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Sub-gradient diffusion operator

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Abstract

The paper deals with stationary equation governed by the operator $-\nabla \cdot A(x, \nabla u) = \mu$ in the case where $A(x, \xi)$ is a maximal monotone graph and μ is a Radon measure. Our main interest concerns the typical situation where $A(x, .)$ is defined only in a bounded region of $I\!R^n$; so that $A(x, .)$ does not satisfy the standard polynomial growth control condition.

1. Introduction

The study of several mechanical and physical structures results in the description locally how changes in a parameter u, usually called density, are related to changes in the flux, that we denote by Φ. Such description is usually conceived in a PDE relating u, Φ and input parameters which are assumed to be known. In our case, the input parameter is a given Radon measure μ, and the PDE use a differential operator in the following form

$$-\nabla \cdot \Phi(x) = \mu(x) \quad \text{and} \quad \Phi(x) = A(x, \nabla u(x)), \quad \text{for } x \in \Omega, \tag{1}$$

where $\Omega \subset I\!R^n$ is an open bounded domain, $n \geq 1$ and $A : \Omega \times I\!R^n \to I\!R^n$. An important class of these PDEs falls into the scope of the so called Leray–Lions operator (cf. [25]). That is A is a Carathéodory function (measurable with respect to $x \in \Omega$ and continuous with respect to

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$\xi \in I\!R^n$) satisfying $A(., 0) = 0$ and, there exists $1 < p < \infty$, such that the following assumptions are fulfilled:

(A1) there exists $C > 0$ such that, for any $\xi \in I\!R^n$ $A(x, \xi).\xi \geq C|\xi|^p$ for a.e. $x \in \Omega$

(A2) for any $\xi, \eta \in I\!R^n$ such that $\xi \neq \eta$ $\big(A(x, \xi) - A(x, \eta)\big).(\xi - \eta) \geq 0$ a.e. $x \in \Omega$

(A3) there exists $\sigma > 0$ and $k \in L^{p'}(\Omega)$ such that $|A(x,\xi)| \leq \sigma(k(x) + |\xi|^{p-1})$ a.e. $x \in \Omega$ and for any $\xi \in I\!R^n$, where $p' = \dfrac{p}{p-1}$.

Our aim here is to study the equation (1) in the peculiar situation where $A(x, .)$ is a maximal monotone graph in $I\!R^n \times I\!R^n$ defined only in a bounded region of $I\!R^n$. In particular, this implies that $A(x, .)$ may grow as fast as the condition (A3) falls to be true. Typical examples appear in the study of non-Newtonian fluids, where A is given by the constitutive law $A(x, \xi) = f(x, |\xi|)\xi$ and $f(x, r)$ is a nonnegative function which becomes very large when r is close to some critical values. This kind of problems appears also in the description of sub-gradient flows dynamics; that is dynamical system governed by a gradient constraint, like for granular matter.

To begin with, recall that the equation (1) corresponds to the Euler–Lagrange equation associated with the optimization problem

$$\min\left\{\int_\Omega J(x, \nabla z) - \int_\Omega f z\,;\ z \in \mathcal{A}\right\}, \tag{2}$$

where $J : \Omega \times \mathbb{R}^n \to [0, \infty]$. Here the minimization problem is related to the energy description of the structure, the set $\mathcal{A}$ is the so called energy space and J is such that $A(x, \xi) = \partial_\xi J(x, \xi)$, where ∂_ξ denotes the sub-differential with respect to ξ. The assumption (A1) and (A2) are related to the coercivity and the convexity of the function $J(x, \xi)$ with respect to ξ, respectively. They incorporate the key ingredient for the existence of a solution to the optimization problem (2). As to the assumption (A3), it insures the behavior of the variation of $J(x, \nabla u)$ for large values of ∇u so that the critical points of (2) turn into a solution of (1).

There is a huge literature concerning the study of existence uniqueness as well as the connection between the problem (1) and (2) under the standard assumptions (A1)–(A3). Driven by diverse applications, some types of nonlinear operators A and functional J appeared beyond the scope of the assumptions (A1)–(A3). They motivated new studies and new developments in the theory of nonlinear elliptic and parabolic PDE. For instance, in the borderline case $p = 1$ ($p' = \infty$), the equation (1) as well as the associate evolution problem appears as a model for heat and mass transfer in turbulent fluids or in the theory of phase transitions (see [5]). Some variant appears also in the context of image denoising and reconstruction (see [5]). In this situation the equation (1) appears as a borderline case with respect to the standard assumptions (A1–A3). Its study has developed many new theoretical and numerical tools (see [5]) currently essential for nonlinear PDEs analysis in the spaces BV, the set of function of bounded variation. Indeed, due to the linear growth condition, the natural energy space to study (1) in this case is the space of functions of bounded variation and the flux is a bounded Lebesgue function. Typical examples for the opposite borderline case $p \to \infty$ ($p' = 1$) appear in the study of sub-gradient flow dynamics. Nowadays, the Monge–Kantorovich equation which corresponds to the limit as $p \to \infty$, in the p-Laplacien operator is extensively used in the study of optimal mass transportation problem (cf. [1,20]) as well as in the optimal mass transfer problem (cf. [8]). It is also used in the description of the dynamic of granular matter like the sandpile (cf. [26,20] and [19]) and also in the deformation of polymer plastic during compression molding (cf. [4]). In this situation $A(x,\xi) = \partial I\!I_{\overline{B}(0,1)}(\xi)$, and due to the gradient constraint, the flow is singular in general. The natural energy space in this case is the space of Lipschitz continuous function and the flux is a vector valued measure. This is a typical example where the assumption (A3) falls to be true. Its study allowed the development of new useful tools like tangential gradient with respect to a Radon measure. The pioneering work in this direction which opened a possible way to manage the difficulties related to PDE with singular flux is [9], where Bouchitté, Buttazzo and Seppecher introduced a new notion of tangent space to a measure on $I\!R^n$. They use these tools in order to model the elastic energy of low-dimensional structures. One can see also the paper [10] where these tools were used for the first time in the study of the limit as $p \to \infty$ in the p-Laplacian equation.

Other situation where the assumption (A3) falls to be true appears when $A(x,.)$ is defined in all $I\!R^n$ and grows rapidly for large values of ξ. Recall that these cases fall into the scope of the theory of elliptic equations in Orlicz–Sobolev space (cf. [17] and [18]). Our aim and approach here are different. To study the problem we handle the equation (1) in the context of nonlinear PDE with singular flux. Indeed, without the assumption (A3), the flux is not a Lebesgue function in general. It is a vector valued Radon measure and we use the theory of tangential gradient to characterize the state equation that gives the connection between the flux and the gradient of the solution. The particular situations where $A(x,\xi) = \partial I\!I_{C(x)}(\xi)$, and $C(x) \subset I\!R^N$ is a bounded closed convex set, the equation (1) with Neumann boundary condition is used in [24] in connection with the optimal mass transport problem. Our aim here is to treat (1) in the general case where $A(x,.)$ is a maximal monotone graph defined in a bounded region of $I\!R^n$.

More precisely, we are interested into the equation

$$\begin{cases} \left.\begin{array}{l} -\nabla\cdot\Phi = \mu \\ \Phi \in A(x,\nabla u) \end{array}\right\} & \text{in } \Omega \\ \\ u = g & \text{on } \partial\Omega, \end{cases} \tag{3}$$

in the case where $\mu \in \mathcal{M}_b(\Omega)$ is a given Radon measure, $g \in \mathcal{C}(\partial\Omega)$ and $A(x,.)$ is a maximal monotone graph in $I\!R^n$ given by

$$A(x,\xi) = \partial_\xi J(x,\xi), \tag{4}$$

where $J : \Omega \times \mathbb{R}^n \to [0,\infty)$; $J(x,\xi)$ is continuous with respect to x, and l.s.c. with respect to ξ, and satisfies $J(x,0) = 0$, for any $x \in \Omega$. Moreover, denoting by $D(x)$ the domain of $J(x,.)$; i.e.

$$D(x) := \mathcal{D}(J(x,.)) := \{\xi \in \mathbb{R}^n : J(x,\xi) < \infty\},$$

we assume that J satisfies the following assumptions

(J1) There exists M in $L^\infty(\Omega)$ such that $D(x) \subseteq \mathcal{B}(0, M(x))$ for any x in Ω.
(J2) For any $x \in \Omega$, $J(x,.)$ is convex.
(J3) There exists $\alpha > 0$, such that $B(0,\alpha) \subseteq Int(D(x))$, for any x in Ω.

Using the assumptions (J1–J2), we prove that the optimization problem (2) has a continuous Lipschitz solution u. Then, assuming moreover the assumption (J3), we prove that there exists a Radon vector-valued measure Φ, such that (u, Φ) is a solution to the PDE (3) in a suitable sense (see Section 3). We combine simultaneously a connection between the regular part of Φ and $A(x, \nabla u)$, and between the singular part of Φ and the support function of the domain of $J(x, .)$. At last, we give the equivalence between the solutions of (2), (3) and a flux maximization problem related to Legendre–Fenchel's duality.

Before to give the plan of the paper, let us take a while to comment the assumptions (J1)–(J3). The convexity assumption (J2) is standard and is connected to the monotonicity condition (A2) for Leray–Lions operator. As to the condition (J1) this is connected to our peculiar situations which describe fast grow behavior of the energy when the gradient approaches critical values, and also the description of subgradient flow phenomena. See here, that while this assumption predicts the coercivity, it makes us lose the suitable control on the flux to perform it Lebesgue integrable. As to the assumption (J3), even if it seems here to be just technically important for the control of the total mass of the flux (see Proof of Lemma 2), we do not know if in general the results of the paper holds to be true or not. Thanks to Remark 2 and the studies of the cases $A(x, \xi) = \partial I\!I_{C(x)}$ (cf. [24]), this assumption seems to be related to the degeneracy of the Finsler metric behind the problem. However, we are not able to make this fact in evidence rigorously in this paper.

In the following section, we begin with some preliminaries recalling the main tools we use to handle a PDE with singular flux, like tangential measure and tangential gradient. Then, we prove two technical results that will be useful for the proof of our main result. In Section 3, we present our main results. Under the assumptions (J1)–(J3), we begin with the characterization of the solution of the optimization problem (2) as a solution of the PDE (1)–(4) with Dirichlet boundary condition. Actually, we show that the flux is a vector valued measure. The regular part (with respect to Lebesgue measure) leaves in $A(x, \nabla u)$ and, the singular part is concentrated on the boundary of $\mathcal{D}(J(x, .))$ and is connected to the tangential gradient of u through the support function of $\mathcal{D}(J(x, .))$. Then, we present equivalent characterization using the notion of variational solution and duality. In Section 4, we give the proof of our main results. We consider a regularization of the problem (2) by taking Yoshida approximation of J, and we use compactness arguments for the proofs.

2. Preliminaries

2.1. Vector valued Radon measure

Let $\Omega \subset I\!R^n$ be a bounded open domain with Lipschitz boundary. We denote by $\mathcal{M}(\overline{\Omega})$ the space of Radon measure in $I\!R^n$ supported in $\overline{\Omega}$; i.e. $\mu(A) = \mu(A \cap \overline{\Omega})$, for any Borel set $A \subseteq I\!R^n$. We recall that $\mathcal{M}(\overline{\Omega})$ can be identified with the dual space of the set of continuous functions defined in $\overline{\Omega}$; i.e. $\mathcal{M}(\overline{\Omega}) = \left(\mathcal{C}(\overline{\Omega})\right)^*$, in the sense that, every $\mu \in \mathcal{M}(\overline{\Omega})$ is equal to $\xi \in \mathcal{C}(\overline{\Omega}) \to \displaystyle\int \xi \, d\mu$. Then $\mathcal{M}(\Omega)$ denotes the space of Radon measures supported in Ω. See here that $\mathcal{M}(\Omega)$ can be identified with the subset of $\mathcal{M}(\overline{\Omega})$ of measures μ such that $|\mu|(\partial\Omega) = 0$. Thus $\mathcal{M}(\Omega)$ can be identified with $\left(\mathcal{C}_0(\Omega)\right)^*$, the dual of the space of continuous functions in Ω that are null on the boundary.

For any $\mu \in \mathcal{M}(\overline{\Omega})$, we denote by μ^+, μ^- and $|\mu|$ the positive part, negative part and the total variation measure associated with μ, respectively. Then we denote, $\mathcal{M}_b(\Omega)$ the space of Radon measures with bounded total variation $|\mu|(\Omega)$. Recall that $\mathcal{M}_b(\overline{\Omega})$ (resp. $\mathcal{M}_b(\Omega)$) equipped with the norm $|\mu|(\overline{\Omega})$ (resp. $|\mu|(\Omega)$) is a Banach space.

We denote by $\mathcal{M}(\overline{\Omega})^n$ the space of $I\!R^n$-valued Radon measures of Ω; i.e. $\mathcal{X} \in \mathcal{M}(\overline{\Omega})^n$ if and only if $\mathcal{X} = (\mathcal{X}_1, ..., \mathcal{X}_n)$ with $\mathcal{X}_i \in \mathcal{M}(\overline{\Omega})$. We recall that the total variation measure associated with $\mathcal{X} \in \mathcal{M}(\overline{\Omega})^n$, denoted again by $|\mathcal{X}|$, is defined by

$$|\mathcal{X}|(B) = \sup\left\{\sum_{i=1}^{\infty} |\mathcal{X}(B_i)| \ ; \ B = \cup_{i=1}^{\infty} B_i, \ B_i \text{ a Borelean set}\right\}$$

and belongs to $\mathcal{M}^+(\overline{\Omega})$, the set of nonnegative Radon measure. The subspace $\mathcal{M}_b(\overline{\Omega})^n$ equipped with the norm $\|\mathcal{X}\| = |\mathcal{X}|(\overline{\Omega})$ is a Banach space. It is clear that $\mathcal{M}(\overline{\Omega})^n$ endowed with the norm $\| \ \|$ is isometric to the dual of $\mathcal{C}(\overline{\Omega})^n$. The duality is given by

$$\left\langle \mathcal{X}, \xi \right\rangle = \sum_{i=1}^{n} \int_{\Omega} \xi_i \, d\mathcal{X}_i,$$

for any $\mathcal{X} = (\mathcal{X}_1, .., \mathcal{X}_n) \in \mathcal{C}(\overline{\Omega})^n$ and $\xi = (\xi_1, \dots, \xi_n) \in \mathcal{C}(\overline{\Omega})^n$. Similarly, $\mathcal{M}(\Omega)^n$ endowed with the norm $\| \ \|$ is isometric to the dual of $\mathcal{C}_0(\Omega)^n$, and the duality is given by

$$\left\langle \mathcal{X}, \xi \right\rangle = \sum_{i=1}^{n} \int_{\Omega} \xi_i \, d\mathcal{X}_i,$$

for any $\mathcal{X} = (\mathcal{X}_1, .., \mathcal{X}_n) \in \mathcal{M}(\Omega)^n$ and $\xi = (\xi_1, ..., \xi_n) \in \mathcal{C}_0(\Omega)^n$.

We denote by $\mathcal{L}^n$ the n-dimensional Lebesgue measure of $I\!R^n$. For any $1 \le p \le +\infty$, $L^p(\Omega)$ denotes the standard Lebesgue space with respect to $\mathcal{L}^n$, and we use $\int u dx$ to denote the Lebesgue integral of u with respect to $\mathcal{L}^n$. Otherwise, we denote by $L^p(\Omega, d\mu)$, the L^p space with respect to the measure μ and $\int u d\mu$ to denote the Lebesgue integral of u with respect to μ. The set $W^{1,p}(\Omega)$ denotes the standard Sobolev space, and for a given $g \in \mathcal{C}(\partial\Omega)$, we'll use the notation

$$W_g^{1,p}(\Omega) := \left\{ u \in W^{1,p}(\Omega) \ : \ u = g \text{ on } \partial\Omega \right\}.$$

For any $\mathcal{X} \in \mathcal{M}_b(\overline{\Omega})^n$ and $\nu \in \mathcal{M}_b(\overline{\Omega})^+$, $\mathcal{X}$ is absolutely continuous with respect to ν; denoted by $\mathcal{X} \ll \nu$, provided $\nu(A) = 0$ implies $|\mathcal{X}|(A) = 0$, for any Borel set $A \subset \overline{\Omega}$. Thanks to Radon–Nikodym decomposition Theorem, we know that for any $\mathcal{X} \in \mathcal{M}_b(\overline{\Omega})^n$ and $\nu \in \mathcal{M}_b(\overline{\Omega})$ such that $\mathcal{X} \ll \nu$, there exists unique bounded $I\!R^n$-valued Radon measure denoted by $\dfrac{d\mathcal{X}}{d\nu}$, such that

$$\mathcal{X}(A) = \int_A \frac{d\mathcal{X}}{d\nu} \, d\nu \quad \text{for any } A \subseteq \Omega.$$

Indeed, $\dfrac{d\mathcal{X}}{d\nu} \in \mathcal{M}_b(\overline{\Omega})^n$ is the density of $\mathcal{X}$ with respect to ν, and can be computed by differentiating. In particular, it is not difficult to see that, for any $\mathcal{X} \in \mathcal{M}_b(\overline{\Omega})^n$, we have $\mathcal{X} \ll |\mathcal{X}|$, $\dfrac{d\mathcal{X}}{d|\mathcal{X}|} \in L^1(\overline{\Omega}, d|\mathcal{X}|)^n$ and $\left|\dfrac{d\mathcal{X}}{d|\mathcal{X}|}\right| = 1$, $|\mathcal{X}|$-a.e. in $\overline{\Omega}$ (see for instance [27]). In connection with the polar factorization, in general $\dfrac{d\mathcal{X}}{d|\mathcal{X}|}$ is denoted by $\dfrac{\mathcal{X}}{|\mathcal{X}|}$. So, for any $\mathcal{X} \in \mathcal{M}_b(\overline{\Omega})^n$, we have

$$\mathcal{X}(A) = \int_A \frac{\mathcal{X}}{|\mathcal{X}|}\, d|\mathcal{X}|, \qquad \text{for any } A \subseteq \overline{\Omega},$$

and, every $\mathcal{X} \in \mathcal{M}_b(\overline{\Omega})^n$ (resp. $\mathcal{X} \in \mathcal{M}_b(\Omega)^n$) can be identified with the linear function

$$\eta \in \mathcal{C}(\overline{\Omega})^n \to \int \frac{\mathcal{X}}{|\mathcal{X}|} \cdot \eta \; d|\mathcal{X}| \quad (\text{resp. } \eta \in \mathcal{C}_0(\Omega)^n \to \int \frac{\mathcal{X}}{|\mathcal{X}|} \cdot \eta \; d|\mathcal{X}|.$$

To simplify the presentation, for any $\eta \in \mathcal{C}(\overline{\Omega})^n$, we shall systematically use the notation

$$\int \eta \, d\mathcal{X}$$

to denote $\int \frac{\mathcal{X}}{|\mathcal{X}|} \cdot \eta \; d|\mathcal{X}|$.

For a given $\mu \in \mathcal{M}_b(\overline{\Omega})$ (resp. $\mu \in \mathcal{M}_b(\Omega)$), we say that $\mathcal{X} \in \mathcal{M}_b(\overline{\Omega})^n$ satisfies the PDE

$$-\nabla \cdot \mathcal{X} = \mu \tag{5}$$

if and only if

$$\int \nabla\xi \, d\mathcal{X} = \int \xi \, d\mu \quad \text{for any } \xi \in \mathcal{C}^1(\overline{\Omega}) \quad (\text{resp. } \xi \in \mathcal{C}_0^1(\Omega)).$$

We denote by $\mathcal{S}(\mu)$ the set of vector valued Radon measure $\mathcal{X} \in \mathcal{M}_b(\overline{\Omega})^n$ satisfying the PDE (5). For any $\mathcal{X} \in \mathcal{M}_b(\overline{\Omega})^n$, we denote by $\mathcal{X}_r\mathcal{L}^n + \mathcal{X}_s$ the Radon–Nikodym decomposition of the vector valued measure $\mathcal{X}$ with respect to $\mathcal{L}^n$. So, for a given $\mu \in \mathcal{M}_b(\overline{\Omega})$ (resp. $\mu \in \mathcal{M}_b(\Omega)$), $\mathcal{X} \in \mathcal{S}(\mu)$ is equivalent to say that

$$\int \nabla\xi \cdot \mathcal{X}_r \, dx + \int \nabla\xi \; d\mathcal{X}_s = \int \xi \; d\mu \quad \text{for any } \xi \in \mathcal{C}^1(\overline{\Omega}) \quad (\text{resp. } \xi \in \mathcal{C}_0^1(\Omega)).$$

2.2. *Tangential measure and tangential gradient*

As we notice in the introduction, the PDE (1) involves Lipschitz continuous functions as an energy space and vector valued measure flux. So, the standard Sobolev space and the standard gradient defined with respect to Lebesgue measure are not enough to handle the state equation (4). To overcome these difficulties, we'll use the notion of tangential gradient introduced by Bouchitté, Buttazzo and Seppecher in [9]. For a given $\Phi \in \mathcal{M}_b(\overline{\Omega})^n$, let us consider $\gamma \in \mathcal{M}_b(\overline{\Omega})^+$ and $\sigma \in L^1(\overline{\Omega}, d\gamma)^n$ be such that $\Phi = \sigma\gamma$. Notice that this is always possible, since one can take $\gamma = |\Phi|$ and $\sigma = \frac{\Phi}{|\Phi|}$. Among the objective of the "tangential gradient" theory is to give a sense to the variation of a Lipschitz continuous function in the Lebesgue space with respect to γ, so that the integration by part formula has a sense Actually, for a suitable vector valued Radon measure Φ such that $\nabla \cdot \Phi =: \nu \in \mathcal{M}_b(\overline{\Omega})$ and a given Lipschitz continuous function u the formula

$$\int u \, d\nu = \int {}''\nabla u'' \cdot \sigma \, d\gamma$$

remains valid for a suitable "∇u"; the tangential gradient of u with respect to γ. Thanks to [9], this is possible if the measure Φ is a tangential measure. That is $\sigma(x) \in \mathcal{T}_\gamma(x)$, γ-a.e., where $\mathcal{T}_\gamma(x) \subseteq I\!R^n$ is the tangential space with respect to γ. In the case where γ coincides with the k-dimensional Hausdorff measure on a smooth k-dimensional manifold $S \subset I\!R^n$, $\mathcal{T}_\gamma(x)$ coincides γ-a.e. with the usual tangent bundle T_S given by differential geometry. In general, it coincides with

$$\mathcal{T}_\gamma(x) = \gamma - ess \cup \left\{ \sigma(x)\, ;\ \sigma \in L^1_\gamma(\overline{\Omega})^n,\ \nabla \cdot (\sigma\,\gamma) \in \mathcal{M}_b(\overline{\Omega}) \right\}.$$

Here, the γ-essential union is defined as a γ-measurable closed multifunction given by

- if $\sigma \in L^1_\gamma(\overline{\Omega})^n$ and $\nabla \cdot (\sigma\gamma) \in \mathcal{M}_b(\overline{\Omega})$, then $\sigma(x) \in \mathcal{T}_\gamma(x)$, for γ-a.e. $x \in \overline{\Omega}$.
- between all the multifunctions with the previous property, the γ essential union is minimal with respect to the inclusion γ-a.e.

Notice that the multifunction $\mathcal{T}_\gamma(x)$ is local on open subsets of $I\!R^n$; i.e. $T_\gamma = T_\nu$, γ-a.e. on Ω, if $\gamma \llcorner \Omega = \nu \llcorner \Omega$. Now, denoting by $P_\gamma(x)$ the orthogonal projection on $\mathcal{T}_\gamma(x)$, for γ-a.e $x \in \overline{\Omega}$, we have

Proposition 1 *(cf. [7]). The linear operator $u \in C^1(\overline{\Omega}) \to P_\gamma(x)\nabla u(x) \in L^\infty_\gamma(\overline{\Omega})^n$ can be extended uniquely to a continuous linear operator:*

$$\nabla_\gamma\ :\ Lip(\overline{\Omega}) \to \nabla_\gamma u \in L^\infty_\gamma(\overline{\Omega})^n$$

where $Lip(\overline{\Omega})$, the set of Lipschitz continuous function in $\overline{\Omega}$, is equipped with the uniform convergence on a bounded subsets of $\overline{\Omega}$, and $L^\infty_\gamma(\overline{\Omega})^n$ is equipped with the weak topology. Then, $\nabla_\gamma u$ is called the tangential gradient of u with respect to γ.*

Then, for the integration by part formula we have

Proposition 2 *(cf. [7]). Let $\gamma \in \mathcal{M}_b(\overline{\Omega})^+$ and $\sigma \in L^1(\overline{\Omega}, d\gamma)^n$ be such that $\sigma(x) \in \mathcal{T}_\gamma(x)$, γ-a.e. and $-\nabla \cdot (\sigma\,\gamma) =: \mu \in \mathcal{M}_b(\overline{\Omega})$. We have*

$$\int u \, d\mu = \int \sigma \cdot \nabla_\gamma u \, d\gamma, \qquad \textit{for any } u \in Lip(\overline{\Omega}).$$

The question now is to identify the set of vector valued Radon measure for which the integration by part formula is true. Thanks to the previous proposition, let us define

$$\mathcal{M}_{\mathcal{T}}(\overline{\Omega}) = \left\{ \sigma\gamma \ ; \ \gamma \in \mathcal{M}_b(\overline{\Omega})^+, \sigma(x) \in \mathcal{T}_\gamma(x), \ \gamma\text{-a.e.} \right\};$$

the so called tangential space of $\overline{\Omega}$.

Proposition 3 *(cf. [7]). Let $\lambda \in \mathcal{M}_b(\overline{\Omega})^n$ be given. Then, $\lambda \in \mathcal{M}_{\mathcal{T}}(\overline{\Omega})$ if and only if there exists $\Phi \in L^1(\Omega)^n$ such that $\nabla \cdot \lambda = \nabla \cdot \Phi$ in $\mathcal{D}'(\overline{\Omega})$.*

Thanks to Proposition 3, we see in particular that if $\lambda \in \mathcal{M}_b(\overline{\Omega})^n$ and $\nabla \cdot \lambda \in \mathcal{M}_b(\overline{\Omega})$, then $\lambda \in \mathcal{M}_{\mathcal{T}}(\overline{\Omega})$ and the integration by part formula of Proposition 2 remains true. Notice here, that formally the outward normal boundary value of a measure $\lambda \in \mathcal{M}_b(\overline{\Omega})^n$ such that $\nabla \cdot \lambda \in \mathcal{M}_b(\overline{\Omega})$ is null. This is not true in general if $\nabla \cdot \lambda \in \mathcal{M}_b(\Omega)$. So, when dealing with test functions that are not null on the boundary, we need to handle the outward normal trace of such vector valued measures. To this aim, we prove the following results.

Proposition 4. *Let $\mu \in \mathcal{M}_b(\Omega)$, $\gamma \in \mathcal{M}_b(\Omega)^+$ and $\Phi := \sigma\gamma \in \mathcal{S}(\mu)$, where $\sigma \in L^1(\Omega, d\gamma)$. We have*

(1) *$\sigma(x) \in \mathcal{T}_\gamma(x)$, γ-a.e. in Ω.*
(2) *For any $g \in \mathcal{C}^1(\partial\Omega)$, there exists $T_g(\Phi) \in I\!R$, such that*

$$\int_\Omega \nabla_\gamma \xi \cdot \sigma \, d\gamma + T_g(\psi) = \int_\Omega \xi \, d\mu, \tag{6}$$

for any $\xi \in W_g^{1,\infty}(\Omega)$.

In particular, $T_0(\Phi) = 0$. And, moreover $\mathcal{X} \in \mathcal{S}(\mu)$ if and only if

$$\int \nabla \xi(x) \cdot \mathcal{X}_r(x) \, dx + \int \nabla_{|\mathcal{X}_s|} \xi \, d\mathcal{X}_s = \int \xi \, d\mu - T_g(\psi), \tag{7}$$

for any $\xi \in W_g^{1,\infty}(\Omega)$.

Proof.

(1) For any $\varphi \in \mathcal{D}(\Omega)$, we see that $\nabla \cdot (\varphi\mathcal{X}) \in \mathcal{M}_b(\overline{\Omega})$. This implies that $\varphi(x)\mathcal{X}(x) \in \mathcal{M}_{\mathcal{T}}(\overline{\Omega})$. So, for any $\varphi \in \mathcal{D}(\Omega)$, $\varphi(x)\sigma(x) \in \mathcal{T}_\gamma(x)$, γ-a.e. in Ω. Thus, $\sigma(x) \in \mathcal{T}_\gamma(x)$, γ-a.e. in Ω.
(2) To prove the second part of the proposition, we define $\eta \in \mathcal{D}'(I\!R^n)$ by

$$\left\langle \eta, \xi \right\rangle = \int \xi \, d\mu - \int \nabla \xi \, d\Phi, \qquad \text{for any } \xi \in \mathcal{D}(I\!R^n).$$

It is not difficult to see that η is a distribution of order 1 and, since $\Phi \in \mathcal{S}(\mu)$, $supp(\eta) \subseteq \partial\Omega$. In particular, for a fixed $g \in \mathcal{C}^1(\partial\Omega)$, considering an arbitrary $\tilde{g} \in \mathcal{C}^1(\Omega)$ such that $\tilde{g} = g$ on $\partial\Omega$, we can define

$$< \eta, \tilde{g} > =: T_g(\Phi).$$

It is clear here, that such definition is independent of the choice of $\tilde{g}$ in Ω, and for any $\xi \in \mathcal{C}^1(\Omega)$ such that $\xi = g$ on $\partial\Omega$, we have

$$\begin{aligned}\int_\Omega \xi \, d\mu &= \int_\Omega \nabla\xi \cdot \sigma \, d\gamma + T_g(\psi) \\ &= \int_\Omega P_\gamma(x)\nabla\xi(x) \cdot \sigma(x) \, d\gamma(x) + T_g(\psi) \\ &= \int_\Omega \nabla_\gamma\xi \cdot \sigma \, d\gamma + T_g(\psi),\end{aligned}$$

where we use the fact that $\sigma(x) \in \mathcal{T}_\gamma(x)$, γ-a.e. in Ω. Then, using Proposition 1, we deduce that (6) remains true for any $\xi \in W^{1,\infty}(\Omega)$, such that $\xi = g$ on $\partial\Omega$.

At last, the last part is a simple consequence of (1), (2) and the fact that $\Phi \in \mathcal{S}(\mu)$ implies that $\Phi_s \in \mathcal{T}_{|\Phi_s|}(x)$, $|\Phi_s|$-a.e. in Ω. □

Remark 1. It is clear that $T_g(\psi)$ is connected to the trace of ψ on $\partial\Omega$. Indeed, formally $T_g(\psi) = \int_{\partial\Omega} g \, ''\psi \cdot n'' d\mathcal{L}^{n-1}$, where n is the outward normal to $\partial\Omega$. Thanks to G.Q. Chen and H. Frid (cf. [12–14]), this trace $''\psi \cdot n''$ can be defined rigorously under regularity assumptions on $\partial\Omega$. However, in general it is not a measure and not even in the dual space of $Lip(\overline{\Omega})$ (unless $\psi \in L^p(\Omega, d\mathcal{L}^n)^n$, with $1 < p \leq \infty$). Since in our situation the test functions have a trace g which is $\mathcal{C}^1$, we choose to work with a $\mathcal{C}^1$-recovery of g in Ω to define the quantity $T_g(\psi)$ and avoid all the technical arguments related to the weak trace in the sense of G.Q. Chen and H. Frid.

2.3. Technical lemmas

Thanks to the assumption (J2), the set $D(x)$ is convex for any $x \in \Omega$. For any $x \in \Omega$, let us denote by $S_{D(x)}$ the support function of $D(x)$, given by

$$S_{D(x)}(p) = \sup\Big\{p \cdot q \; ; \; q \in D(x)\Big\}, \qquad \text{for any } (x, p) \in \Omega \times I\!R^n.$$

Recall that, for any $x \in \Omega$, the function $\xi \in I\!R^n \to S_{D(x)}(\xi)$ is nonnegative, convex and positively homogeneous. So, thanks to [3] (see also [2]), for any $\Phi \in \mathcal{M}_b(\Omega)^n$, the Radon measure $S_{D(.)}(\Phi) \in \mathcal{M}_b(\Omega)$ is well defined by the following formula

$$S_{D(.)}(\Phi)(B) = \int_B S_{D(x)}(\Phi_r(x)) \, dx + \int_B S_{D(x)}\left(\frac{\Phi_s(x)}{|\Phi_s|(x)}\right) d|\Phi_s|(x),$$

for any Borel set $B \subseteq \Omega$.

Moreover, if $\Phi \ll \lambda$, for a given $\lambda \in \mathcal{M}_b(\Omega)^+$, then

$$S_{D(.)}(\Phi)(B) = \int_B S_{D(x)}\left(\frac{d\Phi}{d\gamma}(x)\right) d\gamma(x) \qquad \text{for any Borel set } B \subseteq \Omega.$$

In particular, for any $\Phi \in \mathcal{M}_b(\Omega)^n$, we have

$$S_{D(.)}(\Phi)(B) = \int_B S_{D(x)}\left(\frac{\Phi(x)}{|\Phi|(x)}\right) d|\Phi|(x) \quad \text{for any Borel set } B \subseteq \Omega.$$

The following result which is based on the possibility of approximating functions such that the gradient is in the domain of J by smooth function is important for the proof of our main results.

Proposition 5. *Let $\gamma \in \mathcal{M}_b(\Omega)^+$, $g \in \mathcal{C}^1(\partial\Omega)$ and $\sigma \in L^1(\overline{\Omega}, d\gamma)^n$ be such that $\sigma(x) \in \mathcal{T}_\gamma(x)$, γ-a.e. $x \in \Omega$. If $u \in W_g^{1,\infty}(\Omega)$ is such that $\nabla u(x) \in \overline{D(x)}$, $\mathcal{L}^n$-a.e. $x \in \Omega$, then*

(1) *There exists a sequence $(u_\epsilon)_{\epsilon>0}$ in $\mathcal{C}^1(\overline{\Omega})$ such that $u_\epsilon = g$ on $\partial\Omega$, $\nabla u_\epsilon(x) \in \overline{D(x)}$, for any $x \in \Omega$ and $u_\varepsilon \to u$ in $W^{1,\infty}(\Omega)$-weak*; in the sense that u_ϵ and ∇u_ϵ converges to u and ∇u in $L^\infty(\Omega)$-weak* and $L^\infty(\Omega)^n$-weak*, respectively.*
(2) *We have*

$$\sigma(x) \cdot \nabla_\gamma u(x) \le S_{D(x)}(\sigma(x)), \quad \gamma\text{-a.e. } x \in \Omega. \tag{8}$$

Proof. First, let us prove the result in the case where $g \equiv 0$. Following the same idea of the proof of Lemma 3.2 [23], for a given $\epsilon > 0$, we consider the application $I_\epsilon : I\!R \to I\!R$, defined by

$$I_\epsilon(r) = \begin{cases} 0 & \text{if } |r| \le \epsilon \\ r - sign(r)\,\epsilon & \text{if } |r| > \epsilon. \end{cases}$$

Then, we choose

$$\tilde{u}_\epsilon = I_\epsilon(u), \quad \text{a.e. in } \Omega.$$

One sees that $\tilde{u}_\epsilon$ is compactly supported in Ω. Moreover, there exists $0 < \alpha < 1$ and $\epsilon_0 > 0$, such that

$$z_\epsilon = \tilde{u}_\epsilon * \rho_{\alpha\epsilon} \ \in \mathcal{D}(\omega), \quad \text{for any } 0 < \epsilon < \epsilon_0,$$

where $\omega \subset\subset \Omega$. Now, for any $x \in I\!R^n$, let us consider the dual function of $S_{D(x)}$ given by

$$S^*_{D(x)}(q) = \max\left\{q \cdot p\, ;\ S_{D(x)}(p) \le 1\right\}.$$

Recall that $q \in \overline{D(x)}$ if and only if $S^*_{D(x)}(q) \le 1$. Now, arguing like in the proof of Lemma 1 of [22] (see also the proof of Lemma 3.1 [6]), we consider

$$\omega(\delta) := \sup\left\{\left|S^*_{D(x)}(A) - S^*_{D(y)}(A)\right|\ ;\ |x-y| \le \delta \text{ and } |A| \le \|\nabla u\|_\infty\right\},$$

the uniform modulus of continuity $x \to S^*_{D(x)}(A)$. Then, we set

$$u_\epsilon := \frac{1}{1+\omega(\alpha\epsilon)} z_\epsilon \in \mathcal{D}(\Omega).$$

It is not difficult to see that $u_\varepsilon \to u$ in $W^{1,\infty}(\Omega)$-weak*. Moreover, we have

$$S^*_{D(x)}(\nabla u_\varepsilon(x)) \le 1.$$

Indeed, using Jensen inequality, we have

$$\begin{aligned}
S^*_{D(x)}(\nabla u_\varepsilon(x)) &\le \frac{1}{1+\omega(\alpha\epsilon)} \int \rho_{\alpha\epsilon}(x-y) S^*_{D(x)}(\nabla u(y))\, dy \\
&\le \frac{1}{1+\omega(\alpha\epsilon)} \int \rho_{\alpha\epsilon}(x-y) S^*_{D(y)}(\nabla u(y))\, dy \\
&\quad + \frac{1}{1+\omega(\alpha\epsilon)} \int \rho_{\alpha\epsilon}(x-y) \left(S^*_{D(x)}(\nabla u(y)) - S^*_{D(y)}(\nabla u(y)) \right) dy \\
&\le 1.
\end{aligned}$$

Now, for any open subset $B \subset \Omega$, using Proposition 1 and the fact that $\sigma(x) \in \mathcal{T}_\gamma(x)$, γ-a.e. x, we have

$$\begin{aligned}
\int_B \sigma \cdot \nabla_\gamma u \, d\gamma &= \lim_{\varepsilon\to 0} \int_B \sigma \cdot \nabla_\gamma u_\varepsilon \, d\gamma \\
&= \lim_{\varepsilon\to 0} \int_B \sigma \cdot \nabla u_\varepsilon \, d\gamma \\
&\le \lim_{\varepsilon\to 0} \int_B S_{D(x)}(\sigma(x))\, S^*_{D(x)}(\nabla u_\varepsilon(x))\, d\gamma(x) \\
&\le \int_B S_{D(x)}(\sigma(x)) d\gamma(x).
\end{aligned}$$

For the general case, we consider $\tilde{g} \in \mathcal{C}^1(\Omega)$ be such that $\tilde{g} = g$ on $\partial\Omega$. It is clear that $\tilde{u} = u - \tilde{g} \in W^{1,\infty}_g(\Omega)$ and, for $\mathcal{L}^n$-a.e. $x \in \Omega$, $\nabla\tilde{u}(x) \in \{q - \nabla\tilde{g}(x); q \in D(x)\} =: \tilde{D}(x)$ which is a convex set in turn. Thanks to the first part of the proof, there exists a sequence $(\tilde{u}_\epsilon)_{\epsilon>0}$ in $\mathcal{D}(\Omega)$, such that $\nabla\tilde{u}_\epsilon(x) \in \tilde{D}(x)$, for any $x \in \Omega$ and $\tilde{u}_\varepsilon \to \tilde{u}$ in $W^{1,\infty}(\Omega)$-weak*. Now, taking $u_\epsilon = \tilde{u}_\epsilon + \tilde{g}$, we see that $u_\epsilon \in \mathcal{C}^1(\overline{\Omega})$, $u_\epsilon = g$ on $\partial\Omega$, $\nabla u_\epsilon(x) \in D(x)$, for any $x \in \Omega$ and $u_\varepsilon \to u$ in $W^{1,\infty}(\Omega)$-weak*. Using the definition of $S_{\tilde{D}(x)}$ and $S_{D(x)}$, we see that

$$S_{\tilde{D}(x)}(\sigma(x)) + \sigma(x)\cdot\nabla\tilde{g}(x) = S_{D(x)}(\sigma(x)), \quad \gamma\text{-a.e. } x \in \Omega.$$

Thus

$$\begin{aligned}
\int_B \sigma \cdot \nabla_\gamma u \, d\gamma &= \lim_{\varepsilon \to 0} \int_B \sigma \cdot \nabla u_\varepsilon \, d\gamma \\
&= \lim_{\varepsilon \to 0} \int_B \sigma \cdot (\nabla \tilde{u}_\varepsilon + \nabla \tilde{g}) \, d\gamma \\
&= \lim_{\varepsilon \to 0} \int_B \sigma \cdot \nabla \tilde{u}_\varepsilon \, d\gamma + \int_B \sigma \cdot \nabla \tilde{g} \, d\gamma \\
&\leq \int_B \big(S_{\tilde{D}(x)}(\sigma(x)) + \sigma(x) \cdot \nabla \tilde{g}(x) \big) d\gamma(x) \\
&\leq \int_B S_{D(x)}(\sigma(x)) d\gamma(x).
\end{aligned}$$

This ends up the proof of the proposition. □

Proposition 6. *Let $\gamma \in \mathcal{M}_b(\Omega)^+$ and $\sigma \in L^1(\Omega, d\gamma)^n$ be such that $\sigma(x) \in \mathcal{T}_\gamma(x)$, γ-a.e. $x \in \Omega$. If $u \in W^{1,\infty}(\Omega)$ and $\nabla u(x) \in \overline{D(x)}$, $\mathcal{L}^n$-a.e. $x \in \Omega$, then the following assertions are equivalent:*

(1) $\sigma(x) \cdot \nabla_\gamma u(x) = S_{D(x)}\,(\sigma(x))$, *$\gamma$-a.e. $x \in \Omega$.*
(2) $\int S_{D(x)}\,(\sigma(x)) \, d\gamma(x) \leq \int \nabla_\gamma u \cdot \sigma \, d\gamma.$

Moreover, if $\nabla_\gamma u(x) \in \overline{D(x)}$, $\mathcal{L}^n$-a.e. $x \in \Omega$, then (1) and (2) are equivalent to

$$\sigma(x) \in \partial I\!I_{\overline{D(x)}}(\nabla_\gamma u(x)) \quad \gamma\text{-a.e. } x \in \Omega.$$

Proof. The proof is a simple consequence of Proposition 5 and the definition of $\partial I\!I_{\overline{D(x)}}$. □

3. Main results

Let $\mu \in \mathcal{M}_b(\Omega)$ and $g \in \mathcal{C}^1(\partial\Omega)$ be given, where $\Omega \subset I\!R^n$ is an open bounded domain with Lipschitz boundary. Under the condition (4), the equation (1) with Dirichlet boundary condition reads

$$(P_1) \qquad \begin{cases} \left.\begin{array}{l} -\nabla \cdot \Phi = \mu \\ \Phi \in \partial_\xi J(., \nabla u) \end{array}\right\} & \text{in } \Omega \\ u = g & \text{on } \partial\Omega. \end{cases}$$

To set our first main result, we consider

$$K := \Big\{ z \in W^{1,\infty}_g(\Omega) \,:\, \nabla z(x) \in D(x), \ \mathcal{L}^n\text{-a.e. } x \in \Omega \Big\}.$$

See here that, in general K could be an empty set. Indeed, it is necessary to assume moreover that the function $g \in \mathcal{G}$, where

$$\mathcal{G} := \Big\{ g \in C(\partial\Omega) \ : \ \exists g_0 \in W^{1,\infty}_g(\Omega), \ \nabla g_0(x) \in D(x), \ \mathcal{L}^n\text{-a.e. } x \in \Omega \Big\}.$$

Remark 2.

(1) $\mathcal{G} \neq \emptyset$. Indeed, $0 \in \mathcal{G}$.
(2) Thanks to [21], if $D(x)$ is closed then

$$\mathcal{G} = \Big\{ g \in C(\partial\Omega) \ : \ g(x) - g(y) \leq S_J(y, x) \Big\},$$

where, for any $y, x \in \Omega$,

$$S_J(y, x) = \inf \left\{ \int_0^1 S^*_{D(\varphi(t))}(\dot{\varphi}(t))dt \ : \ \varphi \in L_{y,x} \right\}$$

and

$$L_{y,x} = \Big\{ \varphi \in \mathcal{C}^1([0, 1], \Omega) \ : \ \varphi(0) = y, \varphi(1) = x \Big\}.$$

Indeed, to find $g_0 \in W^{1,\infty}(\Omega)$, such that $\nabla g_0(x) \in D(x)$, a.e. $x \in \Omega$, and $g_0 = g$ on $\partial\Omega$, is equivalent to find a subsolution u to the Hamilton–Jacobi equation (in a viscosity sense)

$$\begin{cases} F(x, \nabla u) = 1 & \text{in } \Omega \\ u = g & \text{on } \partial\Omega, \end{cases} \tag{9}$$

where $F(x, p) = S^*_{D(x)}(p)$, for any $(x, p) \in \Omega \times I\!R^N$. Thanks to [21], u is a subsolution of $F(x, \nabla u) = 1$ in Ω if and only if $u(x) - u(y) \leq S_J(y, x)$, for any $x, y \in \Omega$. So, in one hand, by continuity we have

$$\mathcal{G} \subseteq \Big\{ g \in C(\partial\Omega) \ : \ g(x) - g(y) \leq S_J(y, x) \Big\}.$$

On the other hand, thanks again to [21] (see Proposition 4.7), if $g \in C(\partial\Omega)$ satisfies $g(x) - g(y) \leq S_J(y, x)$, for any $x, y \in \partial\Omega$, then the equation has a subsolution given by the Hopf–Lax explicit formula

$$u(x) = \min \Big\{ g(y) + S_J(y, x) \ : \ y \in \partial\Omega \Big\}.$$

Now, throughout this section we assume that $g \in \mathcal{G} \cap \mathcal{C}^1(\partial\Omega)$, and we fix $g_0 \in \mathcal{C}^1(\Omega)$ such that

$$\nabla g_0(x) \in D(x), \ \mathcal{L}^n\text{-a.e. } x \in \Omega \text{ and } g_0 = g \text{ on } \partial\Omega.$$

Theorem 1. *Assume that J satisfies the assumptions (J1)–(J2). For any $\mu \in \mathcal{M}_b(\Omega)$ and $g \in \mathcal{G} \cap \mathcal{C}^1(\partial\Omega)$, the problem*

$$(P_2) \qquad \min\left\{ \int_\Omega J(x, \nabla z(x))\, dx - \int_\Omega z\, d\mu \; ; \; z \in W^{1,\infty}_g(\Omega) \right\}$$

has a solution u. If, moreover J satisfies (J3), then u is a solution of (P_2) if and only if $u \in K$ and, there exists $\Phi \in \mathcal{M}_b(\Omega)^n$ such that

$$\Phi_r(x) \in \partial_\xi J(x, \nabla u(x)), \quad \mathcal{L}^n\text{-a.e. } x \in \Omega \tag{10}$$

$$\frac{\Phi_s}{|\Phi_s|}(x) \cdot \nabla_{|\Phi_s|} u(x) = S_{D(x)}\left(\frac{\Phi_s}{|\Phi_s|}(x)\right), \quad |\Phi_s|\text{-a.e. in } \Omega \tag{11}$$

and

$$\int_\Omega \Phi_r \cdot \nabla\xi\, dx + \int_\Omega \nabla_{|\Phi_s|}\xi\, d\Phi_s = \int_\Omega \xi\, d\mu, \quad \text{for any } \xi \in C^1_0(\Omega). \tag{12}$$

Thanks to Proposition 6, if $\nabla_{|\Phi_s|} u(x) \in \overline{D(x)}$, $|\Phi_s|$-a.e. $x \in \Omega$, then (11) is equivalent to

$$\frac{\Phi_s}{|\Phi_s|}(x) \in \partial I\!I_{\overline{D(x)}}(\nabla_{|\Phi_s|} u(x)) \quad |\Phi_s|\text{-a.e. } x \in \Omega. \tag{13}$$

Roughly speaking (11) with the fact that $\nabla u(x) \in D(x)$, $\mathcal{L}^n$-a.e. $x \in \Omega$, is a generalized formulation of the standard one (13).

Corollary 1. *Assume that J satisfies the assumptions (J1)–(J3). If, we assume moreover that $J(x, .)$ is symmetric for any $x \in \Omega$, then u is a solution of (P_2) if and only if $u \in K$ and there exists $\Phi \in \mathcal{M}_b(\Omega)^n$ such that (10), (12) and (13) are fulfilled.*

Formally, we can say that the problem (P_1) is governed by the following formulation

$$(P_1') \qquad \begin{cases} \left.\begin{array}{l} -\nabla \cdot \Phi = \mu \\ \Phi_r \in \partial_\xi J(x, \nabla u), \quad \Phi_s \cdot \nabla_{|\Phi_s|} u = S_{D(x)}(\Phi_s) \end{array}\right\} & \text{in } \Omega \\ u = g & \text{on } \partial\Omega. \end{cases}$$

Throughout the paper, the couple $(u, \Phi) \in W^{1,\infty}(\Omega) \times \mathcal{M}_b(\Omega)^n$ given by Theorem 1 will be called a weak solution of (P_1), and (P_1') will be called the weak formulation of (P_1). As to the problem (P_2), thanks to Theorem 2, it is simply the minimization problem associated with (P_1).

Corollary 2. *For any $\mu \in \mathcal{M}_b(\Omega)$ and $g \in \mathcal{G} \cap \mathcal{C}^1(\partial\Omega)$, the problem (P_1) has a weak solution (u, Φ).*

In particular, by using (11) we can deduce the existence of a solution for variational formulation associated with the problem (P_1) as well as its equivalence with a weak formulation and the minimization problem.

Corollary 3. *Under the assumptions (J1–J3), let $\mu \in \mathcal{M}_b(\Omega)$, $g \in \mathcal{G} \cap \mathcal{C}^1(\partial\Omega)$ and $u \in K$ be given. Then, u is a solution of (P_2) if and only if there exists $\Psi \in L^1(\Omega)^n$ such that $\Psi(x) \in \partial_\xi J(x, \nabla u(x))$, $\mathcal{L}^n$-a.e. $x \in \Omega$, and*

$$\int_\Omega \nabla(u(x) - \xi(x)) \cdot \Psi(x)\, dx \le \int_\Omega (u - \xi)\, d\mu, \quad \textit{for any } \xi \in K. \tag{14}$$

The equation (14) will be called the variational formulation associated with (P_1) and $(u, \Phi) \in K \times L^1(\Omega)^n$ given by Corollary 3 is a variational solution of (P_1).

The equivalence between the three formulations is summarized in the following Corollary

Corollary 4. *Under the assumptions (J1–J3), let $\mu \in \mathcal{M}_b(\Omega)$, $g \in \mathcal{G} \cap \mathcal{C}^1(\partial\Omega)$ and $(u, \Phi) \in K \times M_b(\Omega)^n$ be given. The following propositions are equivalent:*

(1) *(u, Φ) is a weak solution of (P_1).*
(2) *(u, Φ_r) is a variational solution of (P_1).*
(3) *u is a solution of the minimizing problem.*

To study Legendre–Fenchel's duality associated with the problem (P_2), we define $J^* : \Omega \times \mathbb{R}^n \to \mathbb{R}$ the dual function of $J(., \xi)$ as follows:

$$J^*(x, y^*) = \sup\left\{\langle y^*, \xi\rangle - J(x, \xi) : \xi \in \mathbb{R}^n\right\}; \quad \text{for any } x \in \Omega.$$

We have

Theorem 2. *Under the assumptions (J1–J3), for any $\mu \in \mathcal{M}_b(\Omega)$ and $g \in \mathcal{G} \cap \mathcal{C}^1(\partial\Omega)$ the problem*

$$(P_3) \qquad \min\left\{\int_\Omega J^*(x, \psi_r(x))\, dx + \int_\Omega S_{D(x)}\left(\frac{\psi_s(x)}{|\psi_s|(x)}\right) d|\psi_s|(x) - T_g(\psi)\,;\ \psi \in \mathcal{S}(\mu)\right\}$$

has a solution $\Phi \in \mathcal{S}(\mu)$. Moreover, Φ is a solution (P_3) if and only if there exists $u \in K$ such that (u, Φ) is a weak solution of (P_1).

4. Proofs of the main results

Thanks to the assumption (J1), it is clear that, for any $p \ge 1$,

$$J(x, \xi) \ge ((|\xi| - M(x))^+)^p, \quad \text{for any } (x, \xi) \in \Omega \times I\!R^n. \tag{15}$$

To prove our main results, we begin by fixing $p > n$ and consider the Yosida approximation of J given by

$$J_\lambda(x,\delta) = \min_{y\in D(x)} \left\{ J(x,y) + \frac{1}{p\lambda^{p-1}} \|y-\delta\|^p \right\}.$$

We note that $J_\lambda(x,.)$ is convex, C^1 and its gradient $\nabla_\xi J_\lambda(x,.)$ is Lipschitz continuous.

4.1. Regularized problem and compactness

First, we begin with the regular minimizing problem:

$$\min \left\{ \int_\Omega J_\lambda(x,\nabla z) - \int_\Omega z d\mu \;;\; z \in W^{1,p}_g(\Omega) \right\}. \tag{16}$$

Lemma 1. *For any $\lambda > 0$, there exists $u_\lambda \in W^{1,\infty}_g(\Omega)$ solution of the problem (16). Moreover $w_\lambda := \nabla_\xi J_\lambda(x,\nabla u) \in L^1(\Omega)^n$ satisfies the PDE*

$$-\nabla \cdot w_\lambda = \mu \quad \text{in } \Omega \tag{17}$$

Proof. Let us consider the functional

$$z \in W^{1,p}(\Omega) \to \mathcal{I}(z) = \int_\Omega J_\lambda(x,\nabla z) - \int_\Omega z d\mu.$$

Since J_λ is convex, C^1 and bounded below, the functional $z \in W^{1,p}(\Omega) \to \int_\Omega J_\lambda(x,\nabla z(x))\,dx$ is lower semi-continuous. Thus $\mathcal{I}$ is l.s.c. Moreover, since J_λ is coercive in the closed set $W^{1,p}_g(\Omega)$, the minimizing problem (16) has a solution $u_\lambda \in W^{1,p}_g(\Omega)$. At last, since the function $J_\lambda(x,.)$ satisfies the growth condition, we deduce the second part of the proof of the lemma (cf. Theorem 3.37 [16]). □

Lemma 2. *The sequences $(u_\lambda)_{0<\lambda<1}$ and $(w_\lambda)_{0<\lambda<1}$ are bounded in $W^{1,p}(\Omega)$ and $L^1(\Omega)^n$, respectively. Moreover, there exists $C(p) = C(\Omega, p, \mu, g_0)$ bounded as $p \to \infty$, such that*

(1) *for any $0 < \lambda < 1$, we have*

$$\int_\Omega (|\nabla u_\lambda| - M(x))^{+p} \le pC(\Omega, p, \mu, g_0). \tag{18}$$

(2) *for any $\xi \in \mathcal{C}_0(\Omega)$, such that $\xi(x) \in D(x)$, for any $x \in \Omega$, we have*

$$\int_\Omega \xi\, d\Phi_\lambda \le \int_\Omega J(.,\xi)\,dx + C(p). \tag{19}$$

Proof. Using the inequality (related to the convexity of $\xi \in I\!R^n \to |\xi|^p$)

$$\frac{1}{2^{p-1}} |\xi|^p \le ((|\xi| - M(x))^+)^p + |M(x)|^p.$$

So, it is clear that by proving the estimate (18), we prove in turn that $(u_\lambda)_{\lambda>0}$ is bounded in $W^{1,p}(\Omega)$. To prove (18), we see first that

$$0 \le \int_\Omega J_\lambda(x, \nabla u_\lambda) \le C_p \left(\|\nabla(u_\lambda - g_0)\|_{L^p(\Omega)} + 1 \right), \tag{20}$$

where $C_p := C(\Omega, p, g_0)$ is a bounded constant as $p \to \infty$. Indeed, since $J_\lambda(x, .)$ is convex, for any $x \in \Omega$, we have

$$J_\lambda(x, \nabla u_\lambda) \le \nabla_\xi J_\lambda(x, \nabla u_\lambda) \cdot \nabla(u_\lambda - g_0) + J_\lambda(x, \nabla g_0), \qquad \text{for any } \lambda > 0.$$

So, using the fact that $u_\lambda - g_0$ in $W_0^{1,p}(\Omega)$, we get

$$\begin{aligned} 0 \le \int_\Omega J_\lambda(x, \nabla u_\lambda) &\le \int_\Omega \nabla_\xi J_\lambda(x, \nabla u_\lambda) \cdot (\nabla u_\lambda - \nabla g_0) + \int_\Omega J_\lambda(x, \nabla g_0) \\ &\le \int_\Omega (u_\lambda - g_0)\, d\mu + \int_\Omega J_\lambda(x, \nabla g_0) \\ &\le C(\Omega, p, g_0)\, \|\nabla(u_\lambda - g_0)\|_{L^p} + \int_\Omega J_\lambda(x, \nabla g_0), \end{aligned}$$

where $C(\Omega, p, g_0)$ depends only on the constant of Poincaré inequality, on the norm of the continuous embedding of $L^\infty(\Omega)$ in $W^{1,p}(\Omega)$ and on $|\mu|(\Omega)$. Then, since $\nabla g_0(x) \in D(x)$, we see that $\int_\Omega J_\lambda(x, \nabla g_0)$ converges to $\int_\Omega J(x, \nabla g_0)$, as $\lambda \to 0$. Thus $\int_\Omega J_\lambda(x, \nabla g_0)$ is bounded and (20) follows. Now, we see that

$$((|\xi| - M(x))^+)^p \le p J_\lambda(x, \xi), \qquad \text{for any } \xi \in I\!R^n \text{ and } 0 < \lambda < 1. \tag{21}$$

Indeed, for a given $x \in \Omega$, any $\xi \in \mathbb{R}^n$, there exist $y \in \mathbb{R}^n$, $|y| \le M(x)$, such that

$$J_\lambda(x, \xi) = J(y) + \frac{1}{p\lambda^{p-1}} |\xi - y|^p .$$

Using the assumption (J1) and the fact that $|y| \le M(x)$, we get:

$$\begin{aligned} J_\lambda(x, \xi) &\ge ((|y| - M(x))^+)^p + \frac{1}{p} \Big| |\xi| - |y| \Big|^p \\ &\ge \frac{1}{p} \Big| |\xi| - |y| \Big|^p \\ &\ge \frac{1}{p} ((|\xi| - M(x))^+)^p . \end{aligned}$$

Thus (21). Using (20), (21) and Young inequality we get

$$\int_\Omega (|\nabla u_\lambda| - M(x))^{+p}\, dx \le pC_p \left(\|\nabla(u_\lambda - g_0)\|_{L^p(\Omega)} + 1\right)$$
$$\le pC_p \left(\left\|(|\nabla u_\lambda| - M)^+\right\|_{L^p} + \||\nabla g_0| + M\|_{L^p} + 1\right)$$
$$\le pC_p \left(\frac{\epsilon^p}{p}\int_\Omega (|\nabla u_\lambda| - M(x))^{+p} + \frac{1}{\epsilon^{p'} p'} + \||\nabla g_0| + M\|_{L^p} + 1\right).$$

Taking $\epsilon^p = \frac{1}{2C_p}$, we have $\epsilon^{p'} = \frac{1}{(2C_p)^{\frac{1}{p-1}}}$ and

$$\int_\Omega (|\nabla u_\lambda| - M(x))^{+p}\, dx \le 2pC_p \left(\frac{(2C_p)^{\frac{1}{p-1}}}{p'} + \||\nabla g_0| + M\|_{L^p} + 1\right) =: pC(p).$$

Since the Poincaré constant $C(\Omega, p)$ is bounded as p tends to $+\infty$ (cf. [15]) and $g_0 \in \mathcal{C}^1(\Omega)$, we deduce that $C(p)$ is bounded as $p \to \infty$. Thus (18). To prove (19), recall that for any $\xi \in I\!R^n$ and a.e. $x \in \Omega$, we have

$$w_\lambda(x) \cdot \xi \le J_\lambda(x, \xi) + w_\lambda(x) \cdot \nabla u_\lambda(x) - J_\lambda(x, \nabla u_\lambda(x))$$
$$\le J_\lambda(x, \xi) + w_\lambda(x) \cdot \nabla u_\lambda(x).$$

This implies that

$$\int_\Omega w_\lambda(x) \cdot \xi\, dx \le \int_\Omega J_\lambda(x, \xi)\, dx + \int_\Omega w_\lambda(x) \cdot \nabla u_\lambda(x) dx$$
$$\le \int_\Omega J_\lambda(x, \xi)\, dx + \int_\Omega (u_\lambda - g_0)\, d\mu.$$

In one hand, thanks to the first part, it is clear that $\int_\Omega (u_\lambda - g_0)\, d\mu$ is less or equal to a constant that we denote again $C(p)$, which is bounded as $p \to \infty$. Let us prove that $(w_\lambda)_{\lambda>0}$ is bounded in $L^1(\Omega)^n$. Thanks to (J3), for any $x \in \Omega$, we see that $\int_\Omega J(x, \xi)\, dx$ is bounded for any $\xi \in B(0, \alpha)$. This implies that $w_\lambda \cdot \xi$ is bounded in $L^1(\Omega)$, for any $\xi \in B(0, \alpha)$. And then, w_λ is bounded in $L^1(\Omega)^n$. Indeed, it's enough to take $\xi = \frac{\alpha w_\lambda}{2|w_\lambda|}$. This ends up the proof of the lemma. □

Lemma 3. *There exists $(u_p, \Phi_p) \in W^{1,p}(\Omega) \times \mathcal{M}_b(\Omega)^n$ and a subsequence that we denote again by $\lambda \to 0$, such that*

$$u_\lambda \to u_p, \quad \text{in } W^{1,p}(\Omega)\text{-weak} \tag{22}$$

and

$$\omega_\lambda \to \Phi_p, \quad \text{in } \mathcal{M}_b(\Omega)^n\text{-weak}^*. \tag{23}$$

Moreover, $u_p = g$ *in* $\partial\Omega$ *and we have*

(1) *The measure* Φ_p *satisfies* $-\nabla\cdot\Phi_p = \mu$, *in* Ω.
(2) *For any* $\xi \in \mathbb{R}^n$, $\varphi \in \mathcal{D}(\Omega)$, $z \in \mathcal{C}^1(\Omega)$ *and* $0 < \lambda_0 < 1$, *we have*

$$\int_\Omega J(x,\xi)\varphi \geq \int_\Omega J_{\lambda_0}(x,\nabla u_p)\varphi + \int_\Omega \varphi\,(\xi - \nabla z) d\Phi_p + \int_\Omega \varphi\,(z - u_p)\,d\mu + \int_\Omega (u_p - z)\,\nabla\varphi\,d\Phi_p. \tag{24}$$

(3) *The sequence* $\big(u_p, \Phi_p\big)_{p\geq n}$ *satisfies, for any* $p \geq n$,

$$\int_\Omega (\left|\nabla u_p\right| - M(x))^{+p} \leq pC(p), \tag{25}$$

and

$$\int_\Omega \xi\,d\Phi_p \leq \int_\Omega J(x,\xi)\,dx + C(p), \qquad \textit{for any } \xi \in D(x), \tag{26}$$

where $C(p)$ *is given by Lemma 2.*

Proof. Thanks to Lemma 2, there exist u_p in $W^{1,p}(\Omega)$, $\Phi_p \in \mathcal{M}_b(\Omega)^n$ and a subsequence such that (22) and (23) are fulfilled. Moreover, $u_p = g$ on $\partial\Omega$ and by using the Rellich–Kondrachov Theorem [11, Theorem 9.16], as $\lambda \to 0$, we have

$$u_\lambda \to u_p, \qquad \text{in } C(\bar{\Omega})$$

and

$$\int_\Omega u_\lambda d\mu \to \int_\Omega u_p\,d\mu.$$

Recall that for any $\varphi \in \mathcal{D}(\Omega)$ such that $\varphi \geq 0$, we have

$$\int_\Omega J(x,\xi)\varphi \geq \int J_\lambda(x,\nabla u_\lambda)\varphi + \int_\Omega \omega_\lambda(\xi - \nabla u_\lambda)\varphi.$$

Since, for any $(x,\xi) \in \Omega \times I\!R^n$, $(J_\lambda(x,\xi))_{\lambda\geq 0}$ is nondecreasing with respect to λ, for any $0 < \lambda \leq \lambda_0 < 1$, we have:

$$\begin{aligned}\int_\Omega J_\lambda(x,\xi)\varphi &\geq \int_\Omega J_{\lambda_0}(x,\nabla u_\lambda)\varphi + \int_\Omega \omega_\lambda\cdot(\xi-\nabla u_\lambda)\varphi \\ &\geq \int_\Omega J_{\lambda_0}(x,\nabla u_\lambda)\varphi + \int_\Omega \omega_\lambda\cdot(\xi-\nabla z)\varphi + \int_\Omega \omega_\lambda\cdot\nabla(\varphi\,(u_\lambda - z)) \\ &\quad + \int_\Omega \omega_\lambda\cdot\nabla\varphi\,(u_\lambda - z) \\ &\geq \int_\Omega J_{\lambda_0}(x,\nabla u_\lambda)\varphi + \int_\Omega \omega_\lambda\cdot(\xi-\nabla z)\varphi - \int_\Omega \varphi\,(u_\lambda - z)d\mu \\ &\quad + \int_\Omega \omega_\lambda\cdot\nabla\varphi\,(u_\lambda - z). \end{aligned} \tag{27}$$

Using the convexity and the l.s.c. of $\xi\in W^{1,p}(\Omega)\to\int J_{\lambda_0}(x,\xi(x))\,dx$, we deduce that

$$\int_\Omega J_{\lambda_0}(x,\nabla u_p)\varphi \leq \liminf_{\lambda\to 0}\int_\Omega J_{\lambda_0}(x,\nabla u_\lambda)\varphi.$$

So, letting $\lambda\to 0$ in (27), we get (24). The last part of the lemma follows by letting $\lambda\to 0$ in (18) and (19). □

Lemma 4. *Let $n\leq q<\infty$, and $(u_p,\Phi_p)_{p\geq q}$ be the sequence given by Lemma 3. There exists $(u,\Phi)\in W^{1,\infty}(\Omega)\times\mathcal{M}_b(\Omega)^n$ and, a subsequence that we denote again by $p\to\infty$, such that*

$$u_p\to u,\qquad \text{in } W^{1,q}(\Omega)\text{-weak}, \tag{28}$$

and

$$\Phi_p\to\Phi,\qquad \text{in } \mathcal{M}_b(\Omega)^n\text{-weak}^*. \tag{29}$$

Moreover, we have

(1) *The measure Φ satisfies $-\nabla\cdot\Phi=\mu$, in Ω*
(2) *For any $\xi\in\mathbb{R}^n$, $\lambda_0>0$ and $\varphi\in\mathcal{D}(\Omega)$ such that $\varphi\geq 0$, we have*

$$\int_\Omega J(x,\xi)\varphi \geq \int_\Omega J_{\lambda_0}(\nabla u)\varphi + \int_\Omega \Phi_r\cdot(\xi-\nabla u)\,\varphi dx + \int_\Omega \varphi(\xi-\nabla_{|\Phi_s|}u)d\Phi_s.$$

Proof. As in the proof of Lemma 2, combining (26) and (J3), we deduce that the sequence $(\Phi_p)_{p\geq q}$ is bounded in $\mathcal{M}_b(\Omega)^n$ and (29) holds to be true. As to the sequence $(u_p)_{p\geq q}$, using Holder inequality and (25) we have

$$\begin{aligned}\int_\Omega (|\nabla u_p| - M(x))^{+q} &\leq \left(\int_\Omega (|\nabla u_p| - M(x))^{+p}\right)^{q/p} |\Omega|^{\frac{p-q}{p}} \\ &\leq (pC(p))^{q/p} |\Omega|^{\frac{p-q}{p}}.\end{aligned} \tag{30}$$

Using the fact that $C(p)$ is bounded as $p \to \infty$, we deduce that $(u_p)_{p\geq q}$ is bounded in $W^{1,q}(\Omega)$ and (28) holds to be true. By using Rellich–Kondrachov Theorem [11, Theorem 9.16] again, we get

$$u_p \to u, \quad \text{in } C(\bar{\Omega})$$

and then

$$\int_\Omega u_p d\mu \to \int_\Omega u\, d\mu.$$

Moreover, we see that letting $p \to \infty$ in (30), we have $u \in W^{1,\infty}(\Omega)$. Now, we take $z = u_\epsilon$ in (24), where $(u_\epsilon)_{\epsilon>0}$ is a sequence of Lipschitz function which converges uniformly to u. Letting $p \to \infty$ and then $\epsilon \to 0$, we get

$$\int_\Omega J(x,\xi)\varphi \geq \int_\Omega J_{\lambda_0}(x,\nabla u)\varphi + \int_\Omega \varphi\Phi_r \cdot (\xi - \nabla u)dx + \liminf_{\epsilon\to 0}\int_\Omega \varphi\,(\xi - \nabla u_\epsilon)d\Phi_s,$$

where $\Phi = \Phi_r\mathcal{L}^n + \Phi_s$ is the Lebesgue decomposition of the measure Φ. Thanks to Proposition 4, $\Phi_s(x) \in \mathcal{T}_{|\Phi_s|}(x)$, $|\Phi_s|$-a.e. $x \in \Omega$. This implies that, for any $\varphi \in \mathcal{D}(\Omega)$,

$$\begin{aligned}\lim_{\epsilon\to 0}\int_\Omega \varphi(\xi - \nabla u_\epsilon)d\Phi_s &= \int_\Omega \varphi\xi d\Phi_s - \lim_{\epsilon\to 0}\int_\Omega \varphi\frac{\Phi_s}{|\Phi_s|}\cdot \nabla u_\epsilon d|\Phi_s| \\ &= \int_\Omega \varphi\xi d\Phi_s - \lim_{\epsilon\to 0}\int_\Omega \varphi\frac{\Phi_s}{|\Phi_s|}\cdot P_{|\Phi_s|}\nabla u_\epsilon d|\Phi_s| \\ &= \int_\Omega \varphi\xi d\Phi_s - \int_\Omega \varphi\frac{\Phi_s}{|\Phi_s|}\cdot \nabla_{|\Phi_s|} u d|\Phi_s| \\ &= \int_\Omega \varphi(\xi - \nabla_{|\Phi_s|}u)d\Phi_s.\end{aligned}$$

And the proof of the lemma is complete. □

4.2. Proof of the main results

The aim of this section is to proceed to the proof of the main results of Section 3.

Lemma 5. *Under the assumptions of Lemma 4, let us consider the couple* $(u, \Phi) \in W^{1,\infty}(\Omega) \times \mathcal{M}_b(\Omega)^n$ *given by Lemma 4. Then,* $u \in K$ *and we have*

(1) $\Phi_r(x) \in \partial J(x, \nabla u(x))$ $\mathcal{L}^n$*-a.e.* Ω
(2) $\frac{\Phi_s}{|\Phi_s|}(x) \cdot \nabla_{|\Phi_s|} u(x) = S_{D(x)}\left(\frac{\Phi_s}{|\Phi_s|}(x)\right)$, $|\Phi_s|$*-a.e.* $x \in \Omega$.

Proof. Thanks to Lemma 4, for any $\xi \in I\!R^n$ and $\varphi \in \mathcal{D}(\Omega)$, we have

$$\int_\Omega J(x,\xi)\varphi \geq \int_\Omega J_{\lambda_0}(x, \nabla u)\varphi + \int_\Omega \varphi \Phi_r \cdot (\xi - \nabla u) dx + \int_\Omega \varphi(\xi - \nabla_{|\Phi_s|} u) d\Phi_s. \tag{31}$$

In particular, this implies that

$$J(x,\xi) \geq J_{\lambda_0}(x, \nabla u(x)) + (\xi - \nabla u(x)) \cdot \Phi_r(x) \qquad \mathcal{L}^n\text{-a.e. } x \in \Omega.$$

Hence, for $\mathcal{L}^n$-a.e. $x \in \Omega$, $J_{\lambda_0}(x, \nabla u(x))$ is bounded in Ω with respect to λ_0. This implies that

$$\nabla u(x) \in D(x), \quad \mathcal{L}^n\text{-a.e. } x \in \Omega. \tag{32}$$

Moreover, letting $\lambda_0 \to 0$ in (31) and using Fatou lemma, we get

$$\int_\Omega J(x,\xi)\varphi \geq \int_\Omega J(x, \nabla u)\varphi + \int_\Omega \varphi \Phi_r \cdot (\xi - \nabla u) dx + \int_\Omega \varphi(\xi - \nabla_{|\Phi_s|} u) d\Phi_s, \tag{33}$$

for any $\varphi \in \mathcal{D}(\Omega)$ and $\xi \in I\!R^n$. In one hand, this implies that, for any $\xi \in I\!R^n$, we have

$$J(x,\xi) \geq J(x, \nabla u(x)) + (\xi - \nabla u(x)) \cdot \Phi_r(x) \quad \mathcal{L}^n\text{-a.e.} \quad x \in \Omega.$$

Thus $\Phi_r(x) \in \partial_\xi J(x, \nabla u)$, $\mathcal{L}^n$-a.e. $x \in \Omega$. On the other hand, (33) implies that for any $\xi \in D(x)$

$$\xi \cdot \frac{\Phi_s}{|\Phi_s|} \leq \nabla_{|\Phi_s|} u \cdot \frac{\Phi_s}{|\Phi_s|}, \quad |\Phi_s|\text{-a.e. in } \Omega.$$

This implies that

$$S_{D(x)}\left(\frac{\Phi_s}{|\Phi_s|}(x)\right) \leq \frac{\Phi_s}{|\Phi_s|}(x) \cdot \nabla_{|\Phi_s|} u(x), \quad |\Phi_s|\text{-a.e. in } \Omega$$

Combining this with the result of Proposition 5, we deduce the second part of the lemma. □

Lemma 6. *For any* $\mu \in \mathcal{M}_b(\Omega)$*, if* $(u, \Phi) \in W^{1,\infty}(\Omega) \times \mathcal{M}_b(\Omega)^n$ *is a weak solution of the problem* (P_1)*, then* Φ *is a solution of problem* (P_3)*.*

Proof. Let us denote by $(u, \Phi) \in W^{1,\infty}(\Omega) \times \mathcal{M}_b(\Omega)^n$ a weak solution of the problem (P_1) and $\psi \in \mathcal{S}(\mu)$. In one hand, thanks to (10) and (11), we have

$$\begin{aligned}
&\int_\Omega J^*(x, \Phi_r) + \int_\Omega J(x, \nabla u) + \int_\Omega S_{D(x)}\left(\frac{\Phi_s(x)}{|\Phi|_s(x)}\right) d|\Phi|_s(x) - T_g(\Phi) \\
&= \int_\Omega \Phi_r \cdot \nabla u + \int_\Omega \nabla_{|\Phi_s|} u \, d\Phi_s - T_g(\Phi) \\
&= \int_\Omega u \, d\mu,
\end{aligned}$$

where we use also Proposition 4 and the fact that $-\nabla \cdot \Phi = \mu$. On the other hand, since $\psi \in \mathcal{S}(\mu)$, by using again Proposition 4 and Proposition 5, we have also

$$\begin{aligned}
\int_\Omega u \, d\mu &= \int_\Omega \psi_r \cdot \nabla u + \int_\Omega \nabla_{|\psi_s|} u \, d\psi_s - T_g(\psi) \\
&\le \int_\Omega J^*(x, \psi_r) + \int_\Omega J(x, \nabla u) + \int_\Omega S_{D(x)}\left(\frac{\psi_s(x)}{|\psi|_s(x)}\right) d|\psi|_s(x) - T_g(\psi).
\end{aligned}$$

This implies that

$$\begin{aligned}
&\int_\Omega J^*(x, \Phi_r) + \int_\Omega S_{D(x)}\left(\frac{\Phi_s(x)}{|\Phi|_s(x)}\right) d|\Phi|_s(x) - T_g(\Phi) \\
&\le \int_\Omega J^*(x, \psi_r) + \int_\Omega S_{D(x)}\left(\frac{\psi_s(x)}{|\psi|_s(x)}\right) d|\psi|_s(x) - T_g(\psi).
\end{aligned}$$

Recall that $\Phi \in \mathcal{S}(\mu)$ and $\psi \in \mathcal{S}(\mu)$ is arbitrary. Thus,

$$\begin{aligned}
&\int_\Omega J^*(x, \Phi_r) + \int_\Omega S_{D(x)}\left(\frac{\Phi_s(x)}{|\Phi|_s(x)}\right) d|\Phi|_s(x) - T_g(\Phi) \\
&= \min_{\psi \in \mathcal{S}(\mu)} \left\{ \int_\Omega J^*(x, \psi_r) + \int_\Omega S_{D(x)}\left(\frac{\psi_s(x)}{|\psi|_s(x)}\right) d|\psi|_s(x) - T_g(\psi) \right\}. \qquad \square
\end{aligned}$$

Lemma 7. *For any $\mu \in \mathcal{M}_b(\Omega)$, if Φ is a solution of (P_3), then there exists u such that the couple $(u, \Phi) \in W^{1,\infty}(\Omega) \times \mathcal{M}_b(\Omega)^n$ is a weak solution of (P_1).*

Proof. Let $(\bar{u}, \bar{\Phi})$ be the couple given by Lemma 4. First, using the fact that Φ is a solution of (P_3), we have

$$\int_\Omega J^*(x,\Phi_r) + \int_\Omega J(x,\nabla\bar u) + \int_\Omega S_{D(x)}\left(\frac{\Phi_s(x)}{|\Phi|_s(x)}\right) d|\Phi|_s(x) - T_g(\Phi)$$
$$\le I := \int_\Omega J^*(x,\bar\Phi_r) + \int_\Omega J(x,\nabla\bar u) + \int_\Omega S_{D(x)}\left(\frac{\bar\Phi_s(x)}{|\bar\Phi|_s(x)}\right) d|\bar\Phi|_s(x) - T_g(\Phi).$$

Moreover, using Proposition 4 and the fact that $(\bar u, \bar\Phi)$ is a solution of (P_2), it is not difficult to see that

$$I = \int \bar u \, d\mu.$$

This implies that

$$\int_\Omega J^*(x,\Phi_r) + \int_\Omega J(x,\nabla\bar u) + \int_\Omega S_{D(x)}\left(\frac{\Phi_s(x)}{|\Phi|_s(x)}\right) d|\Phi|_s(x) - T_g(\Phi) \le \int_\Omega \bar u d\mu. \tag{34}$$

Using again Proposition 4 and the fact that $\Phi \in \mathcal{S}(\mu)$, we have also

$$\int_\Omega \bar u d\mu = \int_\Omega \Phi_r \cdot \nabla\bar u + \int_\Omega \nabla_{|\Phi_s|}\bar u d\Phi_s - T_g(\Phi).$$

Combining this with (34) and the fact that $\Phi_r \cdot \nabla\bar u \le J^*(x,\Phi_r) + J(x,\nabla\bar u)$ a.e. in Ω and $\frac{\Phi_s}{|\Phi_s|}(x) \cdot \nabla_{|\Phi_s|}\bar u(x) \le S_{D(x)}\left(\frac{\Phi_s}{|\Phi_s|}(x)\right)$, $|\Phi_s|$-a.e. in Ω, we deduce that

$$\int_\Omega \Phi_r \cdot \nabla\bar u = \int_\Omega J^*(x,\Phi_r) + \int_\Omega J(x,\nabla\bar u)$$

and

$$\int_\Omega \nabla_{|\Phi_s|}\bar u d\Phi_s = \int_\Omega S_{D(x)}\left(\frac{\Phi_s(x)}{|\Phi|_s(x)}\right) d|\Phi|_s(x).$$

This ends up the proof of the lemma. □

The following uniqueness result will be useful for the proof of Theorem 1.

Lemma 8. *For any $\mu \in \mathcal{M}_b(\Omega)$, the problem*

$$\min\left\{\int_\Omega J(x,\nabla z(x))\,dx + \frac{1}{2}\int_\Omega z^2(x)\,dx - \int_\Omega z\,d\mu\,;\ z \in W^{1,\infty}_g(\Omega)\right\},$$

has at most one solution.

Proof.

$$\mathcal{I}(z) = \int_\Omega J(x, \nabla z(x))\, dx + \frac{1}{2}\int_\Omega z^2(x)\, dx - \int_\Omega z\, d\mu.$$

Suppose that u_1 and u_2 are two solutions of minimizing problem. We denote by $v = \frac{u_1+u_2}{2}$ and we have:

$$\begin{aligned} \mathcal{I}(v) &= \int_\Omega J(x, \nabla \frac{u_1+u_2}{2})\, dx + \frac{1}{2}\int_\Omega (\frac{u_1+u_2}{2})^2(x)\, dx - \int_\Omega \frac{u_1+u_2}{2}\, d\mu \\ &\le \frac{\mathcal{I}(u_1)+\mathcal{I}(u_2)}{2}. \end{aligned}$$

From this we get $u_1 = u_2$ a.e. □

Proof of Theorem 1. First, thanks to Lemma 5, the problem (P_1) has a solution (u, Φ). Moreover, for any $\xi \in D(x)$, we have

$$\begin{aligned} \int_\Omega (\xi - u) d\mu &= \int_\Omega \Phi_r \cdot \nabla(\xi - u) + \int_\Omega \nabla_{|\Phi_s|}(\xi - u) d\Phi_s \\ &\le \int_\Omega J(x, \nabla\xi) - J(x, \nabla u), \end{aligned}$$

where we use the fact that $\Phi_r \in \partial_\xi J(x, \nabla u)$. This implies that u is solution of (P_2). For the converse part, let v be a solution of (P_2) and let us denote by h the measure given by

$$h = \mu + v\,\mathcal{L}^n.$$

Thanks to Lemma 8, it is not difficult to see that v is the unique solution of

$$\min\left\{\int_\Omega J(x, \nabla z(x))\, dx + \frac{1}{2}\int_\Omega z^2(x)\, dx - \int_\Omega z\, dh\, ;\ z \in W_g^{1,\infty}(\Omega)\right\}. \tag{35}$$

For any $\lambda > 0$ and $p > n$, we consider again the regularization J_λ given in Section 2, and the regularized problem

$$\min\left\{\int_\Omega J_\lambda(x, \nabla z(x))\, dx + \frac{1}{2}\int_\Omega z^2(x)\, dx - \int_\Omega z\, dh\, ;\ z \in W_g^{1,p}(\Omega)\right\}. \tag{36}$$

It is not difficult to see that (36) has a solution u_λ, and $w_\lambda := \partial_\xi J(x, \nabla u_\lambda)$ satisfies

$$\begin{cases} -\nabla\cdot\omega_\lambda = h - u_\lambda\,\mathcal{L}^n & \text{in } \Omega \\ \\ u_\lambda = g & \text{on } \partial\Omega. \end{cases}$$

Moreover, one sees that the sequence u_λ is bounded in $L^2(\Omega)$, which implies that $h - u_\lambda\,\mathcal{L}^n$ is bounded in $\mathcal{M}_b(\Omega)$. Now, to let $\lambda \to 0$ and then $p \to \infty$, one follows the same arguments in the same way as in Section 4, except that the second member μ in (17) is replaced here by $h - u_\lambda\,\mathcal{L}^n$ which converges in $\mathcal{M}_b(\Omega)$-weak*. As a consequence, we conclude that there exists $(u, \Phi) \in W^{1,\infty}(\Omega) \times \mathcal{M}_b(\Omega)^n$ such that, $u = g$ on $\partial\Omega$, and setting $\Phi = \Phi_r\mathcal{L}^n + \Phi_s$, we have $\Phi_r(x) \in \partial J(x, \nabla u(x))$, $\mathcal{L}^n$-a.e. $x \in \Omega$, $\frac{\Phi_s}{|\Phi_s|}(x)\cdot\nabla_{|\Phi_s|}u(x) = S_{D(x)}\left(\frac{\Phi_s}{|\Phi_s|}(x)\right)$, $|\Phi_s|$-a.e. in Ω and

$$-\nabla\cdot\Phi = h - u\,\mathcal{L}^n \quad \text{in } \Omega.$$

Thanks to the first part of the proof, it follows that u is a solution of the problem (35). By uniqueness, we get $u = v$, so that $h - u = \mu$ and we conclude that (v, Φ) is solution of (P_1). □

Proof of Theorem 2. Thanks to Theorem 1 there exists $(u, \Phi) \in W^{1,\infty}(\Omega) \times \mathcal{M}_b(\Omega)^n$ a weak solution of (P_1). The proof is a direct consequence of Lemma 6 and Lemma 7. □

Proof of Corollary 1. If $J(x, .)$ is symmetric, then we have

$$\overline{D(x)} = \overline{B(0, R(x))}, \qquad \text{for any } x \in \Omega,$$

where $R : \Omega \to [0, \infty)$. Therefore,

$$|\nabla u(x)| < R(x) \quad \mathcal{L}^n \quad \text{a.e. } x \in \Omega.$$

Using Lemma 1 of [22], there exists u_ϵ a sequence in $D(\Omega)$ such that $u_\epsilon \to u \in C(\Omega)$ and $|\nabla u_\epsilon(x)| < R(x)$ a.e. $x \in \Omega$. In particular, this implies that

$$\left|P_{|\Phi_s|}\nabla u_\epsilon\right| \le R(x) \quad |\Phi_s|\text{-a.e.}$$

By using the $L^\infty(\Omega, d|\Phi_s|)$-weak* continuity of the operator $\nabla_{|\Phi_s|}$ we get

$$\left|\nabla_{|\Phi_s|}u(x)\right| \le R(x), \quad |\Phi_s|\text{-a.e. in } \Omega. \tag{37}$$

This implies that $\nabla_{|\Phi_s|}u(x) \in \overline{D(x)}$ $|\Phi_s|$-a.e. in Ω and the proof of the Corollary follows by using Proposition 6. □

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