary conditions and by Haraux and Kenmochi (cf. [23]) for time dependent Neumann boundary conditions, but the characterization of the limit of \( u(t) \), was left open. Our approach is completely different and is inspired by the works of [7, 25] and independently by the works of [18, 12, 20, 6] concerning singular limits (mesa problem) of porous medium equation.

To give a brief description of our approach, notice that in order to prove the stabilisation results, one need the orbits of the semigroup \( S(t) \), i.e. \( \{S(t)u_0 ; t \geq 0 \} \), to be relatively compact. For this aim, we will use regularity results of [7] for the elliptic problem associated to (P) to show that the resolvent are relatively compact from \( L^\infty(\Omega) \) into \( L^1(\Omega) \). Then, we deduce that the orbits are relatively compact in \( L^1(\Omega) \) by using the same arguments of [25]. On the other hand, to recognize the true value of the limit of \( u(t) \), we will integrate the equation of (P) with respect to \( t \) and pass to the limit in the integrated equation. We also use the decreasing property of the “Mushy regions” of (P) (cf. Proposition 3.) to prove that the limit is a projection on a mesa.

In Section 2, we will give some preliminaries and state our main results. In Section 3, we recall some results of [7] concerning the elliptic problem associated to (P), for general graphs \( \varphi \) and \( \gamma \), and prove the \( L^1 \)-convergence result for a solution \( u(t) \), as \( t \to \infty \). We also give the definition of “Mushy regions” for the problem (P) and prove their decreasing property. In Section 4,
we prove characterization results of the true value of the limit among solutions of the stationary problem.

2 Preliminaries and main results

Throughout this section, $\Omega$ is a bounded domain of $\mathbb{R}^N$ with smooth boundary $\Gamma$, $\varphi$ and $\gamma$ are maximal monotone graphs in $\mathbb{R}$ such that

$(H_1)$ \[ \mathcal{D}(\varphi) = \mathbb{R}, \]

$(H_2)$ either $\mathcal{D}(\gamma) = \mathbb{R}$ or $\mathcal{D}(\gamma) = \{0\}$

and

$(H_3)$ \[ 0 \in \varphi(0) \cap \gamma(0) \]

and, we consider the following evolution problem

\[
P_e(u_0, \varphi, \gamma) \begin{cases} u_t - \Delta w = 0, & w \in \varphi(u) \quad \text{in} \quad Q = (0, \infty) \times \Omega \\ - \frac{\partial w}{\partial \eta} \in \gamma(w) & \text{on} \quad \Sigma = (0, \infty) \times \partial \Omega \\ u(0) = u_0 & \quad \text{in} \quad \Omega, \end{cases}
\]

with $u_0 \in L^1(\Omega)$. Due to general conditions on $\varphi$, $\gamma$ and $u_0$, it is necessary to introduce a suitable class of generalized solutions. In [7], the authors treat $P_e(u_0, \varphi, \gamma)$ in the contest of nonlinear semigroups theory. We work here from this point of view, a solution of $P_e(u_0, \varphi, \gamma)$ will be a generalized solution $u \in \mathcal{C}([0, \infty), L^1(\Omega))$ with $u(0) = u_0$. By definition, $u$ is the limit in $\mathcal{C}([0, \infty), L^1(\Omega))$ of sequence of classical solutions $u_k$ of approximating problems in which $u_0$, $\varphi$ and $\gamma$ are replaced by smooth functions. Actually, it is well known that, under the hypothesis $(H_1) - (H_3)$, $P_e(u_0, \varphi, \gamma)$ has a unique generalized solution $u$, moreover

\[ u(t) = S(t)u_0 \]
where \( S(t) \) is a continuous nonlinear semigroup of order preserving contraction operators in \( L^1(\Omega) \).

We denote by \( \varphi_0 \) the principal part of \( \varphi \), which is the function
\[
\varphi_0(r) = \min \{ s ; s \in \varphi(r) \} \quad \text{for any } r \in \mathbb{R}
\]
and by \( \varphi^{-1} \) the inverse of \( \varphi \), which is the maximal monotone graph given by \( r \in \varphi^{-1}(s) \) if and only if \( s \in \varphi(r) \). We also introduce the set
\[
\mathcal{E} = \{ r \in \mathbb{R} ; \varphi_0^{-1} \text{ is discontinuous on } r \},
\]
where \( \varphi_0^{-1} \) is the principal part of \( \varphi^{-1} \).

Obviously, if \( \varphi \) is a nondecreasing continuous function, then \( \varphi_0 = \varphi \) and \( \varphi^{-1} \) may be multivalued. In particular, \( \varphi^{-1}(r) \) is a subinterval of \( \mathbb{R} \), for any \( r \in \mathcal{E} \).

At last, let us introduce the set \( \mathcal{K}_{\varphi, \gamma} \), defined by
\[
\mathcal{K}_{\varphi, \gamma} = \{ z \in L^1(\Omega) ; \exists c \in \gamma^{-1}(0) \text{ s.t. } z(x) \in \varphi^{-1}(c) \text{ a.e. } x \in \Omega \}.
\]

It is not difficult to see that \( \mathcal{K}_{\varphi, \gamma} \) is a closed subset of \( L^1(\Omega) \) and is contained in the set of stationary solutions of \( Pe(u_0, \varphi, \gamma) \), so that, for any \( z \in \mathcal{K}_{\varphi, \gamma} \),
\[
S(t)z = z, \quad \text{for any } t \geq 0.
\]
To prove the stabilization result, one needs the orbits \( \{ S(t)u_0 ; t \geq 0 \} \) to be relatively compact. Now, it is not possible to obtain this result from the compactness of the semigroup because it is known that if \( \varphi(r) = |r|^{m-1}r \) and \( \gamma \) corresponds to Dirichlet boundary condition, then \( S(t) : L^1(\Omega) \to L^1(\Omega) \) is compact if \( m > \frac{N-2}{N} \) (see [3]) but for \( 0 < m \leq \frac{N-2}{N} \), even the resolvent are not compact (see [10]). However, for general boundary condition and \( \varphi \) any increasing (strictly) continuous function everywhere defined, Mazon and Toledo proved in their work [25] (see also [2]) that \( S(t)u_0 \) is relatively compact in \( L^1(\Omega) \), for any \( u_0 \in L^1(\Omega) \) (one can see also [17] and [1] for Dirichlet and Neumann boundary conditions respectively). The following Theorem is a generalization of those results.

**Theorem 1.** For any \( u_0 \in L^1(\Omega) \), there exists \( u \in \mathcal{K}_{\varphi, \gamma} \), such that
\[
S(t)u_0 \to u \quad \text{in } L^1(\Omega), \quad \text{as } t \to \infty.
\]
Using this Theorem, let us define the operator $L_{\varphi_\gamma}$ in $L^1(\Omega)$, by

\[(2.1) \quad L_{\varphi_\gamma}(u_0) = \lim_{t \to \infty} S(t)u_0, \quad \text{for any } u_0 \in L^1(\Omega).\]

**Corollary 1.** For any $\varphi$ and $\gamma$ such that $(H_1) - (H_3)$ are fulfilled, the operator $L_{\varphi_\gamma}$ is well defined from $L^1(\Omega)$ to $K_{\varphi_\gamma}$ and is an order preserving contraction in $L^1(\Omega)$.

In the following, we will be interested in recognizing the true value of $L_{\varphi_\gamma}(u_0)$ among the elements of $K_{\varphi_\gamma}$. First, we remark that the structure of $K_{\varphi_\gamma}$ is closely connected to $\gamma^{-1}(0)$ and values of $\varphi^{-1}$ on $\gamma^{-1}(0)$. So that, as an immediate consequence of Theorem 1., we have:

**Corollary 2.** If $\varphi_0^{-1}$ is continuous in $\gamma^{-1}(0)$, then, for any $u_0 \in L^1(\Omega)$, there exists $k \in \gamma^{-1}(0)$, such that

\[L_{\varphi_\gamma}(u_0) \equiv \varphi^{-1}(k).\]

In particular, if $\varphi^{-1}(0) = \gamma^{-1}(0) = \{0\}$, then

\[L_{\varphi_\gamma}(u_0) = 0, \quad \text{for any } u_0 \in L^1(\Omega).\]

Now, assume that $\gamma \equiv 0$, it is not difficult to see that $\int S(t)u_0 = \int u_0$, for any $u_0 \in L^1(\Omega)$. This extra property of the solution gives further information on $L_{\varphi_\gamma}(u_0)$, which enable us to characterize its value under additional conditions on $u_0$. This is the aim of the following Theorem.

**Theorem 2.** If $\gamma \equiv 0$, then for any $u_0 \in L^1(\Omega)$, $\int L_{\varphi_\gamma}(u_0) = \int u_0$ and

\[\exists c \in \varphi(\int u_0) \text{ such that } L_{\varphi_\gamma}(u_0) \in \varphi^{-1}(c) \text{ a.e. in } \Omega.\]

If, in addition, $\varphi_0(\int u_0) \notin \mathcal{E}$, then $L_{\varphi_\gamma}(u_0) = \int u_0$. 
Remark 1. Corollary 2. and the second part of Theorem 2. generalize the results of [25] stated in the case \( \varphi \) a strictly increasing function (see also [1] and [17]). Notice also, that the second case of Corollary 2. is the unique one where \( K_{\varphi, \gamma} \) has a single element.

The second part of our main results deals with the characterization of \( L_{\varphi, \gamma}(u_0) \) in the case of linear boundary condition and \( \varphi \) is any maximal monotone graph everywhere defined, with \( 0 \in \varphi(0) \). So, we will assume that \( \gamma \) is such that

\[(H_4) \quad \gamma(r) = \alpha r \text{ for any } r \in \mathbb{R} \]

with \( \alpha \in [0, \infty] \), where the case \( \alpha = \infty \) corresponds to Dirichlet boundary condition. We will also assume that the initial data \( u_0 \in L^1(\Omega) \) satisfies

\[(2.2) \quad \varphi_0(\int u_0) := m_0 \in \mathcal{E} \quad \text{if } \alpha = 0. \]

\[ \text{Theorem 3. Assume that } (H_1), (H_3) \text{ and } (H_4) \text{ are fulfilled and let } u_0 \in L^\infty(\Omega) \text{ satisfying } (2.2). \text{ Setting} \]

\[(2.3) \quad [l, L] = \begin{cases} \varphi^{-1}(0) & \text{if } \alpha > 0 \\ \varphi^{-1}(m_0) & \text{if } \alpha = 0, \end{cases} \]

we have \( l \leq L_{\varphi, \gamma}(u_0) \leq L \) and

\[(2.4) \quad L_{\varphi, \gamma}(u_0) = u_0 + \Delta w \text{ a.e. in } \Omega \]

where \( w \) satisfies

\[(2.5) \quad w \in H^2(\Omega), \quad \frac{\partial w}{\partial \eta} + \gamma(w) \geq 0 \text{ a.e. on } \Gamma \quad \text{and} \quad w = 0 \text{ a.e. in } \{ x \in \Omega ; l < L_{\varphi, \gamma}(u_0)(x) < L \}. \]
Corollary 3. Under the assumptions of Theorem 3., there exists \( A \subseteq [l \leq u_0 \leq L], A_1 \) and \( A_2 \) disjoint subsets of \( \Omega \) such that

\[
L_{\varphi \gamma}(u_0) = u_0 \cdot \chi_A + l \cdot \chi_{A_1} + L \cdot \chi_{A_2}.
\]

Now, for the sequel, let us introduce the following stationary problem

\[
\begin{cases}
v - \Delta w = u_0, & w \in \partial I_C(v) \quad \text{in } \Omega \\
\frac{\partial w}{\partial \eta} + \gamma(w) \ni 0 & \text{on } \Gamma
\end{cases}
\]

where \( C = [l, L] \) is given by (2.3) and \( \partial I_C \) is the sub-differential of \( I_C \), the indicatrice function of \( C \) ( \( I_C(r) = 0 \), if \( r \in C \) and \( I_C(r) = +\infty \), elsewhere). More precisely,

\[
\partial I_C(r) = \begin{cases}
(\infty, 0] & \text{if } r = l \\
0 & \text{if } l < r < L \\
[0, \infty) & \text{if } r = L.
\end{cases}
\]

Using the results of [7], for any \( u_0 \in L^1(\Omega) \) satisfying (2.2), the problem (2.7) has a unique solution that we denote by \( u_0 \), in the following sense

\[
\begin{cases}
\exists w \in W^{1,1}(\Omega), w \in \partial I_C(u_0) \text{ a.e. in } \Omega,
\exists z \in L^1(\Gamma), z \in \gamma(w) \text{ a.e. on } \Gamma \text{ and}
\int_\Omega Dw.D\xi + \int_\Gamma z\xi = \int_\Omega (u_0 - u_0)\xi, \forall \xi \in W^{1,\infty}(\Omega).
\end{cases}
\]

Theorem 4. Assume that \((H_1), (H_3)\) and \((H_4)\) are fulfilled and let \( u_0 \in L^1(\Omega) \) satisfying (2.2). If \( u_0 \geq l \), then \( L_{\varphi \gamma}(u_0) \geq l \) and \( L_{\varphi \gamma}(u_0) = u_0 \), where \( u_0 \) is given by (2.8).
Corollary 4. If, in addition of the hypothesis of Theorem 4., \( u_0 \in L^\infty(\Omega) \), then

\[
L\varphi\gamma(u_0) = u_0\chi_{\{w=0\}} + L\chi_{\{w>0\}},
\]

where \( w \) is the unique solution of

\[
\begin{align*}
&w \in H^2(\Omega), \ w \geq 0, \ l \leq \Delta w + u_0 \leq L, \\
&w(\Delta w + u_0 - L) = 0 \ a.e \ \Omega, \\
&\frac{\partial w}{\partial \eta} + \gamma(w) \ni 0 \ on \ \Gamma.
\end{align*}
\]

(2.9)

Remark 2. In order to give a brief description of our results, let us come back to the example \( \varphi \) given by (1.1), for which the Problem \( Pe(u_0, \varphi, \gamma) \) models a free boundary problem involving a solid-liquid phase change of Stefan type. Then, the functions \( u \) and \( w = \varphi(u) \) represent respectively the Enthalpy and the Temperature of a material assumed to be of \( H_2O \)–based system and for which \( w = 0 \) is the only temperature at which transition of phase takes place. The function \( u \) is sometimes called a phase function, since it characterizes the regions occupied either by the liquid or the solid or the mushy region. Indeed, \( Q_l(t) = [u(t) \geq 1], Q_s(t) = [u(t) \leq 0] \) and \( M(t) = [0 < u(t) < 1] \) represent respectively liquid, solid and mushy region. So, the characterization of the limit of \( u(t) \) in \( L^1(\Omega) \), as \( t \to \infty \), describes the limits of \( Q_l(t), Q_s(t) \) and \( M(t) \).

i - Dirichlet boundary condition : \( \varphi(u) = a \) on \( \Gamma \), with \( a \in \mathbb{R} \).

If \( a \neq 0 \) and \( a = \varphi(\alpha) \), then Corollary 2. implies that \( u(t) \to \alpha \). Indeed, it is sufficient to take \( \psi(r) = \varphi(r + \alpha) - a \) and apply the second part of Corollary 2. with \( \psi \) and \( u_0 - \alpha \) instead of \( \varphi \) and \( u_0 \), respectively. This means that, if we prescribed a liquid (resp. solid) temperature \( a \) on the boundary, then all the material in the domain \( \Omega \) will be in its liquid (resp. solid) phase at the temperature \( a \). All the mushy region is transformed by the diffusion in this case.

If \( a = 0 \), the situation is different, Theorem 4. implies that \( u(t) \to u_0 \), which depends on the initial data \( u_0 \) and is given by (2.6). This means
that if we presibe the temperature of phase transition on the boundary, then all the phases of the material may remains as \( t \to \infty \), we may have liquid at the temperature 1, the solid at the temperature 0 and a part of the initial mushy region may also holds. Corollary 4. characterizes each region in the case where the diffusion in the liquid or the solid is neglected (i.e. one phase Stefan problem).

\( \text{ii - Neumann boundary condition : } \frac{\partial \varphi(u)}{\partial n} = 0 \text{ on } \Gamma \), i.e. there is no exchange of Temperature with the exterior of \( \Omega \) through \( \Gamma \).

If \( \int u_0 \not\in (0,1) \), then Theorem 2. implies that \( u(t) \to u_0 \), as \( t \to \infty \).

This traduces the fact that the Enthalpy average is either big (i.e. \( \int u_0 \geq 1 \)) or small (i.e. \( \int u_0 \leq 0 \)) enough to transform all the material to it’s either a liquid (i.e. \( u \geq 1 \)) or a solid phase (i.e. \( u \leq 0 \)), at the temperature \( \varphi(\int u_0) \).

Conversely, if \( \int u_0 \in (0,L) \), then Enthalpy average is favorable to get the material, as \( t \to \infty \), in it’s three phases and Corollary 4. characterizes each region, as \( t \to \infty \), in function of those at time \( t = 0 \), in the case where \( u_0 \geq 0 \). If we abort sign condition on \( u_0 \) the problem is still open in general, however, under additional condition on \( u_0 \), Theorem 3. can give a characterization of each region, this will be done in forthcoming works.

### 3 Stabilization results

First, let us recall some results of [7] concerning the elliptic problem

\[
Ps(f, \phi, \gamma) \quad \begin{cases}
  v = \Delta w + f, \ w \in \phi(v) \quad \text{in } \Omega \\
  \frac{\partial w}{\partial \eta} + z = 0, \ z \in \gamma(w) \quad \text{on } \Gamma
\end{cases}
\]
where \( \phi \) and \( \gamma \) are maximal monotone graphs in \( \mathbb{R} \). According to [7], for any \( f \in L^1(\Omega) \), \( \phi \) and \( \gamma \) such that \( 0 \in \phi(0) \cap \gamma(0) \) and

\[
B_- < \int_\Omega f < B_+
\]

where \( B_- = |\Omega| \inf \phi^{-1} + |\Gamma| \inf \gamma \) and \( B_+ = |\Omega| \sup \phi^{-1} + |\Gamma| \sup \gamma \), there exists a unique \((u, w, z)\) solution of \( Ps(f, \phi, \gamma) \) in the sense

\[
\begin{cases}
  v \in L^1(\Omega), \ w \in W^{1,1}(\Omega), \ w \in \phi(v) \text{ a.e. in } \Omega, \\
  z \in L^1(\Gamma), \ z \in \gamma(w) \text{ a.e. on } \Gamma \text{ and} \\
  \int_\Omega D w . D \xi + \int_\Gamma z \xi = \int_\Omega (f - v) \xi, \ \forall \xi \in W^{1,\infty}(\Omega)
\end{cases}
\]

and, for any \( f_1, f_2 \in L^1(\Omega) \), if \((v_i, w_i, z_i)\) is the solution of \( Ps(f_i, \phi, \gamma) \) for \( i = 1, 2 \), then

\[
\int_\Omega (v_1 - v_2)^+ + \int_\Gamma (z_1 - z_2)^+ \leq \int_\Omega (f_1 - f_2)^+
\]

and

\[
\int_\Omega |v_1 - v_2| + \int_\Gamma |z_1 - z_2| \leq \int_\Omega |f_1 - f_2|.
\]

Moreover, if \( f \in L^\infty(\Omega) \) then the solution \((v, w, z)\) \( L^\infty(\Omega) \times H^2(\Omega) \times L^2(\Gamma) \) and one has the following estimates

\[
(3.2) \quad \|v\|_\infty \leq \|f\|_\infty
\]

and

\[
(3.3) \quad \|w\|_{H^1(\Omega)} \leq C \|f\|_\infty.
\]

where \( C \) is a constant which depends only on \( \Omega \).

Now, assuming that \( \varphi \) and \( \gamma \) are maximal monotone graphs such that \((H_1)\), \((H_2)\) and \((H_3)\) are full field, we define the operator (possibly multivalued) \( A_{\varphi \gamma} \),
in \( L^1(\Omega) \) by
\[
A_{\varphi \gamma} v = \left\{ f \in L^1(\Omega) ; \exists w \in W^{1,1}(\Omega), w \in \varphi(v) \text{ a.e. in } \Omega \\
\exists z \in L^1(\Gamma) \text{ s.t. } z \in \gamma(w) \text{ a.e. on } \Gamma \text{ and} \\
\int_{\Omega} Dw.D\xi + \int_{\Gamma} z\xi = \int_{\Omega} f\xi \text{ for any } \xi \in W^{1,\infty}(\Omega) \right\}.
\]

Using the preceding arguments it is clear that, for any \( \lambda > 0 \), the resolvent of \( A_{\varphi \gamma} \) defined by
\[
J_\lambda = (I + \lambda A_{\varphi \gamma})^{-1}
\]
is an everywhere defined order preserving contraction in \( L^1(\Omega) \); so that \( A_{\varphi \gamma} \) is m-T-accretive in \( L^1(\Omega) \). Moreover, we have (cf. [7])
\[
\text{D}(A_{\varphi \gamma}) = L^1(\Omega).
\]

So, using the general theory of nonlinear semigroup of evolution equation, \( A_{\varphi \gamma} \) generates a continuous nonlinear semigroup of order preserving contraction \( S(t) \), in \( L^1(\Omega) \). Moreover, for any \( u_0 \in L^1(\Omega) \), \( S(t)u_0 \) is the unique generalized solution of \( Pe(u_0, \varphi, \gamma) \) (cf. Theorem I. of [7]).

By definition of \( S(t) \),
\[
S(t)u_0 = L^1 - \lim_{\varepsilon \to 0} u_\varepsilon(t)
\]
uniformly for \( t \in [0, \tau] \), where for \( \varepsilon > 0 \), \( u_\varepsilon \) is an \( \varepsilon \)- approximate solution corresponding to a subdivision \( t_0 = 0 < t_1 < \ldots < t_{n-1} < \tau \leq t_n \), with \( t_i - t_{i-1} = \varepsilon \) and defined by \( u_\varepsilon(0) = u_0 \), \( u_\varepsilon(t) = u_i \) for \( t \in [t_{i-1}, t_i] \) where \( u_i \in L^1(\Omega) \) satisfies
\[
\frac{u_i - u_{i-1}}{\varepsilon} + A_{\varphi \gamma} u_i \geq 0.
\]

In other words, the generalized solution \( u \) of \( Pe(u_0, \varphi, \gamma) \) is given by the exponential formula
\[
S(t)u_0 = e^{-tA_{\varphi \gamma}}u_0 = \lim_{n \to \infty} J^n_{t/n}u_0.
\]

Proposition 1. If \( u_0 \in L^\infty(\Omega) \), then, the generalized solution \( u \) of \( Pe(u_0, \varphi, \gamma) \) satisfies
\( u \in L^\infty(Q) \cap C([0, \infty); L^1(\Omega)), \exists w \in L^2_{loc}(0, \infty; H^1(\Omega)), \)
\( w \in \varphi(u) \text{ a.e. in } Q, \exists z \in L^2_{loc}(0, \infty; L^2(\Gamma)), z \in \gamma(w) \text{ a.e. in } \Sigma \)
\[ \int_0^\tau \int_\Omega \xi_t u + \int_\Omega \xi(0) u_0 = \int_0^\tau \int_\Omega D w.D \xi + \int_0^\tau \int_\Gamma \xi z + \int_\Omega \xi(\tau) u(\tau), \forall \xi \in C^1([0, \tau] \times \overline{\Omega}) \text{ and } \tau > 0. \]

Moreover, for any \( \tau \geq 0, \)
\[ \| u(\tau) \|_\infty \leq \| u_0 \|_\infty \]
and
\[ \int_\Omega j(u(\tau)) + \int_0^\tau \int_\Omega |Dw|^2 + \int_0^\tau \int_\Gamma zw \leq \int_\Omega j(u_0) \]

where \( j : \mathbb{R} \to [0, \infty] \) is a proper convex s.c.i. function such that \( \varphi = \partial j. \)

**Proof:** This is a quite standard result (cf. [4]). For completeness let give the arguments. Using (3.4) and (3.5), let \( u_\varepsilon \) the \( \varepsilon \)-approximate solution with \( \varepsilon = \frac{\tau}{n} \) and, for \( i = 1, ..., n, \) let \( (w_i, z_i) \in H^2(\Omega) \times L^2(\Gamma) \) such that
\[ \begin{cases} u_i - \varepsilon \Delta w_i = u_{i-1}, & w_i \in \varphi(u_i) \text{ in } \Omega \\ \frac{\partial w_i}{\partial \eta} + z_i = 0, & z_i \in \gamma(w_i) \text{ on } \Gamma. \end{cases} \]
It follows that \( u_i \in L^\infty(\Omega) \) and \( \| u_i \|_\infty \leq \| u_0 \|_\infty \), so that
\[ \| u_\varepsilon \|_\infty \leq \| u_0 \|_\infty. \]
On the other hand, multiplying (3.10) by \( w_i \) and using the fact that
\[ \int_\Omega (u_{i-1} - u_i) w_i \leq \int_\Omega j(u_{i-1}) - \int_\Omega j(u_i) \]
we have
\[ \int_\Omega j(u_i) + \varepsilon \int_\Omega |Dw_i|^2 + \varepsilon \int_\Gamma z_i w_i \leq \int_\Omega j(u_{i-1}). \]
Adding \((3.12)\) from \(i = 1\) to \(n\), we get
\[
(3.13) \quad \int_{\Omega} j(u_\varepsilon(\tau)) + \int_{0}^{T} \int_{\Omega} |Dw_\varepsilon|^2 + \int_{0}^{T} \int_{\Gamma} w_\varepsilon z_\varepsilon \leq \int_{\Omega} j(u_0)
\]
where \(w_\varepsilon : [0, \tau] \to H^1(\Omega)\) (resp. \(z_\varepsilon : [0, \tau] \to L^2(\Gamma)\)) and \(w_\varepsilon(t) = w_i\) (resp. \(z_\varepsilon(t) = z_i\)), for any \(t \in [t_{i-1}, t_i]\), \(i = 1, \ldots, n\). Thanks to \((H_1)\) and \((3.11)\), \(w_\varepsilon\) is bounded in \(L^\infty((0, \tau) \times \Omega)\), then using the fact that \(j \geq 0\) and \(z_\varepsilon w_\varepsilon \geq 0\), a.e. in \([0, \tau] \times \Gamma\), we deduce from \((3.13)\) that \(w_\varepsilon\) is bounded in \(L^2(0, \tau; H^1(\Omega))\), then thanks to \((H_2)\), \(z_\varepsilon\) is bounded in \(L^\infty((0, \tau) \times \Gamma)\).

Let \(w \in L^2(0, \tau; H^1(\Omega))\), \(z \in L^2((0, \tau) \times \Gamma)\) and \(\varepsilon_k \to 0\), such that \(z_{\varepsilon_k} \rightharpoonup z\) weakly in \(L^2((0, \tau) \times \Gamma)\), \(w_{\varepsilon_k} \rightharpoonup w\) weakly in \(L^2(0, \tau; H^1(\Omega))\) and strongly in \(L^2((0, \tau) \times \Omega)\) and in \(L^2((0, \tau) \times \Gamma)\). Since, for any \(t > 0\), \(w_{\varepsilon_k}(t) \in \varphi(u_{\varepsilon_k}(t))\) a.e. in \(\Omega\) (resp. \(z_{\varepsilon_k}(t) \in \gamma(w_{\varepsilon_k}(t))\) a.e. on \(\Gamma\)), then using the strong convergence of \(u_{\varepsilon_k}(t)\) in \(L^1(\Omega)\) (resp. \(w_{\varepsilon_k}(t)\) in \(L^2(\Gamma)\)) and the weak convergence of \(w_{\varepsilon_k}(t)\) in \(L^2(\Omega)\) (resp. \(z_{\varepsilon_k}(t)\) in \(L^2(\Gamma)\)), we obtain \(w(t) \in \varphi(u(t))\) a.e. in \(\Omega\) (resp. \(z(t) \in \gamma(w(t))\) a.e. on \(\Gamma\)).

At last, let \(\tilde{u}_\varepsilon\) be the function from \([0, \tau]\) into \(L^1(\Omega)\), defined by \(\tilde{u}_\varepsilon(t_i) = u_i\), \(\tilde{u}_\varepsilon\) is linear in \([t_{i-1}, t_i]\), then \((3.10)\) implies that
\[
(3.14) \quad \int_{0}^{T} \int_{\Omega} \tilde{u}_\varepsilon \xi_t + \int_{\Omega} \xi(0)u_0 = \int_{0}^{T} \int_{\Omega} Dw_{\varepsilon} \cdot D\xi
\]
\[
+ \int_{0}^{T} \int_{\Gamma} \xi z_\varepsilon + \int_{\Omega} \xi(\tau)u_\varepsilon(\tau)
\]
for any \(\xi \in C^1([0, \tau] \times \Omega)\). Letting \(\varepsilon \to 0\) in \((3.11)\), \((3.13)\) and \((3.14)\), the results of the Proposition follows. \(\blacksquare\)

**Remark 3.** In general we do not know if weak solutions of \(P\varepsilon(u_0, \varphi, \gamma)\), i.e. functions \(u \in C([0, \infty), L^1(\Omega))\) satisfying \((3.7)\) with \(\xi(\tau) = 0\), are unique. However, this is true in the case of linear boundary conditions and also in the case \(\gamma\) and \(\varphi\) locally Lipschitz continuous functions (see for instance [28]).

**Proposition 2.** Under the assumptions \((H_1) - (H_3)\), for any \(u_0 \in L^1(\Omega)\), \(S(t)u_0\) is relatively compact in \(L^1(\Omega)\).
In order to prove this Proposition, let us prove, first, the following Lemma.

**Lemma 1.** Let \( f \in L^\infty(\Omega) \) and \( \lambda > 0 \). For any \( y \in \mathbb{R}^N \) and \( \xi \in C^2(\Omega) \) supported in \( \{ x \in \Omega \colon \text{distance}(x, \Gamma) < |y| \} \), we have

\[
\int_{\Omega} \xi(x) |\mathcal{J}_\lambda f(x + y) - \mathcal{J}_\lambda f(x)| \, dx \leq C |y| \|\Delta \xi\|_\infty \|f\|_\infty + \int_{\Omega} \xi(x) |f(x + y) - f(x)| \, dx
\]

where \( C \) is a constant depending only on \( \Omega \).

**Proof:** Set \( v = \mathcal{J}_\lambda(f) \) and let \( (w, z) \in H^1(\Omega) \times L^2(\Gamma) \), such that \( (v, w, z) \) is the solution of \( Ps(f, \lambda \varphi, \gamma) \).

Using the results of [7], for any \( y \in \mathbb{R}^N \) and \( \xi \in C^2(\Omega) \) supported in \( \{ x \in \Omega \colon \text{distance}(x, \Gamma) < |y| \} \), we have

\[
\int_{\Omega} \xi(x) |v(x + y) - v(x)| \, dx \leq \int_{\Omega} |\Delta \xi| |w(x + y) - w(x)| \, dx + \int_{\Omega} \xi(x) |f(x + y) - f(x)| \, dx
\]

\[
\leq |y| \|\Delta \xi\|_\infty |\Omega|^\frac{1}{2} \|\nabla w\|_2 + \int_{\Omega} \xi(x) |f(x + y) - f(x)| \, dx
\]

then, using (3.3), the result follows. \( \blacksquare \)

**Proof of Proposition 2.** First, using Lemma 1., we see that for any \( \lambda > 0 \) fixed and \( B \) a bounded subset of \( L^\infty(\Omega) \), \( \mathcal{J}_\lambda B \) is a relatively compact subset of \( L^1(\Omega) \). Indeed, for any \( \{f_n\} \subseteq B \), with an appropriate choice of \( \xi \), we have

\[
\lim_{|y| \to 0} \sup_n \int_{\Omega'} |\mathcal{J}_\lambda f_n(x + y) - \mathcal{J}_\lambda f_n(x)| = 0
\]

for any \( \Omega' \subset \Omega \) which implies, with (3.2), that \( \{\mathcal{J}_\lambda f_n\} \) is relatively compact in \( L^1(\Omega) \). Then, the proof of the relative compactness of \( S(t)u_0 \), in \( L^1(\Omega) \), follows exactly in the same way of the proof of Theorem 2.2 in [25] (see also [14] Theorem 3). In fact, one proves, firstly, that \( S(t)u_0 \) is relatively compact for any \( u_0 \in L^\infty(\Omega) \cap \mathcal{D}(A_{\varphi}\gamma) \) by using the inequality

\[
\|S(t)u_0 - \mathcal{J}_\lambda S(t)u_0\|_1 \leq \lambda \inf \{\|v\|_1 \colon v \in A_{\varphi}\gamma u_0\} ;
\]
then, the compactness of a subsequence of $S(t)u_0$, for $u_0 \in L^1(\Omega)$, follows by approximation of $u_0$ and the fact that

$$ (3.15) \quad \sup_{t \geq 0} \inf_{s \geq 0} \| S(t)u_0 - S(s)z \|_1 \leq \| u_0 - z \|_1, \quad \text{for any } z \in L^1(\Omega). $$

Now, we use terminology and notation from topological dynamics: for any $u_0 \in L^1(\Omega)$, we define the $\omega$–limit set of $P e(u_0, \varphi, \gamma)$ by

$$ \omega_{\varphi \gamma}(u_0) = \left\{ u \in L^1(\Omega) \mid u = L^1 - \lim_{t_n \to \infty} S(t_n)u_0 \text{ for some sequence } t_n \right\}. $$

This set is possibly empty. Now, it is well known (see [15]) that if $S(t)u_0$ is relatively compact, then $\omega_{\varphi \gamma}(u_0)$ is a non empty compact and connected subset of $L^1(\Omega)$. Furthermore $\omega_{\varphi \gamma}(u_0)$ is invariant under $S(t)$, i.e., $S(t)\omega_{\varphi \gamma}(u_0) \subseteq \omega_{\varphi \gamma}(u_0)$ for any $t \geq 0$. An equilibrium or stationary point is any $z \in L^1(\Omega)$ such that $\omega_{\varphi \gamma}(z) = \{ S(t)z \} = \{ z \}$.

**Corollary 5.** For any $u_0 \in L^1(\Omega)$, $\omega_{\varphi \gamma}(u_0) \neq \emptyset$.

**Proof of Theorem 1.** : Using the fact that $K_{\varphi \gamma}$ is a closed subset of $L^1(\Omega)$ and the inequality (3.15), one see that it is sufficient to prove the Theorem for any $u_0 \in L^\infty(\Omega)$. So, assume that $u_0 \in L^\infty(\Omega)$ and consider $(w, z) \in L^2_{loc}(0, \infty; H^1(\Omega)) \times L^2_{loc}(0, \infty; L^2(\Gamma))$, such that $(u, w, z)$ satisfy (3.7) with $u(t) = S(t)u_0$. Thanks to (3.9) and since $j \geq 0$ and for any $t \geq 0$ $w(t)z(t) \geq 0$ a.e. in $\Omega$, there exists $t_n \to \infty$,

$$ (3.16) \quad \lim_{t_n \to \infty} \int_\Omega |Dw(t_n)|^2 + \int_\Gamma w(t_n)z(t_n) = 0, $$

then using $(H_1)$, (3.8) and Poincaré inequality, we deduce that $w(t_n)$ is bounded in $H^1(\Omega)$, as $t_n \to \infty$. On the other hand, thanks to $(H_1)$, $(H_2)$ and (3.8), $z(t_n)$ is bounded in $L^\infty(\Gamma)$. Thanks to Proposition 2., let $u \in L^1(\Omega)$ and $t_{nk} \to \infty$, such that $u(t_{nk}) \to u$ in $L^1(\Omega)$ and, let $z \in L^2(\Gamma)$, $w \in H^1(\Omega)$ such that $z(t_{nk}) \to z$ weakly in $L^2(\Gamma)$ and $w(t_{nk}) \to w$ weakly in $L^2(\Omega)$, $w(t_{nk}) \to w$ strongly in $L^2(\Omega)$ and $w(t_{nk}) \to w$ strongly in $L^2(\Gamma)$. Then, as in the proof of Proposition 1., by using standard compactness and monotony arguments, we obtain
(3.17) \( w \in \varphi(u) \) a.e. \( \Omega \) and \( z \in \gamma(w) \) a.e. \( \Gamma \).

Passing to the limit in (3.16), through the subsequence \( t_{nk} \), we get
\[
\int_{\Omega} |Dw|^2 + \int_{\Gamma} w \, z \leq 0.
\]
So, since \( w \, z \geq 0 \) a.e. on \( \Gamma \), then \( Dw \equiv 0 \) a.e. in \( \Omega \) and \( w \, z = 0 \) a.e. on \( \Gamma \);
which implies that there exists \( c \in \mathbb{R} \) such that
\[
(3.18) \quad w \equiv c \text{ a.e. in } \Omega \quad \text{and} \quad zc = 0 \text{ a.e. on } \Gamma.
\]
From this we deduce that \( u \in K_{\varphi, \gamma} \), which implies that \( S(t)u = u \). Then, the
Theorem is an obvious consequence of the contraction property of \( S(t) \).

Before to end up this section, we introduce, the so called “Mushy regions”
in terms of Stefan Problem,
\[
M_r = \left\{ (x, t) \in [0, \infty) \times \Omega ; \; S(t)u_0(x) \in \text{int}(\varphi^{-1}(r)) \right\},
\]
for any \( r \in \mathcal{E} \), and
\[
M_r(t_0) = \{ \, x \in \Omega ; \, (t_0, x) \in M_r \, \}.
\]
The following Proposition is a generalization of results of [33] and [21], and
will be useful in the following section.

**Proposition 3.** Under the hypothesis \((H_1) - (H_3)\), for any \( r \in \mathcal{E} \) and \( u_0 \in L^1(\Omega) \),
\[
(3.19) \quad M_r(t_2) \subseteq M_r(t_1) \quad \text{for any } t_2 > t_1,
\]
in the sense of \( \text{mes}(M_r(t_2) \setminus M_r(t_1)) = 0 \).

**Proof:** Let \( r \in \mathcal{E} \) fixed and set \((a, b) = \text{int}(\varphi^{-1}(r))\). Since, \( S(t)u_0 \) depends
continuously in \( u_0 \), it is sufficient to prove (3.19), for \( u_0 \in L^\infty(\Omega) \). So, assume
that \( u_0 \in L^\infty(\Omega) \) and consider \( u_\varepsilon \), the \( \varepsilon \)--approximate solution in \([0, \tau]\) given
by (3.4) with \( \varepsilon = \frac{\tau}{n} \). We have
\[
(3.20) \quad \{ x \in \Omega ; \; u_\varepsilon(\tau, x) \in (a, b) \} \subseteq \{ x \in \Omega ; \; u_0(x) \in (a, b) \}.
\]
Indeed, by definition of $u_\varepsilon$, for $(w_i, z_i) \in H^1(\Omega) \times L^2(\Gamma)$ given by (3.5), for $i = 0, \ldots, n = \tau/\varepsilon$, we have $\Delta w_i = 0$ and $u_i = u_{i-1}$ a.e. in $\{ x \in \Omega ; u_i(x) \in (a, b) \}$, so that
\[
\{ x \in \Omega ; u_\varepsilon(\tau, x) \in (a, b) \} = \{ x \in \Omega ; u_n(x) \in (a, b) \}
\subseteq \{ x \in \Omega ; u_{n-1}(x) \in (a, b) \}
\subseteq \{ x \in \Omega ; u_i(x) \in (a, b) \},
\]
for any $0 \leq i \leq n - 1$ and $i = 0$. (3.20) follows. Using (3.4) and (3.20), we get
\[
\{ x \in \Omega ; S(\tau)u_0(x) \in (a, b) \} \subseteq \{ x \in \Omega ; u_0(x) \in (a, b) \}
\]
for any $\tau \geq 0$.
At last, since for $0 \leq t_1 < t_2 < \infty$, $S(t_2)u_0 = S(t_2 - t_1)S(t_1)u_0$, then the result follows by replacing $u_0$ by $S(t_1)u_0$, in (3.21).

4 Characterization of the limit

In this section, we assume that $\gamma$ satisfies $(H_4)$, $u_0 \in L^1(\Omega)$ satisfies (2.2) and we introduce the set
\[
S(u_0) = \{ x \in \Omega ; x \text{ is a Lebesgue point of } L\varphi\gamma(u_0) \text{ and } \exists t_n \to \infty, S(t_n)u_0(x) \to L\varphi\gamma(u_0)(x) \}.
\]
Using Theorem 1., we have mes $\{ \Omega \setminus S(u_0) \} = 0$, for any $u_0 \in L^1(\Omega)$. We begin this section by proving a particular case of Theorem 3..

Proposition 4. Assume $\varphi$ and $\gamma$ satisfy $(H_1)$, $(H_3)$ and $(H_4)$, with $\varphi^{-1}(0) = [l, L]$. If $u_0 \in L^\infty(\Omega)$, with $\int u_0 \in [l, L]$ if $\alpha = 0$, then
\[
l \leq L\varphi\gamma(u_0) \leq L \text{ a.e. in } \Omega
\]
and
\[ L_{\varphi \gamma}(u_0) = u_0 + \Delta w \text{ a.e. in } \Omega \]

where

\[ w \in H^2(\Omega), \frac{\partial w}{\partial n} + \gamma(w) \ni 0 \text{ a.e. on } \Gamma \]

and

\[ w = 0 \text{ a.e. in } \{ x \in \Omega ; l < L_{\varphi \gamma}(u_0)(x) < L \} . \]

If, in addition, \( u_0 \geq 0 \), then \( L_{\varphi \gamma}(u_0) \geq 0 \) and there exists \( w \geq 0 \) such that

\[ w = 0 \text{ a.e. in } \{ x \in \Omega ; 0 \leq L_{\varphi \gamma}(u_0)(x) < L \} . \]

In order to prove this proposition let prove the following Lemma:

**Lemma 2.** Let \( f_n \) be a sequence of \( L^1(\Omega) \), \( f \in L^1(\Omega) \), such that \( f_n \to f \) in \( L^1(\Omega) \). If \( x_0 \in \Omega \) is a Lebesgue point of \( f \) such that \( \theta_1 < f(x_0) < \theta_2 \), for \( \theta_1, \theta_2 \in \mathbb{R} \), then, for any \( \delta > 0 \),

\[ \text{mes} \{ x \in B(x_0, \delta) ; \theta_1 < f(x) < \theta_2 \} > 0 \]

and, there exist \( n_0 = n_0(\theta_1, \theta_2, \delta) > 0 \), such that

\[ \text{mes} \{ x \in B(x_0, \delta) ; \theta_1 < f_n(x) < \theta_2 \} > 0 \quad \text{for any } n \geq n_0. \]

**Proof:** Let \( \delta > 0 \) fixed. If (4.7) is not true, then

\[ f(x) \in (-\infty, \theta_1] \cup [\theta_2, \infty), \text{ a.e. } x \in B(x_0, \delta). \]

On the other hand, since \( x_0 \) is a Lebesgue point of \( f \), then (cf. [19])

\[ f(x_0) = \sup \left\{ t \in \mathbb{R} ; \lim_{r \to 0} \frac{|B(x_0, r) \cap [f < t]|}{|B(x_0, r)|} = 0 \right\}, \]
so that, using the fact that \( \theta_1 < f(x_0) < \theta_2 \) and (4.9), we get

\[
f(x_0) = \sup \left\{ t \in [\theta_1, \theta_2] ; \lim_{r \to 0} \frac{|B(x_0, r) \cap [f < t]|}{|B(x_0, r)|} = 0 \right\} = \theta_2,
\]

which contradicts the hypothesis of the Lemma.

Now assume, that (4.8) is not true, then one can construct a sequence \( n_k \to \infty \), such that

\[
\text{mes} \{ x \in B(x_0, \delta) ; \theta_1 < f_{n_k}(x) < \theta_2 \} = 0 \quad \text{for any } n_k,
\]

which implies that

\[
(4.10) \quad f_{n_k}(x) \in (-\infty, \theta_1] \cup [\theta_2, \infty) \quad \text{a.e. } x \in B(x_0, \delta), \text{ for any } n_k.
\]

Since, \( f_n \to f \), in \( L^1(\Omega) \), then there exists a subsequence of \( n_k \), that we denote again by \( n_k \), such that \( f_{n_k} \to f \) a.e. in \( \Omega \) and, then (4.10) implies \( f(x) \in (-\infty, \theta_1] \cup [\theta_2, \infty) \) a.e. \( x \in B(x_0, \delta) \), which contradicts (4.7). This ends up the proof of the Lemma.

\[\]

**Proof of Proposition 4.** : Firstly, we assume that \( \alpha > 0 \). Obviously (4.2) is a direct consequence of Theorem 1.. In order to prove (4.3) and (4.4), we consider \( u(t) = S(t)u_0 \) and \( w \in L^2_{\text{loc}}(0, \infty; H^1(\Omega)) \) given by Proposition 1., such that \( w \in \varphi(u) \) a.e. in \( Q \) and

\[
(4.11) \quad \int_0^\tau \int_{\Omega} \xi \Delta u + \int_{\Omega} \xi(0)u_0 = \int_0^\tau \int_{\Omega} Dw.D \xi + \alpha \int_0^\tau \int_{\Gamma} \xi w + \int_{\Omega} \xi(\tau)u(\tau)
\]

for any \( \xi \in C^1([0, \tau] \times \Omega) \) and \( \tau > 0 \). It is clear that, for any \( t \geq 0 \),

\[
(4.12) \quad W(t) = \int_0^t w(s)ds \in H^1(\Omega)
\]

and, by appropriate choice of \( \xi \) in (4.11), \( W(t) \) is a weak solution of

\[
\begin{cases}
-\Delta W(t) = u_0 - u(t) & \text{in } \Omega \\
\frac{\partial W(t)}{\partial \eta} + \alpha W(t) = 0 & \text{on } \Gamma.
\end{cases}
\]
Since $u_0, u(t) \in L^\infty(\Omega)$, then $W(t) \in H^2(\Omega) \cap C(\Omega)$ and (4.13) is satisfied a.e. in $\overline{\Omega}$. Applying Theorem 1., we have $u(t) \to \underline{u} := \mathcal{L}_{\varphi\gamma}(u_0)$ and, thanks to (4.13), there exists $\overline{w} \in H^2(\Omega)$ and a sequence $t_k \to \infty$, such that

$$W(t_k) \to \overline{w} \text{ weakly in } H^1(\Omega) \text{ and strongly in } L^2(\Omega) \text{ and } L^2(\Gamma),$$

as $t_k \to \infty$, and $\overline{w}$ satisfies

$$\begin{cases} -\Delta \overline{w} = u_0 - \underline{u} \quad \text{a.e. in } \Omega \\ \frac{\partial \overline{w}}{\partial \eta} + \alpha \overline{w} = 0 \quad \text{a.e. on } \Gamma \end{cases}$$

which ends up the proof of (4.3) and (4.4).

Now, let us prove (4.5). For this, we consider $x_0 \in S(u_0)$ fixed such that $l < \underline{u}(x_0) < L$ and we claim that

$$w(x_0) = 0 \quad (4.15)$$

Using Lemma 2., for any $\delta > 0$, there exists $t_0 = t_0(l, L, \delta) \geq 0$, such that

$$\text{mes} \{ x \in B(x_0, \delta) ; l < u(t, x) < L \} > 0 \quad \text{for any } t \geq t_0$$

then, thanks to Proposition 3.,

$$\text{mes} \{ x \in B(x_0, \delta) ; l < u(t, x) < L \} > 0 \quad \text{for any } t \geq 0,$$

which implies, that

$$\text{mes} \{ x \in B(x_0, \delta) ; W(t, x) = 0 \} > 0 \text{ for any } t \geq 0 \quad (4.16)$$

Since $W(t) \in C(\Omega)$ and satisfies (4.16) for any $\delta > 0$, then $W(t, x_0) = 0$, for any $t \geq 0$, which implies (4.15).

In the case $\alpha = 0$, we see that, assuming $l \leq \int u_0 \leq L$, implies with Theorem 2., that $l \leq \mathcal{L}_{\varphi\gamma}(u_0) \leq L$, then all the preceding arguments apply exactly in the same way of the case $\alpha > 0$.

To end up the proof, we see that if $u_0 \geq 0$, then $u(t) = S(t)u_0 \geq 0$ a.e. in $\Omega$ and thanks to $(H_3)$, $w \in L^2_{\text{loc}}(0, \infty; H^1(\Omega))$ given by Proposition 1. is also...
nonnegative. So, using (4.12) and (4.14) we deduce that $w \geq 0$. 

**Proof of Theorem 3.** It is clear that the case where either $\alpha > 0$ or $\alpha = 0$ and $m_0 = 0$ follows directly by Proposition 4. In the case $\alpha = 0$ and $m_0 \neq 0$, we consider the graph $\psi(r) = \varphi(r + \int u_0) - \varphi_0(\int u_0)$ and $v_0 = u_0 - \int u_0$. It is not difficult to see that $L \varphi \gamma (u_0) = L \psi \gamma (v_0) + \int u_0$ and since $\psi(\int v_0) = 0$, then the results of Theorem follows by applying Proposition 4. with $\varphi$ and $u_0$ replaced by $\psi$ and $v_0$.

**Proof of Corollary 3.**: This Corollary is an immediate consequence of the fact that $l \leq L \varphi \gamma (u_0) \leq L$ and $\Delta w = 0$ a.e. in $[w = 0]$ which includes, by (2.5), $[l < L \varphi \gamma (u_0) < L]$.

**Proof of Theorem 4.**: Using the contraction property of $Ps(u_0, \partial I_C, \gamma)$ and $Pe(u_0, \partial I_C, \gamma)$, it is enough to prove the Theorem for $u_0 \in L^\infty(\Omega)$. In the case $\alpha > 0$ and $\alpha = 0$ with $m_0 = 0$, we can assume without loose of generality that $l = 0$, so that the second part of Proposition 4. implies that $L \varphi \gamma (u_0) =: u \geq 0$ and there exists $w \geq 0$ such that (4.3), (4.4) and (4.6) are full filed. On the other hand, it is clear that (4.6) with $u \geq 0$ and $w \geq 0$, implies that $w \in \partial I_C(u)$, with $C = [l, L]$, so that $w$ is a weak solution of $Ps(u_0, \partial I_C, \gamma)$, which is unique.

In the case $\alpha = 0$ and $m_0 \neq 0$, we consider $\psi(r) = \varphi(r + l) - m_0$ and $v_0 = u_0 - l$, so that applying the first part of the proof with $\varphi$ and $\gamma$ replaced respectively by $\psi$ and $v_0$, we deduce that $L \psi \gamma (v_0)$ is the unique solution of $Ps(v_0, \partial I_{[l, L]}(r + l), \gamma)$. At last, we use the fact that $\partial I_{[0, L]}(r) = \partial I_{[l, L]}(r + l)$ for any $r \in [0, L - l]$ and $L \varphi \gamma (u_0) = L \psi \gamma (v_0) + c$ to conclude that $L \varphi \gamma (u_0)$ is the unique solution of $Ps(v_0, \partial I_{[l, L]}), \gamma)$.

**Proof of Corollary 4.**: The Corollary is an obvious consequence of the regularity of the unique solution $(\underline{w}_0, w)$ of $Ps(u_0, \partial I_C, \gamma)$. In fact, if $u_0 \in L^\infty(\Omega)$, then $\underline{w}_0 \in L^\infty(\Omega)$ and $w \in H^2(\Omega)$, so that $\Delta w = 0$ a.e. in $[w = 0]$ and then the Corollary follows.
At last, let us prove the Theorem 2.

**Proof of Theorem 2.** Assuming $\gamma \equiv 0$, we have $\int J_\lambda f = \int f$, for any $\lambda > 0$ and $f \in L^1(\Omega)$. Then, for any $u_0 \in L^1(\Omega)$, (3.6) and Theorem 1. imply

$$\int L_{\varphi \gamma}(u_0) = \int S(t)u_0$$

(4.17)

$$\int u_0$$

(4.18)

and, there exists $c \in \mathbb{R}$ such that $L_{\varphi \gamma}(u_0) \in \varphi^{-1}(c)$. Using the fact that, $\varphi^{-1}(c)$ is a subinterval of $\mathbb{R}$, we get $\int L_{\varphi \gamma}(u_0) \in \varphi^{-1}(c)$, which implies, with (4.18), that $c \in \varphi(\int u_0)$. This ends up the proof of the first part of the Theorem.

If, in addition, we assume that $\varphi_0(\int u_0) \not\in \mathcal{E}$, then $\varphi(\int u_0) = \varphi_0(\int u_0)$ and $\varphi^{-1}(\varphi_0(\int u_0)) = \int u_0$, so that $L_{\varphi \gamma}(u_0) = \int u_0$. ■

**ACKNOWLEDGEMENTS**

This work was realized while the author was a postdoctoral student at the C.M.A.F. / Universidade de Lisboa and supported by CMAF/FCT. The paper was finished while he visited the LAMFA / Université de Picardie Jule-Verne that he wants to thank for the hospitality.

**References**


[18] C.M. Elliot, M.A. Herrero, J.R. King, and J.R. Ockendon. The mesa patterns for \( u_t = \nabla(u^m\nabla u) \) as \( m \to \infty \). *IMA J. Appl. Math.*, 37:147–154, 1986.


[20] A. Friedman and S. Huang. Asymptotic behavior of solutions of \( u_t = \Delta \phi_m(u) \) as \( m \to \infty \) with inconsistent initial values. In *Analyse Mathématique et Applications*. Gauthier-Villars, 1988.


