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Revising uniqueness for a nonlinear diffusion–convection equation

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Abstract

This paper provides a generalized and simplified proof of the uniqueness of a weak solution for nonlinear diffusion–convection problems of Stefan type with homogeneous boundary conditions and continuous convection.

0. Introduction

Throughout the paper, $\Omega \subset \mathbb{R}^N$ is a bounded open domain of $\mathbb{R}^N$, $T > 0$, $Q = (0,T) \times \Omega$ and $\Sigma = (0,T) \times \partial\Omega$. For given $f \in L^1(Q)$ and $u_0 \in L^1(\Omega)$, we are interested in the uniqueness of a (weak) solution to the evolution problem

$$\begin{cases} \partial_t u = \Delta w + \nabla \cdot F(u,w) + f(t,x), \quad w \in \beta(u) \text{ in } Q, \\ w = 0 \quad \text{on } \Sigma, \\ u(0) = u_0 \quad \text{in } \Omega, \end{cases} \tag{E(u_0, f)}$$

under the two assumptions:

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(H1) β is a maximal monotone graph such that $0 \in \beta(0)$;
(H2) $F(p,q) = F_1(q) + pF_2(q)$, for any $(p,q) \in \mathbb{R} \times \mathbb{R}$, with $F_i \in \mathcal{C}(\mathbb{R};\mathbb{R}^N)$ and $F_2(0) = 0$.

Following our references, we assume that Ω is a strongly Lipschitz domain. At the end of the paper, we discuss an extension of the uniqueness results to more general domains and diffusion operators.

There is an extensive literature on this type of problems, since it serves as a mathematical model for a large class of physical problems (cf. [1,10] and the references therein). We are interested in the question of uniqueness of solutions. In general, i.e., without the assumption (H2), one cannot expect that weak solutions are unique. Indeed, the possible degeneracy of β makes the problem to be hyperbolic in some regions of Ω, precisely in the set where u lives in the plane regions of β. To ensure uniqueness of solutions in this case, one requires some additional conditions, called, following Kruzhkov [17], "entropy inequalities," to single out physically relevant solutions. Recall that in the case $\beta \equiv 0$, $\mathrm{E}(u_0, f)$ is the hyperbolic scalar conservation law; the questions of existence and uniqueness were studied in $\mathbb{R}^N$ by Kruzhkov [17], by introducing the notion of an entropy solution. This was generalized to bounded domains by Hil'debrand [11] and Bardos et al. [3] for smooth data, and, more recently, by Otto [20] for general L^∞ data. As to the general case, i.e., if β is a maximal monotone graph, the problem is said to be hyperbolic–elliptic–parabolic. A first attempt to prove that the problems of this kind are well posed was given in [21], and recently, Carillo handled very well the problem by introducing the notion of weak entropy solutions and proved existence and uniqueness for continuous flux functions F (cf. [8]).

For the existence, we refer the reader to the papers [1,2,5,8,13–15], though it has to be bore in mind that under the structure condition (H2), the proof of existence turns out to be nonstandard (one can see [2,5,13]).

Our main interest lays in the uniqueness of a weak solution. The question of uniqueness is well understood if one removes the convection term $\nabla \cdot F(u,w)$ (cf. [6,18]). As to the case of nonnull convection, still the entropy conditions are expected to be superfluous, if firstly, the problem becomes linear in the hyperbolic regions; and secondly, the hyperbolic regions do not touch the boundary. This is exactly the case of our problem under the assumption (H2) and homogeneous Dirichlet boundary condition.

The first proof of uniqueness using duality technics appeared in [18], for the case of sufficiently regular β, β^{-1} and F_1 (one can also see [10]). If β^{-1} is continuous, then the problem is elliptic–parabolic, and thanks to [19] for Lipschitz continuous F_1, to [4] for α-Hölder continuous F_1, with $\alpha \geqslant 1/2$, and to [7] for continuous F_1, the uniqueness of a weak solution is well understood by now. For general nonlinearity of β and continuous F_i, as far as we know, the uniqueness of a weak solution is still open in its generality. In [8], the uniqueness was established under the additional assumption that $\beta^{-1}(0) = \{0\}$. In [12], the authors assumed that F_i is Lipschitz continuous, and in [16], it is assumed that F is continuous and satisfies

$$\|F(u,w)\| \leqslant C\|w\|^2, \quad \text{for any } w \geqslant r_0, \tag{0.2}$$

where r_0 and C are nonnegative constants. It is the purpose of this paper to prove the uniqueness of a weak solution under the structure condition (H2), by assuming only continuity of F_1 and F_2.

As in [8] (see also [12,16]), we tend to establish that weak solutions are weak entropy solutions in the sense of [8], thus they are unique and satisfy the L^1 comparison principle. The main simplification and novelty is that we drop out the unnecessary, for the uniqueness, growth condition (0.2) used in [16] to treat the boundary term. Indeed, we start by showing in the standard way (see [8,12,16]) that a weak solution satisfies the entropy inequality in the interior of Ω (in the sense of [8]). Then, in contrast to [16], where the existence of normal traces of divergence-measure fields (cf. [9]) has been used in order to show that the entropy inequalities still hold up to the boundary, we simply use the weak formulation of the solution in its standard consequence (2.1) combined with the hint of Lemma 1 (see Section 3).

The definition of weak solution we have in mind and our main result are given in Section 1. In Section 2, we give the proofs. In Sections 3 and 4, we briefly indicate the extensions of our results to nonlinear diffusion operators $\nabla \cdot a(\nabla w)$ of Leray–Lions type, and to non-Lipschitz domains Ω.

1. The main result

Definition 1. Given $u_0 \in L^1(\Omega)$ and $f \in L^1(Q)$, a weak solution of $\mathrm{E}(u_0, f)$ is a couple of functions (u, w) such that $u \in L^1(Q)$, $w \in L^2\ (0, T; H_0^1(\Omega))$, $w \in \beta(u)$, $F(u, w) \in (L^2(Q))^N$, and

$$\iint_Q \big(\big(\nabla w + F(u, w)\big) \cdot \nabla\xi - u\xi_t\big) = \iint_Q f\xi - \int_\Omega \xi(0)u_0$$

for any test function $\xi \in \mathcal{D}((-\infty, T) \times \Omega)$.

Our main result is

Theorem 1. *Under the assumptions* (H1) *and* (H2), *if, for* $i = 1, 2$, $u_{0i} \in L^1(\Omega)$, $f_i \in L^1(Q)$ *and* (u_i, w_i) *is a weak solution of* $\mathrm{E}(u_{0i}, f_i)$, *then*

$$\int_\Omega \big(u_1(t) - u_2(t)\big)^+ \leqslant \int_\Omega (u_{01} - u_{02})^+ + \int_0^t \int_\Omega \eta(f_1 - f_2), \tag{1.1}$$

with $\eta \in \mathrm{Sign}^+(u_1 - u_2)$ *a.e. in* Q. *In particular, for given* $u_0 \in L^1(\Omega)$ *and* $f \in L^1(Q)$, *there exists a unique* u *such that the couple* (u, w) *is a weak solution of* $\mathrm{E}(u_0, f)$.

The proof of this theorem will follow as a consequence of a sequence of lemmas that we next present. In fact, we will focus our attention on the problem

$$\begin{cases} \partial_t j(v) = \Delta\varphi(v) + \nabla\cdot F(j(v),\varphi(v)) + f(t,x) & \text{in } Q, \\ \varphi(v) = 0 \quad \text{on } \Sigma, \\ j(v)(0) = u_0 \quad \text{in } \Omega, \end{cases} \tag{E$'(v_0, f)$}$$

where $j, \varphi : \mathbb{R} \to \mathbb{R}$ are nondecreasing continuous functions such that $j(0) = \varphi(0) = 0$. Indeed, by taking $\varphi = (I + \beta^{-1})^{-1}$, $j = (I+\beta)^{-1}$ and $v := u + w$, one sees that $\mathrm{E}(u_0, f)$ and $\mathrm{E}'(u_0, f)$ are equivalent. Next, let us recall the definition of a solution of $\mathrm{E}'(u_0, f)$.

Definition 2. Given $u_0 \in L^1(\Omega)$ and $f \in L^1(Q)$, a weak solution of $\mathrm{E}'(v_0, f)$ is a measurable function v such that the couple (u, v) is a solution of $\mathrm{E}(u_0, f)$, where $u = j(v)$ and $w = \varphi(v)$.

As it is said in the introduction, the notion of entropy solutions is an important ingredient that we will use for the proof of uniqueness of weak solutions. The definition was introduced for the first time in [8]; since then, it was used and adapted for numerous problems and questions. In the following, let us give the definition of [8] and set the corresponding uniqueness result.

Definition 3. Given $u_0 \in L^1(\Omega)$ and $f \in L^1(Q)$, a weak entropy solution of $\mathrm{E}'(v_0, f)$ is a weak solution v satisfying, in addition, the so-called entropy inequalities:

$$\begin{aligned} &\iint_Q \Big\{ -\big(j(v) - j(k)\big)^+ \xi\psi_t \\ &\qquad + \Big(\nabla\varphi(v) + F\big(j(v),\varphi(v)\big) - F\big(j(k),\varphi(k)\big)\Big)\cdot\nabla\xi\,\psi\,\mathrm{Sign}_0^+(v-k)\Big\} \\ &\quad \leqslant \iint_Q f\xi\psi\,\mathrm{Sign}_0^+(v-k) - \int_\Omega \xi\psi(0)(u_0 - k)^+ \end{aligned} \tag{IE$^+$}$$

and

$$\begin{aligned} &\iint_Q \Big\{ -\big(j(v) - j(-k)\big)^- \xi\psi_t \\ &\qquad + \Big(\nabla\varphi(v) + F\big(j(v),\varphi(v)\big) - F\big(j(-k),\varphi(-k)\big)\Big)\cdot\nabla\xi\,\psi\,\mathrm{Sign}_0^+(-k-v)\Big\} \\ &\quad \leqslant \iint_Q f\xi\psi\,\mathrm{Sign}_0^+(-k-v) - \int_\Omega \xi\psi(0)\big(u_0 - j(-k)\big)^- \end{aligned} \tag{IE$^-$}$$

for any $(k, \xi, \psi) \in \mathbb{R} \times H_0^1(\Omega) \times \mathcal{D}(-\infty, T)$ and also for any $(k, \xi, \psi) \in \mathbb{R}^+ \times H^1(\Omega) \times \mathcal{D}(-\infty, T)$.

Theorem 2. (Cf. [8]) *If, for $i = 1, 2$, $u_{0i} \in L^1(\Omega)$, $f_i \in L^1(Q)$ and v_i is a weak entropy solution of* $\mathrm{E}'(u_{0i}, f_i)$*, then*

$$\int_\Omega \big(j(v_1)(t) - j(v_2)(t)\big)^+ \leqslant \int_\Omega (u_{01} - u_{02})^+ + \int_0^t \int_\Omega \eta(f_1 - f_2), \tag{1.2}$$

with $\eta \in \mathrm{Sign}^+(j(v_1) - j(v_2))$ *a.e. in* Q.

Then Theorem 1 is a direct consequence of the following proposition, that we prove in the next section.

Proposition 1. *For given $u_0 \in L^1(\Omega)$ and $f \in L^1(Q)$, if v is a weak solution of* $\mathrm{E}'(v_0, f)$*, then v is a weak entropy solution.*

2. Proofs

Throughout this section, $f \in L^1(Q)$, $u_0 \in L^1(\Omega)$, v is a weak solution of $\mathrm{E}'(u_0, f)$, u and w denote $j(v)$ and $\varphi(v)$, respectively. We will also assume that u_0 is such that there exists a measurable function v_0 such that

$$u_0 = j(v_0) \quad \text{a.e. } \Omega.$$

For any $\varepsilon > 0$, we denote

$$H_\varepsilon(r) = \inf\big(1, r^+/\varepsilon\big) \quad \text{for any } r \in \mathbb{R}.$$

Lemma 1. *For any $k \in \mathbb{R}$, $\psi \in \mathcal{D}((-\infty, T))$, $\psi \geqslant 0$, and $\xi \in H^1(\Omega) \cap L^\infty(\Omega)$ such that $\xi \geqslant 0$ and $H_\varepsilon(w - \varphi(k))\xi \in L^2(0, T; H_0^1(\Omega))$, we have*

$$\begin{aligned} &\lim_{\varepsilon \to 0} \iint_Q \big(\nabla w + F(u, w)\big) \cdot \nabla\big(\xi H_\varepsilon\big(w - \varphi(k)\big)\big)\psi \\ &\quad \leqslant \iint_Q f \xi \psi \, \mathrm{Sign}_0^+\big(w - \varphi(k)\big) + \iint_Q \xi \psi_t \int_k^v \mathrm{Sign}_0^+\big(\varphi(s) - \varphi(k)\big)\, dj(s) \\ &\qquad - \int_\Omega \xi \psi(0) \int_k^{v_0} \mathrm{Sign}_0^+\big(\varphi(s) - \varphi(k)\big)\, dj(s). \end{aligned} \tag{2.1}$$

Proof. The proof is quite standard by now. Indeed, since $H_\varepsilon(w - \varphi(k))\xi\psi \in L^2(0, T; H_0^1(\Omega))$, then using the chain rule lemma (cf. [1,8]), we get

$$\iint_Q \left(-\xi \psi_t \int_k^v H_\varepsilon\big(\varphi(s) - \varphi(k)\big)\, dj(s) + \big(\nabla w + F(u,w)\big) \cdot \nabla\big(H_\varepsilon\big(w - \varphi(k)\big)\xi\big)\psi \right)$$

$$\leqslant \iint_Q H_\varepsilon\big(w - \varphi(k)\big) f \xi - \int_\Omega \xi \psi(0) \int_k^{v_0} H_\varepsilon\big(\varphi(s) - \varphi(k)\big)\, dj(s).$$

Letting $\varepsilon \to 0$ and using Lebesgue's dominated convergence theorem, we get the result. □

Lemma 2. *For any* $(k, \xi, \psi) \in \mathbb{R} \times H_0^1(\Omega) \times \mathcal{D}((-\infty, T))$, (IE^+) *is fulfilled.*

Proof. The proof is given in [8] for the elliptic problem associated with $\mathrm{E}'(v_0, f)$, and in [12,16] for the evolution problem. Here, let us recall the main lines of the proof. We introduce the function

$$\varphi_0^{-1}(x) = \min\big\{\varphi^{-1}(x)\big\}$$

and define the set

$$E = \big\{r \in \mathbb{R}\colon\ \varphi_0^{-1} \text{ is discontinuous at } r\big\}.$$

It is clear that

$$\mathrm{Sign}_0^+\big(\varphi(s) - \varphi(k)\big) = \mathrm{Sign}_0^+(s - k) \quad \text{for any } k \notin E,$$

so that, using (H2) and Lemma 1, one proves that (IE^+) is fulfilled for any $(k, \xi) \in (\mathbb{R} \setminus E) \times H_0^1(\Omega)$. To prove that (IE^+) is fulfilled for $k \in E$ and $\xi \in H_0^1(\Omega)$, take $[m, M] = \varphi^{-1}(r)$. One sees first that (IE^+) remains valid for $k = M$, then derives an entropy inequality for $k = m$ (by taking a sequence $k_n \uparrow m$ such that $\varphi(k_n) \notin E$). Using again (H2) and [8, Lemma 2] exactly in the way of [12], one can pass to the interior of $[m, M]$ and get (IE^+) for any $k \in [m, M]$. We omit here the details of the proof to avoid an unnecessary duplication of arguments. □

Lemma 3. *The entropy inequality* (IE^+) *remains true for any* $(k, \xi, \psi) \in \mathbb{R}^+ \times H^1(\Omega) \times \mathcal{D}([0, T))$.

Proof. Let $(k, \xi, \psi) \in \mathbb{R}^+ \times H^1(\Omega) \times \mathcal{D}([0, T))$ and consider a sequence $(\xi_n)_{n \in \mathbb{N}}$ such that $\xi_n \in H_0^1(\Omega)$, $0 \leqslant \xi_n \leqslant 1$ and $\xi_n \to 1$ in $L^1(\Omega)$. Since $\xi \xi_n \in H_0^1(\Omega)$, then thanks to Lemma 2, we have

$$\iint_Q \big(-\big(u - j(k)\big)^+ \xi \psi_t + \big(\nabla w + F(u, w) - F\big(j(k), \varphi(k)\big)\big) \cdot \nabla\xi\, \mathrm{Sign}_0^+(u - k) \big) \xi_n$$

$$\leqslant -\iint\limits_Q \big(\nabla w + F(u,w) - F\big(j(k),\varphi(k)\big)\big)\cdot\nabla\xi_n\xi \,\mathrm{Sign}_0^+\big(u - j(k)\big)$$

$$+\iint\limits_Q f\xi\xi_n \,\mathrm{Sign}_0^+\big(u - j(k)\big) - \int\limits_\Omega \xi\xi_n\psi(0)\big(u_0 - j(k)\big)^+. \tag{2.2}$$

Let us write the term

$$I_n := -\iint\limits_Q \big(\nabla w + F(u,w) - F\big(j(k),\varphi(k)\big)\big)\cdot\nabla\xi_n\xi \,\mathrm{Sign}_0^+(u-k)$$

as

$$I_n = \lim_{\varepsilon\to 0}\iint\limits_Q \big(\nabla w + F(u,w) - F\big(j(k),\varphi(k)\big)\big)\cdot\nabla(1-\xi_n)\xi H_\varepsilon\big(w-\varphi(k)\big)$$

$$= \lim_{\varepsilon\to 0}\iint\limits_Q \big(\nabla w + F(u,w) - F\big(j(k),\varphi(k)\big)\big)\cdot\nabla\big((1-\xi_n)H_\varepsilon\big(w-\varphi(k)\big)\big)\xi$$

$$- \lim_{\varepsilon\to 0}\iint\limits_Q \big(F(u,w) - F\big(j(k),\varphi(k)\big)\big)\cdot\nabla w H_\varepsilon'\big(w-\varphi(k)\big)\xi(1-\xi_n)$$

$$= \lim_{\varepsilon\to 0} I^1_{n,\varepsilon} - \lim_{\varepsilon\to 0} I^2_{n,\varepsilon}.$$

First, one sees that $I^2_{n,\varepsilon} = \iint_Q \mathcal{F}_\varepsilon\cdot\nabla(\xi(1-\xi_n))$, where

$$\mathcal{F}_\varepsilon = \int\limits_0^w \big(F\big(j\big(\varphi_0^{-1}(r)\big),r\big) - F\big(j(k),\varphi(k)\big)\big)H_\varepsilon'\big(r-\varphi(k)\big)\,dr$$

$$= \frac{1}{\varepsilon}\int\limits_{\min(w,\varphi(k))}^{\min(w,\varphi(k)+\varepsilon)} \big(F\big(j\big(\varphi_0^{-1}(r)\big),r\big) - F\big(j(k),\varphi(k)\big)\big)\,dr,$$

which implies that $\lim_{\varepsilon\to 0} I^2_{n,\varepsilon} = 0$. As to $I^1_{n,\varepsilon}$, we have

$$\lim_{\varepsilon\to 0} I^1_{n,\varepsilon} = \lim_{\varepsilon\to 0}\iint\limits_Q \big(\nabla w + F(u,w) - F\big(j(k),\varphi(k)\big)\big)\cdot\nabla\big((1-\xi_n)H_\varepsilon\big(w-\varphi(k)\big)\xi\big)$$

$$- \iint\limits_Q \big(\nabla w + F(u,w) - F\big(j(k),\varphi(k)\big)\big)\cdot\nabla\xi(1-\xi_n)\,\mathrm{Sign}_0^+\big(w-\varphi(k)\big).$$

Since $k \in \mathbb{R}^+$, we have $H_\varepsilon(w - \varphi(k))(1 - \xi_n)\xi \in L^2(0, T; H_0^1(\Omega))$. Using Lemma 1, we get

$$\lim_{\varepsilon \to 0} I^1_{n,\varepsilon} \leqslant \iint_Q \left(\psi_t \int_k^v \operatorname{Sign}_0^+\big(\varphi(s) - \varphi(k)\big)\, dj(s) + f\psi \operatorname{Sign}_0^+\big(w - \varphi(k)\big) \right)(1 - \xi_n)\xi$$
$$- \iint_Q \big(\nabla w + F(u, w) - F\big(j(k), \varphi(k)\big)\big) \cdot \nabla \xi (1 - \xi_n) \operatorname{Sign}_0^+\big(w - \varphi(k)\big)$$
$$- \int_\Omega \xi(1 - \xi_n)\psi(0) \int_k^{v_0} \operatorname{Sign}_0^+\big(\varphi(s) - \varphi(z)\big)\, dj(s).$$

This implies that $\lim_{n\to\infty} \lim_{\varepsilon\to 0} I^1_\varepsilon \leqslant 0$. Letting $n \to \infty$ in (2.2), we deduce the result of the lemma. □

Proof of Proposition 1. Thanks to Lemmas 2 and 3, v satisfies (IE^+) for any $(k, \xi, \psi) \in \mathbb{R} \times H_0^1(\Omega) \times \mathcal{D}((-\infty, T))$ and also for any $(k, \xi, \psi) \in \mathbb{R}^+ \times H^1(\Omega) \times \mathcal{D}([0, T))$. On the other hand, it is clear that $\tilde{v} := -v$ is a weak solution of the problem

$$\begin{cases} \partial_t \tilde{j}(\tilde{v}) = \Delta\tilde{\varphi}(\tilde{v}) + \nabla \cdot \tilde{F}(\tilde{j}(\tilde{v}), \tilde{\varphi}(\tilde{v})) - f(t, x) & \text{in } Q, \\ \tilde{\varphi}(\tilde{v}) = 0 \quad \text{on } \Sigma, \\ \tilde{u}(0) = -u_0 \quad \text{in } \Omega, \end{cases}$$

with $\tilde{F}(r_1, r_2) = -F(-r_1, -r_2)$ for any $r_1, r_2 \in \mathbb{R}$, and $\tilde{j}(r) = -j(-r)$, $\tilde{\varphi}(r) = -\varphi(-r)$ for any $r \in \mathbb{R}$. Thanks to Lemmas 2 and 3, $\tilde{v}$ satisfies (IE^+) for any $(k, \xi, \psi) \in \mathbb{R} \times H_0^1(\Omega) \times \mathcal{D}((-\infty, T)) \cup \mathbb{R}^+ \times H^1(\Omega) \times \mathcal{D}([0, T))$ with j, φ, F, f and u_0 replaced by $\tilde{j}$, $\tilde{\varphi}$, $\tilde{F}$, $-f$ and $-u_0$, respectively. This implies that v satisfies (IE^-) for any $(k, \xi, \psi) \in \mathbb{R} \times H_0^1(\Omega) \times \mathcal{D}((-\infty, T)) \cup \mathbb{R}^+ \times H^1(\Omega) \times \mathcal{D}([0, T))$, and the proof of the proposition is complete. □

3. The case of nonlinear diffusion

Note that the results of this paper can be extended in a straightforward way to the case of the diffusion–convection equation $\partial_t u = \nabla \cdot a(\nabla w) + \nabla \cdot F(u, w) + f(t, x)$ with nonlinear diffusion flux a of Leray–Lions type (in particular, Δw can be replaced by the p-Laplacian of w). The definition of a weak solution below should be modified accordingly, replacing the requirements $w \in L^p(0, T; W_0^{1,p}(\Omega))$ and $F(u, w) \in (L^2(Q))^N$ by $w \in L^2(0, T; W_0^{1,p}(\Omega))$ and $F(u, w) \in (L^{p'}(Q))^N$. Indeed, the part of the arguments of Carrillo [8] we use only relies upon the monotonicity of $a(\cdot)$ and the fact that $w \mapsto \nabla \cdot a(\nabla w)$ acts from the appropriate Sobolev space into its dual.

4. The case of non-Lipschitz Ω

Following Carrillo [8], we adopt the assumption that the domain Ω is strongly Lipschitz, i.e., for all $x_0 \in \partial\Omega$, there exists a neighborhood B_{x_0} such that $\partial\Omega \cap B_{x_0}$ can be represented by the graph of a Lipschitz continuous function Φ_{x_0}. This assumption is further used in [16] in order to extend the entropy inequalities up to the boundary (in this case, $\partial\Omega$ should be assumed "Lipschitz deformable," cf. [9]).

It is the purpose of this section to indicate an easy generalization of the Carrillo uniqueness result we use (Theorem 2 cited in Section 1 above), dropping the Lipschitz continuity assumption on the functions Φ_{x_0}. More exactly, let us replace the strong Lipschitz assumption on $\partial\Omega$ by the assumption that Ω satisfies the following "two-sided segment property":

(H3) for each $x_0 \in \partial\Omega$, there exists a neighbourhood B_{x_0} of x_0 and a vector $v_{x_0} \in \mathbb{R}^N$ such that if $y \in \overline{\Omega \cap B_{x_0}}$, then $y + tv_{x_0} \in \Omega$ for all $t \in (0,1)$, and if $y \in \overline{\Omega^{\mathbf{c}} \cap B_{x_0}}$, then $y - tv_{x_0} \in \Omega^{\mathbf{c}}$ for all $t \in (0,1)$,

where $\Omega^{\mathbf{c}} \stackrel{\text{def}}{=} \mathbb{R}^N \setminus \Omega$. It is sufficient for the proof of Theorem 2 to construct, for each neighbourhood B_{x_0} in (H3), a sequence of mollifiers $(\rho_n)_{n\in\mathbb{N}}$ defined on $\mathbb{R}^N$ such that, for n large enough and some constant $c > 0$,

$$y \mapsto \rho_n\left(\frac{x-y}{2}\right) \in \mathcal{D}(\Omega) \quad \text{for all } x \in \Omega \cap B_{x_0},$$

$$\chi_n(y) = \int_\Omega \rho_n\left(\frac{x-y}{2}\right) dx \text{ is an increasing sequence} \quad \text{for all } y \in B_{x_0},$$

$$\chi_n(y) = 1 \quad \text{for all } y \in B_{x_0} \text{ such that } \operatorname{dist}\big(y, \Omega^{\mathbf{c}}\big) > c/n.$$

Set

$$R_n = \operatorname{dist}\left(\partial\Omega, \left\{y \pm \frac{1}{n} v_{x_0} \colon\ y \in \overline{\partial\Omega \cap B_{x_0}}\right\}\right),$$

we have $R_n > 0$. The three properties above are satisfied for any sequence of mollifiers ρ_n with support in the ball of radius $R_n/2$ centered at the point $-\frac{1}{n}v_{x_0}$.

Now, note that Lemmas 1, 2 apply for an arbitrary domain, since the test functions ξ_n can be chosen of compact support in Ω. Since the technique of Lemma 3 does not rely upon any regularity of $\partial\Omega$, we conclude that our uniqueness result is valid for an arbitrary bounded domain Ω of $\mathbb{R}^N$ satisfying (H3).

5. The elliptic problem

At the end of this paper, let us give some consequences of the previous results for the stationary problem

$$\begin{cases} u - \Delta w - \nabla \cdot F(u, w) = f, \quad w \in \beta(u) \quad \text{in } \Omega, \\ w = 0 \quad \text{on } \Gamma, \end{cases} \tag{S(f)}$$

by assuming that (H1)–(H3) are fulfilled.

Definition 4. Given $f \in L^1(\Omega)$, a weak solution of $\mathrm{S}(f)$ is a couple of functions (u, w) such that $u \in L^1(\Omega)$, $w \in H_0^1(\Omega)$, $w \in \beta(u)$, a.e. in Ω, $F(u, w) \in (L^2(\Omega))^N$, and

$$\int_\Omega \big(\nabla w + F(u, w)\big) \cdot \nabla \xi = \int_\Omega f \xi$$

for any test function $\xi \in \mathcal{D}(\Omega)$.

Proposition 2. *If, for $i = 1, 2$, $f_i \in L^1(\Omega)$ and u_i is a weak solution of* $\mathrm{S}(f_i)$*, then*

$$\big\|(u_1 - u_2)^+\big\|_1 \leqslant \big\|(f_1 - f_2)^+\big\|_1. \tag{5.2}$$

Proof. This is a simple consequence of the fact that, if (u, w) is a weak solution of $\mathrm{S}(f)$, then by setting $\tilde{u}(t) \equiv u$ and $\tilde{w}(t) \equiv w$, $(\tilde{u}, \tilde{w})$ is a weak solution of $\mathrm{E}(\tilde{u}_0, \tilde{g})$ with $\tilde{u}_0 = u$ and $\tilde{g} = f - u$. So that the result follows by using Theorem 1. □

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