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Research Article

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Optimal mass transportation for costs given by Finsler distances via *p*-Laplacian approximations

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Abstract: In this paper we approximate a Kantorovich potential and a transport density for the mass transport problem of two measures (with the transport cost given by a Finsler distance), by taking limits, as p goes to infinity, to a family of variational problems of p-Laplacian type. We characterize the Euler–Lagrange equation associated to the variational Kantorovich problem. We also obtain different characterizations of the Kantorovich potentials and a Benamou–Brenier formula for the transport problem.

Keywords: Mass transport, Monge–Kantorovich problems, Finsler metric p-Laplacian equation

MSC 2010: 49J20, 49J45, 45G10

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1 Introduction and preliminaries

1.1 Introduction

In this paper our main goal is to show that a family of variational problems of p-Laplacian type allows us to get the Kantorovich potentials and transport densities of the Monge–Kantorovich mass transport problem for general Finsler costs. Moreover, this approach allows us to characterize the Euler–Lagrange equation associated to the variational Kantorovich problem. We will also give different characterizations of the Kantorovich potentials, and a Benamou–Brenier formula of the optimal transport problem

The variational approach using p-Laplacian problems was introduced by Evans and Gangbo [18] to solve the Monge transport problem for the cost given by the Euclidean distance. This limit procedure turned out to be quite flexible and allowed us to deal with different transport problems in which the cost is given by the Euclidean distance or variants of it; for example, optimal matching problems (here one deals with systems of p-Laplacian type), optimal import/export problems (here one considers Dirichlet or Neumann boundary conditions) and optimal transport with the help of a courier (this is related to the double obstacle problem for the p-Laplacian). We refer to [10, 22–26]. Here we extend the previous results considering a more delicate structure, that is given in terms of a Finsler metric that may change from one point to another in the domain (this is what is called a Finsler structure in the literature). Our ideas can also be extended to manifolds but,

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to simplify the presentation, we prefer to state and prove our results just in a bounded smooth domain Ω in $\mathbb{R}^N$. However, at the end of the paper we present how the obtained results read on a Riemannian manifold.

Now, let us introduce some terminology and general results from optimal mass transportation theory. The *Monge transportation problem* consists in moving one distribution of mass into another minimizing a given transport cost. In mathematical terms, the problem can be stated as follows. Let Ω be an open bounded subset of $\mathbb{R}^N$. Given two non-negative compactly supported functions $f^+, f^- \in L^1(\Omega)$ with the same total mass, find a measurable map $T\colon \Omega \to \Omega$ such that $Tf^+ = f^-$, i.e.,

$$\int_{T^{-1}(A)} f^+(x)\,dx = \int_A f^-(x)\,dx \quad \text{for all } A \subset \Omega \text{ measurable},$$

and in such a way that T minimizes the total transport cost, that is,

$$\int_\Omega c(x, T(x)) f^+(x)\,dx = \min_{Sf^+=f^-} \int_\Omega c(x, S(x)) f^+(x)\,dx,$$

where $c\colon \Omega \times \Omega \to \mathbb{R}$ is a given cost function. The map T is called an *optimal transport map*. The difficulties of solving the above problem motivated Kantorovich to introduce a relaxed formulation, called the *Monge–Kantorovich problem*, that consists in looking for plans, that is, non-negative Radon measures μ in $\Omega \times \Omega$ such that $\mathrm{proj}_x(\mu) = f^+(x)\,dx$ and $\mathrm{proj}_y(\mu) = f^-(y)\,dy$. Denoting by $\Pi(f^+, f^-)$ the set of plans, the Monge–Kantorovich problem consists in minimizing the total cost functional

$$\mathcal{K}_c(\mu) := \int_{\Omega\times\Omega} c(x, y)\,d\mu(x, y)$$

in $\Pi(f^+, f^-)$. If μ is a minimizer of the above problem we say that it is an *optimal plan*. When c is continuous, it is well known that

$$\inf_{Tf^+=f^-} \int_\Omega c(x, T(x)) f^+(x)\,dx = \min_{\mu\in\Pi(f^+,f^-)} \mathcal{K}_c(\mu).$$

For notation and general results on Mass Transport Theory we refer to [1, 4, 17, 18, 33, 34]. Below we summarize our main concern in this paper.

Here we will deal with a cost c given by a *Finsler distance* (see Section 1.3 for a precise definition) that can be non-symmetric. However, since the cost satisfies the triangular inequality, the following duality result holds (see [33]):

$$\min\{\mathcal{K}_c(\mu) : \mu \in \Pi(f^+, f^-)\} = \sup\left\{ \int_\Omega v(f^- - f^+) : v \in K_c(\Omega) \right\}, \tag{1.1}$$

where

$$K_c(\Omega) := \{u\colon \Omega \mapsto \mathbb{R} : u(y) - u(x) \le c(x, y)\}.$$

Moreover, there exists $u \in K_c(\Omega)$ such that

$$\int_\Omega u(f^- - f^+) = \sup\left\{ \int_\Omega v(f^- - f^+) : v \in K_c(\Omega) \right\}.$$

Such maximizers are called *Kantorovich potentials*.

When c is symmetric, we have that

$$\min\{\mathcal{K}_c(\mu) : \mu \in \Pi(f^+, f^-)\} = \sup\left\{ \int_\Omega v(f^+ - f^-) : v \in K_c(\Omega) \right\}, \tag{1.2}$$

since $v \in K_c(\Omega)$ if and only if $-v \in K_c(\Omega)$.

In Section 1.4 we state precisely what is our cost function. In order to do this we introduce Finsler structures in Section 1.3, which *grosso modo* are extensions of norms. Basic references in Finsler geometry are [6] and [31].

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1.2 Conditions on the data

From now on, Ω will be a bounded smooth domain in $\mathbb{R}^N$ and $f^+, f^- \in L^2(\Omega)$ are non-negative, compactly supported functions with the same total mass. We also assume that $\operatorname{supp}(f^+) \cup \operatorname{supp}(f^-) \subset\subset \Omega$. Some of the results we will obtain can be obtained for masses $f^\pm$ in larger spaces, e.g., $\mathcal{M}(\Omega)$. Nevertheless, since our objective is to present how the limit procedure works, we will avoid technicalities that could appear with less regular masses.

1.3 Finsler structures

We will denote by $\langle \xi; \eta \rangle$ the Euclidean inner product between ξ and η in $\mathbb{R}^N$ and by $|\xi| = \sqrt{\langle \xi, \xi \rangle}$ the Euclidean norm in $\mathbb{R}^N$.

A Finsler function Φ in $\mathbb{R}^N$ is a function that is non-negative, continuous, convex and positively homogeneous of degree 1. In addition, it has the following property:

$$\Phi(t\xi) = t\Phi(\xi) \quad \text{for any } t \geq 0,\ \xi \in \mathbb{R}^N,$$

and vanishes only at 0. The *dual function (or polar function)* of a Finsler function Φ is defined by

$$\Phi^*(\xi^*) := \sup\{\langle \xi^*; \xi \rangle : \Phi(\xi) \leq 1\} \quad \text{for } \xi^* \in \mathbb{R}^N.$$

It is immediate to verify that Φ^* is also a Finsler function.

Observe that a Finsler function Φ satisfies

$$\alpha|\xi| \leq \Phi(\xi) \leq \beta|\xi| \quad \text{for any } \xi \in \mathbb{R}^N,$$

for some positive constants α, β.

Finsler functions are extensions of norms. In fact, any norm in $\mathbb{R}^N$ is a Finsler function, and any symmetric Finsler function is a norm. Moreover, for any Finsler function, convexity is equivalent to the triangular inequality. In the literature the Finsler functions are also denominated as Minkowski norms.

Set

$$B_\Phi := \{\xi \in \mathbb{R}^N : \Phi(\xi) \leq 1\}.$$

This is a closed bounded convex set with $0 \in \operatorname{int}(B)$. It is symmetric around the origin if Φ is a norm. Conversely, for any closed bounded convex set K with $0 \in \operatorname{int}(K)$, $\phi_K(\xi) := \inf\{\alpha > 0 : \xi \in \alpha K\}$ is a Finsler function with $B_{\phi_K} = K$; when K is centrally symmetric, we have a norm.

The *dual function (or polar function)* of a Finsler function Φ is defined by

$$\Phi^*(\xi^*) := \sup\{\langle \xi^*; \xi \rangle : \xi \in B_\Phi\} \quad \text{for } \xi^* \in \mathbb{R}^N.$$

It is immediate to verify that Φ^* is also a Finsler function; and a norm when Φ is a norm. We also have

$$\Phi^*(\xi^*) = \sup_{\xi \neq 0} \frac{\langle \xi^*; \xi \rangle}{\Phi(\xi)}.$$

Therefore, the following inequality of Cauchy–Schwarz type holds:

$$\langle \xi^*; \xi \rangle \leq \Phi(\xi)\Phi^*(\xi^*). \tag{1.3}$$

If Φ is a norm, we have

$$|\langle \xi^*; \xi \rangle| \leq \Phi(\xi)\Phi^*(\xi^*). \tag{1.4}$$

Now, for general Finsler functions inequality (1.4) is not true. An example of a Finsler function that is not a norm in $\mathbb{R}$ is given by $\Phi(\xi) := a\xi^- + b\xi^+$ with $0 < a < b$.

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It is not difficult to see that

$$\Phi^{**}(\xi) = \Phi(\xi) \quad \text{for all } \xi \in \mathbb{R}^N.$$

Hence,

$$\Phi(\xi) = \sup_{\xi^* \neq 0} \frac{\langle \xi; \xi^* \rangle}{\Phi^*(\xi^*)}.$$

If we assume that the Finsler function Φ is differentiable at ξ, then, by Euler's theorem,

$$\Phi(\xi) = \langle D\Phi(\xi); \xi \rangle. \tag{1.5}$$

Moreover, if we assume that Φ is differentiable in $K \subset \mathbb{R}^N$, then since Φ is convex, it follows from (1.5) that

$$\langle D\Phi(\xi); \eta \rangle \leq \Phi(\eta) \quad \text{for all } \xi, \eta \in K, \tag{1.6}$$

and consequently

$$|\langle D\Phi(\xi); \eta \rangle| \leq \sup\{\Phi(\eta), \Phi(-\eta)\} \leq \beta|\eta| \quad \text{for all } \xi, \eta \in K. \tag{1.7}$$

If we assume that Φ is differentiable in $\mathbb{R}^N \setminus \{0\}$, by Lagrange multipliers, from $\Phi^*(\xi^*) = \sup_{\Phi(\xi)=1} \langle \xi; \xi^* \rangle$, we get that if $\Phi(\xi) = 1$ and $\Phi^*(\xi^*) = \langle \xi; \xi^* \rangle$, then there exists $\lambda \in \mathbb{R}$ such that $\xi^* = \lambda D\Phi(\xi)$. Now, by (1.5), we have that

$$\text{if } \Phi(\xi) = 1 \text{ and } \Phi^*(\xi^*) = \langle \xi; \xi^* \rangle, \quad \text{then } \xi^* = \Phi^*(\xi^*) D\Phi(\xi). \tag{1.8}$$

From (1.5) and (1.6), we also have

$$\Phi^*(D\Phi(\xi)) = 1 \quad \text{for all } \xi \neq 0. \tag{1.9}$$

In this paper, a *Finsler structure* F on an open set D is a continuous function $F\colon D \times \mathbb{R}^N \to \mathbb{R}_+$ such that for any $x \in D$, $F(x, \cdot)$ is a Finsler function in $\mathbb{R}^N$.

For a Finsler structure F on D, we define the dual structure $F^*\colon D \times \mathbb{R}^N \to \mathbb{R}_+$ by

$$F^*(x, \xi) := \sup\{\langle \eta; \xi \rangle : F(x, \eta) \leq 1\}.$$

Note that F^* is also a Finsler structure.

Some important examples of Finsler structures are those of the form $\Phi(B(x)\xi)$, where Φ is a Finsler function and $B(x)$ is a continuous symmetric $N \times N$ positive definite matrix. Such type of Finsler structures are known as *deformations of Minkowski norms*.

1.4 The cost function

Let us now introduce the cost function. Given a Finsler structure F on Ω, we define the following cost function:

$$c_F(x, y) := \inf_{\sigma \in \Gamma_{x,y}^{\Omega}} \int_0^1 F(\sigma(t), \sigma'(t))\, dt \quad \text{for } x, y \in \Omega, \tag{1.10}$$

where

$$\Gamma_{x,y}^{\Omega} := \{\sigma \in C^1([0,1], \Omega),\ \sigma(0) = x,\ \sigma(1) = y\}.$$

We have that c_F is a *Finsler distance*. We emphasize that c_F is not necessary symmetric (i.e., we may have $c_F(x, y) \neq c_F(y, x)$) because F is merely positively homogeneous.

Remark 1.1. In the particular case $F(x, \xi) = \Phi(\xi)$ and Ω convex, we have that

$$c_F(x, y) = \Phi(y - x).$$

In fact, given $\sigma \in \Gamma_{x,y}^{\Omega}$, since Φ is convex, applying Jensen's inequality, we get

$$\Phi(y - x) = \Phi\Big(\int_0^1 \sigma'(t)\, dt\Big) \leq \int_0^1 \Phi(\sigma'(t))\, dt.$$

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Therefore, by taking the infimum, we get $\Phi(y-x) \le c_F(x,y)$. On the other hand, if $\sigma(t) = x + t(y-x)$, we have

$$c_F(x,y) \le \int_0^1 \Phi(\sigma'(t))\,dt = \Phi(y-x).$$

Let us remark that when c_F is not symmetric, (1.2) is not true in general. For example, if $\Phi(\xi) := a\xi^- + b\xi^+$ with $0 < a < b$, then for $f^+ = \chi_{(0,1)}$ and $f^- = \chi_{(1,2)}$, we have that an optimal transport map is $T(x) = x+1$, so

$$\begin{aligned}\min\{\mathcal{K}_{c_F}(\mu) : \mu \in \Pi(f^+, f^-)\} &= \int c(x, T(x)) f^+(x)\,dx\\ &= \int \Phi(T(x) - x) f^+(x)\,dx\\ &= b = \int u(x)(f^-(x) - f^+(x))\,dx,\end{aligned}$$

where $u(x) = bx$ is the Kantorovich potential. On the other hand, an optimal transport map for the transport of f^- to f^+ is $S(x) = x-1$, and consequently

$$\begin{aligned}\sup\Big\{\int_\Omega v(f^+ - f^-) : v \in K_{c_F}(\Omega)\Big\} &= \int c_F(x, S(x)) f^-(x)\,dx\\ &= \int \Phi(S(x) - x)) f^-(x)\,dx\\ &= a = \int u(x)(f^+(x) - f^-(x))\,dx,\end{aligned}$$

where $u(x) = -ax$ is a Kantorovich potential.

1.5 Main results

We will denote by $\mathcal{M}(\overline{\Omega}, \mathbb{R}^N)$ the set of all $\mathbb{R}^N$-valued Radon measures in $\overline{\Omega}$, which, by the Riesz representation theorem, can be identified with the dual of the space $C(\overline{\Omega}, \mathbb{R}^N)$ endowed with the supremum norm.

Given a measure $\mathcal{X} \in \mathcal{M}(\overline{\Omega}, \mathbb{R}^N)$, we define its total variation with respect to the Finsler structure F as follows. For an open set $A \subset \overline{\Omega}$, we define

$$|\mathcal{X}|_F(A) := \sup\Big\{\int_{\overline{\Omega}} \Phi\,d\mathcal{X} : \Phi \in C(\overline{\Omega}, \mathbb{R}^N),\ \operatorname{supp}(\Phi) \subset A,\ \Phi(x) \in B_{F^*(x,\cdot)} \text{ for all } x \in \Omega\Big\}.$$

Its extension to every Borel set of $\overline{\Omega}$ is a Radon measure (see Lemma 3.6).

We will identify the elements $\eta \in L^1(\Omega, \mathbb{R}^N)$ as elements of $\mathcal{M}(\overline{\Omega}, \mathbb{R}^N)$ by means of

$$\langle \eta, \Phi\rangle := \int_{\overline{\Omega}} \langle \Phi(x), \overline{\eta}(x)\rangle\,dx,$$

where

$$\overline{\eta}(x) := \begin{cases} \eta(x) & \text{if } x \in \Omega,\\ 0 & \text{if } x \in \overline{\Omega} \setminus \Omega.\end{cases}$$

We shall suppose that F is a Finsler structure in a bounded open set D with $\Omega \subset\subset D$. We will also suppose the following convexity condition on Ω:

$$c_F(x,y) = c_F^D(x,y) := \inf_{\sigma \in \Gamma_{x,y}^D} \int_0^1 F(\sigma(t), \sigma'(t))\,dt \quad \text{for all } x, y \in \Omega. \tag{1.11}$$

Note that by the continuity of F we can suppose, taking a smaller D if necessary, that

$$\alpha|\xi| \le F^*(x,\xi) \le \beta|\xi| \quad \text{for any } \xi \in \mathbb{R}^N \text{ and } x \in D, \tag{1.12}$$

where α, β are positive constants.

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For the Poincaré disk, that is, the unit disk $D_1(0)$ with the Finsler structure

$$F(x,\xi) = \frac{2|\xi|}{1-|x|^2},$$

since we ask for $\operatorname{supp}(f^+) \cup \operatorname{supp}(f^-) \subset\subset \Omega \subset\subset D$, choosing $\Omega \subset D \subset D_1(0)$ adequately, condition (1.11) is satisfied. In this case, the distance c_F is given by

$$c_F(x,y) = \operatorname{arcosh}\Big(1 + \frac{2|x-y|^2}{(1-|x|^2)(1-|y|^2)}\Big).$$

Our main result reads as follows, remember we are under the data conditions given above and in Section 1.2.

Theorem 1.2. *Suppose $F^*(x,\cdot) \in C^1(\mathbb{R}^N \setminus \{0\})$. The following hold true:*

(1) *For $p > N$, there exists a solution u_p of the variational problem*

$$\min_{u \in S_p} \int_\Omega \frac{[F^*(x,Du)]^p}{p} - \int_\Omega u(f^- - f^+),$$

where $S_p = \{u \in W^{1,p}(\Omega) : \int_\Omega u = 0\}$.

(2) *There exists a subsequence u_{p_j} that converges uniformly to a Lipschitz continuous function u_∞.*

(3) *The function u_∞ is a Kantorovich potential for the mass transport problem of f^+ to f^- with cost given by the Finsler distance c_f given in* (1.10). *Moreover, for*

$$\mathcal{X}_p = [F^*(x,Du_p(x))]^{p-1}\frac{\partial F^*}{\partial \xi}(x,Du_p(x)),$$

there exists a subsequence $\mathcal{X}_{p_{j_k}}$ converging weakly as measures in $\overline{\Omega}$ to $\mathcal{X}_\infty \in \mathcal{M}(\overline{\Omega},\mathbb{R}^N)$ such that*

$$\int_\Omega (f^- - f^+)v = \int_{\overline{\Omega}} Dv\, d\mathcal{X}_\infty \quad \text{for all } v \in C^1(\overline{\Omega}).$$

(4) *We have that*

$$|\mathcal{X}_\infty|_F(\overline{\Omega}) = \int_\Omega u_\infty(f^- - f^+) = \min\{\mathcal{K}_{c_F}(\mu) : \mu \in \Pi(f^+,f^-)\}.$$

(5) *Let μ be the measure $F(x,\mathcal{X}_\infty)$. If $F^*(x,D_\mu u_\infty(x)) \le 1$ μ-a.e. in $\overline{\Omega}$, then*

$$\int_\Omega (f^- - f^+)v = \int_{\overline{\Omega}} \frac{\partial F^*}{\partial \xi}(\cdot,D_\mu u_\infty)\cdot Dv\, d\mu \quad \text{for all } v \in C^1(\overline{\Omega})$$

and

$$F^*(x,D_\mu u(x)) = 1 \quad \mu\text{-a.e. in } \overline{\Omega},$$

where $D_\mu u_\infty$ is the tangential gradient of u_∞ with respect to μ. The measure μ is known as a transport density.

For the particular case of quadratic cost $c(x,y) = |x-y|^2$, Benamou and Brenier [8] introduced the *Eulerian* point of view of the mass transport problem and obtained what is usually known as *Benamou–Brenier formula*. This point of view has been generalized in different directions (see, for instance, [1, 3, 14]). Following Brenier, see [14], we consider the paths $f\colon [0,1] \to \mathcal{M}(\overline{\Omega},\mathbb{R})^+$ and the vector fields $E\colon [0,1] \to \mathcal{M}(\overline{\Omega},\mathbb{R}^N)$ satisfying

$$\begin{cases} \dfrac{d}{dt}\displaystyle\int_{\overline{\Omega}} \phi\, df(t) + \int_{\overline{\Omega}} \nabla\phi\, dE(t) = 0 & \text{in } \mathcal{D}'(0,1) \text{ for all } \phi \in C^1(\overline{\Omega}), \\ f(0) = f^+ \quad \text{and} \quad f(1) = f^-. \end{cases} \tag{1.13}$$

Given a solution (f,E) of (1.13), we define its energy by

$$J_F(f,E) := \int_0^1 |E(t)|_F(\overline{\Omega})\, dt.$$

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We have the following relation between the Monge–Kantorovich problem and the equation (1.13) that provides a Benamou–Brenier formula for this kind of transport problems.

Theorem 1.3. *Assume that $F^*(x,\cdot) \in C^1(\mathbb{R}^N \setminus \{0\})$ and consider $\mathcal{X}_\infty$ the flux given in Theorem 1.2. Then, given $f(t) := f^+ + t(f^- - f^+)$ and $E(t) := \mathcal{X}_\infty$ for $t \in [0,1]$, we have that (f, E) is a solution of problem* (1.13). *Moreover,*

$$\min\{J_F(f,E) : (f,E) \text{ is a solution of } (1.13)\} = \min\{\mathcal{K}_{c_F}(\mu) : \mu \in \Pi(f^+, f^-)\}.$$

The paper is organized as follows. In Section 2 we introduce the p-Laplacian problems that we use to approximate a Kantorovich potential of our mass transport problem, and we prove that we can take limits as $p \to \infty$ along subsequences of the solutions, obtaining in the limit a Lipschitz function. In Section 3 we show that this limit is in fact a Kantorovich potential for our problem and, moreover, we find a PDE, involving a transport density, that is verified by the limit. In Section 4 we see that the results obtained in Section 3 characterize the Kantorovich potentials for the transport problem we study. Section 5 is devoted to get a Benamou–Brenier formula for the problem. Finally, in Section 6 we briefly comment on the extension of our results to a general Riemannian manifold.

2 A *p*-Laplacian problem

We assume the data conditions stated in Section 1.2 and that F and Ω satisfy condition (1.11).

For $p > N$, we consider the variational problem

$$\min_{u \in S_p} \int_\Omega \frac{[F^*(x, Du)]^p}{p} - \int_\Omega uf. \tag{2.1}$$

where $f \in L^2(\Omega)$, $\int_\Omega f = 0$, and $S_p = \{u \in W^{1,p}(\Omega) : \int_\Omega u = 0\}$.

As remarked above we work with $f \in L^2(\Omega)$ to avoid technicalities in the p-Laplacian approach.

Lemma 2.1. *For $p > N$, there exists a continuous solution u_p to the variational problem* (2.1).

This lemma implies statement (1) of Theorem 1.2.

Proof. Note that under the conditions on F^*, we have

$$\alpha|Du| \le F^*(\cdot, Du) \le \beta|Du|. \tag{2.2}$$

Hence, for every $u \in W^{1,p}(\Omega)$,

$$\alpha \int_\Omega \frac{|Du|^p}{p} \le \int_\Omega \frac{[F^*(x, Du)]^p}{p} \le \beta \int_\Omega \frac{|Du|^p}{p},$$

and therefore the functional

$$\Theta_{p,f}(u) = \int_\Omega \frac{[F^*(x, Du)]^p}{p} - \int_\Omega uf,$$

is well defined in the set S_p which is convex, weakly closed and non empty. On the other hand, using the Poincaré inequality, one can prove that $\Theta_{p,f}$ is coercive, bounded below and lower semicontinuous in S_p. Then, there exists a minimizing sequence $u_n \in S_p \subset W^{1,p}(\Omega)$ such that $u_n \rightharpoonup u \in S_p$ and

$$\inf_S \Theta_{p,f} = \liminf_{n \to +\infty} \Theta_{p,f}(u_n) \ge \Theta_{p,f}(u).$$

Hence, the minimum of $\Theta_{p,f}$ in S_p is attained. □

Remark 2.2. When $F^*(x,\cdot)$ is strictly convex, the uniqueness of the solution u_p to (2.1) directly follows from the constraint $\int_\Omega u_p = 0$.

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Assuming that $F^*(x,\cdot) \in C^1(\mathbb{R}^N \setminus \{0\})$, then, via standard arguments like the ones used in [7], we have that u_p is a weak solution of the following problem of p-Laplacian type:

$$\begin{cases} -\operatorname{div}\Big([F^*(x,Du(x))]^{p-1}\dfrac{\partial F^*}{\partial \xi}(x,Du(x))\Big) = f & \text{in } \Omega, \\ [F^*(x,Du(x))]^{p-1}\Big\langle \dfrac{\partial F^*}{\partial \xi}(x,Du(x));\eta\Big\rangle = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.3}$$

Here η is the exterior normal vector on $\partial\Omega$ and $\frac{\partial F^*}{\partial \xi}$ is the gradient of $F^*(x,\xi)$ with respect to the second variable ξ.

Let Φ be a Finsler function and $A(x)$ be a symmetric $N \times N$ positive definite matrix that depends smoothly on x. If $F(x,\xi) = \Phi(A(x)\xi)$, then (2.3) becomes

$$\begin{cases} -\operatorname{div}([\Phi^*(A^{-1}Du)]^{p-1}A^{-1}D\Phi^*(A^{-1}Du)) = f & \text{in } \Omega, \\ [\Phi^*(A^{-1}Du)]^{p-1}\langle A^{-1}D\Phi^*(A^{-1}Du);\eta\rangle = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.4}$$

Note that in the particular case $\Phi(\xi) = |\xi|$ (the Euclidean norm), (2.4) reads

$$\begin{cases} -\operatorname{div}(|A^{-1}Du|^{p-2}A^{-2}Du) = f & \text{in } \Omega, \\ |A^{-1}Du|^{p-2}\langle A^{-2}Du;\eta\rangle = 0 & \text{on } \partial\Omega. \end{cases}$$

Finally, for $A = I$, (2.4) becomes

$$\begin{cases} -\Delta_{p,\Phi^*}u = f & \text{in } \Omega, \\ [\Phi^*(Du)]^{p-1}\langle D\Phi^*(Du);\eta\rangle = 0 & \text{on } \partial\Omega, \end{cases}$$

where

$$\Delta_{p,\Phi^*}u := \sum_{i=1}^N \frac{\partial}{\partial x_i}\Big([\Phi^*(Du)]^{p-1}\frac{\partial\Phi^*}{\partial\xi_i}(Du)\Big).$$

In particular, for Φ^* an ℓ^q-norm, that is,

$$\Phi^*(\xi) = \|\xi\|_q := \Big(\sum_{k=1}^N |\xi_k|^q\Big)^{\frac{1}{q}},$$

the operator Δ_{p,Φ^*} becomes

$$\Delta_{p,\Phi^*}u = \sum_{i=1}^N \frac{\partial}{\partial x_i}\bigg(\bigg[\sum_{k=1}^N \Big|\frac{\partial u}{\partial x_k}\Big|^q\bigg]^{\frac{p-q}{q}}\Big|\frac{\partial u}{\partial x_i}\Big|^{q-2}\frac{\partial u}{\partial x_i}\bigg),$$

and consequently, for $q = 2$, we get the classical p-Laplacian operator

$$\Delta_p u := \operatorname{div}(|Du|^{p-2}Du).$$

Now, let us see that we can extract a sequence of solutions to (2.1) $\{u_{p_j}\}_j$ with $p_j \to \infty$ that converges uniformly as $j \to \infty$.

Lemma 2.3. *Let u_p be a solution to* (2.1) *indexed by p with $p > N$. Then, there exists a subsequence $p_j \to \infty$ such that $u_{p_j} \rightrightarrows u_\infty$ uniformly in $\overline{\Omega}$. Moreover, the limit u_∞ is Lipschitz continuous.*

From this lemma statement (2) of Theorem 1.2 follows.

Proof. Along this proof we will denote by C a constant independent of p that may change from one line to another.

Our first aim is to prove that the L^p-norm of the gradient of u_p is bounded independently of p.

Let v be a fixed Lipschitz function with $F^*(x,Dv(x)) \le 1$ for a.e. $x \in \Omega$ and $\int_\Omega v = 0$. Then we have that $v \in S_p$. Hence, since u_p is a minimizer of the functional $\Theta_{p,f}$ in S_p, we have

$$\int_\Omega \frac{[F^*(x,Du_p(x))]^p}{p} - \int_\Omega fu_p \le \int_\Omega \frac{[F^*(x,Dv(x))]^p}{p} - \int_\Omega fv \le \int_\Omega \frac{1}{p} - \int_\Omega fv.$$

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Consequently,

$$\int_\Omega \frac{[F^*(x, Du_p(x))]^p}{p} \le \frac{1}{p}|\Omega| - \int_\Omega fv + \int_\Omega fu_p.$$

For this calculation, taking $v = 0$ should be enough, nevertheless we will use this later on.

Now, thanks to the fact that $\int_\Omega u_p = 0$ and that the constant in the inequality $\|u_p\|_{L^p(\Omega)} \le C\|Du_p\|_{L^p(\Omega)}$ can be chosen independent of p (see the proof of [23, Theorem 3.5]), we get

$$\int_\Omega fu_p \le C\|u_p\|_{L^p(\Omega)} \le C\|Du_p\|_{L^p(\Omega)},$$

and then we obtain

$$\int_\Omega \frac{[F^*(x, Du_p(x))]^p}{p} \le C + C\|Du_p\|_{L^p(\Omega)}.$$

Then, by (2.2), we get

$$\int_\Omega [F^*(x, Du_p(x))]^p \le pC + pC\bigg(\int_\Omega [F^*(x, Du_p(x))]^p\bigg)^{\frac{1}{p}}.$$

From this inequality we can obtain that there exists C, independent of p, such that

$$\bigg(\int_\Omega [F^*(x, Du_p(x))]^p\bigg)^{\frac{1}{p}} \le (Cp)^{\frac{1}{p-1}}. \tag{2.5}$$

Then, from (2.2), we obtain that there exists C, independent of p, such that

$$\bigg(\int_\Omega |Du_p|^p\bigg)^{\frac{1}{p}} \le C.$$

Now, using this uniform bound, we prove uniform convergence of a sequence u_{p_j}. In fact, for m such that $N < m \le p$, we have

$$\|Du_p\|_{L^m(\Omega)} \le |\Omega|^{\frac{p-m}{pm}} \|Du_p\|_{L^p(\Omega)}.$$

Then $\{u_p\}_{p>N}$ is bounded in $W^{1,m}(\Omega)$, and since we know that $\int_\Omega u_p = 0$, we can obtain a sequence $u_{p_j} \rightharpoonup u_\infty \in W^{1,m}(\Omega)$ with $p_j \to +\infty$. Since $W^{1,m}(\Omega) \hookrightarrow C^{0,\alpha}(\bar\Omega)$ (note that α does not depend on p) and $u_{p_j} \rightharpoonup u_\infty \in W^{1,m}(\Omega)$, we obtain $u_{p_j} \to u_\infty$ in $C^{0,\alpha}(\Omega)$, and in particular $u_{p_j} \rightrightarrows u_\infty$ uniformly in $\overline{\Omega}$. As $u_{p_j} \in C(\overline{\Omega})$, we have that $u_\infty \in C(\overline{\Omega})$.

Finally, let us show that the limit function u_∞ is Lipschitz. In fact, we proved that

$$\bigg(\int_\Omega |Du_\infty|^m\bigg)^{\frac{1}{m}} \le \liminf_{p_j\to+\infty} \bigg(\int_\Omega |Du_{p_j}|^m\bigg)^{\frac{1}{m}} \le C|\Omega|^{\frac{1}{m}} \le C.$$

Now, we take $m \to \infty$, to obtain $\|Du_\infty\|_{L^\infty(\Omega)} \le C$. So, we have proved $u_\infty \in W^{1,\infty}(\Omega)$, that is, u_∞ is a Lipschitz function. □

Remark 2.4. All the results of this section remain true if we assume that $f = f_p$ and

$$f_p \rightharpoonup f \quad \text{weakly in } L^2(\Omega).$$

3 Mass transport interpretation of the limit

3.1 Kantorovich potentials

The goal of this section is to show that for $f = f^- - f^+$, the limit u_∞ of u_p (that we proved to exist in the previous section), up to a subsequence, is a Kantorovich potential for the mass transport problem of f^+ to f^- with the cost given by the Finsler distance $c_F(x, y)$ defined by (1.10).

The key idea is contained in the following result.

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Proposition 3.1. *If F satisfies the conditions given in Section 1.5, then we have the following characterization:*

$$u \in W^{1,\infty}(\Omega) \iff \mathrm{Lip}(u, c_F) < \infty,$$

where

$$\mathrm{Lip}(u, c_F) := \sup\Big\{\frac{u(y) - u(x)}{c_F(x, y)} : x, y \in \Omega,\ x \neq y\Big\}.$$

Moreover,

$$\operatorname*{ess\,sup}_{x\in\Omega} F^*(x, Du(x)) = \mathrm{Lip}(u, c_F).$$

To prove this result we need the following approximation lemma.

Lemma 3.2. *Let $u \in W^{1,\infty}(\Omega)$ be such that $F^*(x, Du(x)) \le 1$ for a.e. $x \in \Omega$. There exists $u_\epsilon \in C^1(\overline{\Omega})$ such that $u_\epsilon \to u$ uniformly in any compact subset K of Ω and*

$$\limsup_{\epsilon\to 0} F^*(x, Du_\epsilon(x)) \le 1 \quad \textit{for any } x \in \Omega.$$

Proof. By means of convolution, let us consider $u_\epsilon := \rho_\epsilon \star u$, where u is extended to 0 outside Ω. Then it is clear that for any $x \in \Omega$ and $x \in \omega \subset\subset \Omega$, there exists $\epsilon_0 > 0$ such that

$$\mathrm{supp}(\rho_\epsilon(x - \cdot)) \subset \Omega \quad \text{for any } 0 < \epsilon \le \epsilon_0.$$

This implies that

$$F^*(y, Du(y))\rho_\epsilon(x - y) \le \rho_\epsilon(x - y) \quad \text{for any } y \in \mathbb{R}^N.$$

Then, by Jensen's inequality, for $0 < \epsilon \le \epsilon_0$, we have

$$\begin{aligned} F^*(x, Du_\epsilon(x)) &\le \int_\Omega F^*(x, Du(y))\rho_\epsilon(x - y)\, dy \\ &= \int_\Omega F^*(x, Du(y))\rho_\epsilon(x - y)\, dy - \int_\Omega F^*(y, Du(y))\rho_\epsilon(x - y)\, dy + \int_\Omega F^*(y, Du(y))\rho_\epsilon(x - y)\, dy \\ &\le \int_\Omega (F^*(x, Du(y)) - F^*(y, Du(y)))\rho_\epsilon(x - y)\, dy + 1. \end{aligned}$$

Letting $\epsilon \to 0$, we deduce that

$$\limsup_{\epsilon\to 0} F^*(x, Du_\epsilon(x)) \le 1. \qquad \square$$

Proof of Proposition 3.1. The first assertion is an easy consequence of (1.12).

First, let us consider $u \in W^{1,\infty}(\Omega)$. Then, for a.e. $x \in \Omega$,

$$\begin{aligned} \frac{\langle Du(x); \xi\rangle}{F(x, \xi)} &= \lim_{h\to 0^+} \frac{u(x + h\xi) - u(x)}{F(x, h\xi)} \\ &\le \mathrm{Lip}(u, c_F) \liminf_{h\to 0^+} \frac{c_F(x, x + h\xi)}{F(x, h\xi)} \\ &\le \mathrm{Lip}(u, c_F) \liminf_{h\to 0^+} \frac{1}{F(x, h\xi)} \int_0^1 F(x + th\xi, h\xi)\, dt \\ &= \mathrm{Lip}(u, c_F). \end{aligned}$$

Consequently, we get the inequality

$$\operatorname*{ess\,sup}_{x\in\Omega} F^*(x, Du(x)) \le \mathrm{Lip}(u, c_F).$$

Let us prove the last inequality in the other direction, which is equivalent to show

$$\operatorname*{ess\,sup}_{x\in\Omega} F^*(x, Du(x)) \le 1 \implies u(x) - u(y) \le c_F(y, x) \quad \text{for any } x, y \in \Omega.$$

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Thanks to Lemma 3.2, we can consider $u_\epsilon \in C^1(\overline{\Omega})$ such that $u_\epsilon \to u$ uniformly in any compact subset K of Ω and

$$\limsup_{\epsilon\to 0} F^*(\cdot, Du_\epsilon(\cdot)) \le 1 \quad \text{everywhere in } \Omega.$$

Given $x, y \in \Omega$, for $\sigma \in \Gamma^\Omega_{x,y}$, we have

$$\begin{aligned} u(y) - u(x) &= \lim_{\epsilon\to 0}(u_\epsilon(y) - u_\epsilon(x)) \\ &= \int_0^1 \langle Du_\epsilon(\sigma(t)); \sigma'(t)\rangle \, dt \\ &\le \lim_{\epsilon\to 0} \int_0^1 F^*(\sigma(t), Du_\epsilon(\sigma(t)))F(\sigma(t), \sigma'(t)) \, dt \\ &\le \int_0^1 \limsup_{\epsilon\to 0} F^*(\sigma(t), Du_\epsilon(\sigma(t)))F(\sigma(t), \sigma'(t)) \, dt \\ &\le \int_0^1 F(\sigma(t), \sigma'(t)) \, dt. \end{aligned}$$

Taking the infimum in $\sigma \in \Gamma^\Omega_{x,y}$, we get $u(y) - u(x) \le c_F(x, y)$. □

Observe that if $F^*(x, \cdot)$ is a norm, then, as usual,

$$\mathrm{Lip}(u, c_F) = \sup\left\{ \frac{|u(y) - u(x)|}{c_F(x, y)} : x, y \in \Omega,\ x \ne y \right\}.$$

Therefore, we have the following corollary.

Corollary 3.3. *Assume that $F^*(x, \cdot)$ is a norm. Then, for $u \in W^{1,\infty}(\Omega)$, we have*

$$F^*(x, Du(x)) \le 1 \quad \text{a.e. in } \Omega \iff |u(x) - u(y)| \le c_F(x, y).$$

As consequence of Proposition 3.1, we have that the set of functions

$$K_{c_F}(\Omega) = \{u \in W^{1,\infty}(\Omega) : u(y) - u(x) \le c_F(x, y)\}$$

coincides with the set

$$K^*_F(\Omega) := \left\{u \in W^{1,\infty}(\Omega) : \operatorname*{ess\,sup}_{x\in\Omega} F^*(x, Du(x)) \le 1\right\}.$$

Hence, (1.1) can be written as follows:

$$\min\{\mathcal{K}_{c_F}(\mu) : \mu \in \Pi(f^+, f^-)\} = \sup\left\{ \int_\Omega v(f^- - f^+) : v \in K^*_F(\Omega)\right\}. \tag{3.1}$$

Remark 3.4. In the case where $F(x, \xi) = |\xi|$ and Ω is convex, c_F coincides with the Euclidean distance. In this case the result of Proposition 3.1 is known. Otherwise, i.e., for $F(x, \xi) = |\xi|$ and Ω not necessarily convex, c_F is not the Euclidean distance, but it is the geodesic distance related to the Euclidean norm inside Ω. Proposition 3.1 asserts that the result for the Euclidean distance in convex sets remains true for Finsler distances in any domain Ω.

Theorem 3.5. *Let u_∞ be the limit of a subsequence $\{u_{p_j}\}_j$ as in Lemma 2.3. Then u_∞ is a Kantorovich potential for the optimal transport problem of f^+ to f^- with the cost given by $c_F(x, y)$. That is, the supremum in (3.1) is attained at u_∞.*

Moreover, if $F^*(x, \cdot) \in C^1(\mathbb{R}^N \setminus \{0\})$, for

$$\mathcal{X}_p := [F^*(x, Du_p(x))]^{p-1} \frac{\partial F^*}{\partial \xi}(x, Du_p(x)), \tag{3.2}$$

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there exists a subsequence $\mathcal{X}_{p_{j_k}}$ *converging weakly* as measures in* $\overline{\Omega}$ *to* $\mathcal{X}_\infty \in \mathcal{M}(\overline{\Omega}, \mathbb{R}^N)$ *such that*

$$\int_\Omega (f^- - f^+)v = \int_{\overline{\Omega}} Dv \, d\mathcal{X}_\infty \quad \textit{for all } v \in C^1(\overline{\Omega}). \tag{3.3}$$

In particular,

$$-\operatorname{div}(\mathcal{X}_\infty) = f^- - f^+ \quad \textit{in the sense of distributions}. \tag{3.4}$$

This theorem gives statement (3) of Theorem 1.2.

Proof. From (2.3), for every $v \in K_F^*(\Omega)$, we have

$$\begin{aligned} -\int_\Omega u_p(f^- - f^+) &\le \int_\Omega \frac{[F^*(x, Du_p(x))]^p}{p} - \int_\Omega u_p(f^- - f^+) \\ &\le \int_\Omega \frac{[F^*(x, Dv(x))]^p}{p} - \int_\Omega v(f^- - f^+) \\ &\le \frac{|\Omega|}{p} - \int_\Omega v(f^- - f^+). \end{aligned}$$

Taking limits as $p_j \to \infty$, we obtain

$$\int_\Omega u_\infty(f^- - f^+) \ge \sup\left\{ \int_\Omega v(f^- - f^+) : v \in K_F^*(\Omega) \right\}.$$

Then, in order to prove that u_∞ is a Kantorovich potential, it only remains to see that

$$u_\infty \in K_F^*(\Omega). \tag{3.5}$$

Now, using again (2.5) from the previous computations, we have that

$$\left(\int_\Omega [F^*(x, Du_p(x))]^p \right)^{\frac{1}{p}} \le (Cp)^{\frac{1}{p-1}}.$$

Then, as above, if we take $N < m \le p$, we get

$$\|F^*(x, Du_p(x))\|_{L^m(\Omega)} \le (Cp)^{\frac{1}{p-1}} |\Omega|^{\frac{p-m}{pm}} \le (C_1 p)^{\frac{1}{p-1}},$$

the constant C_1 being independent of p. Hence, having in mind that $u_{p_j} \rightrightarrows u_\infty$ uniformly in Ω, we can assume that $Du_{p_j} \rightharpoonup Du_\infty$ in $(L^m(\Omega))^N$. Then, by Mazur's theorem [15, Corollary 3.8], there exist $\lambda_i^j \ge 0$, $i = j, \dots, k_j$ with $\sum_{i=j}^{k_j} \lambda_i^j = 1$ such that

$$\sum_{i=j}^{k_j} \lambda_i^j Du_{p_i} \to Du_\infty \quad \text{strongly in } (L^m(\Omega))^N \text{ and a.e. in } \Omega.$$

Then, by the continuity of F^*, we have

$$F^*\left(\cdot, \sum_{i=j}^{k_j} \lambda_i^j Du_{p_i} \right) \to F^*(\cdot, Du_\infty) \quad \text{strongly in } L^m(\Omega) \text{ and a.e. in } \Omega.$$

Therefore,

$$\begin{aligned} \|F^*(\cdot, Du_\infty)\|_{L^m(\Omega)} &\le \liminf_{j\to\infty} \left\| F^*\left(\cdot, \sum_{i=j}^{k_j} \lambda_i^j Du_{p_i} \right) \right\|_{L^m(\Omega)} \\ &\le \liminf_{j\to\infty} \sum_{i=j}^{k_j} \lambda_i^j \|F^*(\cdot, Du_{p_i})\|_{L^m(\Omega)} \\ &\le \liminf_{j\to\infty} \sum_{i=j}^{k_j} \lambda_i^j (C_1 p_i)^{\frac{1}{p_i-1}} \\ &= 1. \end{aligned}$$

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Taking the limit as $m \to \infty$, we get that

$$\|F^*(\cdot, Du_\infty)\|_{L^\infty(\Omega)} \le 1,$$

and we conclude that $u_\infty \in K_F^*(\Omega)$.

Finally, if $F^*(x, \cdot) \in C^1(\mathbb{R}^N \setminus \{0\})$, since u_p is a weak solution of problem (2.3), for $\mathcal{X}_p$ as defined in (3.2), we have that

$$\int_\Omega \langle \mathcal{X}_p; Dv\rangle = \int_\Omega (f^- - f^+)v \quad \text{for all } v \in W^{1,p}(\Omega). \tag{3.6}$$

Let us see that $\{\mathcal{X}_p : p \ge N\}$ is bounded in $L^1(\Omega, \mathbb{R}^N)$. First, taking u_p as test function in (3.6) and having in mind (1.5), we have

$$\int_\Omega [F^*(x, Du_p(x))]^p \, dx \le C_1 \quad \text{for all } p > N.$$

Then, by Hölder's inequality, we get

$$\int_\Omega [F^*(x, Du_p(x))]^{p-1} \, dx \le C_2 \quad \text{for all } p > N. \tag{3.7}$$

On the other hand, from (1.7) and (1.12), we have that

$$\left|\frac{\partial F^*}{\partial \xi}(x, Du_p(x))\right| \le \beta;$$

hence, by (3.7), we have that

$$\int_\Omega |\mathcal{X}_p| \le \beta C_2 \quad \text{for all } p > N.$$

Therefore, there exists $\mathcal{X}_\infty \in \mathcal{M}(\overline{\Omega}, \mathbb{R}^N)$ and a subsequence $\{p_{j_k}\}$ such that

$$\mathcal{X}_{p_{j_k}} \rightharpoonup \mathcal{X}_\infty \quad \text{weakly}^* \text{ as measures in } \overline{\Omega}.$$

Thus, for any $v \in C^1(\overline{\Omega})$, having in mind (3.6), we get

$$\int_\Omega (f^- - f^+)v = \int_\Omega \langle \mathcal{X}_{p_{j_k}}; Dv\rangle \to \int_{\overline{\Omega}} Dv \, d\mathcal{X}_\infty.$$

Hence, we have proved (3.3). □

For the next theorem, we need to introduce for a given measure a new one using the Finsler structure. Let us first prove the following result on $|\cdot|_F$ defined in Section 1.5.

Lemma 3.6. *The extension of $|\mathcal{X}|_F$ to every Borel set $B \subset \overline{\Omega}$ given by*

$$|\mathcal{X}|_F(B) := \inf\{|\mathcal{X}|_F(A) : A \text{ open}, \ B \subset A\}$$

is a Radon measure in $\overline{\Omega}$.

Proof. By the De Giorgi–Letta theorem [2, Theorem 1.53], it is enough to show that $|\mathcal{X}|_F$ is subadditive, superadditive and inner regular. For given open sets $A, B \subset \overline{\Omega}$ and $\Phi \in C(\overline{\Omega}, \mathbb{R}^N)$ such that $\mathrm{supp}(\Phi) \subset A \cup B$ and $\Phi(x) \in B_{F^*(x,\cdot)}$ for all $x \in \Omega$, let $\{\eta_i : i = 1, 2, 3\}$ be a partition of unity such that $\mathrm{supp}(\eta_1) \subset A$, $\mathrm{supp}(\eta_2) \subset B$ and $\mathrm{supp}(\eta_3) \subset \overline{\Omega} \setminus \mathrm{supp}(\Phi)$. Then

$$\int_{\overline{\Omega}} \Phi \, d\mathcal{X} = \int_{\overline{\Omega}} \eta_1 \Phi \, d\mathcal{X} + \int_{\overline{\Omega}} \eta_2 \Phi \, d\mathcal{X} + \int_{\overline{\Omega}} \eta_3 \Phi \, d\mathcal{X} \le |\mathcal{X}|_F(A) + |\mathcal{X}|_F(B).$$

Hence, taking the supremum in Φ, we obtain

$$|\mathcal{X}|_F(A \cup B) \le |\mathcal{X}|_F(A) + |\mathcal{X}|_F(B).$$

The other two properties are easy to prove. □

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Since F is non-negative, positively 1-homogeneous and convex in the second variable, given $\mathcal{X} \in \mathcal{M}(\overline{\Omega}, \mathbb{R}^N)$, we can also define (see, for instance, [2, 5]) the measure $F(x, \mathcal{X})$ by

$$\int_B F(x, \mathcal{X}) := \int_B F(x, \mathcal{X}^a(x))\,dx + \int_B F\Big(x, \frac{d\mathcal{X}^s}{d|\mathcal{X}^s|}(x)\Big)\,d|\mathcal{X}^s| = \int_B F\Big(x, \frac{d\mathcal{X}}{d|\mathcal{X}|}(x)\Big)\,d|\mathcal{X}|$$

for all Borel set $B \subset \overline{\Omega}$, where $\mathcal{X} = \mathcal{X}^a + \mathcal{X}^s$ is the Lebesgue decomposition of $\mathcal{X}$, $|\mathcal{X}|$ is the total variation of $\mathcal{X}$ and $\frac{d\mathcal{X}}{d|\mathcal{X}|}$ is the Radon-Nikodym derivative of $\mathcal{X}$ with respect to $|\mathcal{X}|$. Since $|\mathcal{X}|$ is absolutely continuous with respect to the measure $|\mathcal{X}|_F$, by [2, Proposition 2.37], we have

$$\int_B F(x, \mathcal{X}) = \int_B F\Big(x, \frac{d\mathcal{X}}{d|\mathcal{X}|_F}(x)\Big)\,d|\mathcal{X}|_F \quad \text{for all Borel set } B \subset \overline{\Omega}. \tag{3.8}$$

Having in mind (3.8) and following the proof of the continuity Reshetnyak theorem given in [32], we get the following result.

Lemma 3.7. *Let $\mathcal{X}_n, \mathcal{X} \in \mathcal{M}(\overline{\Omega}, \mathbb{R}^N)$ be such that*

$$\mathcal{X}_n \rightharpoonup \mathcal{X} \quad \text{in } \mathcal{M}(\overline{\Omega}, \mathbb{R}^N) \qquad \text{and} \qquad |\mathcal{X}_n|_F(\overline{\Omega}) \to |\mathcal{X}|_F(\overline{\Omega}).$$

Then

$$\lim_{n\to\infty} \int_{\overline{\Omega}} F(x, \mathcal{X}_n) = \int_{\overline{\Omega}} F(x, \mathcal{X}).$$

We will also use the following approximation result.

Lemma 3.8. *For any $u \in W^{1,\infty}(\Omega)$ such that $F^*(x, Du(x)) \le 1$ for a.e. $x \in \Omega$, there exists $u_\epsilon \in C^1(\overline{\Omega})$ such that $u_\epsilon \to u$ uniformly in any compact subset K of Ω and*

$$\limsup_{\epsilon\to 0} \sup_{\overline{\Omega}} F^*(x, Du_\epsilon(x)) \le 1.$$

Proof. Since $F^*(x, Du(x)) \le 1$ for a.e. $x \in \Omega$ and we are under condition (1.11), if we take the McShane–Whitney extension

$$\overline{u}(x) := \inf_{y\in\Omega}\{u(y) + c_F^D(y, x)\}, \qquad x \in D,$$

then we have that $\overline{u}(x) - \overline{u}(y) \le c_F^D(y, x)$. Take $u_\epsilon = \overline{u} * \rho_\epsilon \in C^1(\overline{\Omega})$ (we can extend $\overline{u}$ as zero outside D). Then $u_\epsilon \to \overline{u}$ uniformly in any compact subset K of D. On the other hand, by continuity, there exists $x_\epsilon \in \overline{\Omega}$ such that

$$\sup_{\overline{\Omega}} F^*(x, Du_\epsilon(x)) = F^*(x_\epsilon, Du_\epsilon(x_\epsilon)).$$

By Lemma 3.1 (that can be applied to D and C_F^D), we have that

$$\operatorname*{ess\,sup}_{x\in D} F^*(x, D\overline{u}(x)) \le 1.$$

Then, by Jensen's inequality, for ϵ small, we have

$$\begin{aligned} F^*(x_\epsilon, Du_\epsilon(x_\epsilon)) &\le \int_D F^*(x_\epsilon, D\overline{u}(y))\rho_\epsilon(x_\epsilon - y)\,dy \\ &= \int_D F^*(x_\epsilon, D\overline{u}(y))\rho_\epsilon(x_\epsilon - y)\,dy - \int_D F^*(y, D\overline{u}(y))\rho(x_\epsilon - y)\,dy + \int_D F^*(y, D\overline{u}(y))\rho_\epsilon(x_\epsilon - y)\,dy \\ &\le \int_D (F^*(x_\epsilon, D\overline{u}(y)) - F^*(y, D\overline{u}(y)))\rho_\epsilon(x_\epsilon - y)\,dy + 1. \end{aligned}$$

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Now, there exists a subsequence such that $x_{\epsilon_n} \to x_0$ and, for this subsequence, we have

$$\int_D (F^*(x_{\epsilon_n}, D\overline{u}(y)) - F^*(y, D\overline{u}(y)))\rho_{\epsilon_n}(x_{\epsilon_n} - y)\,dy \to 0 \quad \text{as } n \to +\infty. \qquad \square$$

Theorem 3.9 (Statement (4) of Theorem 1.2). *Let u_∞ and $\mathcal{X}_\infty$ be as in Theorem 3.5. Then*

$$|\mathcal{X}_\infty|_F(\overline{\Omega}) = \int_{\overline{\Omega}} F(x, \mathcal{X}_\infty) = \int_\Omega u_\infty(f^- - f^+). \tag{3.9}$$

Proof. Let v_ϵ be the approximation given in Lemma 3.8 for $u = u_\infty$, then

$$\begin{aligned}
\int_\Omega (f^- - f^+)u_\infty\,dx &= \lim_{\epsilon\to 0} \int_\Omega (f^- - f^+)v_\epsilon\,dx \\
&= \lim_{\epsilon\to 0} \int_{\overline{\Omega}} Dv_\epsilon\,d\mathcal{X}_\infty \\
&\le \limsup_{\epsilon\to 0} \sup_{\overline{\Omega}} F^*(x, Dv_\epsilon(x))|\mathcal{X}_\infty|_F(\overline{\Omega}) \\
&\le |\mathcal{X}_\infty|_F(\overline{\Omega}).
\end{aligned} \tag{3.10}$$

Let $\{p_i\}_i$ be a sequence such that u_{p_i} and $\mathcal{X}_{p_i}$ converge to u_∞ and $\mathcal{X}_\infty$ in the sense given in the previous results.

Take now $\Phi \in C(\overline{\Omega}, \mathbb{R}^N)$ with $\Phi(x) \in B_{F^*(x,\cdot)}$ for all $x \in \Omega$. By (1.3), we have

$$\int_\Omega \Phi\mathcal{X}_{p_i}\,dx \le \int_\Omega F^*(x, \Phi(x))F(x, \mathcal{X}_{p_i}(x))\,dx \le \int_\Omega F(x, \mathcal{X}_{p_i}(x))\,dx. \tag{3.11}$$

Therefore,

$$\int_{\overline{\Omega}} \Phi\,d\mathcal{X}_\infty = \lim_i \int_\Omega \Phi\mathcal{X}_{p_i} \le \limsup_i \int_\Omega F(x, \mathcal{X}_{p_i}(x))\,dx,$$

and, by taking the supremum in Φ,

$$|\mathcal{X}_\infty|_F(\overline{\Omega}) \le \limsup_i \int_\Omega F(x, \mathcal{X}_{p_i}(x))\,dx. \tag{3.12}$$

Now, applying Hölder's inequality, (1.9), (1.5) and (3.6), we get

$$\begin{aligned}
\limsup_{i\to\infty} \int_\Omega F(x, \mathcal{X}_{p_i}(x))\,dx &= \limsup_{i\to\infty} \int_\Omega [F^*(x, Du_{p_i}(x))]^{p_i-1} F\Big(x, \frac{\partial F^*}{\partial\xi}(x, Du_{p_i}(x))\Big)\,dx \\
&\le \limsup_{i\to\infty} \Big(\int_\Omega [F^*(x, Du_{p_i}(x))]^{p_i}\,dx\Big)^{\frac{p_i-1}{p_i}} \\
&= \limsup_{i\to\infty} \Big(\int_\Omega [F^*(x, Du_{p_i}(x))]^{p_i-1} \Big\langle \frac{\partial F^*}{\partial\xi}(x, Du_{p_i}(x)); Du_{p_i}(x)\Big\rangle\,dx\Big)^{\frac{p_i-1}{p_i}} \\
&= \limsup_{i\to\infty} \Big(\int_\Omega \langle \mathcal{X}_{p_i}; Du_{p_i}\rangle\Big)^{\frac{p_i-1}{p_i}} \\
&= \lim_{i\to\infty} \int_\Omega \langle \mathcal{X}_{p_i}; Du_{p_i}\rangle \\
&= \lim_{i\to\infty} \int_\Omega (f^- - f^+)u_{p_i} \\
&= \int_\Omega (f^- - f^+)u_\infty,
\end{aligned}$$

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that is,

$$\limsup_{i\to\infty}\int_\Omega F(x,\mathcal{X}_{p_i}(x))\,dx \le \int_\Omega (f^- - f^+)u_\infty. \tag{3.13}$$

Then, by (3.10), (3.12) and (3.13),

$$|\mathcal{X}_\infty|_F(\overline{\Omega}) = \int_\Omega (f^- - f^+)u_\infty\,dx. \tag{3.14}$$

Let us see now that

$$|\mathcal{X}_{p_i}|_F(\overline{\Omega}) \to |\mathcal{X}_\infty|_F(\overline{\Omega}). \tag{3.15}$$

By (3.11), taking the supremum in Φ, we have

$$|\mathcal{X}_{p_i}|_F(\overline{\Omega}) \le \int_\Omega F(x,\mathcal{X}_{p_i}).$$

Then, by (3.13) and (3.14), we get

$$\limsup_{i\to\infty}|\mathcal{X}_{p_i}|_F(\overline{\Omega}) \le \limsup_{i\to\infty}\int_\Omega F(x,\mathcal{X}_{p_i}) = \int_\Omega (f^- - f^+)u_\infty = |\mathcal{X}_\infty|_F(\overline{\Omega}).$$

On the other hand, given $\Phi \in C(\overline{\Omega},\mathbb{R}^N)$ with $\Phi(x) \in B_{F^*(x,\cdot)}$ for all $x \in \Omega$, we have

$$\int_\Omega \Phi\mathcal{X}_{p_i} \le |\mathcal{X}_{p_i}|_F(\overline{\Omega}),$$

thus

$$\int_\Omega \Phi\mathcal{X}_\infty \le \liminf_i |\mathcal{X}_{p_i}|_F(\overline{\Omega}),$$

from which, we get that

$$|\mathcal{X}_\infty|_F(\overline{\Omega}) \le \liminf_i |\mathcal{X}_{p_i}|_F(\overline{\Omega}),$$

and the proof of (3.15) is finished.

Finally, since $\mathcal{X}_{p_i} \rightharpoonup \mathcal{X}_\infty$ in $\mathcal{M}(\overline{\Omega},\mathbb{R}^N)$ and we have (3.15), by Lemma 3.7, we get

$$\int_{\overline{\Omega}} F(x,\mathcal{X}_\infty) = \lim_{n\to\infty}\int_{\overline{\Omega}} F(x,\mathcal{X}_{p_i}) = \int_\Omega (f^- - f^+)u_\infty.$$ □

Let us see now that $F(x,\mathcal{X}_\infty)$ is the transport density of the transport problem we are dealing with. To do that we need to recall the concept of tangential derivative with respect to a Radon measure (see, for instance, [11–13]). Given $\mu \in \mathcal{M}(\overline{\Omega})^+$, we define

$$\mathcal{N} := \{\xi \in L^\infty_\mu(\overline{\Omega},\mathbb{R}^N) : \text{there exists } u_n \in C^\infty(\overline{\Omega}) \text{ such that } u_n \to 0 \text{ uniformly and } Du_n \rightharpoonup \xi \text{ in } \sigma(L^\infty_\mu, L^1_\mu)\}.$$

The orthogonal of $\mathcal{N}$ in $L^1_\mu(\overline{\Omega},\mathbb{R}^N)$ is characterized in [13] by

$$\mathcal{N}^\perp = \{\sigma \in L^1_\mu(\overline{\Omega}) : \sigma(x) \in T_\mu(x)\ \mu\text{-a.e.}\},$$

where T_μ is a closed valued μ-measurable multifunction that is called the *tangent space* to the measure μ. For a function $u \in C^1(\overline{\Omega})$, its *tangential gradient* $D_\mu u(x)$ is defined as the projection $P_\mu(x)Du(x)$ on $T_\mu(x)$. In [13] it was proved that the linear operator $u \in C^1(\overline{\Omega}) \mapsto D_\mu u \in L^\infty_\mu(\overline{\Omega},\mathbb{R}^N)$ can be extended in a unique way as a linear continuous operator $D_\mu : \mathrm{Lip}(\overline{\Omega}) \to L^\infty_\mu(\overline{\Omega},\mathbb{R}^N)$, where $\mathrm{Lip}(\overline{\Omega})$ is equipped with the uniform convergence and $L^\infty_\mu(\overline{\Omega},\mathbb{R}^N)$ with the weak star topology. Consequently, there exists $v_\epsilon \in \mathcal{C}^1(\overline{\Omega})$ such that

$$v_\epsilon \to u_\infty \quad \text{uniformly}, \qquad D_\mu v_\epsilon \overset{*}{\rightharpoonup} D_\mu u_\infty \quad \sigma(L^\infty_\mu, L^1_\mu). \tag{3.16}$$

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Following [29], given $u \in W^{1,\infty}(\Omega)$, we define the *μ-tangential gradient* of u with respect to F in the following form: for $x \in \Omega$ such that $D_\mu u(x)$ exists, we define

$$\partial_{F,\mu} u(x) := \left\{ \frac{D_\mu u(x)\cdot \hat{v}}{F(x,\hat{v})^2}\hat{v} \;:\; \hat{v} \in \operatorname*{arg\,max}_{v\in T_\mu(x),\,|v|=1} \frac{D_\mu u(x)\cdot v}{F(x,v)} \right\}.$$

In case $F(x,\cdot)$ is strictly convex, then there is a unique maximum

$$\hat{v} \in \operatorname{arg\,max}\left\{ \frac{D_\mu u(x)\cdot v}{F(x,v)} : v \in T_\mu(x),\ |v| = 1 \right\},$$

and consequently $\partial_{F,\mu} u(x)$ has a unique element, that we denote by $\nabla_{F,\mu} u(x)$, that is called the *μ-tangential gradient* of u at x with respect to F, that is,

$$\nabla_{F,\mu} u(x) = \frac{D_\mu u(x)\cdot \hat{v}}{F(x,\hat{v})^2}\hat{v}.$$

Observe that

$$\partial_{F,\mu} u(x) = \left\{ (D_\mu u(x)\cdot \hat{v})\hat{v} : \hat{v} \in \operatorname*{arg\,max}_{v\in T_\mu(x),\,F(x,v)=1} D_\mu u(x)\cdot v \right\}.$$

Theorem 3.10. *Let u_∞ and $\mathcal{X}_\infty$ be as in Theorem 3.5. If we set $\mu := F(x,\mathcal{X}_\infty)$, then*

$$\int_\Omega (f^- - f^+)v = \int_{\overline{\Omega}} \frac{d\mathcal{X}_\infty}{d\mu}\cdot Dv\, d\mu \quad \text{for all } v \in C^1(\overline{\Omega}),$$

$$\frac{d\mathcal{X}_\infty}{d\mu}(x) \in \partial_{F,\mu} u_\infty(x) \quad \text{and} \quad F\Big(x, \frac{d\mathcal{X}_\infty}{d\mu}(x)\Big) = 1 \quad \mu\text{-a.e. in } \overline{\Omega}.$$

Moreover, if $F(x,\cdot)$ is strictly convex, then

$$\int_\Omega (f^- - f^+)v = \int_{\overline{\Omega}} \nabla_{F,\mu} u_\infty \cdot Dv\, d\mu \quad \text{for all } v \in C^1(\overline{\Omega})$$

and

$$F(x, \nabla_{F,\mu} u_\infty(x)) = 1 \quad \mu\text{-a.e. in } \overline{\Omega}$$

Proof. Since $\mathcal{X}_\infty$ is absolutely continuous with respect to μ, we have that the Radon–Nikodym derivative $\frac{d\mathcal{X}_\infty}{d\mu}$ is in $L^1_\mu(\overline{\Omega}, \mathbb{R}^N)$. On the other hand, by (3.4),

$$-\operatorname{div}\Big(\mu \frac{d\mathcal{X}_\infty}{d\mu}\Big) = f^- - f^+ \quad \text{in the sense of distributions.}$$

Then, from [13, Proposition 3.5], it follows that

$$\frac{d\mathcal{X}_\infty}{d\mu}(x) \in T_\mu(x) \quad \mu\text{-a.e.} \tag{3.17}$$

We claim now that

$$D_\mu u_\infty(x)\cdot v(x) \le F(x, v(x)) \quad \mu\text{-a.e.} \tag{3.18}$$

for any $v(x) \in T_\mu(x)$ μ-a.e. Indeed, let u_ϵ be the function given in Lemma 3.8. Then, by (1.3), if $v(x) \in T_\mu(x)$ μ-a.e., we have

$$D_\mu u_\epsilon(x)\cdot v(x) = Du_\epsilon(x)\cdot v(x) \le F^*(x, Du_\epsilon(x))F(x, v(x))$$

for μ-almost all x. Now, integrating the above inequality over any μ-measurable set A and taking limits as $\epsilon \to 0$, we get

$$\int_A D_\mu u_\infty(x)\cdot v(x)\, d\mu(x) \le \int_A F(x, v(x))\, d\mu(x),$$

which gives (3.18).

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From (3.18) and (3.17), we can write

$$D_\mu u_\infty(x) \cdot \frac{d\mathfrak{X}_\infty}{d\mu}(x) \leq F\Big(x, \frac{d\mathfrak{X}_\infty}{d\mu}(x)\Big) \quad \mu\text{-a.e.} \tag{3.19}$$

Now, since

$$F\Big(x, \frac{d\mathfrak{X}_\infty}{d\mu}(x)\Big) = 1 \quad \mu\text{-a.e.,}$$

inequality (3.19) reads

$$D_\mu u_\infty \cdot \frac{d\mathfrak{X}_\infty}{d\mu} \leq 1 \quad \mu\text{-a.e.} \tag{3.20}$$

Now, taking v_ϵ as in (3.16) and having in mind (3.17), we get

$$\int_{\overline{\Omega}} D_\mu v_\epsilon \frac{d\mathfrak{X}_\infty}{d\mu}\, d\mu = \int_{\overline{\Omega}} Dv_\epsilon \, d\mathfrak{X}_\infty = \int_{\Omega} (f^- - f^+) v_\epsilon.$$

Therefore, taking limits as $\epsilon \to 0$, we obtain

$$\int_{\overline{\Omega}} D_\mu u_\infty \frac{d\mathfrak{X}_\infty}{d\mu}\, d\mu = \int_{\Omega} (f^- - f^+) u_\infty = \int_{\overline{\Omega}} d\mu,$$

where the last equality is a consequence of (3.9). Then, by (3.20),

$$D_\mu u_\infty \cdot \frac{d\mathfrak{X}_\infty}{d\mu} = 1 \quad \mu\text{-a.e.} \tag{3.21}$$

On account of (3.18) and (3.21), we have

$$\frac{d\mathfrak{X}_\infty}{d\mu}(x) \in \operatorname{arg\,max}\{D_\mu u_\infty(x) \cdot v : v \in T_\mu(x),\ F(x, v) = 1\},$$

and consequently

$$\frac{d\mathfrak{X}_\infty}{d\mu}(x) \in \partial_{F,\mu} u_\infty(x).$$

Assuming that $F(x, \cdot)$ is strictly convex, we have

$$\frac{d\mathfrak{X}_\infty}{d\mu}(x) = \nabla_{F,\mu} u_\infty(x),$$

and the proof concludes. □

Corollary 3.11 (Statement (5) of Theorem 1.2). *Let u_∞ and $\mathfrak{X}_\infty$ be as in Theorem 3.5. If in addition we assume that*

$$F^*(x, D_\mu u_\infty(x)) \leq 1 \quad \mu\text{-a.e. in } \overline{\Omega}, \tag{3.22}$$

then

$$\begin{cases} \int_{\Omega} (f^- - f^+) v = \int_{\overline{\Omega}} \frac{\partial F^*}{\partial \xi}(\cdot, D_\mu u_\infty) \cdot Dv \, d\mu & \text{for all } v \in C^1(\overline{\Omega}), \\ F^*(x, D_\mu u_\infty(x)) = 1 & \mu\text{-a.e. in } \overline{\Omega}, \end{cases} \tag{3.23}$$

Proof. Since

$$1 = D_\mu u_\infty \cdot \frac{d\mathfrak{X}_\infty}{d\mu} \leq F^*(x, D_\mu u_\infty(x)) \quad \mu\text{-a.e.,}$$

by (3.22), we have that

$$F^*(x, D_\mu u_\infty(x)) = 1 \quad \mu\text{-a.e.} \tag{3.24}$$

On the other hand,

$$D_\mu u_\infty \cdot \frac{d\mathfrak{X}_\infty}{d\mu} = 1 = F\Big(x, \frac{d\mathfrak{X}_\infty}{d\mu}(x)\Big) \quad \mu\text{-a.e.} \tag{3.25}$$

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Now, having in mind (3.25) and (3.24), and applying (1.8), we deduce that

$$\frac{d\mathcal{X}_\infty}{d\mu}(x) = F\Big(x, \frac{d\mathcal{X}_\infty}{d\mu}(x)\Big)\frac{\partial F^*}{\partial \xi}(x, D_\mu u_\infty(x)) = \frac{\partial F^*}{\partial \xi}(x, D_\mu u_\infty(x)) \quad \mu\text{-a.e.}$$

Then, by the above theorem we get (3.23). □

Remark 3.12. If $F(x,\xi) = |A(x)\xi|$ with $A(x)$ a symmetric positive definite matrix, then

$$D_\mu u_\infty(x) \in B_{F^*(x,\cdot)} \quad \mu\text{-a.e. in } \overline{\Omega}.$$

In fact, we have $F^*(x,\xi) = |A(x)^{-1}\xi|$. Then, since $A(x)^{-1}$ preserves the orthogonality and $D_\mu u_\infty(x)$ and $Du_\infty(x) - D_\mu u_\infty(x)$ are orthogonal (remember that $D_\mu u_\infty(x)$ is defined as the projection $P_\mu(x)Du_\infty(x)$ on $T_\mu(x)$), using Pythagoras' theorem, we have

$$|A(x)^{-1}Du_\infty(x)|^2 = |A(x)^{-1}D_\mu u_\infty(x)|^2 + \big|A(x)^{-1}(Du_\infty(x) - D_\mu u_\infty(x))\big|^2.$$

Therefore,

$$F^*(x, D_\mu u_\infty(x)) \le F^*(x, Du_\infty(x)) \le 1.$$

Let us remark that in this case, it is known that, in fact, μ is absolutely continuous with respect to the Lebesgue measure since $f^+ \in L^1(\Omega)$ (see [16, 19, 29, 30]), and thus $D_\mu u_\infty = Du_\infty$.

In the case $F(\cdot, \mathcal{X}_\infty(\cdot)) \in L^1(\Omega)$, we have the following result.

Corollary 3.13. *Let u_∞ and $\mathcal{X}_\infty$ be as in Theorem 3.5. If $F(\cdot, \mathcal{X}_\infty(\cdot)) \in L^1(\Omega)$, then for almost every x,*

$$F(x, \mathcal{X}_\infty(x)) > 0 \implies F^*(x, Du_\infty(x)) = 1, \tag{3.26}$$

and

$$\int_\Omega F(x, \mathcal{X}_\infty(x))\Big\langle \frac{\partial F^*}{\partial \xi}(x, Du_\infty(x)); Dv(x)\Big\rangle\, dx = \int_\Omega (f^-(x) - f^+(x))v(x)\, dx$$

for all $v \in C^1(\overline{\Omega})$. In particular,

$$-\operatorname{div}\Big(F(\cdot, \mathcal{X}_\infty(\cdot))\frac{\partial F^*}{\partial \xi}(\cdot, Du_\infty)\Big) = (f^- - f^+) \quad \textit{in the sense of distributions,} \tag{3.27}$$

and

$$\int_\Omega F(x, \mathcal{X}_\infty(x))\, dx = \int_\Omega u_\infty(x)(f^-(x) - f^+(x))\, dx. \tag{3.28}$$

Remark 3.14. Let us give an interpretation of equation (3.27) in terms of the Finsler manifold (Ω, F). For this we need to recall the concept of gradient vector in a Finsler manifold (see, for example, [28]). Let us suppose that $\frac{1}{2}F^2(x,\cdot)$ is differentiable for $\xi \neq 0$. Let $J\colon \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ be the transfer map of the Finsler structure F, defined in $\alpha \in \mathbb{R}^N$ as the unique maximizer of the function $\xi \mapsto \langle \alpha, \xi\rangle - \frac{1}{2}F^2(x,\xi)$. The vector $J(x,\alpha)$ can be given as follows:

$$J(x,\alpha) = F^*(x,\alpha)\frac{\partial F^*}{\partial \xi}(x,\alpha).$$

The gradient vector in the Finsler manifold (Ω, F) of a smooth function $u\colon \Omega \to \mathbb{R}$ is defined by

$$\nabla u(x) := J(x, Du(x)) = F^*(x, Du(x))\frac{\partial F^*}{\partial \xi}(x, Du(x)).$$

Let us remark that the gradient vector ∇u coincides with $\nabla_{F,\mu} u$ when μ is absolutely continuous with respect to the Lebesgue measure.

Then, setting $a(x) = F(x, \mathcal{X}_\infty)$, by (3.26), we have

$$a(x)\nabla u_\infty(x) = a(x)\frac{\partial F^*}{\partial \xi}(x, Du_\infty(x)).$$

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Therefore, we can write equation (3.27) as follows:

$$-\operatorname{div}(a\nabla u_\infty) = f^- - f^+ \quad \text{in the sense of distributions},$$

with $\operatorname{ess\,sup}_{x\in\Omega} F(x, \nabla u_\infty(x)) \le 1$. Moreover, for almost every x,

$$a(x) > 0 \implies F(x, \nabla u_\infty(x)) = 1.$$

Indeed, by (1.9),

$$\begin{aligned} F(x, \nabla u_\infty(x)) &= F\Big(x, F^*(x, Du_\infty(x)) \frac{\partial F^*}{\partial \xi}(x, Du_\infty(x))\Big) \\ &= F^*(x, Du_\infty(x)) F\Big(x, \frac{\partial F^*}{\partial \xi}(x, Du_\infty(x))\Big) \\ &= F^*(x, Du_\infty(x)), \end{aligned}$$

and, by (3.26), we have that $F(x, \nabla u_\infty(x)) = F^*(x, Du_\infty(x)) = 1$ for almost every x such that $a(x) > 0$.

We have been dealing with a mass transport problem in the Finsler metric space (Ω, F, dx), with a quite general Finsler structure F, for the distance induced by such structure. This general structure includes the case $F(x, \xi) = \Phi(A(x)\xi)$, where Φ is a Finsler function and $A(x)$ is a symmetric $N \times N$ positive definite matrix that depends smoothly on x; in particular, the Riemannian structures $F(x, \xi) = |A(x)\xi|$, where $|\cdot|$ is the Euclidean norm. Let us see how these results can be interpreted in the context of optimal transportation on Riemannian manifolds.

3.2 Example

In the particular case in which $F(x, \xi) = |A(x)\xi|$, where $|\cdot|$ is the Euclidean norm and $A(x)$ is a symmetric $N \times N$ positive definite matrix that depends smoothly on $x \in \overline{\Omega}$, we have

$$c_F(x, y) = \inf_{\sigma\in\Gamma^\Omega_{x,y}} \int_0^1 \sqrt{\langle A(\sigma(t))\sigma'(t); A(\sigma(t))\sigma'(t)\rangle}\, dt = \inf_{\sigma\in\Gamma^\Omega_{x,y}} \int_0^1 \sqrt{\langle A^2(\sigma(t))\sigma'(t); \sigma'(t)\rangle}\, dt.$$

Therefore, writing $A^2(z) = (g_{i,j}(z))_{i,j} =: g(z)$, the cost function c is given by

$$c_F(x, y) = d_{g,\Omega}(x, y) := \inf_{\sigma\in\Gamma^\Omega_{x,y}} \int_0^1 \sqrt{\sum_{i,j} g_{i,j}(\sigma(t))\sigma_i'(t)\sigma_j'(t)}\, dt.$$

That is, in this case the cost function c is the distance induced by the metric tensor g.

When $A(z) = b(z)I_N$ (here I_N denotes the $N \times N$ identity matrix), we have

$$c_F(x, y) = \inf_{\sigma\in\Gamma^\Omega_{x,y}} \int_\sigma b(z)\, ds.$$

This case has been studied in [26].

The results obtained can be interpreted in the context of optimal transportation on Riemannian manifolds with cost function the distance induced by the metric tensor. Let us illustrate this with the following example.

N-dimensional parametrized manifolds in $\mathbb{R}^M$. Let $\mathcal{S}$ be an N-dimensional parametrized manifold in $\mathbb{R}^M$ ($M \ge N$), that is, $\mathcal{S} = \psi(\Omega)$, where Ω is an open bounded set of $\mathbb{R}^N$ and $\psi\colon \Omega \to \mathbb{R}^M$ is a smooth map such that for each $x \in \Omega$, the $M \times N$ Jacobian matrix $J_\psi(x)$ has rank N. We denote by g the metric tensor $g := J_\psi^t \cdot J_\psi$ and by $|g|$ the determinant of g. Consider in $\mathcal{S}$ the Riemannian distance induced by the Euclidean distance in $\mathbb{R}^M$, i.e.,

$$d_{I_M,\mathcal{S}}(\xi, \eta) = \inf_{\sigma\in\Gamma^{\mathcal{S}}_{\xi,\eta}} \int_0^1 |\sigma'(t)|\, dt,$$

where I_M is the $M \times M$ identity matrix.

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One can think on $\mathbb{S}$, for example, the sphere of radius R in $\mathbb{R}^3$, parametrized by $\psi :]0, 2\pi[\times]0, \pi[\to \mathbb{R}^3$, given by

$$\psi(\theta, \phi) = (R\cos\theta\sin\phi,\ R\sin\theta\sin\phi,\ R\cos\phi),$$

which is a non Euclidean Riemannian manifold with metric g defined by

$$g(\theta, \phi) = \begin{pmatrix} R^2\sin^2\phi & 0 \\ 0 & R^2 \end{pmatrix}.$$

Suppose that we have two functions $\tilde{f}^{\pm} \in L^1(\mathbb{S}, d\mathrm{vol})$, both with equal mass, i.e.,

$$\int_{\mathbb{S}} \tilde{f}^+(z)\, d\mathrm{vol}(z) = \int_{\Omega} \sqrt{|g|(x)}\tilde{f}^+(\psi(x))\, dx = \int_{\mathbb{S}} \tilde{f}^-(z)\, d\mathrm{vol}(z) = \int_{\Omega} \sqrt{|g|(x)}\tilde{f}^-(\psi(x))\, dx,$$

and we want to transport $\tilde{f}^+$ to $\tilde{f}^-$ on $\mathbb{S}$ with cost function the distance $d_{I_M,\mathbb{S}}$. If we take

$$f^{\pm}(x) = \sqrt{|g|(x)}\tilde{f}^{\pm}(\psi(x)),$$

then we have

$$\int_{\Omega} f^+(x)\, dx = \int_{\Omega} f^-(x)\, dx.$$

A simple calculation shows that

$$d_{I_M,\mathbb{S}}(\xi, \eta) = d_{g,\Omega}(\psi^{-1}(\xi), \psi^{-1}(\eta)) \quad \text{for all } \xi, \eta \in \mathbb{R}^M. \tag{3.29}$$

Moreover, if $\tilde{T}\tilde{f}^+ = \tilde{f}^-$ and $T := \psi^{-1} \circ \tilde{T} \circ \psi$, then $Tf^+ = f^-$ and

$$\begin{aligned}\int_{\mathbb{S}} d_{I_M,\mathbb{S}}(\xi, \tilde{T}(\xi))\tilde{f}^+(\xi)\, d\mathrm{vol}(\xi) &= \int_{\Omega} \sqrt{|g|(x)}d_{g,\Omega}(x, \psi^{-1}(\tilde{T}(\psi(x)))\tilde{f}^+(\psi(x)))\, dx \\ &= \int_{\Omega} d_{g,\Omega}(x, T(x))f^+(x)\, dx.\end{aligned}$$

Similarly, if $Tf^+ = f^-$ and $\tilde{T} := \psi \circ T \circ \psi^{-1}$, then $\tilde{T}\tilde{f}^+ = \tilde{f}^-$ and

$$\int_{\Omega} d_{g,\Omega}(x, T(x))f^+(x)\, dx = \int_{\mathbb{S}} d_{I_M,\mathbb{S}}(\xi, \tilde{T}(\xi))\tilde{f}^+(\xi)\, d\mathrm{vol}(\xi).$$

Therefore, for the Monge problems, we have

$$\min_{\tilde{T}\tilde{f}^+ = \tilde{f}^-}\left\{\int_{\mathbb{S}} d_{I_M,\mathbb{S}}(\xi, \tilde{T}(\xi))\tilde{f}^+(\xi)\, d\mathrm{vol}(\xi)\right\} = \min_{Tf^+ = f^-}\left\{\int_{\Omega} d_{g,\Omega}(x, T(x))f^+(x) dx\right\}.$$

Consider now the Kantorovich potential u_∞ obtained in Theorem 3.5 for $F^*(x, \xi) = |A^{-1}(x)\xi|$, A a square root of g and the masses $f^{\pm}$. Then

$$\begin{aligned}\sup\left\{\int_{\Omega} v(x)(f^-(x) - f^+(x))\, dx : v \in K_{d_{g,\Omega}}(\Omega)\right\} &= \int_{\Omega} u_\infty(x)(f^-(x) - f^+(x))\, dx \\ &= \int_{\Omega} u_\infty(\psi^{-1}(\psi(x)))\Big(\sqrt{|g|(x)}\tilde{f}^-(\psi(x)) - \sqrt{|g|(x)}\tilde{f}^+(\psi(x))\Big)\, dx \\ &= \int_{\mathbb{S}} u_\infty(\psi^{-1}(z))(\tilde{f}^-(z) - \tilde{f}^+(z))\, d\mathrm{vol}(z).\end{aligned}$$

On the other hand, by (3.29), it is easy to see that

$$v \in K_{d_{g,\Omega}}(\Omega) \iff v(\psi^{-1}(z)) \in K_{d_{I_M,\mathbb{S}}}(\mathbb{S}).$$

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Thus,

$$\sup\left\{\int_\Omega v(x)(f^-(x)-f^+(x))\,dx : v \in K_{d_{g,\Omega}}(\Omega)\right\} = \sup\left\{\int_{\mathcal{S}} w(z)(\tilde f^-(z)-\tilde f^+(z))\,d\mathrm{vol}(z) : w \in K_{d_{I_M,\mathcal{S}}}(M)\right\}.$$

Consequently, for $\tilde u_\infty(z) := u_\infty(\psi^{-1}(z))$,

$$\int_{\mathcal{S}} \tilde u_\infty(z)(\tilde f^-(z)-\tilde f^+(z))\,d\mathrm{vol}(z) = \sup\left\{\int_{\mathcal{S}} w(z)(\tilde f^-(z)-\tilde f^+(z))\,d\mathrm{vol}(z) : w \in K_{d_{I_M,\mathcal{S}}}(\mathcal{S})\right\},$$

and $\tilde u_\infty$ is a Kantorovich potential for the transport of $\tilde f^+$ to $\tilde f^-$ on the manifold $\mathcal{S}$ with respect to the Riemannian distance $d_{I_M,\mathcal{S}}$.

When $N = M$ we consider a change of variables. In this case, $\sqrt{|g(x)|} = |J_\psi(x)|$. Now a square root A of g can be J_ψ among others.

Corollary 3.13 reads now as follows. Let us denote with $a(x)$ the transport density $F(x, \mathcal{X}_\infty(x))$. Then

$$-\operatorname{div}(a g^{-1} D u_\infty) = f^- - f^+ \quad \text{in } \Omega, \tag{3.30}$$

and for a.e. x,

$$a(x) > 0 \implies \langle g^{-1}(x) D u_\infty(x); D u_\infty(x)\rangle = 1. \tag{3.31}$$

If we define

$$\tilde a := \frac{a}{\sqrt{|g|}} \circ \psi^{-1},$$

then from (3.28) we have

$$\begin{aligned}\int_{\mathcal{S}} \tilde u_\infty(z)(\tilde f^-(z)-\tilde f^+(z))\,dS &= \int_\Omega \sqrt{|g|(x)}\,\tilde u_\infty(\psi(x))(\tilde f^-(\psi(x))-\tilde f^+(\psi(x)))\,dx\\ &= \int_\Omega u_\infty(x)(f^-(x)-f^+(x))\,dx\\ &= \int_\Omega a(x)\,dx\\ &= \int_{\mathcal{S}} \tilde a(z)\,d\mathrm{vol}(z).\end{aligned}$$

Recall that $w \in W^{1,\infty}(\mathcal{S})$ if $w\circ\psi \in W^{1,\infty}(\Omega)$. For $w \in W^{1,\infty}(\mathcal{S})$, the gradient of w at $z \in \mathcal{S}$ is denoted by $\nabla w(z) \in T_z\mathcal{S}$ and is defined, for $v \in T_z\mathcal{S}$, by

$$\langle \nabla w(z), v\rangle = \frac{d}{dt}(w\circ\alpha)\Big|_{t=0},$$

where $\alpha :]-\epsilon,\epsilon[\to \mathcal{S}$ is a smooth path such that $\alpha(0) = z$ and $\alpha'(0) = v$. Then we have

$$\langle \nabla w(\psi(x)), J_\psi(x) u\rangle = \langle D(w\circ\psi)(x), u\rangle \quad \text{for all } x \in \Omega,\ u \in \mathbb{R}^N. \tag{3.32}$$

In fact, if we define $\alpha(t) := \psi(x+tu) = (\psi\circ r)(t)$, then by applying the chain rule, we have

$$\langle \nabla w(\psi(x)), J_\psi(x) u\rangle = \frac{d}{dt}(w\circ\alpha)\Big|_{t=0} = \frac{d}{dt}((w\circ\psi)\circ r)\Big|_{t=0} = \langle D(w\circ\psi)(x), u\rangle.$$

Given $\varphi \in W^{1,\infty}(\mathcal{S})$, multiplying (3.30) by $\varphi\circ\psi$ and integrating by parts, we get

$$\begin{aligned}\int_\Omega a(x)\langle g^{-1}(x) D u_\infty(x); D(\varphi\circ\psi)(x)\rangle\,dx &= \int_\Omega \varphi(\psi(x))(f^-(x)-f^+(x))\,dx\\ &= \int_{\mathcal{S}} \varphi(z)(\tilde f^-(z)-\tilde f^+(z))\,d\mathrm{vol}(z).\end{aligned}$$

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On the other hand, applying two times (3.32), we get

$$\begin{aligned}\int_\Omega a(x)\langle g^{-1}(x)Du_\infty(x); D(\varphi\circ\psi)(x)\rangle\,dx &= \int_\Omega a(x)\langle J_\psi(x)(J_\psi(x)^t J_\psi(x))^{-1}Du_\infty(x); \nabla\varphi(\psi(x))\rangle\,dx\\ &= \int_\Omega a(x)\langle J_{\psi^{-1}}(\psi(x))Du_\infty(x); \nabla\varphi(\psi(x))\rangle\,dx\\ &= \int_\Omega \sqrt{|g|(x)}\tilde a(\psi(x))\langle\nabla\tilde u_\infty(\psi(x)); \nabla\varphi(\psi(x))\rangle\,dx\\ &= \int_S \tilde a(z)\langle\nabla\tilde u_\infty(z); \nabla\varphi(z)\rangle\,d\mathrm{vol}(z).\end{aligned}$$

Consequently,

$$-\operatorname{div}(\tilde a\nabla\tilde u_\infty) = \tilde f^- - \tilde f^+ \quad \text{in the weak sense.}$$

Moreover, by (3.31), if $\tilde a(z) > 0$ then $\langle\nabla\tilde u_\infty(z); \nabla\tilde u_\infty(z)\rangle = 1$. Observe that this is the formulation given in [19].

3.3 Optimal mass transport maps

Let us point out that Feldman and McCann in [19], using Kantorovich potentials, found an optimal transport map $\tilde T_0 : \mathcal{S} \to \mathcal{S}$, which solves the following Monge's problem:

$$\min_{\tilde T\tilde f^+ = \tilde f^-}\left\{\int_{\mathcal{S}} d_{I_M,\mathcal{S}}(\xi, \tilde T(\xi))\tilde f^+(\xi)\,d\mathrm{vol}(\xi)\right\}.$$

Here we have presented a way to obtain Kantorovich potentials by taking limits of p-Laplacian type problems, by using the idea of Evans and Gangbo in [18].

On the existence of optimal transport maps, see also [9], and for Tonelli Lagrangians with superlinear growth, see [20]. The existence of an optimal transport map in Finsler manifolds is obtained in [27] in the case where the Finsler structure is independent of x and for quadratic cost functions. The Lagrangian $F(x, \xi)$ treated here has not superlinear growth.

4 Characterization of the Kantorovich potentials

In this section we shall see that the results obtained in Section 3 characterize the Kantorovich potentials for the transport problem we are dealing here. Similar results have been obtained by Pratelli in [29], with different methods, in the context of Riemannian manifolds and for symmetric Finsler restructures.

Remark 4.1. Observe that the statements (3.5), (3.3) and (3.9) remain true if we assume that $f^- - f^+ = f_p$ with

$$f_p \rightharpoonup f \quad \text{weakly in } L^2(\Omega).$$

Lemma 4.2. *Assume $p > N$. Let $g \in L^2(\Omega)$ with $\int_\Omega g = 0$. Then there exists a solution v_p of*

$$\begin{cases} v_p - \operatorname{div}\Big([F^*(x, Dv_p(x))]^{p-1}\dfrac{\partial F^*}{\partial\xi}(x, Dv_p(x))\Big) = g & \text{in } \Omega,\\ [F^*(x, Dv_p(x))]^{p-1}\Big\langle\dfrac{\partial F^*}{\partial\xi}(x, Dv_p(x)); \eta\Big\rangle = 0 & \text{on } \partial\Omega\end{cases}\tag{4.1}$$

such that for a subsequence $p_j \to \infty$,

$$v_{p_j} \to v_\infty = \boldsymbol{P}_{K_F^*(\Omega)}(g) \quad \text{uniformly in } \Omega,$$

where $\boldsymbol{P}_{K_F^*(\Omega)}$ is the projection in $L^2(\Omega)$ on the convex set $K_F^*(\Omega)$.

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Proof. Working as in the proof of Lemma 2.1, we have that the minimizer v_p of

$$\tilde{\Theta}_{p,g}(v) = \int_{\Omega} \frac{[F^*(x, Dv)]^p}{p} + \frac{1}{2}\int_{\Omega} |v - g|^2$$

is a solution of (4.1). It is easy to see that v_p is bounded in $L^2(\Omega)$, and so there exists a subsequence $p_j \to +\infty$ such that $v_{p_j} \rightharpoonup v_\infty$ weakly in $L^2(\Omega)$.

Now, working as in the proof of (3.5) and (3.3) (see Remark 4.1), we have that

$$v_\infty \in K_F^*(\Omega), \tag{4.2}$$

and also that there exists $\mathcal{X} \in \mathcal{M}(\overline{\Omega}, \mathbb{R}^N)$ such that

$$\int_{\Omega} (g - v_\infty)v = \int_{\overline{\Omega}} Dv\, d\mathcal{X} \quad \text{for all } v \in C^1(\overline{\Omega}). \tag{4.3}$$

On the other hand, working as in the proof of (3.9) (see Remark 4.1), we get

$$\int_{\overline{\Omega}} F(x, \mathcal{X}) = \int_{\Omega} (g - v_\infty)v_\infty.$$

From (4.3), for $v \in K_F^*(\Omega)$, we obtain (after a regularization approach using Lemma 3.8)

$$\int_{\Omega} (g - v_\infty)v \le \int_{\overline{\Omega}} F(x, \mathcal{X}) = \int_{\Omega} (g - v_p)v_\infty. \tag{4.4}$$

Now, (4.2) and (4.4) gives $v_\infty = \boldsymbol{P}_{K_F^*(\Omega)}g$, as we wanted to show. □

Theorem 4.3. *The following assertions are equivalent:*

(1) *u is a Kantorovich potential for the mass transport problem of f^+ to f^-, with the cost being the Finsler distance given in* (1.10).

(2) *$u \in K_{F^*}$, and there exists $\mathcal{X} \in \mathcal{M}(\overline{\Omega}, \mathbb{R}^N)$ satisfying*

$$\begin{cases} \int_\Omega (f^- - f^+)v = \int_{\overline{\Omega}} Dv\, d\mathcal{X} & \textit{for all } v \in C^1(\overline{\Omega}), \\ \int_\Omega (f^- - f^+)u = \int_{\overline{\Omega}} F(x, \mathcal{X}). \end{cases} \tag{C1}$$

(3) *$u \in K_{F^*}$, and there exist $\nu \in \mathcal{M}(\overline{\Omega})^+$ and $\Lambda \in L^1_\nu(\overline{\Omega}, \mathbb{R}^N)$ such that*

$$\begin{cases} \int_\Omega (f^- - f^+)v = \int_{\overline{\Omega}} \Lambda \cdot Dv\, d\nu & \textit{for all } v \in C^1(\overline{\Omega}), \\ \Lambda(x) \in \partial_{F,\nu} u(x) \quad \textit{and} \quad F(x, \Lambda(x)) = 1 & \nu\textit{-a.e. in } \overline{\Omega}. \end{cases} \tag{C2}$$

Proof. First of all observe that

$$u \text{ is a Kantorovich potential} \iff u = \boldsymbol{P}_{K_F^*}(f + u). \tag{4.5}$$

(1) implies (2). Take v_p a weak solution of the following problem of p-Laplacian type:

$$\begin{cases} v_p - \operatorname{div}\Big([F^*(x, Dv_p(x))]^{p-1}\dfrac{\partial F^*}{\partial \xi}(x, Dv_p(x))\Big) = f + u & \text{in } \Omega, \\ [F^*(x, Dv_p(x))]^{p-1}\Big\langle \dfrac{\partial F^*}{\partial \xi}(x, Dv_p(x)); \eta \Big\rangle = 0 & \text{on } \partial\Omega. \end{cases}$$

Then, by Lemma 4.2 and (4.5), we have that

$$\lim_{p\to\infty} v_p(x) = \boldsymbol{P}_{K_{F^*}}(u + f) = u \quad \text{uniformly in } \Omega.$$

Observe also that in the proof of Lemma 4.2, we had $\mathcal{X} \in \mathcal{M}(\overline{\Omega}, \mathbb{R}^N)$ satisfying condition (C1).

(2) implies (1). From (C1), using Lemma 3.8, it is not difficult to see that

$$\int_\Omega (f^- - f^+)v \le \int_{\overline{\Omega}} F(x, \mathcal{X}) = \int (f^- - f^+)u \quad \text{for all } v \in K_{F^*},$$

and thus u is a Kantorovich potential.

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(3) implies (2). If we set $\mathcal{X} := \Lambda\nu$, it is enough to show that

$$\int_{\overline{\Omega}} F(x, \mathcal{X}) = \int_{\overline{\Omega}} F(x, \Lambda)\, d\nu = \int (f^- - f^+)u.$$

By (3.16), there exist smooth functions v_ϵ such that

$$v_\epsilon \to u \quad \text{uniformly}, \qquad D_\nu v_\epsilon \rightharpoonup D_\nu u \quad \sigma(L^\infty_\nu, L^1_\nu).$$

Then, taking $v = v_\epsilon$ in (C2), we have

$$\int_\Omega (f^- - f^+)v_\epsilon = \int_{\overline{\Omega}} \Lambda \cdot Dv_\epsilon\, d\nu = \int_{\overline{\Omega}} \Lambda \cdot D_\nu v_\epsilon\, d\nu,$$

and by taking limits, we get

$$\int_\Omega (f^- - f^+)u = \int_{\overline{\Omega}} \Lambda \cdot D_\nu u\, d\nu. \tag{4.6}$$

Now, working as in the proof of (3.18), we obtain

$$D_\nu u(x) \cdot v(x) \le F(x, v(x)) \quad \nu\text{-a.e.}$$

for any $v(x) \in T_\nu(x)$ ν-a.e. This implies that

$$F(x, \Lambda(x)) = \Lambda(x) \cdot D_\nu u(x) \quad \nu\text{-a.e. in } \overline{\Omega}.$$

Going back to (4.6) and using again (C2), we get

$$\int_\Omega (f^- - f^+)u = \int_{\overline{\Omega}} F(x, \Lambda)\, d\nu.$$

(2) implies (3). Take $\nu = F(x, \mathcal{X})$ and $\Lambda = \frac{d\mathcal{X}}{d\nu}$. We only need to show that

$$\Lambda(x) \in \partial_{F,\nu} u(x) \quad \text{and} \quad F(x, \Lambda(x)) = 1 \quad \nu\text{-a.e. in } \overline{\Omega}.$$

This can be proved as Theorem 3.10, by replacing u_∞ with u. □

5 The Benamou–Brenier approach

Proof of Theorem 1.3. By (3.4), we have that (f, E) is a solution of problem (1.13) for $f(t) := f^+ + t(f^- - f^+)$ and $E(t) := \mathcal{X}_\infty$ for $t \in [0, 1]$. Then, from (3.9), it follows that

$$\min\{J_F(f, E) : (f, E) \text{ is a solution of (1.13)}\} \le |\mathcal{X}_\infty|_F(\overline{\Omega}) = \min\{\mathcal{K}_{c_F}(\mu) : \mu \in \Pi(f^+, f^-)\}.$$

To prove the reverse inequality, consider v_ϵ the approximation given in Lemma 3.8 for $u = u_\infty$. Then, given (f, E) a solution of (1.13), we have

$$\begin{aligned}
\min\{\mathcal{K}_c(\mu) : \mu \in \Pi(f^+, f^-)\} &= \int_\Omega u_\infty (f^- - f^+) = -\int_\Omega \int_0^1 u_\infty \frac{\partial f}{\partial t} \\
&= -\lim_{\epsilon \to 0} \int_\Omega \int_0^1 v_\epsilon \frac{\partial f}{\partial t} = \lim_{\epsilon \to 0} \int_0^1 \int_{\overline{\Omega}} \nabla v_\epsilon\, dE(t) \\
&\le \int_0^1 |E(t)|_F(\overline{\Omega}) \le J_F(f, E),
\end{aligned}$$

and consequently

$$\min\{\mathcal{K}_{c_F}(\mu) : \mu \in \Pi(f^+, f^-)\} \le \min\{J_F(f, E) : (f, E) \text{ is a solution of (1.13)}\}.$$ □

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We say that the Finsler structure F is *geodesically complete* if for any $x, y \in \Omega$ there exists $\sigma_{x,y} \in \Gamma^{\Omega}_{x,y}$ such that

$$c_F(x,y) = \inf_{\sigma\in\Gamma^{\Omega}_{x,y}} \int_0^1 F(\sigma(t),\sigma'(t))\,dt = \int_0^1 F(\sigma_{x,y}(t),\sigma'_{x,y}(t))\,dt.$$

Theorem 5.1. *Assume $F^*(x,\cdot) \in C^1(\mathbb{R}^N \setminus \{0\})$ and also that F is geodesically complete. For any transport plan $\gamma \in \Pi(f^+, f^-)$ we define the measures*

$$f(t) := \pi_t\,\gamma, \quad E(t) := \pi_t(\sigma'_{x,y}(t)\gamma)$$

with $\pi_t(x,y) := \sigma_{x,y}(t)$. Then (f, E) is a solution of (1.13). *Moreover, if γ is an optimal transport plan, then*

$$J_F(f,E) = \min\{\mathcal{K}_{c_F}(\mu) : \mu \in \Pi(f^+, f^-)\}. \tag{5.1}$$

Proof. Let $\gamma \in \Pi(f^+, f^-)$ be a transport plan. Given $\phi \in C^1(\overline{\Omega})$,

$$\frac{d}{dt}\int_{\overline{\Omega}} \phi\,df(t) = \frac{d}{dt}\int_{\overline{\Omega}\times\overline{\Omega}} \phi(\sigma_{x,y}(t))\,d\gamma(x,y) = \int_{\overline{\Omega}\times\overline{\Omega}} \nabla\phi(\sigma_{x,y}(t))\sigma'_{x,y}(t)\,d\gamma(x,y) = \int_{\overline{\Omega}} \nabla\phi\,dE(t),$$

hence (f, E) is a solution of (1.13). Suppose now that γ is an optimal transport plan. Then

$$\begin{aligned}|E(t)|_F(\overline{\Omega}) &= \sup\Big\{\int_{\overline{\Omega}} \Phi\,dE(t) : \Phi \in C(\overline{\Omega},\mathbb{R}^N) \text{ with } \Phi(x) \in B_{F^*(x,\cdot)} \text{ for all } x \in \Omega\Big\}\\ &= \sup\Big\{\int_{\overline{\Omega}\times\overline{\Omega}} \langle\Phi(\sigma_{x,y}(t)),\sigma'_{x,y}(t)\rangle\,d\gamma(x,y) : \Phi \in C(\Omega,\mathbb{R}^N) \text{ with } \Phi(x) \in B_{F^*(x,\cdot)} \text{ for all } x \in \Omega\Big\}.\end{aligned}$$

Now, by (1.3), we have

$$\langle\Phi(\sigma_{x,y}(t)),\sigma'_{x,y}(t)\rangle \le F(\sigma_{x,y}(t),\sigma'_{x,y}(t))F^*(\sigma_{x,y}(t),\Phi(\sigma_{x,y}(t))) \le F(\sigma_{x,y}(t),\sigma'_{x,y}(t)).$$

Thus,

$$|E(t)|_F(\overline{\Omega}) \le \int_{\overline{\Omega}\times\overline{\Omega}} F(\sigma_{x,y}(t),\sigma'_{x,y}(t))\,d\gamma(x,y),$$

and then

$$\begin{aligned}J_F(f,E) &= \int_0^1 |E(t)|_F(\overline{\Omega})\,dt\\ &\le \int_0^1\int_{\overline{\Omega}\times\overline{\Omega}} F(\sigma_{x,y}(t),\sigma'_{x,y}(t))\,d\gamma(x,y)\\ &= \int_{\overline{\Omega}\times\overline{\Omega}} c_F(x,y)\,d\gamma(x,y)\\ &= \min\{\mathcal{K}_{c_F}(\mu) : \mu \in \Pi(f^+, f^-)\}.\end{aligned}$$

Therefore, by Theorem 1.3, we get (5.1). □

6 Extensions to Riemannian manifolds

In this section we briefly comment on the extension of our results to the case in which the optimal transport problem takes place on a Riemannian manifold. For such extension, we use ingredients of the general theory of Sobolev spaces on Riemannian manifolds, and we refer to [21] for details.

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We deal with a Riemannian manifold M of dimension N with a metric tensor g_{ij} and a compatible measure μ (that is, a measure such that the measure of a geodesic ball of radius r is comparable with r^N). The manifold M is assumed to be compact with or without boundary. We also have that $\mathrm{vol}_\mu(M) = \int_M d\mu$ is finite.

On this manifold we have a Finsler structure, that is, a function $F(x, \xi)$ that for each $x \in M$ is a Finsler function on $\xi \in T_x M$. Using the Riemannian inner product in the tangent plane, we can define the dual Finsler structure $F^*(x, \xi)$ (that gives also a Finsler function on $T_x M$ for every $x \in M$). Associated to this Finsler structure, we can define the cost c_F exactly as we did before. Given $x, y \in M$, set

$$\Gamma_{x,y}^M := \{\sigma \in C^1([0,1], M) : \sigma(0) = x,\ \sigma(1) = y\},$$

and define

$$c_F(x, y) := \inf_{\sigma\in\Gamma_{x,y}^M} \int_0^1 F(\sigma(t), \sigma'(t))\, dt. \tag{6.1}$$

Now, our mass transport problem reads as follows: given f^+ and f^- with the same total mass, find T an optimal transport map, that is, a minimizer of

$$\min_{Tf^+=f^-} \int_M c_F(x, T(x)) f^+(x)\, d\mu.$$

In this setting we can consider the following variational problem: for $p > N$, minimize

$$\int_M \frac{[F^*(x, Du)]^p}{p}\, d\mu - \int_M uf\, d\mu.$$

in the set $S_p = \{u \in W^{1,p}(M) : \int_M u\, d\mu = 0\}$. Here, as before, $f = f^- - f^+$.

For minimizers of this functional (that can be proved to exists as in Lemma 2.1), one can show with the same computations of Lemma 2.3 that there exists a subsequence $p_j \to \infty$ such that $u_{p_j} \rightrightarrows u_\infty$ uniformly in M. Moreover, the limit u_∞ is Lipschitz continuous.

In addition, it can be proved, as in Section 3, that u_∞ is a Kantorovich potential for the mass transport problem of f^+ to f^- with cost being the Finsler distance given in (6.1), that is, u_∞ maximizes

$$\int_M v(f^- - f^+)\, d\mu,$$

in the set $K_{c_F}(M) := \{u : M \mapsto \mathbb{R} : u(y) - u(x) \le c_F(x, y)\}$.

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