Elliptic-Parabolic Equation with Absorption of Obstacle type

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Abstract

This paper is concerned with existence and uniqueness of solutions for a doubly nonlinear degenerate parabolic problem of the type $\beta(w)_t - \text{div}\, a(x, Dw) + \partial j\big(., \beta(w)\big) \ni f$, where a is a Leray-Lions operator, β is a nondecreasing continuous function and $\partial j(., r)$ is a maximal monotone graph with respect to r defined on a closed interval of $I\!R$. Particular cases of j correspond to the so called obstacle problem.
Keywords : Obstacle poblem, Elliptic-parabolic problem, L^1 theory, Semigroup of contraction, accretive operator, weak solution, integral solution.

1 Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with smooth boundary Γ, $p > 1$ and $a : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ a Leray-Lions operator, i.e. a is a Caratheodory function (i.e is measurable in $x \in \Omega$ for all $\xi \in \mathbb{R}^N$ and continuous in $\xi \in \mathbb{R}^N$ for a.e. $x \in \Omega$) with $a(., 0) = 0$, satisfying

(H_1) there exists $\alpha > 0$ such that for all $\xi \in \mathbb{R}^N$ $\quad a(x, \xi).\xi \geq \alpha |\xi|^p$ for a.e. $x \in \Omega$

(H_2) for any $\xi, \eta \in \mathbb{R}^N$ such that $\xi \neq \eta$ $\ (a(x, \xi) - a(x, \eta)).(\xi - \eta) > 0$ a.e. $x \in \Omega$

(H_3) there exists $\sigma > 0$ and $k \in L^{p'}(\Omega)$ such that $|a(x, \xi)| \leq \sigma(k(x) + |\xi|^{p-1})$ a.e. $x \in \Omega$

and for any $\xi \in \mathbb{R}^N$, where $p' = \dfrac{p}{p-1}$.

In $(0, T) \times \Omega$, we consider the elliptic-parabolic problem of the type

$$P^{\beta,j}(u_0,\ f) \begin{cases} u_t - \text{div}\, a(x,\ Dw) + \partial j(x, u) \ni f & u = \beta(w) \quad \text{in } Q := (0,T) \times \Omega \\ a(x,\ Dw).\vec{n} + z = 0 & z \in \gamma(w) \quad \text{in } \Sigma := (0,T) \times \Gamma \\ u(0) = u_0 & \text{in } \Omega, \end{cases}$$

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where $\vec{n}$ is the unit outward normal of Γ, $f \in L^{p'}(Q)$ and $\beta : \mathbb{R} \to \mathbb{R}$ is a nondecreasing continuous function with $\beta(0) = 0$ and

$$(H_4) \qquad R(\beta) = \mathbb{R}.$$

For a.e. $x \in \Omega$, $\partial j(x,.)$ is a maximal monotone graph in $\mathbb{R}$ and γ is a maximal monotone graph in $\mathbb{R} \times \mathbb{R}$ such that $0 \in \gamma(0)$ and

$$(H_5) \qquad \mathcal{D}(\gamma) = \mathbb{R} \ \text{ or } \ \mathcal{D}(\gamma) = \{0\}.$$

The assumptions $(H_1 - H_3)$ are classical in the study of nonlinear operators in divergence form (see for instance [7] and [18]). The nonlinearity β appears in the study of nonlinear diffusion problems like the filtration equation (cf. [19] and the reference therein). We are interested into the existence and uniqueness of a solution. A standard weak solution of $P^{\beta,j}(u_0,\ f)$ is a function $u \in \mathcal{C}\Big([0,T],\ L^1(\Omega)\Big)$ such that $u(0) = u_0$, and there exists $(w, z, \eta) \in L^p\Big(0,T;\ W^{1,p}(\Omega)\Big) \times L^1(\Sigma) \times L^1(Q)$ such that $u = \beta(w)$, $\eta \in \partial j(.,\ u)$ a.e. in Q, $z \in \gamma(w)$ a.e. on Σ and, for any $\xi \in \mathcal{C}^1(\overline{\Omega})$,

$$\int_\Omega a(.,\ Dw(t))D\xi + \int_\Omega \eta(t)\xi + \int_\Gamma z(t)\xi = \int_\Omega f(t)\xi + \frac{d}{dt}\int_\Omega u(t)\xi \quad \text{in } \ \mathcal{D}'(0,T). \tag{1}$$

The main difficulty when treating this type of problem is due to the (non-smooth) x-dependence of the absorption term $\partial j(x,u)$. If $\beta = Id_{\mathbb{R}}$, $\mathcal{D}(\gamma) = \{0\}$ and either j is independent of x or $j(.,r) \in L^\infty(\Omega) + L^1(\Omega)$, for any $r \in \mathbb{R}$, it is well- known by now that (for L^∞ data) the problem admits a unique weak solution (cf. [9]). However, in the general case (including the case of $x-$dependent obstacles that we treat in this paper) the absorption term $\partial j(x,u)$ gives rise to a measure term μ and there only exists some generalized weak solution to the equation for which the condition $\mu \in \partial j(.,u)$ has to be interpreted in some appropriate way. For the particular case $\beta = Id_{\mathbb{R}}$ and homogeneous Dirichlet boundary condition ($\mathcal{D}(\gamma) = \{0\}$) existence and uniqueness of a generalized weak solution has been proved in [25] for the elliptic problem and in [3] for the corresponding parabolic problem. The proofs in [3] and [25] rely on rather technical and, especially in the parabolic case, sophisticated arguments from capacity and measure theory. To our knowledge the case where $\beta \neq Id_{\mathbb{R}}$ and $\mathcal{D}(\gamma) \neq \{0\}$ is still open. In this paper we treat a peculiar case for which the condition $j(.,r) \in L^\infty(\Omega) + L^1(\Omega)$ fails to be true and the assumptions $(H_1 - H_5)$ are fulfilled. We treat the case where j is such that

$$j(x,r) = \tilde{j}(x,r) + \mathbb{I}_{[\psi_-(x),\,\psi_+(x)]}(r), \tag{2}$$

where $\tilde{j} : \Omega \times \mathbb{R} \longrightarrow [0,\infty]$ is convex, l.s.c in $r \in \mathbb{R}$, $\tilde{j}(.,r) \in L^1(\Omega)$ for all $r \in \mathbb{R}$ with $\tilde{j}(.,0) = 0$ and ψ_-, ψ_+ are two given measurable functions. The main application we have in mind is the so called obstacle problem (cf. [6], [13], [16], [22] and [24]). Our approach is different and new, we develop a new notion of solution for the nonlinear elliptic-parabolic problem $P^{\beta,j}(u_0,\ f)$. We use entropic inequality (cf. [7]) with test functions satisfying the obstacle condition. This notion permits to handle the problem with L^1 data. However, even for bounded solution the truncation seems to be necessary for the uniqueness of solution for the evolution problem. More precisely, we prove that under the assumptions $(H_1 - H_5)$, for any $f \in L^\infty(Q)$ and $u_0 \in L^\infty(\Omega)$ such that $\psi_-(x) \leq u_0(x) \leq \psi_+(x)$, for a.e. $x \in \Omega$, the problem $P^{\beta,j}(u_0, f)$ has a unique solution in the sense that $u \in \mathcal{C}\Big([0,T), L^1(\Omega)\Big)$, $u(0) = u_0$ and there exists $w \in L^p\Big(0,T;\, W^{1,p}(\Omega)\Big)$, $\tilde\eta \in L^1(Q)$, $z \in L^1(\Sigma)$ such that $u = \beta(w)$, $\tilde\eta \in \partial\tilde{j}(x,u)$ a.e. in Q, $z \in \gamma(w)$ a.e. Σ and

$$\begin{aligned} \frac{d}{dt}\int_\Omega \int_0^w T_l(r-\xi)d\beta(r) \quad &+ \int_\Omega a(.,\, Dw) DT_l(w-\xi) + \int_\Omega \tilde\eta T_l(w-\xi) \\ + \int_\Gamma z T_l(w-\xi) \quad &\leq \int_\Omega f T_l(w-\xi) \quad \text{ in } \mathcal{D}'(0,T),\ \forall l > 0 \end{aligned}$$

for any $\xi \in \mathcal{C}^1(\overline{\Omega})$, such that $\psi_- \leq \beta(\xi) \leq \psi_+$ a.e. $x \in \Omega$, where T_l is the truncation function at level l defined by

$$T_l(s) := max\{-l,\, min\{l,\, s\}\},\ s \in \mathbb{R}.$$

Notice that, our approach can be extended naturally to L^1 data. But, we focus our attention here into bounded solution.

In the next section, we prove existence and uniqueness of the solution of the stationary problem associated to $P^{\beta,\, j}(u_0,\, f)$. For that, we approximate ∂j by a sequence of absorptions $\partial j_{\lambda,\,\mu}$ defined everywhere on $\mathbb{R}$, and we give also some preliminary estimates that will be used afterwards. In the third section, we use the nonlinear semigroup theory and we establish existence result for the problem without obstacle. Then, we approach the obstacle problem by a elliptic-parabolic problem $P^{\beta_{\lambda,\,\mu},\, j_{\lambda,\,\mu}}(u_0,\, f)$ without obstacle and we pass to limit, thus proving existence. The last section is devoted to the proof of uniqueness. We use the concept of integral solution and we show that the solutions of $P^{\beta,\, j}(u_0,\, f)$ are integral solutions, thus they are unique.

2 Elliptic problem

In order to study the problem in the framework of nonlinear semigroup theory, we consider the stationary problem associated with $P^{\beta,\, j}(u_0,\, f)$ defined by

$$S^{\beta,\, j}(f) \quad \begin{cases} v - \operatorname{div} a(x,\, Dw) + \partial j(.,\, v) \ni f, & v = \beta(w) \quad \text{ in } \Omega \\ \\ a(x,\, Dw).\vec{n} + z = 0 & z \in \gamma(w) \quad \text{ in } \Gamma. \end{cases}$$

The norm in $L^p(\Omega)$ is denoted by $||.||_p$, $1 \leq p \leq \infty$. The space $W^{1,p}(\Omega)$ denotes the classical Sobolev space endowed with the norm $||.||_{1,p}$.

2.1 Elliptic problem without obstacle: $j(.,\, r) \in L^1(\Omega)\ \forall r \in \mathbb{R}$

Throughout this section, we assume that

$$j(.,\, r) \in L^1(\Omega) \quad \text{for all} \quad r \in \mathbb{R} \tag{3}$$

and we prove existence and uniqueness of a weak solution of the problem $S^{\beta,j}(f)$. Our main result in this section is

Proposition 2.1 *Given $f \in L^\infty(\Omega)$, there exists a unique weak solution v for the problem $S^{\beta,j}(f)$ in the sense that $v \in L^1(\Omega)$ and there exists $w \in W^{1,p}(\Omega)$, $\eta \in L^1(\Omega)$ and $z \in L^1(\Gamma)$ such that $\eta \in \partial j(.,\, v)$, $v = \beta(w)$ a.e. Ω, $z \in \gamma(w)$ a.e. Γ and*

$$\int_\Omega v\,\xi + \int_\Omega a(x,\, Dw)\, D\,\xi + \int_\Omega \eta\,\xi + \int_\Gamma z\,\xi \;=\; \int_\Omega f\,\xi \tag{4}$$

for any $\xi \in W^{1,p}(\Omega)$. Moreover, v, w and z are bounded in L^∞ and we have the following estimates:

$$\int_\Omega |\, v\,| + \int_\Omega |\,\eta\,| + \int_\Gamma |\,z\,| \le \int_\Omega |\,f\,|, \tag{5}$$

$$||\, v\,||_\infty \le ||\, f\,||_\infty, \quad ||\, w\,||_\infty \le C_\beta(||\, f\,||_\infty), \quad ||\, z\,||_\infty \le C_{\gamma,\beta}(||\, f\,||_\infty), \tag{6}$$

$$\int_\Omega |Dw|^p \le C_1 \quad \text{and} \quad \Big(\int_\Omega |a(.,\; Dw)|^{p'}\Big)^{\frac{1}{p'}} \le C_2. \tag{7}$$

where, C_1, C_2 are constants that depends only on Ω, p, N, $||f||_\infty$, α and $||\, k\,||_{L^{p'}(\Omega)}$.

The existence, uniqueness, contraction and order preserving properties are equivalent to the fact that the operator $\mathcal{A}_{\beta,\, j}$ defined in $L^1(\Omega)$ by

$$f \in \mathcal{A}_{\beta,\, j} u \Leftrightarrow \begin{cases} u,\, f \in L^{p'}(\Omega),\, \exists\, w \in W^{1,p}(\Omega),\, \exists\, \eta \in L^{p'}(\Omega),\, \exists\, z \in L^{p'}(\Gamma) \\[2ex] u = \beta(w),\;\; \eta \in \partial j(.,\, u) \;\text{ a.e. } \Omega,\;\; z \in \gamma(w) \;\text{ a.e. } \Gamma \\[2ex] \displaystyle\int_\Omega a(.,\; Dw) D\xi + \int_\Omega \eta\xi + \int_\Gamma z\xi = \int_\Omega f\xi,\; \forall \xi \in W^{1,p}(\Omega) \end{cases}$$

is T-accretive in $L^1(\Omega)$ and $\mathcal{R}(I + \varepsilon\mathcal{A}_{\beta,\, j}) \supseteq L^\infty(\Omega)$ for any $\varepsilon > 0$. These results are well known by now in the case, where $j = 0$ (see for instance [5]). To treat the case $j \neq 0$, let us consider B_j the operator defined by

$$B_j = \Big\{(u,\, \eta) \;\in\; L^1(\Omega) \times L^1(\Omega);\; \eta \;\in\; \partial j(.,\, u)\Big\},$$

and we write

$$\mathcal{A}_{\beta,\, j} = \mathcal{A}_{\beta,\, 0} + B_j.$$

Recall that the Yoshida approximation of $j(x,\, .)$ is given by

$$j_\lambda(x, .) = \lambda\Big(I - (I + \frac{1}{\lambda} j(x, .))^{-1}\Big).$$

Lemma 2.2 *i) The operator $\overline{\mathcal{A}}_{\beta,\, j_\lambda}$ is m-T-accretive in $L^1(\Omega)$.*
ii) Given $f \in L^q(\Omega) \cap L^{p'}(\Omega)$ and $1 \le q \le \infty$, for any $\varepsilon > 0$ we have

$$||\, (I + \varepsilon\mathcal{A}_{\beta,\, j_\lambda})^{-1} f\, ||_{L^q(\Omega)} \le ||\, f\, ||_{L^q(\Omega)}.$$

Proof: *i*) First, recall (cf. [5]) that for any $(u, v) \in \mathcal{A}_{\beta,\,0}$, we have

$$\int_\Omega p(u)\, v \geq 0 \quad \text{ for any } p \in P_0, \tag{8}$$

where $P_0 := \Big\{ p \in Lip(\mathbb{R});\ p \text{ nondecreasing },\ p(0) = 0 \ \text{ and } \ supp(p') \text{ compact} \Big\}.$
So, since B_{j_λ} is continuous and T-accretive, then $\mathcal{A}_{\beta,\,j_\lambda}$ is also T-accretive in $L^1(\Omega)$. On the other hand, thanks to [5] we know that for any $\varepsilon > 0$ we have $\mathcal{R}(I + \varepsilon \mathcal{A}_{\beta,\,0}) \supseteq L^{p'}(\Omega)$, then by using the corollary 3.1 of [4], we deduce that $\mathcal{R}(I + \varepsilon\, \mathcal{A}_{\beta,\,j_\lambda}) \supseteq L^{p'}(\Omega)$ and $\overline{\mathcal{A}}_{\beta,\,j_\lambda}$ is m-T-accretive in $L^1(\Omega)$.

ii) Thanks to [8], it is enough to prove that $\mathcal{A}_{\beta,\,j_\lambda}$ satisfies (8). Let $(u_\lambda,\ f) \in \mathcal{A}_{\beta, j_\lambda}$. Thanks to (8), we have

$$\int_\Omega p(u_\lambda)\, (f - \partial j_\lambda(.,\ u_\lambda)) \geq 0.$$

Using the fact that $p(u_\lambda)\, \partial j_\lambda(.,\ u_\lambda) \geq 0$ a.e. in Ω, we deduce that the property (8) is satisfied for the operator $\mathcal{A}_{\beta,\,j_\lambda}$ and *ii*) follows. ■

Proof of Proposition 2.1. The proof of this proposition is standard. For completeness, let us give the main arguments.
Uniqueness: For $i = 1, 2$; let $f_i \in L^\infty(\Omega)$ and v_i the solution of $S^{\beta,j}(f_i)$ in the sense of Proposition 2.1, then

$$\int_\Omega (v_1 - v_2)\xi + \int_\Omega \Big(a(.,\ Dw_1) - a(.,\ Dw_2) \Big) D\xi + \int_\Omega (\eta_1 - \eta_2)\xi + \int_\Gamma (z_1 - z_2)\xi$$

$$= \int_\Omega (f_1 - f_2)\xi$$

for any $\xi \in W^{1,p}(\Omega)$. Taking $\dfrac{1}{l}\, T_l(w_1 - w_2)$, for $l > 0$ as a test function in the preceding equality, using the monotonicity and letting $l \to 0$, we deduce that

$$\int_\Omega |v_1 - v_2| \leq \int_\Omega |f_1 - f_2|. \tag{9}$$

Existence: Thanks to Lemma 2.2 there exists a unique solution $v_\lambda \in L^1(\Omega)$ of $S^{\beta,\,j_\lambda}(f)$. So, there exists $w_\lambda \in W^{1,p}(\Omega)$, $\eta_\lambda \in L^1(\Omega)$ and $z_\lambda \in L^1(\Gamma)$ such that $\eta_\lambda = \partial j_\lambda(x,\ v_\lambda)$, $v_\lambda = \beta(w_\lambda)$ a.e. Ω, $z_\lambda = \gamma(w_\lambda)$ a.e. Γ, and

$$\int_\Omega v_\lambda\, \xi + \int_\Omega a(x,\ Dw_\lambda) D\, \xi + \int_\Omega \eta_\lambda\, \xi + \int_\Gamma z_\lambda\, \xi = \int_\Omega f\, \xi, \tag{10}$$

for any $\xi \in W^{1,p}(\Omega)$. Thanks to *ii*) of Lemma 2.2 we have

$$||\, v_\lambda\, ||_{L^\infty(\Omega)} \leq ||\, f\, ||_{L^\infty(\Omega)}. \tag{11}$$

In addition, since $R(\beta)=\mathbb{R}$ and $\mathcal{D}(\gamma)=\mathbb{R}$, then

$$|| \, w_\lambda \, ||_{L^\infty(\Omega)} \leq \max\Big(\beta^{-1}(\, || \, f \, ||_{L^\infty(\Omega)} \,) \cup (-\beta^{-1})(- \, || \, f \, ||_{L^\infty(\Omega)} \,)\Big) =: C_\beta(|| \, f \, ||_\infty) \quad (12)$$

and

$$|| \, z_\lambda \, ||_{L^\infty(\Gamma)} \leq \max\Big(\gamma(\, || \, w_\lambda \, ||_{L^\infty(\Omega)} \,) \cup (-\gamma)(- \, || \, w_\lambda \, ||_{L^\infty(\Omega)} \,)\Big) \leq C_{\gamma,\beta}(|| \, f \, ||_\infty). \quad (13)$$

Now, taking w_λ as a test function in (10), using (H_1), (12) and monotonicity, we deduce that

$$\int_\Omega |Dw_\lambda|^p \leq \frac{1}{\alpha}||f||_{L^1(\Omega)}||w_\lambda||_{L^\infty(\Omega)} =: C_1. \quad (14)$$

Thanks to (12) and (14), there exists a subsequence, that we denote again by w_λ, such that

$$w_\lambda \to w \quad \text{in } \; W^{1,p}(\Omega)\text{- weak} \quad \text{and} \quad w_\lambda \to w \quad \text{a.e. in } \Omega.$$

Since β is continuous and v_λ is bounded in $L^\infty(\Omega)$ then

$$v_\lambda \to v \quad \text{in } \; L^p(\Omega) \; \text{ and } \; v=\beta(w) \quad \text{a.e. in} \quad \Omega.$$

By using (13) and (14) we deduce that

$$w_\lambda \to w \quad \text{in } \; L^1(\Gamma), \; z_\lambda \to z \quad \text{in } \; L^1(\Gamma)\text{-weak and } \; z\in\gamma(w) \quad \text{a.e. on} \quad \Gamma.$$

For the passage to the limit in the term $\partial j_\lambda(.,v_\lambda)$, we use the same arguments of [9] to deduce that

$$\partial j_\lambda(., \, v_\lambda) \to \eta \quad \text{in} \quad L^1(\Omega)\text{-weak}. \quad (15)$$

Indeed, setting $\varrho_\lambda = (I+\dfrac{1}{\lambda}\partial j(x,.))^{-1}v_\lambda$, we have

$$|| \, \varrho_\lambda \, ||_{L^\infty(\Omega)} \leq || \, v_\lambda \, ||_{L^\infty(\Omega)} \leq || \, f \, ||_{L^\infty(\Omega)}.$$

In addition, since $\partial j_\lambda(x, \, v_\lambda)\in\partial j(x,\varrho_\lambda)$ a.e. in Ω, then using the definition of ∂j, we get

$$-j(x,-|| \, f \, ||_{L^\infty(\Omega)}-1) \leq \partial j_\lambda(x, \, v_\lambda) \leq j(x, || \, f \, ||_{L^\infty(\Omega)}+1) \qquad \text{a.e. in } \Omega,$$

and (15) follows by using (3). Then, thanks to Lemma 1.6 of [9], we deduce that $\eta\in\partial j(.,v)$ a.e. Ω. Now, let us prove that, as $\lambda\to 0$,

$$a(x, \, Dw_\lambda) \to h \text{ in } [L^{p'}(\Omega)]^N\text{-weak} \quad \text{and} \quad \operatorname{div} h = \operatorname{div} a(x, \, Dw) \;\text{ in }\; \mathcal{D}'(\Omega). \quad (16)$$

Using (H_2), (14) and Minkowski inequality, we have

$$\begin{aligned}
(\int_\Omega |a(., \, Dw_\lambda)|^{p'})^{\frac{1}{p'}} &\leq \sigma\Big(\int_\Omega (|k|+|Dw_\lambda|^{p-1})^{p'}\Big)^{\frac{1}{p'}} \\
&\leq \sigma\Big((\int_\Omega |k(x)|^{p'})^{\frac{1}{p'}} + (\int_\Omega |Dw_\lambda|^{p'})^{\frac{1}{p'}}\Big) \\
&\leq \sigma\Big(||k||_{L^{p'}(\Omega)} + ||Dw_\lambda||^{\frac{p}{p'}}_{L^p(\Omega)}\Big) \\
&\leq \sigma\Big(||k||_{L^{p'}(\Omega)} + C_1^{\frac{1}{p'}}\Big) := C_2,
\end{aligned} \quad (17)$$

so that, there exists a subsequence that we denote again by λ, such that

$$a(x,\ Dw_\lambda) \to h \text{ in } [L^{p'}(\Omega)]^N\text{-weak} \quad \text{as } \lambda \to 0.$$

Using Minty-Browder's monotonicity arguments (cf. [10]), one can actually prove $\operatorname{div} a(x, Dw) = \operatorname{div} h$ in $\mathcal{D}'(\Omega)$. Indeed, taking $(w_\lambda - w)$ as a test function in (10), we obtain

$$\int_\Omega a(.,\ Dw_\lambda)D(w_\lambda - w) \leq \int_\Omega (f - v_\lambda)(w_\lambda - w) - \int_\Omega G_\lambda(.,\ v_\lambda)(w_\lambda - w) - \int_\Gamma z_\lambda(w_\lambda - w),$$

Letting $\lambda \to 0$ and using previous convergence and $L^\infty(\Omega)$ estimates, we get

$$\limsup_{\lambda \to 0} \int_\Omega a(.,\ Dw_\lambda)D(w_\lambda - w) \leq 0$$

and

$$\limsup_{\lambda \to 0} \int_\Omega a(.,\ Dw_\lambda)Dw_\lambda \leq \int_\Omega hDw, \tag{18}$$

which is the key inequality that allows to deduce then (16) by the standard monotonicity arguments. At last, letting $\lambda \to 0$ in (10), (11), (12), (13), (14) and (17) the results of the proposition follow. ∎

2.2 Elliptic obstacle problem

Now, we assume that

$$j(x, r) = \tilde{j}(x, r) + \mathbb{I}_{[\psi_-(x),\ \psi_+(x)]}(r),$$

where $\tilde{j} : \Omega \times \mathbb{R} \longrightarrow [0, \infty]$ is convexe, l.s.c in $r \in \mathbb{R}$, $\tilde{j}(., r) \in L^1(\Omega)$ for all $r \in \mathbb{R}$ with $\tilde{j}(., 0) = 0$ and ψ_-, ψ_+ are two given measurable functions. Let K be given by

$$K = \Big\{ w \in L^\infty(\Omega);\ \psi_-(x) \leq \beta(w) \leq \psi_+(x) \ \text{ a.e. } \ x \in \Omega \Big\}.$$

Proposition 2.3 *Let $f \in L^\infty(\Omega)$, then there exists a unique solution of $S^{\beta,j}(f)$ in the sense that $v \in L^1(\Omega)$, there exists $(w, z, \eta) \in W^{1,p}(\Omega) \times L^1(\Gamma) \times L^1(\Omega)$ such that $\tilde{\eta}(x) \in \partial\tilde{j}(x,\ v(x))$, $v(x) = \beta(w(x))$ a.e. $x \in \Omega$, $z(x) = \gamma(w(x))$ a.e. $x \in \Gamma$, and*

$$\int_\Omega v(w - \xi) + \int_\Omega a(.,\ Dw)D(w - \xi) + \int_\Omega \tilde{\eta}(w - \xi) + \int_\Gamma z(w - \xi) \leq \int_\Omega f(w - \xi) \tag{19}$$

for any $\xi \in W^{1,p}(\Omega) \cap K$.

In order to prove Proposition 2.3, we approximate j by a sequence of functions satisfying the assumptions of the preceeding section. In addition, in order to handle the evolution problem, we choose a monotone approximation. More precisely, we consider the problem

$$S^{\beta_{\lambda,\mu},\, j_{\lambda,\mu}}(f)\begin{cases} v_{\lambda,\mu} - \operatorname{div} a(x,\, Dw_{\lambda,\mu}) + \partial j_{\lambda,\mu}(.,\, v_{\lambda,\mu}) \ni f, & v_{\lambda,\mu} = \beta_{\lambda,\mu}(w_{\lambda,\mu}) \quad \text{in } \Omega \\ a(x,\, Dw_{\lambda,\mu}).\vec{n} + z_{\lambda,\mu} = 0 & z_{\lambda,\mu} \in \gamma(w_{\lambda,\mu}) \quad \text{in } \Gamma \end{cases}$$

where $j_{\lambda,\mu}$ and $\beta_{\lambda,\mu}$ are monotones approximations of j and β, respectively given by

$$j_{\lambda,\mu}(x,\, r) = \tilde{j}(x,\, r) + \frac{1}{2\lambda}\Big(\big(\, r - \psi_+(x)\big)^+\Big)^2 - \frac{1}{2\mu}\Big(\big(\, r - \psi_-(x)\big)^-\Big)^2 \quad \text{a.e. } x \in \Omega \tag{20}$$

and

$$\beta_{\lambda,\mu}(r) = \beta(r) \,-\, \lambda\; r^- + \mu\; r^+.$$

Proof of Proposition 2.3: Existence Thanks to Proposition 2.1, the problem $S^{\beta_{\lambda,\mu},\, j_{\lambda,\mu}}(f)$ has a unique solution $v_{\lambda,\mu} \in L^1(\Omega)$ and there exists $(w_{\lambda,\mu}, z_{\lambda,\mu}, \delta_{\lambda,\mu}) \in W^{1,p}(\Omega) \times L^1(\Gamma) \times L^1(\Omega)$ such that $v_{\lambda,\mu} = \beta_{\lambda,\mu}(w_{\lambda,\mu})$ a.e. Ω, $z_{\lambda,\mu} \in \gamma(w_{\lambda,\mu})$ a.e. Γ, $\delta_{\lambda,\mu} \in \partial j_{\lambda,\mu}(.,\, v_{\lambda,\mu})$ a.e. Ω and, for any $\xi \in W^{1,p}(\Omega)$,

$$\int_\Omega v_{\lambda,\mu}\, \xi + \int_\Omega a(x,\, Dw_{\lambda,\mu})\, D\,\xi + \int_\Omega \delta_{\lambda,\mu}\, \xi \,+ \int_\Gamma z_{\lambda,\mu}\, \xi \;=\; \int_\Omega f\, \xi.$$

Moreover, $v_{\lambda,\mu}$, $w_{\lambda,\mu}$, $z_{\lambda,\mu}$ and $\delta_{\lambda,\mu}$ satisfies the estimates (5)-(7). Setting

$$\eta_{\lambda,\mu} = \frac{1}{2\lambda}\Big(\big(\, v_{\lambda,\mu} - \psi_+(x)\big)^+\Big)^2 - \frac{1}{2\mu}\Big(\big(\, v_{\lambda,\mu} - \psi_-(x)\big)^-\Big)^2 \quad \text{a.e. } x \in \Omega,$$

we have $\tilde{\eta}_{\lambda,\mu} := \delta_{\lambda,\mu} - \eta_{\lambda,\mu} \in L^1(\Omega)$ and $\tilde{\eta}_{\lambda,\mu} \in \partial\tilde{j}(.,\, v_{\lambda,\mu})$. Using the estimates (5)-(7) in the same way as in the proof of Proposition 2.1 and applying a diagonal process, we deduce that there exists $v \in L^1(\Omega)$, $w \in W^{1,p}(\Omega)$ and $z \in L^1(\Gamma)$, and a subsequence $\lambda(\mu)$ such that $\lambda(\mu) \longrightarrow 0$; as $\mu \longrightarrow 0$ and

$$v_{\lambda(\mu),\mu} =: v_\mu \;\longrightarrow\; v \;\text{ in }\; L^1(\Omega), \quad w_{\lambda(\mu),\mu} =: w_\mu \to w \;\text{ in } W^{1,p}(\Omega)\text{-weak},$$

$$z_{\lambda(\mu),\mu} =: z_\mu \to z \;\text{ in } L^1(\Gamma)\text{-weak and }\; a(.,\, Dw_\mu) \to \phi \;\text{ in } [L^{p'}(\Omega)]^N\text{-weak}$$

with $v = \beta(w)$ a.e. Ω and $z \in \gamma(w)$ a.e. Γ. In addition, using the definition of $\partial\tilde{j}$, we get

$$|\partial\tilde{j}(x,\, v_\mu)| \leq \tilde{j}(x, v_\mu - Sign_0(v_\mu)) - \tilde{j}(x, v_\mu) \qquad \text{a.e. in } \Omega.$$

Since $\tilde{j}(., r) \in L^1(\Omega)\ \forall r \in \mathbb{R}$ and $||\, v_\mu\, ||_{L^\infty(\Omega)} \leq ||\, f\, ||_{L^\infty(\Omega)}$, then

$$\partial\tilde{j}(.,\, v_\mu) \;\to\; \tilde{\eta} \;\text{ in } L^1(\Omega)\text{-weak},$$

and (the Lemma 1.6 of [9]) $\tilde{\eta} \in \partial\tilde{j}(.,\, v)$ a.e. Ω.
Now, let us show that $(v,\, w,\, z, \tilde{\eta})$ satisfy (19). Taking $w_\mu - \xi \in W^{1,p}(\Omega)$ as a test function in $S^{\beta_{\lambda(\mu),\mu},\, j_{\lambda(\mu),\mu}}(f)$, we get

$$\int_\Omega v_\mu(w_\mu - \xi) + \int_\Omega a(x,\, Dw_\mu)D(w_\mu - \xi) + \int_\Omega \delta_\mu(w_\mu - \xi) + \int_\Gamma z_\mu(w_\mu - \xi) = \int_\Omega f(w_\mu - \xi).$$

Assuming that $\xi \in K$, we check easily that

$$\Big(\delta_\mu - \tilde{\eta}_\mu\Big)(w_\mu - \xi) \ \geq 0 \quad \text{a.e.}\, \Omega$$

and, then

$$\begin{aligned}\int_\Omega v_\mu (w_\mu - \xi) + \int_\Omega a(.,\, Dw_\mu) D(w_\mu - \xi) + \int_\Omega \tilde{\eta}_\mu (w_\mu - \xi) \\ + \int_\Gamma z_\mu (w_\mu - \xi) \leq \int_\Omega f(w_\mu - \xi).\end{aligned}$$

Letting $\mu \ \to \ 0$, we get

$$\begin{aligned}\int_\Omega v(w - \xi) + \liminf_{\mu \to 0} \int_\Omega a(.,\, Dw_\mu) D(w_\mu - \xi) + \int_\Omega \tilde{\eta}(w - \xi) \\ + \int_\Gamma z(w - \xi) \leq \int_\Omega f(w - \xi).\end{aligned} \tag{21}$$

Moreover, we have $w \in K$. Indeed, thanks to (5) we have

$$\int_\Omega \mid \frac{1}{\lambda(\mu)} (v_\mu - \psi_+(x))^+ - \frac{1}{\mu}(v_\mu - \psi_-(x))^- \mid \leq \int_\Omega \mid f \mid, \tag{22}$$

then letting $\mu \ \longrightarrow \ 0$, we deduce $\psi_- \ \leq v \ \leq \psi_+$ a.e. Ω; hence $w \in K$. Now, let us prove that

$$\liminf_{\mu \to 0} \int_\Omega a(x,\, Dw_\mu) D(w_\mu - \xi) \geq \int_\Omega a(x,\, Dw) D(w - \xi). \tag{23}$$

Then, taking $\xi = w$ in (21), we get

$$\liminf_{\mu \to 0} \int_\Omega a(x,\, Dw_\mu) Dw_\mu \leq \int_\Omega \phi Dw. \tag{24}$$

At last, using Minty-Browder arguments as in the proof of Proposition 2.1 the proof is finished. ∎

Uniqueness: Let v_1, v_2 are two solutions in the sense of (19). Taking $w_2 - T_l(w_2 - w_1)$ (respectively $w_1 + T_l(w_2 - w_1)$) as a test function in the inequality satisfied by v_2 (respectively v_1) adding the two inequalities and using (H_2), we obtain

$$\int_\Omega (v_2 - v_1) T_l(w_2 - w_1) + \int_\Omega (\tilde{\eta}_2 - \tilde{\eta}_1) T_l(w_2 - w_1) + \int_\Gamma (z_2 - z_1) T_l(w_2 - w_1) \leq \int_\Omega (f_2 - f_1) T_l(w_2 - w_1).$$

Then, multiplying by $\dfrac{1}{l}$, using the monotonicity and letting $l \to 0$, we get

$$\int_\Omega |v_2 - v_1| \leq \int_\Omega |f_2 - f_1| \tag{25}$$

and the result follows. ∎

As an immediate consequence of the proof of Proposition 2.1, we have the following convergence result of the operator $\mathcal{A}_{\beta_{\lambda,\,\mu}, j_{\lambda,\,\mu}}$

Corollary 2.4 *Under assumptions of Proposition 2.3, there exists a subsequence* $\lambda(\mu)$ *such that as* $\mu \longrightarrow 0$ *we have* $\lambda(\mu) \longrightarrow 0$ *and the operator* $\mathcal{A}_{\beta_{\lambda(\mu),\,\mu},\, j_{\lambda(\mu),\,\mu}}$ *converges in* $L^1(\Omega)$*, in the sense of the resolvent to the T-accretive operator* $\mathcal{A}_{\beta,\,j}$*, defined by*

$$f \in \mathcal{A}_{\beta,\,j}\, v \Leftrightarrow \begin{cases} v \in L^{p'}(\Omega),\ \exists w \in W^{1,p}(\Omega) \cap K,\ \exists \eta \in L^1(\Omega),\ \exists z \in L^1(\Gamma) \\ v = \beta(w),\ \tilde{\eta} \in \partial\tilde{j}(.,\, v)\ \text{a.e. } \Omega,\ z \in \gamma(w)\ \text{a.e. } \Gamma, \text{ and} \\ \displaystyle\int_\Omega a(.,\, Dw) D(w-\xi) + \int_\Omega \tilde{\eta}(w-\xi) + \int_\Gamma z(w-\xi) \leq \int_\Omega f(w-\xi) \\ \text{for any } \ \xi \in W^{1,p}(\Omega) \cap K. \end{cases} \tag{26}$$

3 Evolution problem

To treat $P^{\beta,\,j}(u_0, f)$, we consider in $L^1(\Omega)$ the Cauchy problem

$$CP^{\beta,\,j}(u_0,\ f) \begin{cases} u_t + \mathcal{A}_{\beta,\,j}\, u \ni f & \text{in } \ (0,T) \\ u(0) = u_0. \end{cases}$$

Lemma 3.1 $\overline{\mathcal{D}(\mathcal{A}_{\beta,\,j})} = \Big\{ z \in L^1(\Omega)\ ;\ \psi_-(x) \leq z(x) \leq \psi_+(x)\ \text{a.e. } \Omega \text{ and } \exists\, w \in W^{1,p}(\Omega) \text{ such that } \ z = \beta(w) \Big\} =: X.$

Proof: By density and the definition of the operator $\mathcal{A}_{\beta,\,j}$ we have $\overline{\mathcal{D}(\mathcal{A}_{\beta,\,j})} \subseteq X$. To prove that $X \subseteq \overline{\mathcal{D}(\mathcal{A}_{\beta,\,j})}$, it is enough to prove that $X \cap L^\infty(\Omega) \subseteq \overline{\mathcal{D}(\mathcal{A}_{\beta,\,j})}$. Let $u \in X \cap L^\infty(\Omega)$ and u_ε be the solution of

$$\begin{cases} u_\varepsilon - \varepsilon \operatorname{div} a(x, Dw_\varepsilon) + \varepsilon \partial j(., u_\varepsilon) \ni u & u_\varepsilon = \beta(w_\varepsilon) \quad \text{in } \ \Omega \\ a(x, Dw_\varepsilon).\vec{n} + z_\varepsilon = 0 & z_\varepsilon \in \gamma(w_\varepsilon) \quad \text{on } \ \Gamma. \end{cases} \tag{27}$$

In the sense that, there exists $(w_\varepsilon, z_\varepsilon, \delta_\varepsilon) \in .....$ **.....**

We recall that

$$||u_\varepsilon||_{L^q(\Omega)} \leq ||u||_{L^q(\Omega)} \qquad \text{for all } \ q \in [1,\ \infty],$$

$$||\ w_\varepsilon\ ||_{L^\infty(\Omega)} \leq \max\Big(\beta^{-1}(\ ||\ u\ ||_{L^\infty(\Omega)}\ ) \cup (-\beta^{-1})(-\ ||\ u\ ||_{L^\infty(\Omega)}\ ) \Big)$$

and

$$||\ z_\varepsilon\ ||_{L^\infty(\Gamma)} \leq \max\Big(\gamma(\ ||\ w_\varepsilon\ ||_{L^\infty(\Omega)}\ ) \cup (-\gamma)(-\ ||\ w_\varepsilon\ ||_{L^\infty(\Omega)}\ ) \Big)$$

Then, by taking subsequence $\varepsilon_k \to 0$ if necessary, we have

$$\varepsilon z_\varepsilon \to 0 \text{ in } L^\infty(\Gamma),$$

and, there exists $(\tilde{u},\tilde{w}) \in L^\infty(\Omega)\times L^\infty(\Omega)$ such that

$$u_\varepsilon \to \tilde{u} \text{ in } L^{p'}(\Omega)\text{-weak}$$

and

$$w_\varepsilon \to \tilde{w} \text{ in } L^{p'}(\Omega)\text{-weak} .$$

Our aim now, is to prove that $\tilde{u} = u$ a.e. in Ω and the convergence of u_ε holds to be true in $L^1(\Omega)$. Taking $\xi = 0$ as a test function in the definition of the solution of problem (27) in the sense of Proposition 2.3 and using (H_1) we get

$$\int_\Omega |Dw_\varepsilon|^p \leq \frac{C(||u||_\infty, ||w||_\infty),}{\varepsilon\alpha} := \frac{C_4}{\varepsilon}.$$

Applying Minkowski inequality and (H_3), we get

$$\begin{aligned}(\int_\Omega |\varepsilon a(.,\, Dw_\varepsilon)|^{p'})^{\frac{1}{p'}} &\leq \sigma\Big(\varepsilon||k||_{L^{p'}(\Omega)} + \varepsilon(\int_\Omega |Dw_\varepsilon|^p)^{\frac{1}{p'}}\Big)\\ &\leq \sigma\Big(\varepsilon||k||_{L^{p'}(\Omega)} + \varepsilon^{\frac{1}{p}} C_4^{\frac{1}{p'}}\Big),\end{aligned}$$

so that

$$\varepsilon a(.,\, Dw_\varepsilon) \to 0 \text{ in } [L^{p'}(\Omega)]^N\text{-weak} .$$

Now, set $\delta_\varepsilon = \tilde{\eta}_\varepsilon + \eta_\varepsilon$, **with**

$$\eta_\varepsilon(x) \in \partial\mathbb{I}_{[\psi_-(x),\, \psi_+(x)]}(u_\varepsilon(x)), \quad \tilde{\eta}_\varepsilon(x) \in \partial\tilde{j}(x, u_\varepsilon(x)) \quad \textbf{a.e. } x\in\Omega$$

and $\tilde{\eta}_\varepsilon \in L^1(\Omega)$**.** Using the same arguments of the proof of Proposition 2.3, we have

$$\varepsilon\tilde{\eta}_\varepsilon \to 0 \text{ in } L^1(\Omega)\text{-weak}$$

Passing to the limit in the definition of the solution of problem (27), we obtain

$$\lim_{\varepsilon\to 0}\int_\Omega (u_\varepsilon - u)(w_\varepsilon - \xi) \leq 0 \qquad \forall\, \xi \in W^{1,p}(\Omega)\cap K. \tag{28}$$

This implies that

$$\lim_{\varepsilon\to 0}\int_\Omega u_\varepsilon w_\varepsilon \leq \int_\Omega u\tilde{w} + \int_\Omega \tilde{u}\xi - \int_\Omega u\xi \qquad \forall\, \xi \in W^{1,p}(\Omega)\cap K, \tag{29}$$

so that

$$\lim_{\varepsilon\to 0}\int_\Omega u_\varepsilon w_\varepsilon \leq \int_\Omega \tilde{u}\tilde{w}.$$

Indeed, since $w_\varepsilon \in W^{1,p}(\Omega)\cap K$, it is enough to take $\xi = w_{\varepsilon'}$ in (29) and to let $\varepsilon' \to 0$. Now, by the standard monotonicity arguments we deduce that $\tilde{u} = \beta(\tilde{w})$ and $\lim_{\varepsilon\to 0}\int_\Omega u_\varepsilon w_\varepsilon = \int_\Omega \tilde{u}\tilde{w}$ in $L^1(\Omega)$. Then (28) implies

$$\int_\Omega (\tilde{u} - u)(\tilde{w} - \xi) \leq 0 \qquad \forall\, \xi \in W^{1,p}(\Omega)\cap K. \tag{30}$$

At last, taking $\xi = \tilde{w} - T_l(\tilde{w} - w)$ in (30), multiplying the inequality by $\frac{1}{l}$ and letting $l \to 0$, we obtain

$$||\tilde{u} - u||_{L^1(\Omega)} = \int_\Omega (\tilde{u} - u) Sign_0(\tilde{w} - w) \leq 0,$$

then $\tilde{u} = u$ a.e. $x \in \Omega$ and u_ε converges weakly to u in $L^{p'}(\Omega)$. Moreover,

$$||u_\varepsilon||_{L^{p'}(\Omega)} \leq ||u||_{L^{p'}(\Omega)},$$

then we deduce that u_ε converge strongly to u in $L^{p'}(\Omega)$ and then in $L^1(\Omega)$. ■

In order to treat the evolution problem, let us consider again the monotone approximations

$$j_{\lambda,\,\mu}(x,\, r) = \tilde{j}(x,\, r) + \frac{1}{2\lambda}\Big(\big(\, r - \psi_+(x)\big)^+\Big)^2 - \frac{1}{2\mu}\Big(\big(\, r - \psi_-(x)\big)^-\Big)^2 \quad \text{a.e. } x \in \Omega$$

and

$$\beta_{\lambda,\,\mu}(r) = \beta(r) \;-\; \lambda\; r^- + \mu\; r^+.$$

Proposition 3.2 *If $f \in L^\infty(Q)$ and $u_0 \in L^\infty(\Omega)$, then the mild solution $u_{\lambda,\,\mu}$ of $CP^{\beta_{\lambda,\,\mu},\, j_{\lambda,\,\mu}}(u_0, f)$ is the unique weak solution of $P^{\beta_{\lambda,\,\mu},\, j_{\lambda,\,\mu}}(u_0,\, f)$; i.e. $u_{\lambda,\,\mu} \in L^1(Q)$ and there exists $(w_{\lambda,\,\mu}, z_{\lambda,\,\mu}, \delta_{\lambda,\,\mu}) \in L^p\Big(0, T;\, W^{1,p}(\Omega)\Big) \times L^1(\Sigma) \times L^1(Q)$ such that, $u_{\lambda,\,\mu} = \beta_{\lambda,\,\mu}(w_{\lambda,\,\mu})$ and $\delta_{\lambda,\,\mu} \in \partial j_{\lambda,\,\mu}(.,\, u_{\lambda,\,\mu})$ a.e. in Q, $z_{\lambda,\,\mu} \in \gamma(w_{\lambda,\,\mu})$ a.e. on Σ and*

$$\begin{aligned} -\int_0^\tau\int_\Omega u_{\lambda,\,\mu}\xi_t + \int_0^\tau\int_\Omega a(.,\, Dw_{\lambda,\,\mu})D\xi + \int_0^\tau\int_\Omega \delta_{\lambda,\,\mu}\xi + \int_0^\tau\int_\Gamma z_{\lambda,\,\mu}\xi \\ = \int_0^\tau\int_\Omega f\xi + \int_\Omega u_0\xi(0) \end{aligned} \tag{31}$$

for any $\xi \in \mathcal{C}^1([0,\tau] \times \overline{\Omega})$ with $\xi(.,\tau) \equiv 0$. Moreover, for any $\tau \geq 0$

$$||u_{\lambda,\,\mu}(\tau)||_{L^\infty(Q)} \leq ||u_0||_{L^\infty(\Omega)} + T||f||_{L^\infty(Q)} \tag{32}$$

and

$$\int_0^\tau\int_\Omega |Dw_{\lambda,\,\mu}|^p \;\leq\; C \tag{33}$$

where C is a constant independent of m and n. Moreover, for any $\lambda' \geq \lambda > 0$ and $\mu' \geq \mu > 0$, we have

$$w_{\lambda',\,\mu} \leq w_{\lambda,\,\mu} \leq w_{\lambda,\,\mu'} \qquad \text{a.e. in } Q.$$

To prove this proposition we need the following lemma (chain rule) which is is well known by now in the field of nonlinear degenerate parabolic problem (see for instance [1], [2], [11] and [23]).

Lemma 3.3 *Let u be a weak solution of the problem $P^{\beta_{\lambda,\,\mu},j_{\lambda,\,\mu}}(u_0,f)$ in the sense of Proposition 3.2. Then, we have*

$$\int\int_Q \sigma a(.,\, Dw)DT_l(w-\xi) + \int\int_Q \sigma\delta T_l(w-\xi) + \int\int_\Sigma \sigma z T_l(w-\xi)$$
$$= \int\int_Q \sigma f T_l(w-\xi) + \int\int_Q \sigma_t \int_{w_0}^{w(t)} T_l(r-\xi)d\beta(r), \quad \forall\, l>0$$

for any $\xi \in W^{1,p}(\Omega)$ and $\sigma \in \mathcal{D}(0,T)$.

Proof: The lemma is a simple consequence of Chain Rule Lemma (see for instance [1], [2], [11] and [23]). ■

Proof of Proposition 3.2 : Let $\varepsilon = T/k$, with $k \in \mu$ and let us consider the subdivision $t_0 = 0 < t_1 < ... < t_{k-1} < \tau \le t_k$, with $t_i - t_{i-1} = \varepsilon,\ f^1,...,f^k \in L^\infty(\Omega)$ and $\displaystyle\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i} ||f(t)-f^i||_{L^1(\Omega)} \le \varepsilon$. We define the ε-approximate solution by $u^\varepsilon_{\lambda,\,\mu}(0) = u_0$ and

$$u^\varepsilon_{\lambda,\,\mu}(t) = u^i_{\lambda,\,\mu} \qquad \text{for } t \in]t_{i-1},t_i],\ i=1,...,k,$$

where $u^i_{\lambda,\,\mu}$ is given by

$$u^i_{\lambda,\,\mu} - u^{i-1}_{\lambda,\,\mu} + \varepsilon\mathcal{A}_{\beta_{\lambda,\,\mu},\,j_{\lambda,\,\mu}}\, u^i_{\lambda,\,\mu} \ni \varepsilon f^i.$$

That is, there exists $(w^i_{\lambda,\,\mu}, z^i_{\lambda,\,\mu}, \delta^i_{\lambda,\,\mu}) \in W^{1,p}(\Omega)\times L^1(\Gamma)\times L^1(\Omega)$ such that $\delta^i_{\lambda,\,\mu} \in \partial j_{\lambda,\,\mu}(.,\, u^i_{\lambda,\,\mu})$, $u^i_{\lambda,\,\mu} = \beta_{\lambda,\,\mu}(w^i_{\lambda,\,\mu})$ a.e. Ω, $z^i_{\lambda,\,\mu} \in \gamma(\, w^i_{\lambda,\,\mu})$ a.e. Γ and

$$\int_\Omega u^i_{\lambda,\,\mu}\xi + \int_\Omega a(.,\, Dw^i_{\lambda,\,\mu})D\xi + \int_\Omega \delta^i_{\lambda,\,\mu}\xi + \int_\Gamma z^i_{\lambda,\,\mu}\xi = \int_\Omega u^{i-1}_{\lambda,\,\mu}\xi + \int_\Omega f^i\xi \tag{34}$$

for any $\xi \in W^{1,p}(\Omega)$. As in this proposition m and n are fixed; we omit the notation with respect to λ and μ. We denote w^ε, z^ε, δ^ε and f^ε the functions defined by $w^\varepsilon(t) = w^i_{\lambda,\,\mu}$, $\delta^\varepsilon(t) = \delta^i$, $z^\varepsilon(t) = z^i$, $f^\varepsilon(t) = f^i$ for $t \in]t_{i-1},t_i]$. Thanks to Proposition 2.1, it follows that

$$||u^i||_{L^\infty(\Omega)} \le ||u_0||_{L^\infty(\Omega)} + \varepsilon\sum_{j=1}^{i}||f^j||_{L^\infty(\Omega)},$$

so that

$$||u^\varepsilon(t)||_{L^\infty(\Omega)} \le ||u_0||_{L^\infty(\Omega)} + \int_0^T ||f(t)||_{L^\infty(\Omega)}dt := M_1, \quad \forall t\in[0,T]. \tag{35}$$

Using (H_4), (H_5) and the definition of ∂j, we have

$$||w^\varepsilon(t)||_{L^\infty(\Omega)} \le \max\Big(\beta^{-1}(M_1)\cup(-\beta^{-1})(-M_1)\Big) =: M_2 \quad \forall\, t\in(0,\,T), \tag{36}$$

$$||z^\varepsilon(t)||_{L^\infty(\Gamma)} \le \max\Big(\gamma(M_2)\cup(-\gamma)(-M_2)\Big) =: M_2' \quad \forall\, t\in(0,\,T), \tag{37}$$

and

$$|\partial j(x,\,u^\varepsilon)| \le j(x,u^\varepsilon - Sign_0(u^\varepsilon)) - j(x,u^\varepsilon) \qquad \text{a.e. in } \ Q. \tag{38}$$

Now, taking w^i as a test function in (34) and using the fact that, for any $i = 1,...,k$,

$$\int_\Omega (u^{i-1}-u^i)w^i \le \int_\Omega \mathcal{J}(u^{i-1}) - \int_\Omega \mathcal{J}(u^i), \quad \eta^i w^i \ge 0 \text{ and } \ z^i w^i \ge 0,$$

where $\mathcal{J} : \mathbb{R} \longmapsto [0,\infty]$ is given by $\mathcal{J}(\beta(q)) = \displaystyle\int_0^q s d\beta(s)$. We get

$$\int_\Omega \mathcal{J}(u^i) + \varepsilon\int_\Omega a(.,\,Dw^i)Dw^i \le \varepsilon\int_\Omega f^i w^i + \int_\Omega \mathcal{J}(u^{i-1}). \tag{39}$$

So, adding (39) for $i = 1,...,k$, we get

$$\int_\Omega \mathcal{J}(u^\varepsilon(\tau)) + \int_0^\tau\int_\Omega a(.,\,Dw^\varepsilon)Dw^\varepsilon \le \int_\Omega \mathcal{J}(u_0) + \int_0^\tau\int_\Omega f^\varepsilon w^\varepsilon. \tag{40}$$

Using (H_1), (36) and the positivity of $\mathcal{J}$, we have

$$\alpha\int_0^\tau\int_\Omega |Dw^\varepsilon|^p \quad\le\quad \int_\Omega \mathcal{J}(u_0) + ||w^\varepsilon||_{L^\infty(Q)}||f_\varepsilon||_{L^1(Q)} =: C \tag{41}$$

We recall that, by nonlinear semigroup theory, as $\varepsilon \to 0$, u^ε converges in $\mathcal{C}([0,T],\,L^1(\Omega))$ to the *mild solution* u of the Cauchy problem $CP^{\beta,j}(u_0,\,f)$. Thanks to (36), (41) and (H_3), there exists $w \in L^p(0,T,\,W^{1,p}(\Omega))$, $\chi \in [L^{p'}(Q)]^N$ such that, as $\varepsilon_k \longrightarrow 0$, we have $w^{\varepsilon_k} \longrightarrow w$, in $L^p\Big(0,T;\,W^{1,p}(\Omega)\Big)$-weak and $a(.,\,Dw^{\varepsilon_k}) \longrightarrow \chi$ in $[L^{p'}(Q)]^N$-weak. On the other hand, we have $j(x,.) \in L^1(Q)$, using (35) and (38), then $\delta^{\varepsilon_k} \to \delta$ in $L^1(Q)$-weak. Thanks to (37), $z^{\varepsilon_k} \to z$ in $L^1(\Sigma)$-weak. Applying Lemma 1.6 of [9] we have also $\delta \in \partial j(.,\,u)$ a.e. Q and $z \in \gamma(w)$ a.e. Σ. The proof of $\operatorname{div}\chi = \operatorname{div} a(.,\,Dw)$ follows in a standard way (cf. [5] and [15]). Combining all estimates and passing to the limit with $\varepsilon_k \longrightarrow 0$ in the weak formulation, (35) and (41) we obtain (31), (32) and (33).
Now, let $\lambda' \ge \lambda > 0$ and $\mu > 0$. Thanks to Lemma 4.3 (see the Appendix), for $i = 1,...,k$, we have $w^i_{\lambda,\,\mu} \ge w^i_{\lambda',\,\mu}$ a.e. Q, then $w^\varepsilon_{\lambda,\,\mu} \ge w^\varepsilon_{\lambda',\,\mu}$ a.e. Q . Passing to the limit as ε go to 0, we deduce that $w_{\lambda,\,\mu} \ge w_{\lambda',\,\mu}$ a.e. Q and for $\mu \ge \mu' > 0$, $\lambda > 0$; $w_{\lambda,\,\mu} \le w_{\lambda,\,\mu'}$ a.e. Q. ■

Theorem 3.4 *Given* $f \in L^\infty(Q)$, $u_0 \in L^\infty(\Omega)\cap\overline{\mathcal{D}(\mathcal{A}_{\beta,\,j})}$. *Let* u *be the* mild solution *of* $CP^{\beta,\,j}(u_0,\,f)$. *Then, there exists* $w \in L^p\Big(0,T;\,W^{1,p}(\Omega)\Big)$, $\tilde\eta \in L^1(Q)$ *and* $z \in L^1(\Sigma)$ *such that* $u = \beta(w)$, $\tilde\eta \in \partial\tilde{j}(x,\,u)$ *a.e.* Q, $z \in \gamma(w)$ *a.e.* Σ *and*

$$\begin{aligned}\frac{d}{dt}\int_\Omega\int_0^w T_l(r-\xi)d\beta(r) \quad &+\int_\Omega a(.,\ Dw)DT_l(w-\xi)+\int_\Omega \tilde{\eta}T_l(w-\xi)\\ +\int_\Gamma zT_l(w-\xi) \quad &\leq \int_\Omega fT_l(w-\xi) \qquad in\ \ \mathcal{D}'(0,T)\end{aligned} \tag{42}$$

for any $\xi\in\mathcal{C}^1(\overline{\Omega})\cap K$.

Proof: Let $u_{\lambda,\,\mu}$ be the mild solution of $CP^{\beta_{\lambda,\,\mu},\,j_{\lambda,\,\mu}}(u_0,\ f)$ and $(w_{\lambda,\,\mu},\ ,z_{\lambda,\,\mu},,\ ,\delta_{\lambda,\,\mu})$ as given by Proposition 3.2. Thanks to Proposition 3.2 and the monotonicity of $w_{\lambda,\,\mu}$, there exists a subsequence $\lambda(\mu)$ such that as $\mu\longrightarrow 0$; $\lambda(\mu)\longrightarrow 0$ and $u_\mu =: u_{\lambda(\mu),\,\mu}\to u$ in $\mathcal{C}\Big([0,T);\ L^1(\Omega)\Big)$, $u\in\overline{D(\mathcal{A}_{\beta,\,j})}$ and there exists $(w_{\lambda(\mu),\,\mu},\ z_{\lambda(\mu),\,\mu},\ \delta_{\lambda(\mu),\,\mu})$ such that as $\mu\to 0$, we have $w_{\lambda(\mu),\,\mu} =: w_\mu \to w$ in $L^1(Q)$ and in $L^p\Big(0,T;W^{1,p}(\Omega)\Big)$-weak , $a(.,Dw_\mu)\to\chi$ in $[L^{p'}(Q)]^N$-weak and $z_{\lambda(\mu),\,\mu} =: z_\mu\to z$ in $L^p(\Sigma)$-weak. Since for a.e. $t\in(0,T)$, $u(t)\in\overline{D(\mathcal{A}_{\beta,\,j})}$ then $w(t)\in K$ for a.e. $t\in(0,T)$. Setting $\delta_{\lambda(\mu),\,\mu} =: \delta_\mu$ and

$$\tilde{\eta}_\mu=\delta_\mu-\frac{1}{\lambda(\mu)}(u_\mu-\psi_+(x))^+ +\frac{1}{\mu}(u_\mu-\psi_-(x))^-,$$

we have, $\tilde{\eta}_\mu\in\partial\tilde{j}(.,u_\mu)$ and

$$\Big(\delta_\mu-\tilde{\eta}_\mu\Big)T_l(w_\mu-\xi)\geq 0 \qquad \text{for any } \xi\in K;$$

so that,

$$\begin{aligned}&\int\int_Q \sigma a(.,\ Dw_\mu)DT_l(w_\mu-\xi)+\int\int_Q\sigma\tilde{\eta}_\mu T_l(w_\mu-\xi)+\int\int_\Sigma\sigma z_\mu T_l(w_\mu-\xi)\\ &\leq\int\int_Q\sigma_t\int_{w_0}^{w_\mu(t)}T_l(r-\xi)d\beta_\mu(r)+\int\int_Q\sigma fT_l(w_\mu-\xi)\quad\forall\xi\in K.\end{aligned} \tag{43}$$

Using (32) with the definition $\partial\tilde{j}$ and the fact that $\tilde{j}(x,\ u_\mu)\in L^1(Q)$, we get $\tilde{\eta}_\mu\to\tilde{\eta}$ in $L^1(Q)$-weak and $\tilde{\eta}\in\partial\tilde{j}(.,\ u)$ a.e. in Q (see Lemma 1.6 of [9]). Letting $\mu\to 0$ in (43), we get

$$\begin{aligned}\liminf_{\mu\to 0}&\int\int_Q\sigma a(.,\ Dw_\mu)DT_l(w_\mu-\xi)+\int\int_Q\sigma\tilde{\eta}T_l(w-\xi)+\int\int_\Sigma\sigma zT_l(w-\xi)\\ &\leq\int\int_Q\sigma_t\int_{w_0}^{w(t)}T_l(r-\xi)d\beta(r)+\int\int_Q\sigma fT_l(w-\xi),\ \forall\xi\in K.\end{aligned} \tag{44}$$

To prove that $(u,w,z,\tilde{\eta})$ satisfies (42), we need to show that

$$\liminf_{\mu\to 0}\int\int_Q\sigma a(.,\ Dw_\mu)DT_l(w_\mu-\xi)\geq\int\int_Q\sigma a(x,Dw)DT_l(w-\xi).$$

To this aim, it is enough to prove that

$$\liminf_{\mu\to 0}\int_Q \sigma a(x,\ Dw_\mu)DS(w_\mu-\xi)\geq \int_Q \sigma a(x,\ Dw)DS(w-\xi) \tag{45}$$

for any $S\in\mathcal{P}$ where $\mathcal{P}:=\{p\in C^1(\mathbb{R});\ p(0)=0,\ 0\leq p'\leq 1,\ Supp(p')\ \text{ is compact}\}$. Thanks to (H_2), we have

$$\begin{aligned}\int_Q \sigma a(x,\ Dw_\mu)DS(w_\mu-\xi) &\geq \int_Q \sigma a(x,\ Dw_\mu)DwS'(w_\mu-\xi)\\ +\int_Q \sigma a(x,\ Dw)D(w_\mu-w)S'(w_\mu-\xi) &- \int_Q \sigma a(x,\ Dw_\mu)D\xi S'(w_\mu-\xi).\end{aligned} \tag{46}$$

Since, $S'(w_\mu-\xi)\to S'(w-\xi)$ a.e. on Q, $Dw_\mu\rightharpoonup Dw$ in $[L^p(Q)]^N$-weak and $a(x,Dw_\mu)\rightharpoonup\chi$ in $[L^{p'}(Q)]^N$-weak, then letting $\mu\to 0$ in (46) we obtain

$$\liminf_{\mu\to 0}\int_Q \sigma a(x,Dw_\mu)DS(w_\mu-\xi)\geq \int_Q \sigma\Big(a(x,Dw)Dw-\chi D\xi\Big)S'(w-\xi). \tag{47}$$

So, by using Minty argument, (45) follows by proving

$$\liminf_{\mu\to 0}\int\int_Q \sigma a(.,\ Dw_\mu)Dw_\mu \leq \int\int_Q \sigma\chi Dw. \tag{48}$$

Our aim now is to prove (48). First we see that by using Lemme 3.3, we have

$$\begin{aligned}&\int\int_Q \sigma a(.,\ Dw_\mu)Dw_\mu = \int\int_Q \sigma f w_\mu\\ &+\int\int_Q \sigma_t\int_0^{w_\mu} r d\beta_\mu(r) - \int\int_Q \sigma\tilde{\eta}_\mu w_\mu - \int\int_\Sigma \sigma z_\mu w_\mu,\end{aligned}$$

and by letting $\mu\to 0$, we get

$$\begin{aligned}\liminf_{\mu\to 0}\int\int_Q \sigma a(.,\ Dw_\mu)Dw_\mu &= \int\int_Q \sigma f w + \int\int_Q \sigma_t\int_0^w r d\beta(r)\\ -\int\int_Q \sigma\tilde{\eta}w - \int\int_\Sigma \sigma z w &- \liminf_{\mu\to 0}\int\int_Q \sigma(\delta_\mu-\tilde{\eta}_\mu)w_\mu.\end{aligned} \tag{49}$$

To prove that the right hand side of (49) is less or equal to $\int\int_Q \sigma\chi Dw$ we use the Landes regularization in time of w (cf. [17]) defined by

$$\bar{w_k}(x,t):=k\int_{-\infty}^t e^{k(s-t)}w(x,s)ds$$

for a.e. (x,t), where extend w by 0 for $s<0$. Observe that $\bar{w_k}\in L^p\Big(0,T;W^{1,p}(\Omega)\Big)\cap L^\infty(Q)$, $\dfrac{\partial\bar{w_k}}{\partial t}=k(w-\bar{w_k})\in L^p\Big(0,T;\ W^{1,p}(\Omega)\Big)\cap L^\infty(Q)$, $\bar{w_k}(0)=0$ a.e. in Ω, and

$\bar{w_k} \longrightarrow w \in L^p\Big(0,T;W^{1,p}(\Omega)\Big)$ as $k \to \infty$. Taking $\sigma \bar{w_k}$ as a test function in the definition of the solution of problem $P^{\beta_\mu,\,j_\mu}(u_0,\ f)$, we obtain

$$\int\int_Q \sigma a(.,\ Dw_\mu)D\bar{w_k} + \int\int_Q \sigma\delta_\mu \bar{w_k} + \int\int_\Sigma \sigma z_\mu \bar{w_k} = \int\int_Q \sigma f \bar{w_k} + \int\int_Q u_\mu(\sigma\bar{w_k})_t.$$

Letting $\mu \to 0$, we get

$$\int\int_Q \sigma\chi D\bar{w_k} + \int\int_\Sigma \sigma z\bar{w_k} = \int\int_Q \sigma f\bar{w_k} + \int\int_Q u(\sigma\bar{w_k})_t - \liminf_{\mu\to 0}\int\int_Q \sigma\delta_\mu\bar{w_k}. \quad (50)$$

Letting $k \to \infty$, we get

$$\int\int_Q \sigma\chi Dw + \int\int_\Sigma \sigma zw = \int\int_Q \sigma fw + \lim_{k\to\infty}\int\int_Q u(\sigma\bar{w_k})_t - \lim_{k\to\infty}\liminf_{\mu\to 0}\int\int_Q \sigma\delta_\mu\bar{w_k}. (51)$$

For the second term of the right hand of (50), we observe that

$$\begin{aligned}
\int\int_Q u(\sigma\bar{w_k})_t &= \int\int_Q u\sigma_t\bar{w_k} + k\int\int_Q u\sigma(w-\bar{w_k}) \\
&= \int\int_Q u\sigma_t\bar{w_k} + k\int\int_Q \sigma(u-\beta(\bar{w_k}))(w-\bar{w_k}) \\
&+ \ k\int\int_Q \sigma\beta(\bar{w_k})(w-\bar{w_k})
\end{aligned}$$

Since β is monotone, then $(u-\beta(\bar{w_k}))(w-\bar{w_k}) \geq 0$ and

$$\begin{aligned}
\int\int_Q u(\sigma\bar{w_k})_t &\geq \int\int_Q u\sigma_t\bar{w_k} + k\int\int_Q \sigma\beta(\bar{w_k})(w-\bar{w_k}) \\
&\geq \int\int_Q u\sigma_t\bar{w_k} + \int\int_Q \sigma\beta(\bar{w_k})(\bar{w_k})_t \\
&\geq \int\int_Q u\sigma_t\bar{w_k} - \int\int_Q \sigma_t\int_0^{\bar{w_k}}\beta(r)dr \\
&\geq \int\int_Q \sigma_t\int_0^{\bar{w_k}} rd\beta(r).
\end{aligned}$$

So, letting $k \to \infty$, we obtain

$$\liminf_{k\to\infty}\int\int_Q u(\sigma\bar{w_k})_t \geq \int\int_Q \sigma_t\int_0^w rd\beta(r). \quad (52)$$

As to the last term, it is not difficult to see that $\bar{w_k} \in K$ and

$$\sigma(\delta_\mu - \tilde{\eta}_\mu)(w_\mu - \bar{w_k}) \geq 0 \quad \text{a.e.}\, Q.$$

So, using the monotonicity of $\partial \tilde{j}$, we have

$$\begin{aligned} &\int\int_Q \sigma\delta_\mu \bar{w_k} = \int\int_Q \sigma\tilde{\eta}_\mu \bar{w_k} + \int\int_Q \sigma(\delta_\mu - \tilde{\eta}_\mu)\bar{w_k} \\ &= \int\int_Q \sigma\tilde{\eta}_\mu \bar{w_k} + \int\int_Q \underbrace{\sigma(\delta_\mu - \tilde{\eta}_\mu)(\bar{w_k} - w_\mu)}_{\leq 0} \\ &+ \int\int_Q \sigma(\delta_\mu - \tilde{\eta}_\mu) w_\mu \\ &\leq \int\int_Q \sigma\tilde{\eta}_\mu \bar{w_k} + \int\int_Q \sigma(\delta_\mu - \tilde{\eta}_\mu) w_\mu \end{aligned} \tag{53}$$

and

$$\liminf_{\mu\to 0} \int\int_Q \sigma\delta_\mu \bar{w_k} \leq \int\int_Q \sigma\tilde{\eta}\bar{w_k} + \liminf_{\mu\to 0}\int\int_Q \sigma(\delta_\mu - \tilde{\eta}_\mu) w_\mu. \tag{54}$$

Then (51) implies

$$\begin{aligned} \int\int_Q \sigma\chi Dw \quad &\geq \int\int_Q \sigma f w + \int\int_Q \sigma_t \int_0^w r d\beta(r) - \int\int_Q \sigma\tilde{\eta} w \\ &-\liminf_{\mu\to 0}\int\int_Q \sigma(\delta_\mu - \tilde{\eta}_\mu) w_\mu - \int\int_\Sigma \sigma z w \end{aligned} \tag{55}$$

and

$$\int\int_Q \sigma\chi Dw \geq \lim_{\mu\to 0}\int\int_Q \sigma a(.,\, Dw_\mu) Dw_\mu.$$

Consequently (48) holds. ■

4 Uniqueness

To prove uniqueness, we use the concept of integral solution, which is well known in the context of the abstract Cauchy problem (cf. [8]). This concept was previously used in [9] for the proof of uniqueness of weak solution of elliptic-parabolic problems, with homogeneous Dirichlet boundary conditions.

Definition 4.1 *A function $u \in \mathcal{C}\Big([0,T],\ L^1(\Omega)\Big)$ is an integral solution of $CP^{\beta,\,j}(u_0,\ f)$ if for any $\hat{f} \in \mathcal{A}_{\beta,\,j}\hat{u}$, we have*

$$\frac{d}{dt}\int_\Omega |u(t)-\hat{u}| \leq \int_\Omega (f-\hat{f})Sign_0(u(t)-\hat{u}) + \int_{[u(t)=\hat{u}]} |f-\hat{f}| \quad \textit{in } \mathcal{D}'(0,T),$$

and $u(0)=u_0$.

Since, $\mathcal{A}_{\beta,\,j}$ is accretive in $L^1(\Omega)$, it is well known (cf. [8], [9]) that a *mild solutions* and integral solutions of problem $CP^{\beta,\,j}(u_0,\ f)$ coincide. To get the uniqueness, we prove with the following proposition that a solution of $P^{\beta,\,j}(u_0,\ f)$ is an integral solution of $CP^{\beta,\,j}(u_0,\ f)$. Therefore, by the nonlinear semigroup theory (cf. [9]), the *mild solution* of $P^{\beta,\,j}(u_0,\ f)$ is the unique solution of problem $P^{\beta,\,j}(u_0,\ f)$.

Proposition 4.2 *Let* $u_0 \in L^\infty(\Omega)$ *and* $f \in L^\infty(Q)$. *If* $u \in L^\infty(Q)$ *is a solution of* $P^{\beta,\,j}(u_0,\ f)$ *in the sense of Theorem 3.4, then* u *is an integral solution of the problem* $CP^{\beta,\,j}(u_0,\ f)$.

Proof. Let $\hat{f} \in \mathcal{A}_{\beta,\,j}\hat{u}$, $\hat{\eta} \in \partial\tilde{j}(.,\ \hat{u})$, $\hat{u}=\beta(\hat{w})$ a.e. Ω and $\hat{z} \in \gamma(\hat{w})$ a.e. Γ be as given by the definition of the operator $\mathcal{A}_{\beta,\,j}$. Taking $\hat{w}$ as a test function in (42) and multiplying the inequality by $\dfrac{1}{l}$, we obtain

$$\begin{aligned}&\int\int_Q \sigma a(.,\ Dw)D\frac{1}{l}T_l(w-\hat{w}) + \int\int_Q \sigma\tilde{\eta}\frac{1}{l}T_l(w-\hat{w})\\ &\leq \int\int_Q \sigma_t \int_{w_0}^{w(t)} \frac{1}{l}T_l(r-\hat{w})d\beta(r) + \int\int_Q \sigma f\frac{1}{l}T_l(w-\hat{w}).\end{aligned} \tag{56}$$

Passing to limit as $l \to 0$, we get

$$\begin{aligned}\int_{w_0}^{w(x,t)} \frac{1}{l}T_l(r-\hat{w}(x))d\beta(r) \quad &\to \int_{w_0}^{w(x,t)} Sign_0(r-\hat{w}(x))d\beta(r)\\ &= |u(x,t)-\beta(\hat{w}(x))| - |u_0(x)-\beta(\hat{w}(x))|\end{aligned}$$

a.e. $(t,x) \in Q$. Moreover,

$$|\int_0^{w(x,t)} \frac{1}{l}T_l(r-\hat{w}(x))d\beta(r)| \leq |u(x,t)|$$

so that by Lebesgue's dominated convergence theorem, the first term of the right hand of inequality (56) converges to $\int\int_Q (|u(x,t)-\beta(\hat{w})|-|u_0-\beta(\hat{w})|)\sigma_t$. Obviously, $\int\int_Q f\sigma\frac{1}{l}T_l(w-\hat{w})$, $\int\int_Q \sigma\tilde{\eta}\frac{1}{l}T_l(w-\hat{w})$ and $\int\int_\Sigma \sigma z\frac{1}{l}T_l(w-\hat{w})$ converges, respectively to $\int\int_Q f\sigma Sign_0(w-\hat{w})$, $\int\int_Q \sigma\tilde{\eta}Sign_0(w-\hat{w})$ and $\int\int_\Sigma \sigma z Sign_0(w-\hat{w})$. As to the first term on the left-hand side, note that using the definition of the operator $\mathcal{A}_{\beta,\,j}$, we have

$$\int\int_Q \sigma a(.,\ Dw) D\frac{1}{l}T_l(w-\hat{w}) \geq I_l^1 + I_l^2,$$

where

$$I_l^1 = \int\int_Q \sigma(a(.,\ Dw) - a(.,\ D\hat{w}))D\frac{1}{l}T_l(w-\hat{w})$$

and

$$I_l^2 = \int\int_Q \hat{f}\sigma\frac{1}{l}T_l(w-\hat{w}) - \int\int_Q \sigma\hat{\eta}\frac{1}{l}T_l(w-\hat{w}) - \int\int_\Sigma \sigma\hat{z}\frac{1}{l}T_l(w-\hat{w}).$$

Clearly, I_l^2 converges to

$$\int\int_Q \hat{f}\sigma Sign_0(w-\hat{w}) - \int\int_Q \sigma\hat{\eta} Sign_0(w-\hat{w}) - \int\int_\Sigma \sigma\hat{z} Sign_0(w-\hat{w}).$$

On the other hand, we have

$$\liminf_{l\to 0} I_l^1 \geq \lim_{l\to 0}\int\int_Q \sigma\frac{1}{l}T_l'(w-\hat{w})(a(.,\ Dw) - a(.,\ D\hat{w}))D(w-\hat{w}) \geq 0.$$

So, letting $l \to 0$ in (56), we get

$$\begin{aligned}
-\int\int_Q (|u(x,t)-\beta(\hat{w})| - |u_0(x)-\beta(\hat{w})|)\sigma_t + \int\int_Q |\tilde{\eta}-\hat{\eta}|\sigma + \int\int_\Sigma |z-\hat{z}|\sigma \\
\leq \int\int_Q \sigma(f-\hat{f})Sign_0(w-\hat{w}) \\
\leq \int_0^T \Big\{ \int_\Omega \sigma(f-\hat{f})Sign_0(u(t)-\beta(\hat{w})) + \int_{[u(t)=\beta(\hat{w})]} \sigma|f-\hat{f}|\Big\}.
\end{aligned}$$

This implies that for any $t \in [0,T)$, we have

$$\int_\Omega |u(t)-\hat{u}| \leq \int_\Omega |u_0-\hat{u}| + \int_0^t \Big\{ \int_\Omega (f-\hat{f})Sign_0(u(t)-\hat{u}) + \int_{[u(t)=\hat{u}]} |f-\hat{f}|\Big\}$$

and the proof is finished. ■

Appendix

Lemma 4.3 *Let $v_{\lambda,\,\mu}$, $v_{\lambda',\,\mu}$, $v_{\lambda,\,\mu'}$ be the weak solutions in the sense of Proposition (2.1) of $S^{\beta_{\lambda,\,\mu},\,j_{\lambda,\,\mu}}(f_{\lambda,\,\mu})$, $S^{\beta_{\lambda',\,\mu},\,j_{\lambda',\,\mu}}(f_{\lambda',\,\mu})$ and $S^{\beta_{\lambda,\,\mu'},\,j_{\lambda,\,\mu'}}(f_{\lambda,\,\mu'})$ respectively. If, for $\lambda' \geq \lambda > 0$ and $\mu' \geq \mu > 0$, we have $f_{\lambda',\,\mu} \leq f_{\lambda,\,\mu} \leq f_{\lambda,\,\mu'}$ a.e. Ω, then*

$$w_{\lambda',\,\mu} \leq w_{\lambda,\,\mu} \leq w_{\lambda,\,\mu'} \qquad a.e.\ \Omega$$

where, $w_{\lambda,\,\mu} = \beta^{-1}_{\lambda,\,\mu}(v_{\lambda,\,\mu})$, $w_{\lambda',\,\mu} = \beta^{-1}_{\lambda',\,\mu}(v_{\lambda',\,\mu})$ and $w_{\lambda,\,\mu'} = \beta^{-1}_{\lambda,\,\mu'}(v_{\lambda,\,\mu'})$ a.e. Ω.

Proof: Using the monotonicity of ∂j, a and γ, we get

$$\int_\Omega (v_{\lambda',\mu} - v_{\lambda,\mu}) Sign_0^+(w_{\lambda',\mu} - w_{\lambda,\mu}) \ \leq \ \int_\Omega (f_{\lambda',\mu} - f_{\lambda,\mu})^+.$$

Thanks to the definition of $\beta_{\lambda,\mu}$, we have

$$\int_\Omega (v_{\lambda',\mu} - v_{\lambda,\mu})\, Sign_0^+(w_{\lambda',\mu} - w_{\lambda,\mu}) = \int_\Omega (\beta(w_{\lambda',\mu}) - \beta(w_{\lambda,\mu})) Sign_0^+(w_{\lambda',\mu} - w_{\lambda,\mu})$$
$$+\mu \int_\Omega (w^+_{\lambda',\mu} - w^+_{\lambda,\mu}) Sign_0^+(w_{\lambda',\mu} - w_{\lambda,\mu}) + (\lambda - \lambda') \int_\Omega w^-_{\lambda,\mu} Sign_0^+(w_{\lambda',\mu} - w_{\lambda,\mu})$$
$$-\lambda' \int_\Omega (w^-_{\lambda',\mu} - w^-_{\lambda,\mu}) Sign_0^+(w_{\lambda',\mu} - w_{\lambda,\mu}).$$

Since, $r \to \pm r^\pm$ and β are nondecreasing, we deduce that

$$\begin{aligned} +\mu \int_\Omega (w^+_{\lambda',\mu} - w^+_{\lambda,\mu}) Sign_0^+(w_{\lambda',\mu} - w_{\lambda,\mu}) \\ -\lambda' \int_\Omega (w^-_{\lambda',\mu} - w^-_{\lambda,\mu}) Sign_0^+(w_{\lambda',\mu} - w_{\lambda,\mu}) \leq \int_\Omega (f_{\lambda',\mu} - f_{\lambda,\mu})^+. \end{aligned} \tag{57}$$

This implies that for $f_{\lambda,\mu} \geq f_{\lambda',\mu}$, we have $\pm\, w^\pm_{\lambda,\mu} \geq \pm\, w^\pm_{\lambda',\mu}$ thus $w_{\lambda,\mu} \ \geq \ w_{\lambda',\mu}$.
In the same way, for $\mu \geq \mu' > 0$, $\lambda > 0$, we prove that

$$\begin{aligned} -\lambda \int_\Omega (w^-_{\lambda,\mu} - w^-_{\lambda,\mu'}) Sign_0^+(w_{\lambda,\mu} - w_{\lambda,\mu'}) \\ +\mu \int_\Omega (w^+_{\lambda,\mu} - w^+_{\lambda,\mu'}) Sign_0^+(w_{\lambda,\mu} - w_{\lambda,\mu'}) \leq \int_\Omega (f_{\lambda,\mu} - f_{\lambda,\mu'})^+ \end{aligned} \tag{58}$$

and then for $f_{\lambda,\mu'} \geq f_{\lambda,\mu}$, we deduce $w_{\lambda,\mu} \leq w_{\lambda,\mu'}$. ∎

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