Discrete Collapsing Sandpile Model[*]

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Abstract

Our main goal is to introduce and study a discrete model for the collapsing of a pile of cubes. This is a typical example of Self-organized critical phenomena exhibited by a critical slop. We prove existence and uniqueness of the solution for the model. Then by using dual arguments we study the numerical computation of the solution and we present some numerical simulations.

1 Introduction

The dynamics of granular materials has been studied quite intensively due to their importance in various naturally occurring phenomena such as landslides, rockfalls, desert dunes evolution, sediment transport in rivers, ... and engineering transportation applications. The description of such flows still represents a major challenge for the theory. In the last decade, several studies have been devoted to the mathematical and numerical studies of granular system. Different models have been proposed using kinetic approach (cf. [6, 7]), cellular automata (cf. [11, 24, 16, 21]) or partial differential equations (cf. [1, 2, 3, 23, 8, 18, 4, 5, 12, 15, 13, 17, 10, 20]).

Granular materials are complex objects and it is important to understand their behavior by using simple prototypes. Actually, it is known that one of the approach that may be relevant for their study is based on modeling the dynamic of pile of cubes. That is, to imagine that the matter at the microscopic level consists of particles similar to cubes (in some cases, a particle can be likened to a certain volume of material) arranged on a regular grid. The principle after consists in establishing simple rules across the unit cell and repeat until the interplay between cells occurs by itself coherent structures or organized forms at the macroscopic scale. Of course, the elementary constituents of a material are so numerous that the study at the microscopic level needs probabilistic methods. However, appropriate scaling of time enables a transition to deterministic models of nonlocal type (see for instance [16] and [19]). These rescaling takes into account rigorously the fact that there is a very large number of particles and there is a significant gap between the time scales of microscopic and macroscopic.

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A typical example is the growing pile of cubes (cf. [16]) which corresponds to the evolution of stack of unit cubes resting on the plane when new cubes are being added to the pile. In [16], Evans and Rezakhanlou introduce a stochastic description of the dynamics and proved that, if we randomly add more and more, smaller and smaller cubes, we obtain an interesting continuum limit, which is an evolution governed by the sub differential of a convex functional that is very connected to Prigozhin model for sandpile [23]. To that aim, they introduce an intermediate nonlinear discrete dynamic of nonlocal type at the level of cubes. By using Partial Integro-Differential Equation, N. Igbida shows in [19] that this discrete nonlocal equation gives a right deterministic description of the dynamic of a growing pile of granular structure when the component are not very small. Our aim here is to show how to use this kind of discrete equation to model the collapsing of an unstable pile of cubes.

The paper is organized as follows: in the next section we establish our discrete model and study the existence and uniqueness of the solution. In section 3 we develop a numerical study of the model based on duality argument. At last, we give numerical simulations showing the stabilization of unstable discrete structures.

2 The discrete model for the collapse of a pile

It is well known by now, that the collapsing phenomena in granular materials can be described by nonlinear evolution equations governed by nondecreasing critical angles. In the continuous case, recall that combining the continuity equation of fluid dynamic and phenomenological equation N. Igbida introduce in [20] (see also [15] and [13]) a sub-gradient flow for variational problems with time dependent gradient constraints. The gradient constraints are interpreted as critical angle of sandpile. In particular, the continuous model [20] produces an evolution in time of avalanches in a drying of a sandpile, rather than instantaneous collapse. Our aim here is to introduce a discrete non local model that we can associate with such phenomena.

2.1 The discrete model.

We consider the surface of the pile be divided into cubes of integer point $i \in \mathbb{Z}^n, n = 1$ or 2. So, a suck of cubes can be described by an application $u : \mathbb{Z}^n \to \mathbb{R}$, where $u(i)$ describes the density of cubes at the position i.

The collapse produced when the slope of the surface exceeds an angle of stability. In the discrete case the stability condition for a profile u reads (cf. [16] and [19])

$$|u(i) - u(j)| \leq 1 \quad \text{for} \quad i \sim j, \tag{1}$$

where we use $i \sim j$ to describe $|i - j| \leq 1$. Assume that, we start with unstable configuration represented by $u_0 : \mathbb{Z}^n \to \mathbb{R}$ such that

$$|u_0(i) - u_0(j)| > 1 \quad \text{for} \quad i \sim j.$$

To reach a stable configuration, we assume a suitable of various avalanches are produced, so as to stabilize the pile. More precisely, we assume that the pile tends to stabilizes itself by taking a continuous sequence of intermediate profile characterized by

$$|u(i) - u(j)| \leq c(t) \quad \text{for} \quad i \sim j, \tag{2}$$

where $c : [0,T) \to I\!R^+$ is a given non increasing function satisfying

$$\lim_{t\to T} c(t) = 1.$$

Here, the stability constraint, forces the pile to rearrange itself to reach a stable profile. So, a suitable of various unstable configurations are produced with non increasing angle of stability that converges to 1, as $t \to T \leq \infty$.

The dynamic of the height $u(t,i)$ of the pile at a fixed point $i \in Z\!\!Z^n$, can be derived as follows. For a small time Δt, the evolution of u is given by:

$$u(t+\Delta t, i) \simeq u(t,i) + \Delta t\, Q(t,i),$$

where $Q(t,i)$ is the rate of material arriving at the position i. We can express Q as follows

$$Q(t,i) = I(t,i) - O(t,i),$$

where, $I(t,i)$ records the material arriving to the position i from the neighborhood positions and $O(t,i)$ records the material leaving the position i towards neighborhood positions. We have

$$I(t,i) = \sum_{j:j\sim i} \alpha(t,j,i) \qquad \text{and} \qquad O(t,i) = \sum_{j:j\sim i} \alpha(t,i,j),$$

where $\alpha(t,i,j)$ records the material arriving to the position j from the neighborhood positions i. This implies that

$$\frac{u(t+\Delta t,i) - u(t,i)}{\Delta t} + \sum_{j:j\sim i} (\alpha(t,i,j) - \alpha(t,j,i)) \simeq 0.$$

At each time $t > 0$, we put

$$\sigma(t,i,j) = \alpha(t,i,j) - \alpha(t,j,i).$$

Obviously, σ is anti-symmetric, i.e

$$\sigma(t,i,j) = -\sigma(t,j,i).$$

Letting $\Delta t \to 0$, we obtain

$$\partial_t u(t,i) + \sum_{j:j\sim i} \sigma(t,i,j) = 0.$$

To complete the model we have to give the connection between σ and u. Since the dynamic is induced by the discrete constraint (2). Then, we can assume that the cubes move only when the limiting condition is turning to be exceeded. So, the dynamic in turn is concentrated on the set $X_{c(t)}(u(t))$, where

$$X_r(v) := \{(i,j) \in Z\!\!Z^n \times Z\!\!Z^n : |v(i) - v(j)| = r \text{ and } i \sim j\}$$

for a given $r > 0$ and a given application $v\ :\ \mathbb{Z}^n \to \mathbb{R}$, so that

$$support(\sigma(t,.,.)) \subseteq X_{c(t)}(u(t)).$$

Finally, our model is the following system:

$$(DM) \qquad \begin{cases} \partial_t u(t,i) + \displaystyle\sum_{j:j\sim i} \sigma(t,i,j) = 0,\ t>0,\ i \in \mathbb{Z}^n, \\ \\ |u(t,i) - u(t,j)| \le c(t) \text{ for } i \sim j, \\ \\ \sigma(t,i,j) = -\sigma(t,j,i) \ \text{ and } \ support(\sigma(t,.,.)) \subseteq X_{c(t)}(u(t)). \end{cases}$$

In the case where $c(t) = 1$, for any $t \in [0,T)$, and the equation is subject to a non null source term. This model describes a growing sandpile with respect to an external source of cube. Indeed, in this case the system (DM) is the discrete model that we can associate with the continuous nonlocal model for discrete structures in $\mathbb{R}^2$ (see [19] for more details).

To study this problem, we recall the infinite-dimensional ℓ^p spaces defined by

$$\ell^p(\mathbb{Z}^n) = \begin{cases} \Big\{\eta :\ \mathbb{Z}^n \longrightarrow \mathbb{R}\ ;\ \|\eta\|_p := \Big(\displaystyle\sum_{i\in\mathbb{Z}^n} |\eta(i)|^p\Big)^{1/p} < \infty\Big\}, & \text{for } 1 \le p < \infty \\ \\ \Big\{\eta : : \mathbb{Z}^n \longrightarrow \mathbb{R}\ ;\ \|\eta\|_\infty := \displaystyle\max_{i\in\mathbb{Z}^n} |\eta(i)| < \infty\Big\}, & \text{for } p = \infty\ . \end{cases}$$

For a given $r > 0$, we introduce the convex set

$$K(r) = \{z \in \ell^2(\mathbb{Z}^n) : |z(i) - z(j)| \le r \quad \text{for} \quad i \sim j\}.$$

2.2 Existence and uniqueness of a solution.

For $\lambda > 0$, we consider $\Big(t_l\Big)_{l=1,\dots n}$ a $\lambda-$ discretization of $[0,T)$, that is $t_0 = 0 < t_1 < \dots < t_{n-1} < T = t_n$. For any $\lambda > 0$, we say that u_λ is a $\lambda-$approximate solution of (DM), if there exists $\Big(t_l\Big)_{l=1,\dots n}$ a $\lambda-$ discretization of $[0,T)$, such that

$$(3) \qquad u_\lambda(t) = \begin{cases} u_0 & \text{for } t \in\ [0,t_1], \\ \\ u_l & \text{for } t \in\ ]t_{l-1},t_l],\ l = 2,\dots n \end{cases}$$

and u_l solves the Euler implicit time discretization of (DM)

$$(4) \qquad \left.\begin{cases} u_l(i) + \displaystyle\sum_{j:j\sim i} \sigma_l(i,j) = u_{l-1}(i), \quad i \in \mathbb{Z}^n \\ \\ u_l \in K(c(t_l)),\ \ \sigma_l(i,j) = -\sigma_l(j,i), \quad i,j \in \mathbb{Z}^n \\ \\ \text{and } \ support(\sigma_l) \subseteq X_{c(t_l)}(u_l), \end{cases}\right\} \quad l = 1,\dots,n.$$

See that the generic problem is given by

$$(DSP) \qquad \begin{cases} v(i) + \sum_{j:j\sim i} \sigma(i,j) = g(i) & \text{for any } i \in \mathbb{Z}^n, \\ \\ v \in K(r), \quad \sigma(i,j) = -\sigma(j,i) & \text{for any } (i,j) \in \mathbb{Z}^n \times \mathbb{Z}^n, \\ \\ \text{ and } \ support(\sigma) \subseteq X_r(v), \end{cases}$$

where $r \geq 1$ is a given constant and $g : \mathbb{Z}^2 \to \mathbb{R}$ is a given application. In this paper, we prove

Theorem 1 *Let $g \in \ell^2(\mathbb{Z}^n)$ and $v \in K(r)$. Then $v = \mathbb{P}_{K(r)}(g)$ if and only if, there exists $\sigma \in \ell^1(\mathbb{Z}^n \times \mathbb{Z}^n)$, such that the couple (v, σ) satisfies (DSP).*

In the Theorem $\mathbb{P}_{K(r)}(g)$ denote the standard projection onto the convex set $K(r)$. Remember that $v = \mathbb{P}_{K(r)}(g)$ if and only if $v \in K(r)$ and

$$J(v) = \frac{1}{2}\|v - g\|^2_{\ell^2(\mathbb{Z}^n)} = \min_{z \in K(r)} J(z). \tag{5}$$

Now, let us consider $\mathbb{I}_{K(r)}$ the convex indicator function of $K(r)$ given as

$$\mathbb{I}_{K(r)}(z) = \begin{cases} 0 & \text{if } z \in K(r) \\ \\ +\infty & \text{otherwise.} \end{cases}$$

As a consequence of Theorem 1, the characterization of $\partial \mathbb{I}_K$ in terms of a discrete equation is given by the following Corollary:

Corollary 1 *Let $g \in \ell^2(\mathbb{Z}^n)$ and $v \in K(r)$. Then, $g \in \partial \mathbb{I}_{K(r)}(v)$ if and only if there exists $\sigma \in \ell^1(\mathbb{Z}^n \times \mathbb{Z}^n)$, such that the couple (v, σ) satisfies*

$$\begin{cases} \sum_{j:j\sim i} \sigma(i,j) = g(i) & \textit{for any } i \in \mathbb{Z}^n, \\ \\ v \in K(r), \quad \sigma(i,j) = -\sigma(j,i) & \textit{for } (i,j) \in \mathbb{Z}^n \times \mathbb{Z}^n, \\ \\ \textit{ and } \ support(\sigma) \subseteq X_r(v). \end{cases}$$

In particular, this corollary gives the connexion between the evolution problem (DM) and the nonlinear dynamic

$$\begin{cases} u_t(t) + \partial \mathbb{I}_{K(c(t))}(u(t)) \ni 0 \text{ for } t \in (0,T) \\ \\ u(0) = u_0. \end{cases} \tag{6}$$

Again, for any $\lambda > 0$, we say that u_λ is a $\lambda-$approximate solution of (6), if there exists $\Big(t_l\Big)_{l=1,...n}$ a $\lambda-$ discretization of $[0,T)$, such that

$$u_\lambda(t) = \begin{cases} u_0 & \text{for } t \in\ [0,t_1], \\ \\ u_l & \text{for } t \in\ ]t_{l-1}, t_l],\ l = 2, ...n \end{cases}$$

and u_l solves the Euler implicit time discretization of (6), that is

$$u_l = I\!P_{K(c(t_l))} u_{l-1}, \qquad \text{for } l = 1, ...n.$$

It is clear that this problem is a particular case of the stationary problem

$$(7) \qquad v + \partial I\!I_{K(r)}(v) \ni g \qquad \text{i.e.} \sum_{i \in \mathbb{Z}^n} (g - v)\, v = \max_{\xi \in K(r)} \sum_{i \in \mathbb{Z}^n} (g - v)\, \xi.$$

Applying Corollary 1, for any $g \in \ell^2(\mathbb{Z}^n)$, there exists a solution u of the problem (DSP) in the sense that $u \in K(r)$ and

$$\sum_i \Big(g(i) - u(i)\Big)\Big(u(i) - \xi(i)\Big) \geq 0 \quad \text{for any } \xi \in K(r).$$

For the existence and uniqueness of the solution of (DM), we prove the following results

Theorem 2 *Assume that $c \in W^{1,\infty}(0,\ T)$, $u_0 \in K(c(0))$ and $0 < T < \infty$. Then the problem (DM) has a unique solution $u \in W^{1,1}\Big(0, T;\ \ell^2(\mathbb{Z}^n)\Big)$ and u satisfies*

$$(8) \qquad \begin{cases} u_t(t) + \partial I\!I_{K(c(t))}(u(t)) \ni 0 \text{ for } t \in (0,T) \\ \\ u(0) = u_0. \end{cases}$$

Moreover, if u_λ is a λ- approximate solution, then

$$u_\lambda \longrightarrow u \quad \text{in} \quad \mathcal{C}([0,T); \ell^2(\mathbb{Z}^n)) \qquad \text{as } \lambda \longrightarrow 0.$$

Proof: It is not difficult to see that u is a solution of (8) if and only if $v(t) := u(t)/c(t)$ is a solution of

$$(9) \qquad \begin{cases} v_t(t) + \partial \mathrm{I\!I}_{K(1)}(v(t)) + F(t, v(t)) \ni 0 \text{ for } t \in (0,T) \\ \\ v(0) = u_0/c(0), \end{cases}$$

where $F(t,r) = \dfrac{c'(t)}{c(t)} r$. It is clear that, F is measurable in t and Lipschitz continuous with respect to r. Since $\dfrac{c'(t)}{c(t)} \in L^\infty(0,\ T)$, thanks to Proposition 3.13 of [9], the problem (9) has a unique solution $v \in W^{1,\infty}\Big(0, T;\ \ell^2(\mathbb{Z}^n)\Big)$. Then, using similar arguments of [20] combining v and u_λ, one can prove that the $\lambda-$approximate solution converges to u and this ends up the proof of the theorem. ∎

Proposition 1 *Assume that, there exists (σ, v) such that $(\sigma, v) \in \ell^1(\mathbb{Z}^n \times \mathbb{Z}^n) \times K(r)$ satisfying*

$$(10) \qquad \begin{cases} v(i) + \displaystyle\sum_{j:j\sim i} \sigma(i,j) = g(i) & \text{for } i \in \mathbb{Z}^n, \\ \sigma(i,j) = -\sigma(j,i) & \text{for } (i,j) \in \mathbb{Z}^n \times \mathbb{Z}^n, \\ \text{and } \ \sigma(i,j) \neq 0 \Rightarrow |v(i) - v(j)| = r & \text{for } (i,j) \in \mathbb{Z}^n \times \mathbb{Z}^n, \end{cases}$$

then v is solution of the problem (7).

Proof: Let $z \in K(r)$. First we see that

$$\begin{aligned} I &= \sum_i (g(i) - v(i))(v(i) - z(i)) \\ &= \sum_i \sum_{j:j\sim i} \sigma(i,j)(v(i) - z(i)) \\ &= \sum_i \sum_{j:j\sim i} \sigma(i,j)(v(i) - v(j) + v(j) - z(j) + z(j) - z(i)) \\ &= \sum_i \sum_{j:j\sim i} \sigma(i,j)(v(i) - v(j) + z(j) - z(i)) + \sum_i \sum_{j:j\sim i} \sigma(i,j)(v(j) - z(j)) \\ &= \sum_i \sum_{j:j\sim i} \sigma(i,j)(v(i) - v(j) + z(j) - z(i)) - \sum_i \sum_{j:j\sim i} \sigma(j,i)(v(j) - z(j)). \end{aligned}$$

Since $(\sigma, v) \in \ell^1(\mathbb{Z}^n \times \mathbb{Z}^n) \times K(r)$ and using Fubini's theorem, we have

$$\sum_i \sum_{j:j\sim i} \sigma(j,i)(v(j) - z(j)) = \sum_j \sum_{i:i\sim j} \sigma(j,i)(v(j) - z(j)).$$

Then

$$\begin{aligned} I &= \sum_i \sum_{j:j\sim i} \sigma(i,j)(v(i) - v(j) + z(j) - z(i)) - \sum_j \sum_{i:i\sim j} \sigma(j,i)(v(j) - z(j)) \\ &= \sum_i \sum_{j:j\sim i} \sigma(i,j)(v(i) - v(j) + z(j) - z(i)) - I, \end{aligned}$$

which implies that

$$2I = \sum_i \sum_{j:j\sim i} \sigma(i,j)(v(i) - v(j) + z(j) - z(i)).$$

Now, using the condition $\sigma(i,j) \neq 0 \Rightarrow |v(i) - v(j)| = r$, then we obtain

- If $v(i) - v(j) = r$ we get $v(i) - v(j) + z(j) - z(i) \geq 0$ and $\sigma(i,j) > 0$, then

$$\sigma(i,j)(v(i) - v(j) + z(j) - z(i)) \geq 0.$$

- If $v(i)-v(j)=-r$ we have $v(i)-v(j)+z(j)-z(i)\leq 0 \quad$ and $\quad \sigma(i,j)<0$, then

$$\sigma(i,j)(v(i)-v(j)+z(j)-z(i))\geq 0.$$

Consequently, $I\geq 0$ and this completes the proof of Proposition. $\blacksquare$

This proposition gives a first part of the proof of Theorem 1. The second and final part of the proof is given at the end of section 3.

3 Numerical study

Now, our aim is to study numerical approximation of $P_{K(r)}g$. Thanks to theorem 1, the problem (DSP) has a unique solution v satisfying

$$v+\partial I\!I_{K(r)}(v)\ni g. \tag{11}$$

To give a numerical method for the approximation of the solution of (7), we use dual arguments. Thanks to Theorem 1, a solution of (DSP) is given by

$$v=I\!P_{K(r)}g, \tag{12}$$

3.1 Dual formulation

To study the numerical approximation of $P_{K(r)}g$, we use dual arguments. To this aim, we introduce the set of anti-symmetric bounded sequence defined on $Z\!\!\!Z^n\times Z\!\!\!Z^n$ by:

$$\ell^1_{as}(Z\!\!\!Z^n\times Z\!\!\!Z^n)=\Big\{\hat\mu\in\ell^1(Z\!\!\!Z^n\times Z\!\!\!Z^n)\ ;\hat\mu(i,j)=-\hat\mu(j,i),\ \text{for any } (i,j)\in Z\!\!\!Z^n\times Z\!\!\!Z^n\Big\}$$

and the set $\mathcal{S}_{as}(Z\!\!\!Z^n\times Z\!\!\!Z^n)$ of sequences of $\ell^1_{as}(Z\!\!\!Z^n\times Z\!\!\!Z^n)$ concentrated on the set $\Big\{(i,j)\in Z\!\!\!Z^n\times Z\!\!\!Z^n\ ;\ :i\sim j\Big\}$; i.e.

$$\mathcal{S}_{as}=\Big\{\hat\mu\in\ell^1_{as}(Z\!\!\!Z^n\times Z\!\!\!Z^n)\ ;\hat\mu(i,j)=0 \text{ for } |i-j|>1\Big\}.$$

Considering the operator $\Lambda:\ell^2(Z\!\!\!Z^n)\longrightarrow\mathcal{C}_0(Z\!\!\!Z^n\times Z\!\!\!Z^n)$ defined by

$$\begin{array}{rll}\Lambda z: & Z\!\!\!Z^n\times Z\!\!\!Z^n & \longrightarrow I\!R^+\\ & (i,j) & \longrightarrow \Lambda z(i,j)=\left\{\begin{array}{ll} z(i)-z(j) & \text{if } i\sim j\\ 0 & \text{otherwise,}\end{array}\right.\end{array}$$

the problem (5) can be rewritten as

$$\min_{z\in\ell^2(Z\!\!\!Z^n)}\Big\{J(z)+H(\Lambda z)\Big\}, \tag{13}$$

where the function $H:\mathcal{C}_0(Z\!\!\!Z^n\times Z\!\!\!Z^n)\longrightarrow I\!R^+$ is given by

$$H(\Lambda z) = \begin{cases} 0 & \text{if } \|\Lambda z\|_\infty \le r \\ +\infty & \text{otherwise .} \end{cases}$$

Using standard duality argument (cf. [14]), we compute the dual problem associated to (5). This is the aim of the following proposition:

Proposition 2 *Let $g \in \ell^2(\mathbb{Z}^n)$. Then, the dual problem of (5) is given by*

$$\max_{\eta \in \ell^1(\mathbb{Z}^n \times \mathbb{Z}^n)} G(\eta),$$

where $G:\ \ell^1(\mathbb{Z}^n \times \mathbb{Z}^n) \longrightarrow \mathbb{R}$ is defined by

$$(14)\, G(\eta) = -\frac{1}{2}\sum_i \Big(\sum_{j:j\sim i} \eta(i,j) - \eta(j,i) \Big)^2 - \sum_i \Big(\sum_{j:j\sim i} \eta(i,j) - \eta(j,i) \Big)\, g(i) - r \sum_{i,j} |\eta(i,j)|.$$

Proof: Thanks to Theorem 4.2 of [14], the dual problem of (13) can be written as:

$$\max_{\eta^* \in \left(\mathcal{C}_0(\mathbb{Z}^n \times \mathbb{Z}^n)\right)^*} \Big\{ - J^*(\Lambda^*\eta^*) - H^*(-\eta^*) \Big\}, \tag{15}$$

where J^*, H^* and Λ^* are the conjugate of J, H and Λ, respectively. Recall that $\Big(\mathcal{C}_0(\mathbb{Z}^n \times \mathbb{Z}^n)\Big)^* = \ell^1(\mathbb{Z}^n \times \mathbb{Z}^n)$. First, we see that, for any $\eta \in \ell^1(\mathbb{Z}^n \times \mathbb{Z}^n)$, we have

$$\begin{aligned} \langle \Lambda^*\eta, z\rangle &= \langle \eta, \Lambda z\rangle \\ &= \sum_i \sum_j \eta(i,j)\Lambda z(i,j) \\ &= \sum_i \sum_{j:j\sim i} \eta(i,j)\Big((z(i) - z(j)\Big). \\ &= \sum_i \sum_{j:j\sim i} \eta(i,j) z(i) - \sum_j \sum_{i:i\sim j} \eta(j,i) z(i) \\ &= \sum_i \sum_{j:j\sim i} \Big(\eta(i,j) - \eta(j,i)\Big) z(i), \end{aligned}$$

where we have used the fact that $(i,j) \longrightarrow \eta(i,j)z(j)$ is in $\ell^1(\mathbb{Z}^n \times \mathbb{Z}^n)$. This implies that

$$\begin{aligned} \Lambda^* :\ \ \ell^1(\mathbb{Z}^n \times \mathbb{Z}^n) &\longrightarrow \ \ell^2(\mathbb{Z}^n) \\ \eta &\longrightarrow \ (\Lambda^*\eta)(i) = \sum_{j:j\sim i} \Big(\eta(i,j) - \eta(j,i)\Big). \end{aligned}$$

On the other hand, for any $z^* \in \ell^2(\mathbb{Z}^n)$, we have

$$\begin{aligned}
J^*(z^*) &= \sup_{z\in\ell^2(\mathbb{Z}^n)} \sum_i z^*(i)z(i) - \frac{1}{2}\sum_i |z(i)-g(i)|^2 \\
&= \sup_{z\in\ell^2(\mathbb{Z}^n)} \sum_i z^*(i)(z(i)-g(i)) - \frac{1}{2}\sum_i |z(i)-g(i)|^2 + \sum_i z^*(i)g(i) \\
&= \sup_{y\in\ell^2(\mathbb{Z}^n)} \sum_i z^*(i)y(i) - \frac{1}{2}\sum_i |y(i)|^2 + \sum_i z^*(i)g(i) \\
&= \frac{1}{2}\sum_i |z^*(i)|^2 + \sum_i z^*(i)g(i).
\end{aligned}$$

At last, for any $p^* \in \ell^1(\mathbb{Z}^n \times \mathbb{Z}^n)$, we have

$$\begin{aligned}
H^*(p^*) &= \sup_{p\in\ell^1(\mathbb{Z}^n\times\mathbb{Z}^n)} \sum_{i,j} p^*(i,j)p(i,j) - H(p) \\
&= \sup_{p\in\ell^1(\mathbb{Z}^n\times\mathbb{Z}^n)} \sum_{i,j} p^*(i,j)p(i,j) \text{ if } ||p||_\infty \le r \\
&= r\sum_{i,j} |p^*(i,j)|,
\end{aligned}$$

and the proof is complete. ∎

Moreover, we have

Lemma 1 *Assume that, there exists $w \in \ell^1(\mathbb{Z}^n \times \mathbb{Z}^n)$ such that*

$$G(w) = \max_{\eta\in\ell^1(\mathbb{Z}^n\times\mathbb{Z}^n)} G(\eta),$$

then $w \in \mathcal{S}_{as}$.

Proof: We assume, by contradiction, that the maximum w of G satisfies $w(i_0, j_0) \neq 0$ at a point $(i_0 \nsim j_0)$, and take

$$\bar{w} = \begin{cases} w & \text{if } (i,j) \neq (i_0, j_0) \\ 0 & \text{if } (i_0, j_0). \end{cases}$$

A simple calculation, gives

$$\begin{aligned}
G(w) &= -\frac{1}{2}\sum_i\Big(\sum_{j:j\sim i} w(i,j) - w(j,i)\Big)^2 - \sum_i\Big(\sum_{j:j\sim i} w(i,j) - w(j,i)\Big)\, g(i) - r\sum_{i,j}|w(i,j)| \\
&= G(\bar{w}) - r|w(i_0,j_0)|,
\end{aligned}$$

then $G(\bar{w}) > G(w)$ and we get the contradiction with the maximality of G at w. Now, taking $\tilde{w} \in \mathcal{S}_{as}$ as the following:

$$\begin{aligned}
\tilde{w}(i,j) &= \frac{1}{2}\Big(w(i,j) - w(j,i)\Big) \\
\tilde{w}(j,i) &= \frac{1}{2}\Big(w(j,i) - w(i,j)\Big),
\end{aligned}$$

we see that $\tilde{w}(i,j) - \tilde{w}(j,i) = w(i,j) - w(j,i)$. On the other hand, we have

$$|\tilde{w}(i,j)| \leq \frac{1}{2}|w(i,j)| + \frac{1}{2}|w(j,i)|,$$

then $-\sum |\tilde{w}(i,j)| \geq -\sum |w(i,j)|$ and we deduce that

$$G(\tilde{w}) \geq G(w).$$

From this, we get

$$\max_{\eta \in \ell^1(\mathbb{Z}^n \times \mathbb{Z}^n)} G(\eta) = \max_{\eta \in \mathcal{S}_{as}} G(\eta).$$

This completes the proof of Lemma. ∎

As consequence of Proposition 2 and Lemma 1, we have the following result:

Theorem 3 *Let $g \in \ell^2(\mathbb{Z}^n)$ and $v := \mathbb{P}_{K(r)}(g)$. Then, there exists $w \in \mathcal{S}_{as}$ and $v \in K(r)$ such that*

$$G(w) = \max_{\eta \in \mathcal{S}_{as}} G(\eta) = \min_{z \in K(r)} J(z) = J(v).$$

Moreover, for any $i \in \mathbb{Z}^n$,

$$v(i) = g(i) + \sum_{j : j \sim i} \Big(w(i,j) - w(j,i) \Big).$$

Proof: Thanks to proposition 2, we have

$$J(v) = \min_{z \in K(r)} J(z) = \max_{\eta \in \ell^1(\mathbb{Z}^n \times \mathbb{Z}^n)} G(\eta).$$

Using lemma 1, we obtain

$$\max_{\eta \in \ell^1(\mathbb{Z}^n \times \mathbb{Z}^n)} G(\eta) = \max_{\eta \in \mathcal{S}_{as}} G(\eta) = G(w).$$

Thanks to the extremality relation between v and w, we have

(16) $$(\Lambda^* w, -w) \in \Big(\partial J(v), \partial H(\Lambda v) \Big).$$

Since $\Lambda^* w \in \partial J(v)$, then we have

$$\sum_{j : j \sim i} w(i,j) - w(j,i) = v(i) - g(i), \quad \text{for any } i \in \mathbb{Z}^n.$$

We deduce, that

$$v(i) = g(i) + \sum_{j : j \sim i} \Big(w(i,j) - w(j,i) \Big), \quad \text{for any } i \in \mathbb{Z}^n.$$

and the proof is finished. $\blacksquare$

Proof of Theorem 1 finished: Thanks to Proposition 1, we know that, if there exists (σ, v) such that $(\sigma, v) \in \ell^1(\mathbb{Z}^n \times \mathbb{Z}^n) \times K(r)$ satisfies (10), then $v = \mathbb{P}_{K(r)}(g)$. Now, taking $g \in \ell^2(\mathbb{Z}^n)$ and $v \in K(r)$ satisfies $v + \partial \mathbb{I}_{K(r)}(v) \ni g$. Thanks to Theorem 3, we have

$$v(i) + \sum_{j:j\sim i} \sigma(i,j) = g(i), \quad \text{for } i \in \mathbb{Z}^n,$$

where $\sigma(i,j) = w(i,j) - w(j,i) = 2w(i,j)$. Now we prove that, if $\sigma(i,j) \neq 0$ then $|v(i) - v(j)| = r$ for any $(i,j) \in \mathbb{Z}^n \times \mathbb{Z}^n$. Indeed, thanks to (16), we have $-w \in \partial H(\Lambda v)$, then we get

$$H(\Lambda v) + H^*(-w) = < -w, \Lambda v >$$

which implies that

$$r \sum_{i,j} |w(i,j)| = -\sum_i \sum_{j:j\sim i} w(i,j)\Big(v(i) - v(j)\Big),$$

and therefore, we have $w(i,j) = 0$ for $i \nsim j$ and

$$r|w(i,j)| = -w(i,j)\Big(v(i) - v(j)\Big) \quad \text{for } i \sim j.$$

Consequently

$$\sigma(i,j) \neq 0 \Rightarrow |v(i) - v(j)| = r \text{ for } (i,j) \in \mathbb{Z}^n \times \mathbb{Z}^n$$

and the proof is finished. $\blacksquare$

3.2 Numerical results and simulations

To compute numerically the solution of the problem (6), we attempt to discretize it by the Euler implicit scheme. Let us denote by Δt the time step and $u^n(i)$ the approximate solution at time $t = n\Delta t$ for $n \in \mathbb{N}$. Then the study of the problem (6) becomes to solve a sequence of stationary equations. Starting with u^0, we need to compute u^{n+1} satisfying :

$$u^{n+1} + \partial \mathbb{I}_{K(r)}(u^{n+1}) \ni u^n + \Delta t f^n := g^n \tag{17}$$

where $f^n(i) = f(n\Delta t)$ for $n = 0, 1, 2, ..., K$; where $K \in \mathbb{N}$ is given. The problem (17) can be rewritten as

$$J(u^{n+1}) = \min_{z \in K(r)} J(z). \tag{18}$$

where $J(z) = \dfrac{1}{2}\|z - g^n\|^2_{\ell^2(\mathbb{Z}^2)}$. Thanks to Theorem 3, it is clear that the problem is equivalent to find a numerical method to minimize the functional $\tilde{G} : \mathcal{S}_{as} \longrightarrow \mathbb{R}$ defined by

$$\tilde{G}(\eta) = \frac{1}{2} \sum_{i \in A_N} \Big(\sum_{j:j\sim i} \eta(i,j) - \eta(j,i) \Big)^2 + \sum_{i \in A_N} \Big(\sum_{j:j\sim i} \eta(i,j) - \eta(j,i) \Big) g^n(i) + r \sum_{(i,j) \in A_N \times A_N} |\eta(i,j)|$$

for a given $g^n(i)$, where $A_N := \Big\{ i = (i_1, i_2) \in \mathbb{Z}^2 \quad \text{such that } -N \leq i_1, i_2 \leq N \Big\}$ with N is a given large (enough) integer.
Thanks to the fact that $\sigma \in \mathcal{S}_{as}$ the functional can be rewritten as

$$\begin{aligned}
\tilde{G}(\eta) \; &= 2 \sum_{-N\leq i_1,i_2\leq N} \Big(\sum_{k\in\{-1,1\}} \eta(i_1,i_2,i_1+k,i_2) + \sum_{l\in\{-1,1\}} \sigma(i_1,i_2,i_1,i_2+l) \Big)^2 \\
&+2 \sum_{-N\leq i_1,i_2\leq N} \Big(\sum_{k\in\{-1,1\}} \eta(i_1,i_2,i_1+k,i_2) + \sum_{l\in\{-1,1\}} \sigma(i_1,i_2,i_1,i_2+l) \Big) g^n(i_1,i_2) \\
&+r \sum_{-N\leq i_1,i_2\leq N} \Big(\sum_{k\in\{-1,1\}} |\eta(i_1,i_2,i_1+k,i_2)| + \sum_{l\in\{-1,1\}} |\sigma(i_1,i_2,i_1,i_2+l)| \Big)
\end{aligned}$$

and for convenience taking $\eta(i_1,i_2,j_1,j_2) = 0$ for $\max\{j_1,j_2\} > N$ or $\min\{j_1,j_2\} < -N$.

Since, the functional $\tilde{G}$ is non-differentiable, we use a relaxation algorithm (cf. [22]). Denoting the cartesian basis vectors by e_j for $j = 1,...,M$ with $M = 8\,N^2$; the algorithm can be written as follows :

1. Initiate the algorithm with w^0, set $k = 0$
2. For $j = 1,...,M$, we solve the one-dimensional subproblems $\min\limits_{\xi\in\mathbb{R}} \Psi_{jk}(\xi)$ where Ψ_{jk} is defined as:
$$\begin{aligned} \Psi_{jk} : \quad & \mathbb{R} \longrightarrow \mathbb{R} \\ & \xi \longmapsto \tilde{G}(w^k + \sum_{i<j} \xi_{ik} + \xi e_j). \end{aligned}$$
Since Ψ_{jk} is the sum of a polynomial of degree two and an absolute value, we are used a Newton algorithm to find ξ^*_k when Ψ_{jk} is differentiable, and computing directly ξ^*_k otherwise. After, taking $\xi_{jk} = \xi^*_k\, e_j$.
3. Take $w^{k+1} = w^k + \lambda \sum\limits_{j=1,...,M} \xi_{jk}$, where $\lambda \in (0,2)$ is an over-relaxation parameter.
4. As stopping criterion we use : $||w^k - w^{k+1}||_{\ell^2(\mathbb{R}^M)} < tol$, for a given convergence tolerance tol.

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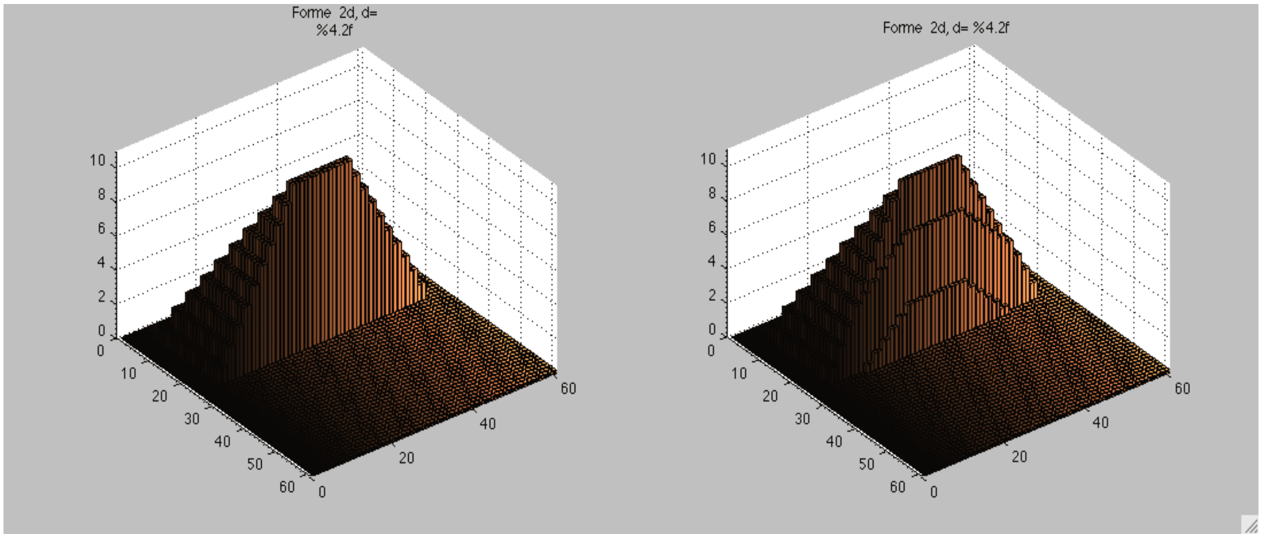
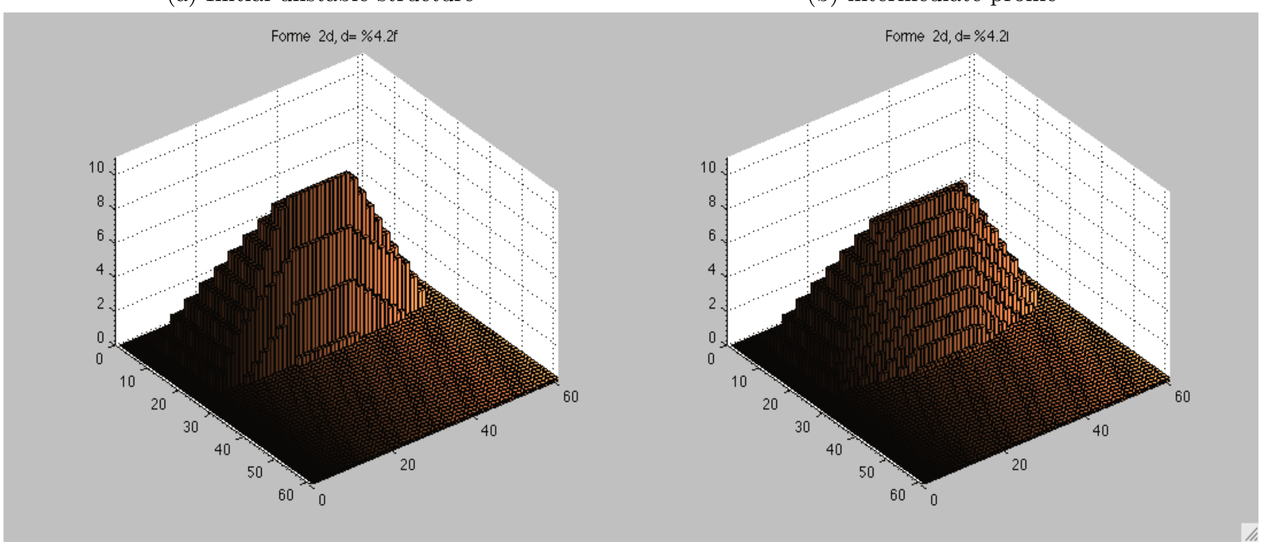


(a) Initial unstable structure

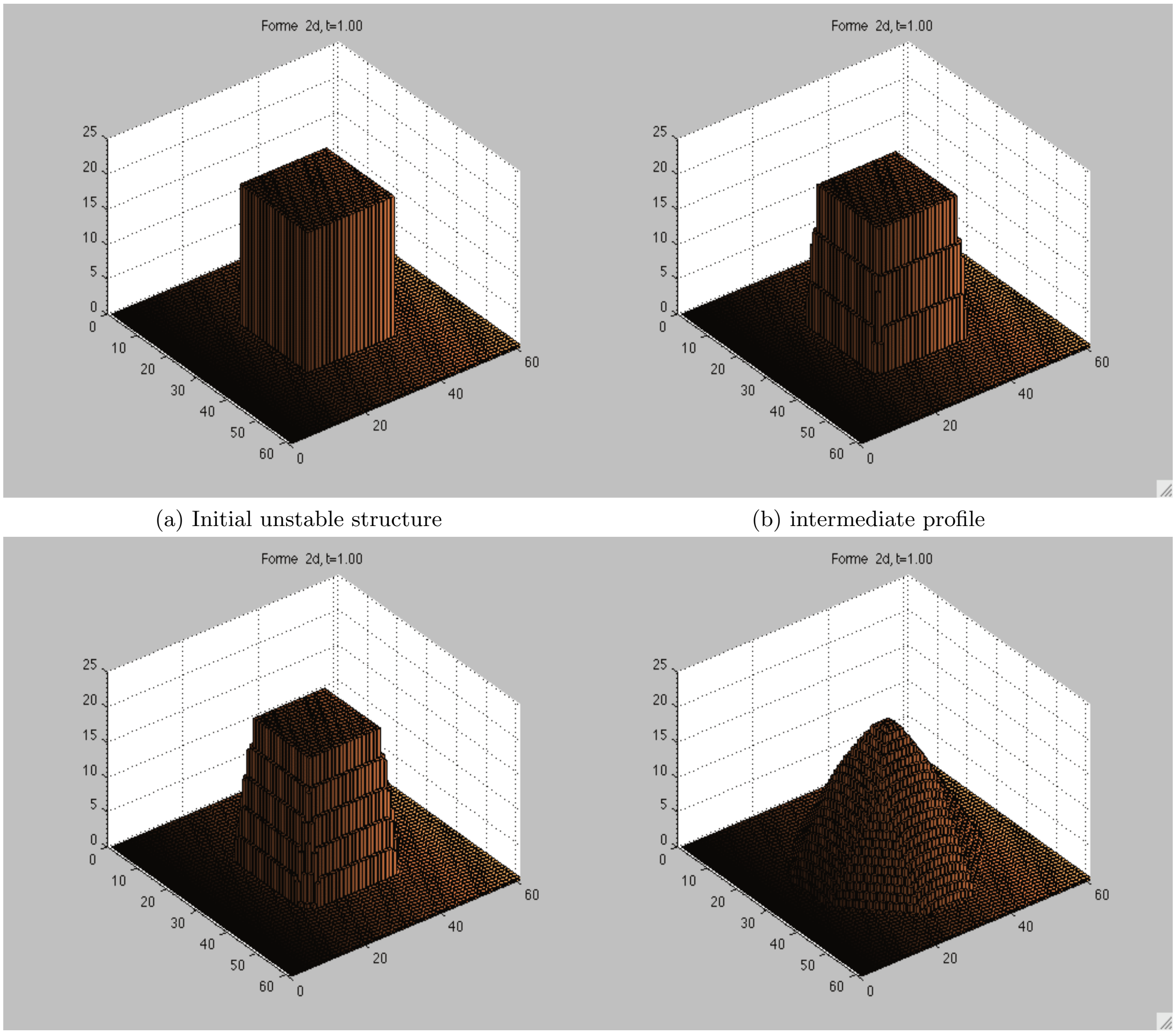
(b) intermediate profile



(c) intermediate profile

(d) Final stable structure

Figure 1: Stabilization of unstable discrete structure.



(a) Initial unstable structure

(b) intermediate profile

(c) intermediate profile

(d) Final stable structure

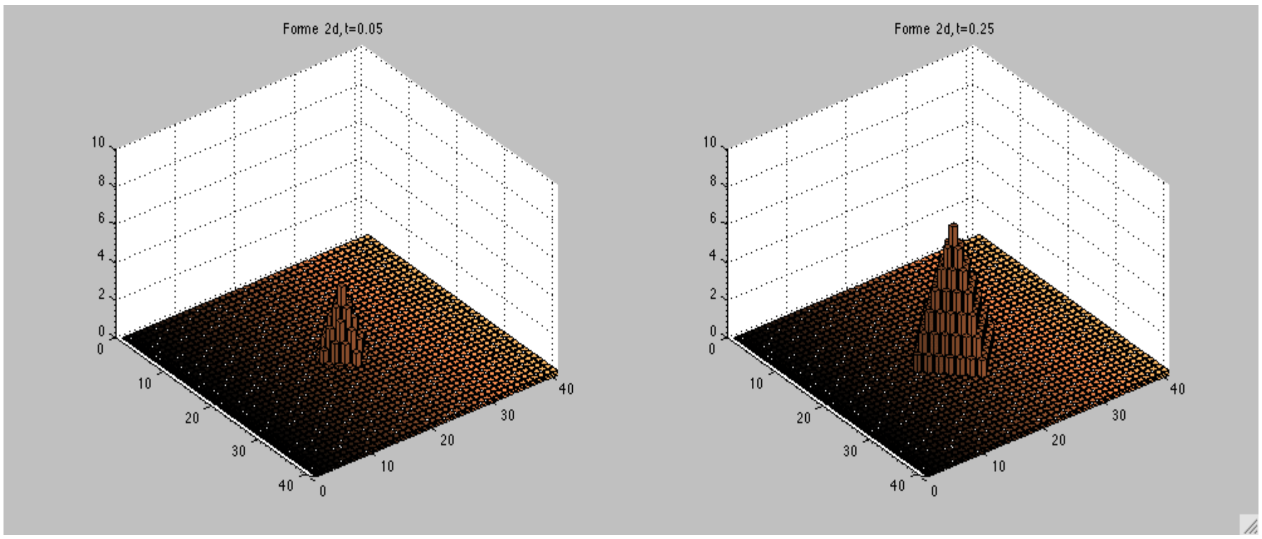
Figure 2: Stabilization of unstable discrete structure.

Our numerical algorithm enables us also to simulate the growing of a discrete structure when we have a source of distribution of materials. Recall that in this case, the problem may be written as:

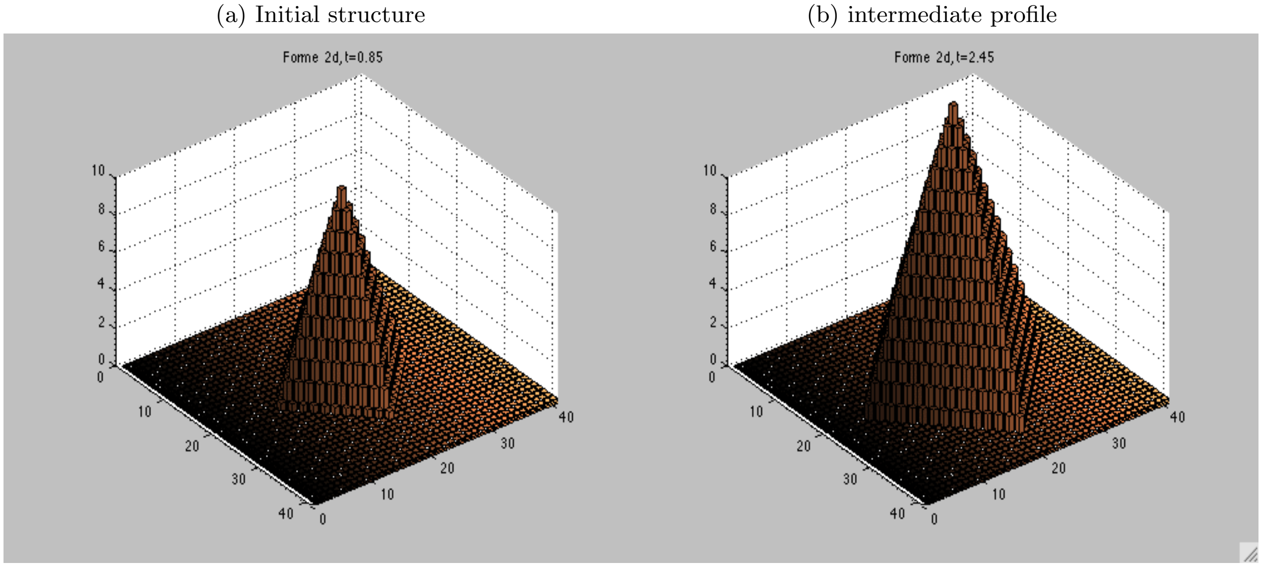
$$\begin{cases} u_t(t) + \partial \mathbb{I}_{K(1)}(u(t)) \ni f \text{ for } t \in (0,T) \\ \\ u(0) = u_0, \end{cases}$$

where f modeling the source term.
The idea is to keep the stability condition; *i.e.* $c \equiv 1$ and using the Euler implicit time

discretization of the problem, we compute successive projection of terms including the material that the source add per time. We present some numerical experiments of a growing pile of cubes. In all the simulation below, we have chosen relaxation parameter $\lambda = 1.2$, and convergence tolerance $tol = 10^{-6}$. The first case is devoted to the constant source term $f(t, i)$ distributed on the subdomain.

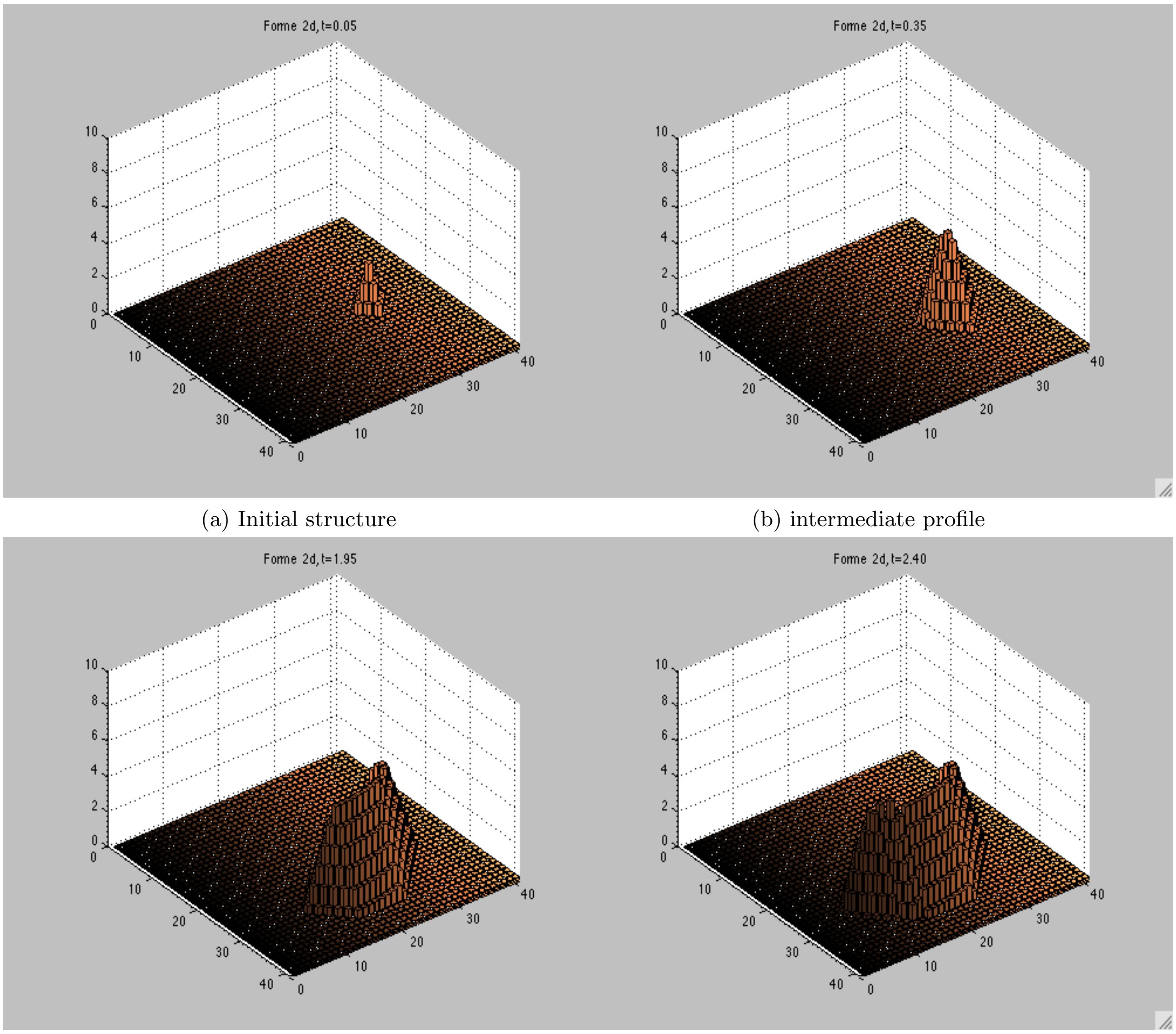


(a) Initial structure (b) intermediate profile



(c) intermediate profile (d) Final structure

Figure 3: Growing discrete structure with central source.



(a) Initial structure (b) intermediate profile

(c) intermediate profile (d) Final structure

Figure 4: Growing discrete structure with moving source.

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