Elliptic Problem Involving Non-Local Boundary Conditions

Noureddine Igbida[†] and Soma Safimba[‡]

ABSTRACT. In this paper, we study existence and uniqueness of a solution for a nonlinear elliptic problem subject to nonlocal boundary condition. Moreover, we prove the equivalence between this kind of problem and nonlinear problem with very large diffusion around the boundary.

1. Introduction and assumptions

Let Ω be $\mathcal{C}^1-$open bounded domain in $\mathbb{R}^N$, $(N \geq 2)$, $\partial\Omega = \Gamma_D \cup \Gamma_N$ with $\Gamma_D \cap \Gamma_N = \emptyset$ and β a maximal monotone graph in $\mathbb{R} \times \mathbb{R}$. We consider the nonlinear elliptic problem

$$\beta(u) - \nabla \cdot a(x, \nabla u) \ni f \quad \text{in } \Omega, \tag{1.1}$$

where a is a Leray-Lions operator and f is a function in $L^\infty(\Omega)$.

Equation (1.1) has been widely studied in the literature with standard boundary conditions like Dirichlet, Neumann, Robin, etc. (see [**5**], [**3, 4**] and the references therein). In contrast of the standard case where the condition on the boundary is given on the local values of the flux, nonlocal boundary conditions acts on the average of the flux on the boundary. More precisely, we shall ask u to satisfy the condition

$$\rho(u) + \int_{\Gamma_N} a(., \nabla u) \cdot \eta \ni d \quad \text{on } \Gamma_N, \tag{1.2}$$

where η is the unit outward normal vector on $\partial\Omega$, $d \in \mathbb{R}$ is given and ρ is a maximal monotone graph in $\mathbb{R} \times \mathbb{R}$. In addition, we'll assume that u satisfies Dirichlet boundary on Γ_D ; i.e.

$$u = 0, \quad \text{on } \Gamma_D, \tag{1.3}$$

It is not difficult to see that under the conditions (1.3) and (1.2) the problem is ill-posed. To close the problem, we ask u to be a constant function (unknown) on Γ_N. Beside the mathematical interest for the theoretical development of nonlinear PDE, nonlocal boundary condition appears naturally in concrete situations where one can not reach the local values of the flux on the boundary and neither can control it. For instance, this type of boundary condition appears

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[†]E-mail : `noureddine.igbida@unilim.fr` .

[‡]E-mail : SOMA SAFIMBA <somasaf2000@yahoo.fr.

in petroleum engineering model for well modeling in a 3D stratified petroleum reservoir with arbitrary geometry (see [**10**] and [**11**] for details).

More precisely, our aim in this paper is to study existence and uniqueness of a solution to the problem (1.1) subject to the boundary conditions (1.3) and (1.2) in the case where $f \in L^\infty(\Omega)$ and the nonlinearities β and γ satisfy

$$0 \in \beta(0) \cap \rho(0) \tag{1.4}$$

and

$$\mathcal{D}(\beta) = \mathcal{D}(\rho) = Im(\beta) = Im(\rho) = \mathbb{R}. \tag{1.5}$$

Moreover, we prove that the nonlocal boundary condition on Γ_N is closely connected to the problem where the domain Ω is extended around Γ_D and the new extension region is subject to a huge diffusion with an adequate local boundary condition.

In the papers [**10**] and [**11**], the authors considers the case of linear operator. Existence is proved by Schauder theorem and the uniqueness is obtained under more restricted conditions. In this paper, we study the general case of doubly-nonlinear elliptic problem. By using compactness and monotonicity technics we prove existence and uniqueness of a solution. Moreover, we prove that the problem can be handled by reorganizing the nonlocal boundary condition into a large diffusion around the boundary Γ_N.

This paper is organized as follows. In Section 2, we state our main results of existence and uniqueness as well as the equivalence with the problem of large diffusion around Γ_N. In Section 3, we study a regular problem where we proceed by extending the domain Ω around Γ_N, smoothing the nonlinearities β and ρ, and parameterizing the diffusion outside Ω to be proportional to $\dfrac{1}{\epsilon}$, for a given $\epsilon > 0$. In Section 3, we prove that letting $\epsilon \to 0$, we get the existence of a solution to the original problem (1.1) subject to nonlocal boundary condition (1.2). At last, in Section 4, we prove a L^1-contraction principle to show the uniqueness.

2. Main results

Recall that a Leray-Lions type operator is a Caratheodory function $a(x,\xi) : \Omega \times \mathbb{R}^N \longrightarrow \mathbb{R}^N$ (i.e. $a(x,\xi)$ is continuous in ξ for a.e. $x \in \Omega$ and measurable in x for every $\xi \in \mathbb{R}^N$) satisfying, there exists $p \in (1, +\infty)$ such that :

- There exists $C > 0$ such that, for any $\xi \in \mathbb{R}^N$

$$|a(x,\xi)| \leq C(j(x) + |\xi|^{p-1}), \quad \text{for a.e. } x \in \Omega, \tag{2.1}$$

  where $j \in L^{p'}(\Omega)$ with $\dfrac{1}{p} + \dfrac{1}{p'} = 1$.

- For any $\xi, \eta \in \mathbb{R}^N$, with $\xi \neq \eta$, we have

$$(a(x,\xi) - a(x,\eta)) \cdot (\xi - \eta) > 0, \quad \text{for a.e. } x \in \Omega. \tag{2.2}$$

- There exists $C' > 0$ such that, for any $\xi \in \mathbb{R}^N$

$$\frac{1}{C'}|\xi|^p \leq a(x,\xi).\xi \quad \text{for a.e. } x \in \Omega. \tag{2.3}$$

Throughout the paper, we assume that β and γ are maximal monotone graphs in $\mathbb{R}\times\mathbb{R}$ satisfying the conditions (1.4) and (1.5). Let us consider the problem

$$S^{\beta,\rho}_{f,d} \qquad \begin{cases} \beta(u) - \nabla\cdot a(x,\nabla u) \ni f & \text{in } \Omega,\\ u = 0 & \text{on } \Gamma_D\\ \left.\begin{array}{l} \rho(u) + \displaystyle\int_{\Gamma_N} a(.,\nabla u)\cdot\eta \ni d \\ u \equiv \text{cste} \end{array}\right\} & \text{on } \Gamma_N \end{cases}$$

where $f \in L^\infty(\Omega)$ and $d \in \mathbb{R}$ are given. To study this problem we consider the following functionals spaces :

$$W^{1,p}_D(\Omega) = \Big\{\varphi \in W^{1,p}(\Omega)/\varphi = 0 \text{ on } \Gamma_D\Big\} \text{ and } W^{1,p}_N(\Omega) = \Big\{\varphi \in W^{1,p}_D(\Omega)/\varphi \equiv \text{cste on } \Gamma_N\Big\}.$$

Moreover, for any $v \in W^{1,p}_N(\Omega)$, we set

$$v_N := v|_{\Gamma_N} \quad \text{(in the sense of trace on } \Gamma_N).$$

The concept of solution for $S^{\beta,\rho}_{f,d}$ is given as follow:

Definition 2.1. *A solution of $S^{\beta,\rho}_{f,d}$ is a triplet $(u,w,v) \in W^{1,p}_N(\Omega)\times L^1(\Omega)\times\mathbb{R}$ satisfying*

$$(2.4) \qquad \begin{cases} w \in \beta(u) \text{ a.e. in } \Omega,\ v \in \rho(u_N) \text{ and}\\ \displaystyle\int_\Omega a(x,\nabla u)\cdot\nabla\varphi dx + \int_\Omega w\varphi dx = \int_\Omega f\varphi dx + \Big(d - v\Big)\varphi_N,\\ \text{for any } \varphi \in W^{1,p}_N(\Omega)\cap L^\infty(\Omega). \end{cases}$$

Our first main result is the following theorem :

Theorem 2.1. *For any $f \in L^\infty(\Omega)$ and $d \in \mathbb{R}$ the problem $S^{\beta,\rho}_{f,d}$ admits a solution (u,w,v) in the sense of Definition 2.1. Moreover, if (u_1,w_1,v_1) and (u_2,w_2,v_2) are two solutions of $S^{\beta,\rho}_{f,d}$, then*

$$\begin{cases} w_1 = w_2 \text{ a.e. in } \Omega,\\ v_1 = v_2. \end{cases}$$

Now, for a fixed arbitrary $\delta > 0$, we consider the open bounded domain $\widetilde{\Omega} \supset \Omega$, given by

$$\widetilde{\Omega} = \Omega \cup \Big\{x \in \mathbb{R}^N / \operatorname{dist}(x,\Gamma_N) < \delta\Big\}.$$

Here, we consider $\delta > 0$ small enough such that $\partial\widetilde{\Omega}$ is Lipschitz and

$$(2.5) \qquad \Gamma_D \subset \partial\widetilde{\Omega}.$$

Then, let us denote by

$$\widetilde{\Gamma}_N = \partial\widetilde{\Omega}\setminus\Gamma_D.$$

In the extended domain $\widetilde{\Omega}$, we consider the following nonlinear elliptic problem with standard (local) nonlinear boundary condition

$$P_\epsilon(\widetilde{\beta}_\epsilon, \widetilde{\rho}_\epsilon, \widetilde{f}, \widetilde{d}) \qquad \begin{cases} \widetilde{\beta}_\epsilon(x, u_\epsilon) - \nabla \cdot \widetilde{a}(x, \nabla u_\epsilon) = \widetilde{f} & \text{in } \widetilde{\Omega}, \\ u_\epsilon = 0 & \text{on } \Gamma_D \\ \widetilde{\rho}_\epsilon(u_\epsilon) + \widetilde{a}(x, \nabla u_\epsilon).\eta = \widetilde{d} & \text{on } \widetilde{\Gamma}_N, \end{cases}$$

where the modified Leray-Lions operator is given by

$$\widetilde{a}(x,\xi) := a(x,\xi)\chi_\Omega(x) + \frac{1}{\epsilon^p}|\xi|^{p-2}\xi\chi_{\widetilde{\Omega}\setminus\Omega}(x),\ \forall (x,\xi) \in \widetilde{\Omega} \times \mathbb{R}^N.$$

The non linearities $\widetilde{\beta}_\epsilon : \widetilde{\Omega} \times \mathbb{R} \to \mathbb{R}$ and $\widetilde{\rho}_\epsilon : \widetilde{\Omega} \times \mathbb{R} \to \mathbb{R}$ are given by

$$\widetilde{\beta}_\epsilon(x,s) = \beta_\epsilon(s)\chi_\Omega(x),\ \forall (x,s) \in \widetilde{\Omega} \times \mathbb{R},$$

and

$$\widetilde{\rho}_\epsilon(s) = \frac{1}{|\widetilde{\Gamma}_N|}\rho_\epsilon(s),\ \forall s \in \mathbb{R}$$

where β_ϵ and ρ_ϵ are the regularized Yosida approximation of β and ρ respectively, and $|\widetilde{\Gamma}_N|$ denotes the $(N-1)-$ Hausdorff measure of $\widetilde{\Gamma}_N$. The function $\widetilde{d} \in L^\infty(\widetilde{\Gamma}_N)$ is given such that

$$\int_{\widetilde{\Gamma}_N} \widetilde{d} = d, \tag{2.6}$$

and $\widetilde{f}$ is given by

$$\widetilde{f} = f\chi_\Omega.$$

Using standard arguments, one can prove that the problem $P_\epsilon(\widetilde{\beta}_\epsilon, \widetilde{\rho}_\epsilon, \widetilde{f}, \widetilde{d})$ has at least one solution u_ϵ in the sense that

$$\begin{cases} u_\epsilon \in W^{1,p}_D(\widetilde{\Omega}),\ \beta_\epsilon(u_\epsilon) \in L^1(\Omega),\ \widetilde{\rho}_\epsilon(u_\epsilon) \in L^1(\Gamma_N) \text{ and} \\ \displaystyle\int_{\widetilde{\Omega}} \widetilde{a}(x, \nabla u_\epsilon) \cdot \nabla\widetilde{\varphi} dx + \int_\Omega \beta_\epsilon(u_\epsilon)\widetilde{\varphi} dx = \int_\Omega f\widetilde{\varphi} dx + \int_{\widetilde{\Gamma}_N} \Big(\widetilde{d} - \widetilde{\rho}_\epsilon(u_\epsilon)\Big)\widetilde{\varphi} \\ \text{for any } \widetilde{\varphi} \in W^{1,p}_D(\widetilde{\Omega}) \cap L^\infty(\Omega). \end{cases} \tag{2.7}$$

Formally, one sees here that letting ϵ goes to 0, the nonlinearity a_ε becomes very large on the region $\widetilde{\Omega} \setminus \Omega$, and forces u to be constant on this region. Thanks to (2.5), this implies that u is a constant function on Γ_N and we prove that u satisfies the condition (1.2). More precisely, we have the following result

THEOREM 2.2. *Let $f \in L^\infty(\Omega)$, $d \in \mathbb{R}$ and, for any $\epsilon > 0$, let us consider u_ϵ a solution of the problem $P_\epsilon(\widetilde{\beta}_\epsilon, \widetilde{\rho}_\epsilon, \widetilde{f}, \widetilde{d})$. As $\varepsilon \to 0$, up to a subsequence if necessary, we have*

$$u_\varepsilon \overset{*}{\rightharpoonup} u \text{ and } \beta_\varepsilon(u_\varepsilon) \overset{*}{\rightharpoonup} w \quad \text{in } \ L^\infty(\Omega)$$

$$\widetilde{\rho}_\varepsilon(u_\varepsilon) \overset{*}{\rightharpoonup} \widetilde{v} \quad \text{in } L^\infty(\widetilde{\Gamma}_N),$$

and the triplet (u, w, v) is a solutions of $S^{\beta,\rho}_{f,d}$ where $v := \displaystyle\int_{\widetilde{\Gamma}_N} \widetilde{v} d\sigma$.

3. The approximated problem corresponding to $S^{\beta,\rho}_{f,d}$

Our aim here is to study the problem $P_\epsilon(\widetilde{\beta}_\epsilon, \widetilde{\rho}_\epsilon, \widetilde{f}, \widetilde{d})$ for a given fixed $\epsilon >$ and $\delta > 0$. We prove the following result

PROPOSITION 3.1. *For any $f \in L^\infty(\Omega)$ and $\epsilon > 0$, the problem $P_\epsilon(\widetilde{\beta}_\epsilon, \widetilde{\rho}_\epsilon, \widetilde{f}, \widetilde{d})$ has at least one solution in the sense of (2.7). Moreover, we have*

$$\left\{ \begin{array}{l} |\beta_\epsilon(u_\epsilon)| \le \theta_1 := \max\left\{ \|f\|_\infty, (\beta_\epsilon \circ \rho_\epsilon^{-1}) \left(|\widetilde{\Gamma}_N| \|\widetilde{d}\|_\infty \right) \right\} \ a.e. \ in \ \Omega \\ and \\ |\widetilde{\rho}_\epsilon(u_\epsilon)| \le \theta_2 := \max\left\{ \|\widetilde{d}\|_\infty, (\widetilde{\rho}_\epsilon \circ \beta_\epsilon^{-1}) (\|f\|_\infty) \right\} \ a.e. \ on \ \widetilde{\Gamma}_N. \end{array} \right. \tag{3.1}$$

To simplify the presentation, we withdraw the subscript ϵ and δ throughout this section. To prove Proposition 3.1, we consider the problem

$$P_k(\widetilde{\beta}, \widetilde{\rho}, \widetilde{f}, \widetilde{d}) \qquad \left\{ \begin{array}{ll} T_k\left(\widetilde{\beta}(x, u_k)\right) - \nabla \cdot \widetilde{a}(x, \nabla u_k) = \widetilde{f} & \text{in } \widetilde{\Omega}, \\ \\ u_k = 0 & \text{on } \Gamma_D \\ \\ T_k\left(\widetilde{\rho}(u_k)\right) + \widetilde{a}(x, \nabla u_k).\eta = \widetilde{d} & \text{on } \widetilde{\Gamma}_N \end{array} \right.$$

where the truncation function T_k is define as

$$\forall s \in \mathbb{R}, \ T_k(s) = \left\{ \begin{array}{l} -k \text{ if } s < -k, \\ s \text{ if } |s| \le k, \\ k \text{ if } s > k. \end{array} \right.$$

We have

PROPOSITION 3.2. *Assume $\widetilde{\beta}, \widetilde{\rho}, \widetilde{f}$ and $\widetilde{d}$ as above. Then for any $k > 0$ the problem $P_k(\widetilde{\beta}, \widetilde{\rho}, \widetilde{f}, \widetilde{d})$ admits at least one solution u_k in the sense that $u_k \in W^{1,p}_D(\widetilde{\Omega})$ and*

$$\int_{\widetilde{\Omega}} \widetilde{a}(x, \nabla u_k).\nabla \widetilde{\varphi} dx + \int_\Omega T_k\left(\beta(u_k)\right) \widetilde{\varphi} dx = \int_\Omega f \widetilde{\varphi} dx + \int_{\widetilde{\Gamma}_N} \left(\widetilde{d} - T_k\left(\widetilde{\rho}(u_k)\right) \right) \widetilde{\varphi}, \tag{3.2}$$

for any $\widetilde{\varphi} \in W^{1,p}_D(\widetilde{\Omega})$. Furthermore, for any k large enough

$$|\beta(u_k)| \le \theta_1 \ a.e. \ in \qquad and \qquad |\widetilde{\rho}(u_k)| \le \theta_2 \ a.e. \ on \ \widetilde{\Gamma}_N. \tag{3.3}$$

Proof. For any $k>0$, let us introduce the following operator $\Lambda_k : W^{1,p}_D(\widetilde{\Omega}) \longrightarrow \Big(W^{1,p}_D(\widetilde{\Omega})\Big)'$ given by, for any $(u,v)\in W^{1,p}_D(\widetilde{\Omega})\times W^{1,p}_D(\widetilde{\Omega})$,

$$\langle \Lambda_k(u),v\rangle = \int_{\widetilde{\Omega}} \widetilde{a}(x,\nabla u).\nabla v dx + \int_{\Omega} T_k\left(\beta(u)\right) v dx + \int_{\widetilde{\Gamma}_N} T_k\left(\widetilde{\rho}(u)\right) v d\sigma. \tag{3.4}$$

Let us prove that, for any $k>0$, Λ_k is surjective. To this aim, it is enough to show that the operator Λ_k is bounded, coercive and of type M, that is if $u_n \rightharpoonup u$ in $W^{1,p}_D(\widetilde{\Omega})$, $\Lambda_k(u_n)\rightharpoonup \chi_k$ in $\Big(W^{1,p}_D(\widetilde{\Omega})\Big)'$, $\limsup\limits_{n\to+\infty}\langle \Lambda_k(u_n),u_n\rangle$ and $\leq \langle \chi_k,u\rangle$, then $\Lambda_k(u)=\chi_k$.

(i) Boundedness of Λ_k**.** For any $(u,v)\in W^{1,p}_D(\widetilde{\Omega})\times W^{1,p}_D(\widetilde{\Omega})$, using (2.1), we have

$$\begin{aligned}
\Big|\langle \Lambda_k(u),v\rangle\Big| &\leq \int_{\widetilde{\Omega}} |\widetilde{a}(x,\nabla u)||\nabla v| dx + k\int_{\Omega}|v|dx + k\int_{\widetilde{\Gamma}_N}|v|d\sigma.\\
&\leq \int_{\widetilde{\Omega}} |\widetilde{a}(x,\nabla u)||\nabla v| dx + kC_1(|\Omega|,p)\|v\|_{L^p(\Omega)} + kC_2(|\widetilde{\Gamma}_N|,p)\|v\|_{L^p(\widetilde{\Gamma}_N)}\\
&\leq \int_{\widetilde{\Omega}} C\Big(j(x)+|\nabla u|^{p-1}\Big)|\nabla v|\, dx + kC_1(|\Omega|,p)\|v\|_{L^p(\Omega)}\\
&\quad + kC_2(|\widetilde{\Gamma}_N|,p)\|v\|_{L^p(\widetilde{\Gamma}_N)}\\
&\leq \int_{\widetilde{\Omega}} Cj(x)|\nabla v|\, dx + \int_{\widetilde{\Omega}} C|\nabla u|^{p-1}|\nabla v|\, dx + kC_1(|\Omega|,p)\|v\|_{L^p(\Omega)}\\
&\quad + kC_2(|\widetilde{\Gamma}_N|,p)\|v\|_{L^p(\widetilde{\Gamma}_N)}\\
&\leq C_3(j,p)\|\nabla v\|_{L^p(\widetilde{\Omega})} + C_4\|\nabla u\|_{L^p(\widetilde{\Omega})}\|\nabla v\|_{L^p(\widetilde{\Omega})} + kC_1(|\Omega|,p)\|v\|_{L^p(\Omega)}\\
&\quad + kC_2(|\widetilde{\Gamma}_N|,p)\|v\|_{L^p(\widetilde{\Gamma}_N).}
\end{aligned}$$

Using the continuous injection of $W^{1,p}_D(\widetilde{\Omega})$ into $L^p(\widetilde{\Gamma}_N)$, we deduce that

$$\begin{aligned}
\Big|\langle \Lambda_k(u),v\rangle\Big| &\leq C_3(j,p)\|v\|_{W^{1,p}_D(\widetilde{\Omega})} + C_4\|u\|_{W^{1,p}_D(\widetilde{\Omega})}\|v\|_{W^{1,p}_D(\widetilde{\Omega})} + kC_1(|\Omega|,p)\|v\|_{W^{1,p}_D(\widetilde{\Omega})}\\
&\quad + kC_2(|\widetilde{\Gamma}_N|,p)\|v\|_{W^{1,p}_D(\widetilde{\Omega})}\\
&\leq \Big(C_3(j,p)+C_4\|u\|_{W^{1,p}_D(\widetilde{\Omega})} + kC_1(|\Omega|,p) + kC_2(|\widetilde{\Gamma}_N|,p)\Big)\|v\|_{W^{1,p}_D(\widetilde{\Omega})}.
\end{aligned}$$

This implies that Λ_k maps bounded subsets of $W^{1,p}_D(\widetilde{\Omega})$ to bounded subsets of $\Big(W^{1,p}_D(\widetilde{\Omega})\Big)'$. Then Λ_k is bounded on $W^{1,p}_D(\widetilde{\Omega})$.

(ii) Coerciveness of Λ_k. For any $k>0$, let us prove that

$$\frac{\langle \Lambda_k(u),u\rangle}{\|u\|_{W^{1,p}_D(\widetilde{\Omega})}} \longrightarrow +\infty \text{ as } \|u\|_{W^{1,p}_D(\widetilde{\Omega})} \longrightarrow +\infty.$$

For any $u \in W_D^{1,p}(\widetilde{\Omega})$, we have

$$\langle \Lambda_k(u), u\rangle = \int_{\widetilde{\Omega}} a(x,\nabla u).\nabla u dx + \int_{\Omega} T_k\left(\beta(u)\right) u dx + \int_{\widetilde{\Gamma}_N} T_k\left(\widetilde{\rho}(u)\right) u d\sigma. \tag{3.5}$$

In one hand, we see that the two last terms in the right hand side of (3.5) are nonnegative. On the other hand, using the assumption (2.3), we obtain

$$\int_{\widetilde{\Omega}} a(x,\nabla u).\nabla u dx \geq \frac{1}{C'}\|\nabla u\|^p_{L^p(\widetilde{\Omega})}$$

Therefore, (3.5) implies that

$$\langle \Lambda_k(u), u\rangle \quad \geq \frac{1}{C'}\|\nabla u\|^p_{L^p(\widetilde{\Omega})}.$$

Since $u \in W_D^{1,p}(\widetilde{\Omega})$, by using Poincaré's inequality, we deduce that Λ_k is coercive.

(iii)The operator Λ_k verifies the M-property. To this aim, we use the following known lemma.

Lemma 3.1. *(cf. [**16**])*
Let $\mathcal{A}$ and $\mathcal{B}$ be two operators. If $\mathcal{A}$ is of type M and $\mathcal{B}$ is monotone and weakly continuous, then $\mathcal{A}+\mathcal{B}$ is of type M.

We see that setting

$$\langle \mathcal{A}u, v\rangle := \int_{\widetilde{\Omega}} \widetilde{a}(x,\nabla u).\nabla v dx \text{ and } \langle \mathcal{B}_k u, v\rangle := \int_{\Omega} T_k\left(\beta(u)\right) v dx + \int_{\widetilde{\Gamma}_N} T_k\left(\widetilde{\rho}(u)\right) v d\sigma,$$

we have

$$\Lambda_k = \mathcal{A} + \mathcal{B}_k, \qquad \text{for any } k > 0.$$

In one hand, it is well known that the operator $\mathcal{A}$ is of the type M. On the other, using the monotonicity of β, $\widetilde{\rho}$ and the map T_k, it is clear that $\mathcal{B}_k$ is monotone, for any $k>0$. Moreover, for any sequence $(u_n)_{n\in\mathbb{N}} \subset W_D^{1,p}(\widetilde{\Omega})$ such that $u_n \rightharpoonup u$ in $W_D^{1,p}(\widetilde{\Omega})$, we see that $\mathcal{B}_k u_n \rightharpoonup \mathcal{B}_k u$, as $n \to +\infty$, $\forall k > 0$. Thus the operator $\mathcal{B}_k$ is weakly continuous.
Indeed, let $\phi \in W_D^{1,p}(\widetilde{\Omega})$. In one hand, we have

$$\lim_{n\longrightarrow+\infty} \int_{\Omega} T_k\left(\beta(u_n)\right)\phi dx = \int_{\Omega} T_k\left(\beta(u)\right)\phi dx. \tag{3.6}$$

On the other hand, since $u_n \rightharpoonup u$ in $W_D^{1,p}(\widetilde{\Omega})$, up to a subsequence, we have $u_n \longrightarrow u$ in $L^p(\partial\widetilde{\Omega})$ and a.e on $\partial\widetilde{\Omega}$ and we deduce

$$\lim_{n\longrightarrow+\infty} \int_{\widetilde{\Gamma}_N} T_k\left(\widetilde{\rho}(u_n)\right)\phi d\sigma = \int_{\widetilde{\Gamma}_N} T_k\left(\widetilde{\rho}(u)\right)\phi d\sigma. \tag{3.7}$$

From (3.6) and (3.7), it yields

$$\forall k>0, \lim_{n\longrightarrow+\infty} \langle \mathcal{B}_k u_n, \phi\rangle = \langle \mathcal{B}_k u, \phi\rangle, \text{ which means that } \mathcal{B}_k u_n \rightharpoonup \mathcal{B}_k u.$$

So, by using Lemma 3.1, we conclude that the operator Λ_k is the type M. This implies that, for any $L \in \left(W_D^{1,p}(\widetilde{\Omega})\right)'$, there exists $u_k \in W_D^{1,p}(\widetilde{\Omega})$ such that $\Lambda_k(u_k) = L$. Taking $L \in \left(W_D^{1,p}(\widetilde{\Omega})\right)'$

defined by $L(v) := \displaystyle\int_\Omega fv dx + \int_{\widetilde{\Gamma}_N} \widetilde{d}v \ d\sigma, \ \forall v \in W^{1,p}_D(\widetilde{\Omega})$, we deduce the existence of a solution of $P_k(\widetilde{\beta}, \widetilde{\rho}, \widetilde{f}, \widetilde{d})$. Now, let us prove (3.3).

Let us introduce the following function H_ϵ for any $\epsilon > 0$:

$$\forall s \in \mathbb{R}, \ H_\epsilon(s) = \begin{cases} 0 \text{ if } s < 0, \\ \dfrac{s}{\epsilon} \text{ if } 0 \le s \le \epsilon, \\ 1 \text{ if } s > \epsilon. \end{cases}$$

In (3.2), we set $\widetilde{\varphi} = H_\epsilon(u_k - M), \ \epsilon > 0$ where $M > 0$ is a constant to be fixed later. We get

$$\begin{aligned} &\int_{\widetilde{\Omega}} \widetilde{a}(x, \nabla u_k).\nabla H_\epsilon(u_k - M)dx + \int_\Omega T_k\left(\beta(u_k)\right) H_\epsilon(u_k - M)dx = \int_\Omega fH_\epsilon(u_k - M)dx \\ &+ \int_{\widetilde{\Gamma}_N} \Big(\widetilde{d} - T_k\left(\widetilde{\rho}(u_k)\right)\Big) H_\epsilon(u_k - M) \ d\sigma. \end{aligned} \tag{3.8}$$

It is not difficult to see that the first term in (3.8) is non negative. This implies that

$$\int_\Omega T_k\left(\beta(u_k)\right) H_\epsilon(u_k - M)dx \le \int_\Omega fH_\epsilon(u_k - M)dx + \int_{\widetilde{\Gamma}_N} \Big(\widetilde{d} - T_k\left(\widetilde{\rho}(u_k)\right)\Big) H_\epsilon(u_k - M) \ d\sigma,$$

and

$$\begin{aligned} &\int_\Omega \Big(T_k\left(\beta(u_k)\right) - T_k\left(\beta(M)\right)\Big) H_\epsilon(u_k - M)dx + \int_{\widetilde{\Gamma}_N} \Big(T_k\left(\widetilde{\rho}(u_k)\right) - T_k\left(\widetilde{\rho}(M)\right)\Big) H_\epsilon(u_k - M) \ d\sigma \\ &\le \int_\Omega \Big(f - T_k\left(\beta(M)\right)\Big) H_\epsilon(u_k - M)dx + \int_{\widetilde{\Gamma}_N} \Big(\widetilde{d} - T_k\left(\widetilde{\rho}(M)\right)\Big) H_\epsilon(u_k - M) \ d\sigma. \end{aligned}$$

Letting ε goes to 0, we get

$$\begin{aligned} &\int_\Omega \Big(T_k\left(\beta(u_k)\right) - T_k\left(\beta(M)\right)\Big) \mathrm{sign}_0^+(u_k - M)dx + \int_{\widetilde{\Gamma}_N} \Big(T_k\left(\widetilde{\rho}(u_k)\right) - T_k\left(\widetilde{\rho}(M)\right)\Big) \mathrm{sign}_0^+(u_k - M) \ d\sigma \\ &\le \int_\Omega \Big(f - T_k\left(\beta(M)\right)\Big) \mathrm{sign}_0^+(u_k - M)dx + \int_{\widetilde{\Gamma}_N} \Big(\widetilde{d} - T_k\left(\widetilde{\rho}(M)\right)\Big) \mathrm{sign}_0^+(u_k - M) \ d\sigma, \end{aligned}$$

which is equivalent to

$$\begin{aligned} &\int_\Omega \Big(T_k\left(\beta(u_k)\right) - T_k\left(\beta(M)\right)\Big)^+ dx + \int_{\widetilde{\Gamma}_N} \Big(T_k\left(\widetilde{\rho}(u_k)\right) - T_k\left(\widetilde{\rho}(M)\right)\Big)^+ \ d\sigma \\ &\le \int_\Omega \Big(f - T_k\left(\beta(M)\right)\Big) \mathrm{sign}_0^+(u_k - M)dx + \int_{\widetilde{\Gamma}_N} \Big(\widetilde{d} - T_k\left(\widetilde{\rho}(M)\right)\Big) \mathrm{sign}_0^+(u_k - M) \ d\sigma. \end{aligned}$$

Now, thanks to (1.5), we can take $M = M_0 := \max\Big\{\beta^{-1}(\|f\|_\infty), \rho^{-1}\Big(|\widetilde{\Gamma}_N|\|\widetilde{d}\|_\infty\Big)\Big\}$. Then, for any $k > \max\Big\{\|f\|_\infty, \|\widetilde{d}\|_\infty\Big\}$, it follows that

$$\int_\Omega \Big(T_k\left(\beta(u_k)\right) - T_k\left(\beta(M_0)\right)\Big)^+ dx + \int_{\widetilde{\Gamma}_N} \Big(T_k\left(\widetilde{\rho}(u_k)\right) - T_k\left(\widetilde{\rho}(M_0)\right)\Big)^+ \ d\sigma \le 0. \tag{3.9}$$

This yields

$$\left\{\begin{array}{l} T_k\left(\beta(u_k)\right) \leq T_k\left(\beta(M_0)\right) \text{ a.e. in } \Omega \\ \text{ and} \\ T_k\left(\widetilde{\rho}(u_k)\right) \leq T_k\left(\widetilde{\rho}(M_0)\right) \text{ a.e. on } \widetilde{\Gamma}_N. \end{array}\right. , \tag{3.10}$$

So, for any $k > k_0 := \max\left\{\|f\|_\infty, \|\widetilde{d}\|_\infty, \beta(M_0), \widetilde{\rho}(M_0)\right\}$,

$$\beta(u_k) \leq \beta(M_0) \text{ a.e. in } \Omega \quad \text{ and } \quad \widetilde{\rho}(u_k) \leq \widetilde{\rho}(M_0) \text{ a.e. on } \widetilde{\Gamma}_N.$$

In particular, this implies that

$$\left\{\begin{array}{l} \beta(u_k) \leq \max\left\{\|f\|_\infty, (\beta\circ\rho^{-1})\left(|\widetilde{\Gamma}_N|\|\widetilde{d}\|_\infty\right)\right\} \text{ a.e. in } \Omega \\ \text{and} \\ \widetilde{\rho}(u_k) \leq \max\left\{\|\widetilde{d}\|_\infty, (\widetilde{\rho}\circ\beta^{-1})\left(\|f\|_\infty\right)\right\} \text{ a.e. on } \widetilde{\Gamma}_N. \end{array}\right. \tag{3.11}$$

At last, one see that $(-u_k)$ is a solution of

$$P_k(\widehat{\beta}, \widehat{\rho}, \widehat{f}, \widehat{d}\,) \left\{\begin{array}{ll} T_k\left(\widehat{\beta}(x,u)\right) - \nabla\cdot\widehat{a}(x,\nabla u) = \widehat{f} & \text{in } \widetilde{\Omega}, \\ u = 0 & \text{on } \Gamma_D \\ T_k\left(\widehat{\rho}(u)\right) + \widehat{a}(x,\nabla u).\eta = \widehat{d} & \text{on } \widetilde{\Gamma}_N. \end{array}\right.$$

where

$$\widehat{a}(x,\xi) = -\widetilde{a}(x,-\xi),\ \widehat{\beta}(x,s) = -\widetilde{\beta}(x,-s),\ \widehat{\rho}(s) = -\widetilde{\rho}(-s),\ \widehat{f} = -\widetilde{f} \text{ and } \widehat{d} = -\widetilde{d}.$$

So, using the same arguments we deduce that for k large enough, we have

$$\left\{\begin{array}{l} \beta(u_k) \geq -\max\left\{\|f\|_\infty, (\beta\circ\rho^{-1})\left(|\widetilde{\Gamma}_N|\|\widetilde{d}\|_\infty\right)\right\} \text{ a.e. in } \Omega \\ \widetilde{\rho}(u_k) \geq -\max\left\{\|\widetilde{d}\|_\infty, (\widetilde{\rho}\circ\beta^{-1})\left(\|f\|_\infty\right)\right\} \text{ a.e. on } \widetilde{\Gamma}_N. \end{array}\right. \tag{3.12}$$

Thus (3.3). $\square$

Proof of Proposition 3.2. Considering $k = 1 + \max\{\theta_1, \theta_2\}$, and using (3.3) we see that a solution u_k of $P_k(\widetilde{\beta}, \widetilde{\rho}, \widetilde{f}, \widetilde{d})\Big)$ given by Proposition 3.2 is a solution of $P(\widetilde{\beta}, \widetilde{\rho}, \widetilde{f}, \widetilde{d})\Big)$ satisfying (3.1). This ends up the proof of Proposition 3.2.

$\square$

4. Large diffusion : letting $\epsilon \to 0$

Now, coming back to the problem $P_\epsilon(\widetilde{\beta}_\varepsilon, \widetilde{\rho}_\varepsilon, \widetilde{f}, \widetilde{d})$, our aim is to pass to the limit as $\epsilon \to 0$. Thanks to Proposition 3.1, there exists a measurable function $u_\epsilon : \widetilde{\Omega} \longrightarrow \mathbb{R}$ such that

$$(4.1)\qquad \begin{cases} u_\epsilon \in W_D^{1,p}(\widetilde{\Omega}),\ \beta_\varepsilon(u_\epsilon) \in L^1(\Omega) \text{ and } \forall \widetilde{\varphi} \in W_D^{1,p}(\widetilde{\Omega}) \cap L^\infty(\Omega), \\ \displaystyle\int_\Omega a(x, \nabla u_\epsilon).\nabla\widetilde{\varphi}dx + \int_{\widetilde{\Omega}\setminus\Omega} \frac{1}{\epsilon^p}|\nabla u_\epsilon|^{p-2}\nabla u_\epsilon.\nabla\widetilde{\varphi}dx + \int_\Omega \beta_\varepsilon(u_\epsilon)\widetilde{\varphi}dx \\ \qquad\qquad = \displaystyle\int_\Omega f\widetilde{\varphi}dx + \int_{\widetilde{\Gamma}_N} \Big(\widetilde{d} - \widetilde{\rho}_\varepsilon(u_\epsilon)\Big)\widetilde{\varphi}\, d\sigma. \end{cases}$$

Moreover by (3.3) we have

$$(4.2)\qquad \begin{cases} |\beta_\varepsilon(u_\varepsilon)| \le \theta_3 := \max\Big\{\|f\|_\infty, (\beta_\varepsilon \circ \rho_\varepsilon^{-1})\Big(|\widetilde{\Gamma}_N| \|\widetilde{d}\|_\infty\Big)\Big\} \text{ a.e. in } \Omega \\ \text{and} \\ |\widetilde{\rho}_\varepsilon(u_\varepsilon)| \le \theta_4 := \max\Big\{\|\widetilde{d}\|_\infty, (\widetilde{\rho}_\varepsilon \circ \beta_\varepsilon^{-1})\Big(\|f\|_\infty\Big)\Big\} \text{ a.e. on } \widetilde{\Gamma}_N. \end{cases}$$

First, we see that

Lemma 4.2. *There exists $C_1, C_2, C_3, C_4 \in I\!R^+$ independent of ϵ, such that*

$$(4.3)\qquad \begin{cases} |\beta_\varepsilon(u_\varepsilon)| \le C_1 \textit{ a.e. in } \Omega \textit{ and } |\widetilde{\rho}_\varepsilon(u_\varepsilon)| \le C_2 \textit{ a.e. on } \widetilde{\Gamma}_N \\ \\ |u_\varepsilon| \le C_3 \textit{ a.e. in } \Omega \textit{ and } |u_\varepsilon| \le C_4 \textit{ a.e. on } \widetilde{\Gamma}_N. \end{cases}$$

Proof. Thanks to the assumption (1.5), it clear that $\theta_3 = (\beta_\varepsilon \circ \rho_\varepsilon^{-1})\Big(|\widetilde{\Gamma}_N| \|\widetilde{d}\|_\infty\Big)$ and $\theta_4 = \big(\widetilde{\rho}_\varepsilon \circ \beta_\varepsilon^{-1}\big)\Big(\|f\|_\infty\Big)$ are bounded. This follows from the fact that the assumptions (1.5) implies that ρ_ε^{-1} and β_ε^{-1} are bounded in bounded sets of $I\!R$. Using the same arguments combined with the fact that

$$(4.4)\qquad \begin{cases} |u_\varepsilon| \le \theta_5 := \max\left\{\beta_\varepsilon^{-1}(M_1), -\beta_\varepsilon^{-1}(-M_1)\right\} \text{ a.e. in } \Omega \\ \\ |u_\varepsilon| \le \theta_6 := \max\left\{\widetilde{\rho}_\varepsilon^{-1}(M_2), -\widetilde{\rho}_\varepsilon^{-1}(-M_2)\right\} \text{ a.e. on } \widetilde{\Gamma}_N \end{cases}$$

where $M_1 := \max\{\|f\|_\infty, C_1\}$ and $M_2 := \max\Big\{\|\widetilde{d}\|_\infty, C_2\Big\}$, we deduce the result of the lemma. $\square$

Remark 4.1. *If u_ϵ is a solution of $P_\varepsilon(\widetilde{\beta}_\varepsilon, \widetilde{\rho}_\varepsilon, \widetilde{f}, \widetilde{d})$, then, by using test functions $\widetilde{\varphi} \in W_N^{1,p}(\Omega) \cap L^\infty(\Omega)$ such that $\widetilde{\varphi} \equiv cste$ on $\widetilde{\Omega} \setminus \Omega$, we see that*

$$(4.5)\qquad \int_\Omega a(x, \nabla u_\epsilon).\nabla\widetilde{\varphi}dx + \int_\Omega \beta_\varepsilon(u_\epsilon)\widetilde{\varphi}dx = \int_\Omega f\widetilde{\varphi}dx + \Big(d - \int_{\widetilde{\Gamma}_N} \widetilde{\rho}_\varepsilon(u_\epsilon)d\sigma\Big)\widetilde{\varphi}_N.$$

The next result gives us a priori estimates on the solution u_ϵ of the problem $P_\varepsilon(\widetilde{\beta}_\varepsilon, \widetilde{\rho}_\varepsilon, \widetilde{f}, \widetilde{d})$.

Lemma 4.3. *Let u_ϵ be a solution of $P_\varepsilon(\widetilde{\beta}_\varepsilon, \widetilde{\rho}_\varepsilon, \widetilde{f}, \widetilde{d})$. Then the following statements hold to be true:*
(i) $\displaystyle\int_\Omega |\nabla u_\epsilon|^p\,dx \;+\frac{1}{\epsilon^p}\int_{\widetilde{\Omega}\setminus\Omega}|\nabla u_\epsilon|^p dx \le C\times\Big(\|\widetilde{d}\|_{L^1(\widetilde{\Gamma}_N)}+\|f\|_{L^1(\Omega)}\Big)$, *where C is a positive constant independent of ϵ.*
(ii) $\displaystyle\int_\Omega |\beta_\varepsilon(u_\epsilon)|\,dx+\int_{\widetilde{\Gamma}_N}|\widetilde{\rho}_\varepsilon(u_\epsilon)|\;d\sigma \le \|\widetilde{d}\|_{L^1(\widetilde{\Gamma}_N)}+\|f\|_{L^1(\Omega)}$.

Proof. First, we see that taking $\widetilde{\varphi}=u_\epsilon$ in (4.1), we have

$$\begin{aligned}\int_\Omega a(x,\nabla u_\epsilon).\nabla u_\epsilon dx+\int_{\widetilde{\Omega}\setminus\Omega}\frac{1}{\epsilon^p}|\nabla u_\epsilon|^{p-2}\nabla u_\epsilon.\nabla u_\epsilon dx+\int_\Omega \beta_\varepsilon(u_\epsilon)u_\epsilon dx\\ =\int_\Omega f u_\epsilon dx+\int_{\widetilde{\Gamma}_N}\Big(\widetilde{d}-\widetilde{\rho}_\varepsilon(u_\epsilon)\Big)u_\epsilon\;d\sigma.\end{aligned}\tag{4.6}$$

(i) Obviously, we have $\displaystyle\int_\Omega \beta_\varepsilon(u_\epsilon)u_\epsilon dx\ge 0,\ \int_{\widetilde{\Omega}\setminus\Omega}\frac{1}{\epsilon^p}|\nabla u_\epsilon|^{p-2}\nabla u_\epsilon.\nabla u_\epsilon dx\ge 0$ and $\displaystyle\int_\Omega f u_\epsilon dx\le C_3\|f\|_{L^1(\Omega)}$. For the last term in (4.6) we have

$$\begin{aligned}\int_{\widetilde{\Gamma}_N}\Big(\widetilde{d}-\widetilde{\rho}_\varepsilon(u_\epsilon)\Big)u_\epsilon\;d\sigma &= \int_{\widetilde{\Gamma}_N}\widetilde{d}u_\epsilon\;d\sigma-\int_{\widetilde{\Gamma}_N}\widetilde{\rho}_\varepsilon(u_\epsilon)u_\epsilon\;d\sigma\\ &\le \int_{\widetilde{\Gamma}_N}\widetilde{d}u_\epsilon\;d\sigma\\ &\le \int_{\widetilde{\Gamma}_N}|\widetilde{d}||u_\epsilon|\;d\sigma\\ &\le C_4\|\widetilde{d}\|_{L^1(\widetilde{\Gamma}_N)}.\end{aligned}$$

Having in mind the relation $a(x,\xi).\xi\ge\dfrac{1}{C'}|\xi|^p$ we get

$$\int_\Omega a(x,\nabla u_\epsilon).\nabla u_\epsilon dx\ge\frac{1}{C'}\int_\Omega|\nabla u_\epsilon|^p\,dx.$$

So, using the inequalities above and (4.3), we deduce the first part of the lemma.
(ii) We set $\widetilde{\varphi}=T_k(u_\epsilon),\ k>0$ in (4.1) to get

$$\begin{aligned}\int_\Omega a(x,\nabla u_\epsilon).\nabla T_k(u_\epsilon)dx+\int_{\widetilde{\Omega}\setminus\Omega}\frac{1}{\epsilon^p}|\nabla u_\epsilon|^{p-2}\nabla u_\epsilon.\nabla T_k(u_\epsilon)dx+\int_\Omega\beta_\varepsilon(u_\epsilon)T_k(u_\epsilon)dx\\ +\int_{\widetilde{\Gamma}_N}\widetilde{\rho}_\varepsilon(u_\epsilon)T_k(u_\epsilon)\;d\sigma=\int_\Omega fT_k(u_\epsilon)dx+\int_{\widetilde{\Gamma}_N}\widetilde{d}T_k(u_\epsilon)\;d\sigma.\end{aligned}\tag{4.7}$$

The two first terms in (4.7) are nonnegative. For the term in the right hand side of (4.7), we have

$$\int_\Omega fT_k(u_\epsilon)dx+\int_{\widetilde{\Gamma}_N}\widetilde{d}T_k(u_\epsilon)\;d\sigma\quad\le\quad k\left(\int_\Omega|f|dx+\int_{\widetilde{\Gamma}_N}|\widetilde{d}|\;d\sigma.\right)$$

$$= \quad k\left(\|\widetilde{d}\|_{L^1(\widetilde{\Gamma}_N)} + ||f||_{L^1(\Omega)}\right)$$

Then, from (4.7), we have

$$\int_\Omega \beta_\varepsilon(u_\epsilon)T_k(u_\epsilon)dx + \int_{\widetilde{\Gamma}_N} \widetilde{\rho}_\varepsilon(u_\epsilon)T_k(u_\epsilon)\ d\sigma \le k\left(\|\widetilde{d}\|_{L^1(\widetilde{\Gamma}_N)} + ||f||_{L^1(\Omega)}\right).$$

We divide by k and let k goes to zero, we get

$$\int_\Omega |\beta_\varepsilon(u_\epsilon)|dx + \int_{\widetilde{\Gamma}_N} |\widetilde{\rho}_\varepsilon(u_\epsilon)|\ d\sigma \le \left(\|\widetilde{d}\|_{L^1(\widetilde{\Gamma}_N)} + ||f||_{L^1(\Omega)}\right)$$

□

Now, we state our convergences results.

Lemma 4.4. *There exists $u \in W^{1,p}_N(\widetilde{\Omega})$, such that, up to a subsequence if necessary, as $\varepsilon \to 0$*

$$u_\epsilon \rightharpoonup u \ \text{in}\ W^{1,p}_D(\widetilde{\Omega}) \quad \text{and} \quad u_\epsilon \longrightarrow u \ \text{a.e. in}\ \widetilde{\Omega}\ \text{and a.e. on}\ \widetilde{\Gamma}_N. \tag{4.8}$$

Moreover,

$$u \in W^{1,p}_N(\Omega) \ \text{ and }\ a(x,\nabla u_\epsilon) \rightharpoonup a(x,\nabla u) \ \text{in}\ \left(L^{p'}(\Omega)\right)^N.$$

Proof. First, we see that

$$\begin{aligned}\int_{\widetilde{\Omega}} |\nabla u_\epsilon|^p\,dx &= \int_\Omega |\nabla u_\epsilon|^p\,dx + \int_{\widetilde{\Omega}\setminus\Omega} |\nabla u_\epsilon|^p dx \\ &\le \int_\Omega |\nabla u_\epsilon|^p\,dx + \int_{\widetilde{\Omega}\setminus\Omega} \frac{1}{\epsilon^p}|\nabla u_\epsilon|^p dx,\end{aligned}$$

for any $0 < \varepsilon < 1$. Then thanks to Proposition 4.3-(i), the sequence $(\nabla u_\epsilon)_{\epsilon>0}$ is bounded in $L^p(\widetilde{\Omega})^N$. This implies that there exists $u \in W^{1,p}_N(\widetilde{\Omega})$ such that (4.8) is fulfilled. Thanks to Lemma 4.3, $\displaystyle\int_{\widetilde{\Omega}\setminus\Omega} \frac{1}{\epsilon^p}|\nabla u_\epsilon|^p dx$ is bounded. Then, it is clear that u is a constant function in $\widetilde{\Omega}\setminus\Omega$. Thus $u \in W^{1,p}_N(\Omega)$. For the last part of the lemma, we recall that u_ε is bounded in $W^{1,p}_D(\Omega)$. So, $(a(x,\nabla u_\epsilon))_{\varepsilon>0}$ is bounded in $\left(L^{p'}(\Omega)\right)^N$ and, we can extract a subsequence such that $a(x,\nabla u_\epsilon) \rightharpoonup \Phi$ in $\left(L^{p'}(\Omega)\right)^N$. Using standard monotonicity arguments, one can see that

$$\Phi = a(x,\nabla u) \text{ a.e. in } \Omega.$$

Indeed, taking $\widetilde{\varphi} = u_\epsilon - u$ as a test function in (4.1), we have

$$\begin{aligned}&\int_\Omega a(x,\nabla u_\epsilon).\nabla(u_\epsilon - u)dx + \int_{\widetilde{\Omega}\setminus\Omega} \frac{1}{\epsilon^p}|\nabla u_\epsilon|^{p-2}\nabla u_\epsilon.\nabla(u_\epsilon - u)dx + \int_\Omega \beta_\varepsilon(u_\epsilon)(u_\epsilon - u)dx \\ &+ \int_{\widetilde{\Gamma}_N} \widetilde{\rho}_\varepsilon(u_\epsilon)(u_\epsilon - u)\ d\sigma = \int_\Omega f(u_\epsilon - u)dx + \int_{\widetilde{\Gamma}_N} \widetilde{d}(u_\epsilon - u)\ d\sigma.\end{aligned} \tag{4.9}$$

So, using the first part and Lebesgue dominated convergence Theorem, we obtain

$$\limsup_{\epsilon\to 0}\int_\Omega a(x,\nabla u_\epsilon).\nabla\left(u_\epsilon-u\right)dx\le 0. \tag{4.10}$$

Then, by using standard monotonicity arguments, this implies that $\Phi=a(x,\nabla u)$ a.e. in Ω. For completeness let us give the arguments. Let $\varphi\in\mathcal{D}(\Omega)$ and $\lambda\in\mathbb{R}^*$. Using (4.10) and (2.2), we get

$$\begin{aligned}\lambda\lim_{\epsilon\to 0}\int_\Omega a(x,\nabla u_\epsilon).\nabla\varphi\,dx &\ge \limsup_{\epsilon\to 0}\int_\Omega a(x,\nabla u_\epsilon).\nabla\Big(u_\epsilon-u+\lambda\varphi\Big)\,dx\\ &\ge \limsup_{\epsilon\to 0}\int_\Omega a(x,\nabla(u-\lambda\varphi)).\nabla\Big(u_\epsilon-u+\lambda\varphi\Big)\,dx.\end{aligned}$$

Hence

$$\lambda\lim_{\epsilon\to 0}\int_\Omega a(x,\nabla u_\epsilon).\nabla\varphi\,dx\ge\ \lambda\int_\Omega a(x,\nabla(u-\lambda\varphi)).\nabla\varphi\,dx. \tag{4.11}$$

Dividing (4.11) by $\lambda>0$ and by $\lambda<0$ respectively, passing to the limit with $\lambda\to 0$ it follows that

$$\lim_{\epsilon\to 0}\int_\Omega a(x,\nabla u_\epsilon).\nabla\varphi\,dx=\int_\Omega a(x,\nabla u).\nabla\varphi\,dx.$$

This means that $\displaystyle\int_\Omega \Phi.\nabla\varphi\,dx=\int_\Omega a(x,\nabla u).\nabla\varphi\,dx$ and so $\mathrm{div}(\Phi)=$ div $a(x,\nabla u)$ in $\mathcal{D}'(\Omega)$.

Hence $\Phi=a(x,\nabla u)$ a.e. in Ω and we have $a(x,\nabla u_\epsilon)\rightharpoonup a(x,\nabla u)$ in $\Big(L^{p'}(\Omega)\Big)^N$ as $\varepsilon\to 0$.

$\square$

Proof of Theorem 2.2. Thanks to Lemma 4.4, we know that there exists $u\in W_N^{1,p}(\Omega)$, such that, up to a subsequence if necessary, as $\varepsilon\to 0$

$$u_\epsilon\rightharpoonup u \text{ in } W_D^{1,p}(\widetilde{\Omega})\quad\text{ and }\quad a(x,\nabla u_\epsilon)\rightharpoonup a(x,\nabla u)\text{ in }\Big(L^{p'}(\Omega)\Big)^N.$$

Using moreover (4.3), we can assert that $u\in L^\infty(\Omega)$ and there exists $w\in L^\infty(\Omega)$ and $\widetilde{v}\in L^\infty(\widetilde{\Gamma}_N)$ such that as $\varepsilon\to 0$

$$\begin{cases}\beta_\varepsilon(u_\varepsilon)\overset{*}{\rightharpoonup} w \quad\text{ in }\ L^\infty(\Omega)\\ \text{ and}\\ \widetilde{\rho}_\varepsilon(u_\varepsilon)\overset{*}{\rightharpoonup}\widetilde{v}\quad\text{ in } L^\infty(\widetilde{\Gamma}_N).\end{cases} \tag{4.12}$$

Thanks to Remark 4.1

$$\int_\Omega a(x,\nabla u_\epsilon).\nabla\varphi dx+\int_\Omega\beta_\varepsilon(u_\epsilon)\varphi dx=\int_\Omega f\varphi dx+\left(d-\int_{\widetilde{\Gamma}_N}\widetilde{\rho}_\varepsilon(u_\epsilon)\ d\sigma\right)\varphi_N, \tag{4.13}$$

for any $\varphi\in W_N^{1,p}(\Omega)\cap L^\infty(\Omega)$. So, letting $\epsilon\to 0$, we get

$$\int_\Omega a(x,\nabla u).\nabla\varphi dx+\int_\Omega w\varphi dx\ =\ \int_\Omega f\varphi dx+(d-v)\,\varphi_N.$$

where $v := \displaystyle\int_{\widetilde{\Gamma}_N} \widetilde{v} d\sigma \in \mathbb{R}$. Recall that β_ϵ and $\widetilde{\rho}_\varepsilon$ converge to β and $\dfrac{1}{|\widetilde{\Gamma}_N|}\rho$, in the sense of graphs, respectively. So, since $u_\varepsilon \longrightarrow u$ in $L^p(\Omega)$ as $\varepsilon \to 0$, $\beta_\varepsilon(u_\varepsilon) \overset{*}{\rightharpoonup} w$ in $L^\infty(\Omega)$ and $\widetilde{\rho}_\varepsilon(u_\varepsilon) \rightharpoonup \widetilde{v}$ in $L^p(\widetilde{\Gamma}_N)$ as $\varepsilon \to 0$, we deduce that (cf. [**8**]) that

$$w \in \beta(u) \text{ a.e. in } \Omega \quad \text{and} \quad v \in \rho(u_N) \text{ a.e. on } \widetilde{\Gamma}_N.$$

At last, since $u_N \in \mathbb{R}$, $v \in \rho(u_N)$ a.e. on $\widetilde{\Gamma}_N$ and $\rho(u_N)$ is an interval, we deduce that $v \in \rho(u_N)$. This ends up the proof of the existence of a solution. □

5. Uniqueness

The uniqueness concerning the solution of $S^{\beta,\rho}_{f,d}$ is a straightforward consequence of the contraction property in the following lemma :

Lemma 5.5. *Assume that (u_1, w_1, v_1) and (u_2, w_2, v_2) are two solutions of the problems $S^{\beta,\rho}_{f_1,d_1}$ and $S^{\beta,\rho}_{f_2,d_2}$, respectively. Then*

$$\int_\Omega (w_1 - w_2)^+ \, dx + (v_1 - v_2)^+ \leq \int_\Omega (f_1 - f_2)^+ \, dx + (d_1 - d_2)^+ . \tag{5.1}$$

Proof. Recall that, for $i = 1, 2$, we have $w_i \in \beta(u_i)$ a.e. in Ω, $v_i \in \rho\big((u_i)_N\big)$ and

$$\int_\Omega a(x, \nabla u_i).\nabla\varphi dx + \int_\Omega w_i \varphi dx = \int_\Omega f_i \varphi dx + (d_i - v_i)\, \varphi_N \tag{5.2}$$

for any $\varphi \in W^{1,p}_N(\Omega) \cap L^\infty(\Omega)$.
Let $\xi \in W^{1,p}_N(\Omega)$. Taking $\varphi = H_\varepsilon(u_1 - u_2 + \varepsilon\xi)$ as a test function for each $i = 1, 2$ and subtracting the resulting equations we get

$$\begin{aligned}
&\int_\Omega \Big(a(x, \nabla u_1) - a(x, \nabla u_2)\Big).\nabla H_\varepsilon(u_1 - u_2 + \varepsilon\xi) dx + \int_\Omega (w_1 - w_2)\, H_\varepsilon(u_1 - u_2 + \varepsilon\xi) dx \\
&+ (v_1 - v_2)\, (H_\varepsilon(u_1 - u_2 + \varepsilon\xi))_N = \int_\Omega (f_1 - f_2)\, H_\varepsilon(u_1 - u_2 + \varepsilon\xi) dx \\
&+ (d_1 - d_2)\, (H_\varepsilon(u_1 - u_2 + \varepsilon\xi))_N .
\end{aligned} \tag{5.3}$$

It is clear that,

$$\int_\Omega (f_1 - f_2)\, H_\varepsilon(u_1 - u_2 + \varepsilon\xi) dx + (d_1 - d_2)\, (H_\varepsilon(u_1 - u_2 + \varepsilon\xi))_N \leq \int_\Omega (f_1 - f_2)^+ \, dx + (d_1 - d_2)^+ .$$

Moreover, thanks to (2.2), the first term in (5.3) satisfies

$$\limsup_{\varepsilon \to 0} \int_\Omega \Big(a(x, \nabla u_1) - a(x, \nabla u_2)\Big).\nabla H_\varepsilon(u_1 - u_2 + \varepsilon\xi) dx \geq 0.$$

Indeed, we have

$$\int_\Omega \Big(a(x, \nabla u_1) - a(x, \nabla u_2)\Big).\nabla H_\varepsilon(u_1 - u_2 + \varepsilon\xi) dx$$

$$
\begin{aligned}
&= \frac{1}{\varepsilon}\int_{\Omega\cap[0\le u_1-u_2+\varepsilon\xi\le\varepsilon]} \Big(a(x,\nabla u_1)-a(x,\nabla u_2)\Big).\left(\nabla u_1-\nabla u_2\right)dx \\
&\qquad + \int_{\Omega\cap[0\le u_1-u_2+\varepsilon\xi\le\varepsilon]} \Big(a(x,\nabla u_1)-a(x,\nabla u_2)\Big).\nabla\xi dx \\
&\ge \int_{\Omega\cap[0\le u_1-u_2+\varepsilon\xi\le\varepsilon]} \Big(a(x,\nabla u_1)-a(x,\nabla u_2)\Big).\nabla\xi dx,
\end{aligned}
$$

and letting $\epsilon\to 0$, we have

$$
\begin{aligned}
&\limsup_{\varepsilon\to 0}\int_\Omega \Big(a(x,\nabla u_1)-a(x,\nabla u_2)\Big).\nabla H_\varepsilon(u_1-u_2+\varepsilon\xi)dx \\
&\qquad\qquad \ge \lim_{\varepsilon\to 0}\int_{\Omega\cap[0\le u_1-u_2+\varepsilon\xi\le\varepsilon]} \Big(a(x,\nabla u_1)-a(x,\nabla u_2)\Big).\nabla\xi dx. \\
&\qquad\qquad = \int_{\Omega\cap[u_1=u_2]} \Big(a(x,\nabla u_1)-a(x,\nabla u_2)\Big).\nabla\xi dx \\
&\qquad\qquad = 0.
\end{aligned}
$$

So, from (5.3) we obtain

$$
\begin{aligned}
&\limsup_{\varepsilon\to 0}\int_\Omega (w_1-w_2)\,H_\varepsilon(u_1-u_2+\varepsilon\xi)dx + \limsup_{\varepsilon\to 0}\,(v_1-v_2)\left(H_\varepsilon(u_1-u_2+\varepsilon\xi)\right)_N \\
&\qquad\qquad \le \int_\Omega (f_1-f_2)^+\,dx + (d_1-d_2)^+\,.
\end{aligned}
\tag{5.4}
$$

See that, for any $a,b\in\mathbb{R}$, as $\epsilon\to 0$,

$$
H_\epsilon(a-b+\epsilon\xi)\to \mathrm{sign}_0^+(a-b)+\xi\chi_{[a=b]}.
$$

So, by using Lebesgue dominated convergence Theorem, we obtain

$$
\begin{aligned}
\lim_{\varepsilon\to 0}\int_\Omega (w_1-w_2)\,H_\varepsilon(u_1-u_2+\varepsilon\xi)dx &= \int_\Omega (w_1-w_2)\,\xi\chi_{[u_1=u_2]}dx \\
&\quad + \int_\Omega (w_1-w_2)\,\mathrm{sign}_0^+(u_1-u_2)\chi_{[u_1\ne u_2]}dx
\end{aligned}
$$

and

$$
\begin{aligned}
&\lim_{\varepsilon\to 0}\Big((v_1)_N-(v_2)_N\Big)\left(H_\varepsilon(u_1-u_2+\varepsilon\xi)\right)_N = (v_1-v_2)\,\xi_N\chi_{[(u_1)_N=(u_2)_N]} \\
&\qquad\qquad + (v_1-v_2)\,\mathrm{sign}_0^+\left((u_1)_N-(u_2)_N\right)\chi_{[(u_1)_N\ne(u_2)_N]}.
\end{aligned}
$$

So, (5.4) implies that

$$
\begin{aligned}
&\int_\Omega (w_1-w_2)\,\xi\chi_{[u_1=u_2]}dx + \int_\Omega (w_1-w_2)\,\mathrm{sign}_0^+(u_1-u_2)\chi_{[u_1\ne u_2]}dx \\
&+ (v_1-v_2)\,\xi_N\chi_{[(u_1)_N=(u_2)_N]} + (v_1-v_2)\,\mathrm{sign}_0^+\left((u_1)_N-(u_2)_N\right)\chi_{[(u_1)_N\ne(u_2)_N]} \\
&\le \int_\Omega (f_1-f_2)^+\,dx + (d_1-d_2)^+\,.
\end{aligned}
\tag{5.5}
$$

By density, we can take the function ξ_0 defined by :

$$\xi_0 = \begin{cases} \operatorname{sign}_0^+(w_1 - w_2) & \text{in } \Omega \\ \operatorname{sign}_0^+(v_1 - v_2) & \text{on } \Gamma_N \\ 0 \text{ in } \mathbb{R}^N \setminus \Omega. \end{cases}$$

as a test function in (5.5). This implies that

$$\begin{aligned} &\int_\Omega (w_1 - w_2)\operatorname{sign}_0^+(w_1 - w_2)\chi_{[u_1=u_2]}dx + \int_\Omega (w_1 - w_2)\operatorname{sign}_0^+(u_1 - u_2)\chi_{[u_1\neq u_2]}dx \\ &+ (v_1 - v_2)\operatorname{sign}_0^+(v_1 - v_2)\chi_{[(u_1)_N=(u_2)_N]} + (v_1 - v_2)\operatorname{sign}_0^+((u_1)_N - (u_2)_N)\chi_{[(u_1)_N\neq(u_2)_N]} \\ &\leq \int_\Omega (f_1 - f_2)^+ dx + (d_1 - d_2)^+ \end{aligned}$$

and then

$$\begin{aligned} &\int_\Omega (w_1 - w_2)^+ \chi_{[u_1=u_2]}dx + \int_\Omega (w_1 - w_2)^+ \chi_{[u_1\neq u_2]}dx \\ &+ (v_1 - v_2)^+ \chi_{[(u_1)_N=(u_2)_N]} + (v_1 - v_2)^+ \chi_{[(u_1)_N\neq(u_2)_N]} \\ &\leq \int_\Omega (f_1 - f_2)^+ dx + (d_1 - d_2)^+ . \end{aligned}$$

Thus

$$\int_\Omega (w_1 - w_2)^+ dx + (v_1 - v_2)^+ \leq \int_\Omega (f_1 - f_2)^+ dx + (d_1 - d_2)^+ .$$

□

Proof of Theorem 2.1. It is clear that the existence is a consequence of Theorem 2.2. The uniqueness follows from the L^1-comparaison principle in Lemma 5.5. □

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(**Noureddine IGBIDA**) Institut de recherche XLIM-DMI, UMR-CNRS 6172, Faculté des Sciences et Techniques, Université de Limoges, France.

E-mail address: noureddine.igbida@unilim.fr

(**Soma Safimba**) Université OUAGA I, Professeur Joseph KI-ZERBO, LAboratoire de Mathématique et Informatique (LAMI), Unité de Formation et de Recherches en Sciences Exactes et Appliquées, Département de Mathématiques, 03 BP 7021 Ouaga 03 Ouagadougou, Burkina Faso

E-mail address: somasaf2000@yahoo.fr