EQUIVALENT FORMULATIONS FOR NONHOMOGENEOUS NEUMANN MONGE-KANTOROVICH EQUATION

N. IGBIDA, S. OUARO AND U. TRAORE

ABSTRACT. In this work we give some equivalent formulations for the optimization problem $\max\left\{\int_{\Omega}\xi d\mu+\int_{\Gamma_N}\xi d\nu\ ;\ \xi\in W^{1,\infty}(\Omega)\, s.t\, \xi_{/\Gamma_D}=0, |\nabla\xi(x)|\leq 1\ a.e.\, x\in\Omega\right\}$, where the boundary of Ω is $\Gamma=\Gamma_N\cup\Gamma_D$.

1. Introduction and main result

Let Ω be a bounded open Lipschitz domain of $\mathbb{R}^N$ with $\mathcal{C}^1$ smooth boundary Γ. We assume that Γ is divided into two parts Γ_N, Γ_D such that $\Gamma_D\cap\Gamma_N=\emptyset$ and the measure area of Γ_D is positive.
We set

$$W_D^{1,\infty}(\Omega)=\left\{z\in W^{1,\infty}(\Omega)\,;\ z=0 \text{ on } \Gamma_D\right\}.$$

and

$$K=\left\{z\in W_D^{1,\infty}(\Omega)\,;\ |\nabla z(x)|\leq 1 \text{ a.e. } x\in\Omega\right\}.$$

We are interested in the study of the optimization problem

$$\max\left\{\int_{\Omega}z\mathrm{d}\mu+\int_{\Gamma_N}z\mathrm{d}\nu\ ;\ z\in K\right\}, \tag{1.1}$$

where μ and ν are a bounded Radon measures concentrated respectively in and Ω and Γ. If we put $\widetilde{K}=\left\{\xi\in W_0^{1,1}(\Omega); |\nabla z(x)|\leq k(x) \text{ a.e. } x\in\Omega\right\}$ where $k\in\mathcal{C}\left(\bar{\Omega}\right)$and $\Gamma_N=\emptyset$, we get from (1.1) the following optimization problem

$$\max\left\{\int_{\Omega}\xi\mathrm{d}\mu\ ;\ \xi\in\widetilde{K}\right\}, \tag{1.2}$$

which is the so-called dual equation of Monge-Kantorovich problem. It is of wide interest for Monge optimal mass transport problem (see[1, 11] and the references therein). It was used by Kantorovich for the study of existence of a solution for is relaxed formulation of the original Monge problem.
In [14], the author showed the equivalence between problem (1.2) and the following three formulations in divergence form

$$\begin{cases} -\nabla\cdot\phi &= \mu \text{ in } \mathcal{D}'(\Omega)\\ k\phi &= |\phi|\,\nabla_{|\phi|}u, \end{cases} \tag{1.3}$$

Key words and phrases. Nonlinear PDE, Tangential gradient, $\mathbb{R}^N$-valued Radon measure flux, Optimization problem.

$$\left\{ \begin{array}{rcl} -\nabla \cdot \phi &=& \mu \text{ in } \mathcal{D}'(\Omega) \\ \int_\Omega k\mathrm{d}\,|\phi| &\leq& \int_\Omega u\mathrm{d}\mu \end{array} \right. \tag{1.4}$$

and

$$\left\{ \begin{array}{rcl} \int_\Omega kd\,|\phi| &=& \min\left\{ \int_\Omega kd\,|\nu|\,;\ -\nabla\cdot\nu = \mu \text{ in } \mathcal{D}'(\Omega) \right\} \\ &=& \int_\Omega ud\mu, \end{array} \right. \tag{1.5}$$

where $|\phi|$ denotes the total variation measure of ϕ and $\nabla_{|\phi|}$ denotes the tangential gradient with respect to $|\phi|$ ([3, 4, 5, 6]).
The main interest in the formulation (1.2)-(1.4) is their connection with the Monge optimal mass transport problem (see [1, 11, 13, 17] and the references therein) as well as mass optimization (see [3, 18]) and sandpile (see [8, 10, 11, 19]). The formulation (1.3) is called in the litterature Monge-Kantorovich equation, it appears in the study of optimal transport problem (see [5]). The relation between the formulation (1.5) (called dual formulation of (1.2)) and (1.3) is given in [5] within the context of mass optimization problem. The formulation (1.5) also appears in the context of optimal transport problem (see [5, 18]). Concerning the formulation (1.4) , it appears in [5] and it is used in the study of the evolution problem associated with the Monge-Kantorovich equation and sandpile problem (see [2, 15]).
In this paper, we prove the equivalence between (1.1) and the following formulations :

$$\left\{ \begin{array}{c} \phi \in (\mathcal{M}_b(\Omega))^N, v \in K \text{ such that} \\ \int_\Omega \nabla\xi.\frac{\phi}{|\phi|}\mathrm{d}\,|\phi| = \int_\Omega \xi\mathrm{d}\mu + \int_{\Gamma_N}\xi\mathrm{d}\nu \\ |\phi|(\Omega) = \int_\Omega v\mathrm{d}\mu + \int_{\Gamma_N} v\mathrm{d}\nu \end{array} \right. \tag{1.6}$$

and

$$\left\{ \begin{array}{c} \phi \in (\mathcal{M}_b(\Omega))^N, v \in K \text{ such that} \\ \int_\Omega \nabla\xi.\frac{\phi}{|\phi|}d\,|\phi| = \int_\Omega \xi\mathrm{d}\mu + \int_{\Gamma_N}\xi\mathrm{d}\nu \\ \phi = |\phi|\,\nabla_{|\phi|}v \end{array} \right. \tag{1.7}$$

and

$$\left\{ \begin{array}{l} \phi \in (\mathcal{M}_b(\Omega))^N, v \in K \text{ such}\,; \text{that} \\ |\phi|(\Omega) = \min\left\{ |\Phi|(\Omega) : \int_\Omega \nabla\xi.\frac{\Phi}{|\Phi|}d\,|\Phi| = \int_\Omega \xi\mathrm{d}\mu + \int_{\Gamma_N}\xi\mathrm{d}\nu \right\} \\ \qquad = \int_\Omega v\mathrm{d}\mu + \int_{\Gamma_N} v\mathrm{d}\nu, \end{array} \right. \tag{1.8}$$

where $\xi \in \mathcal{C}^1(\Omega) \cap W_D^{1,\infty}(\Omega)$.
Since K is bounded, the problem (1.1) admits at least one solution for all bounded Radon measures μ and ν. Let's recall that the question that we treat in this paper has already been landed in other articles in which the authors use non trivial techniques (see [1, 3, 6]). Here we look at a simple case with less complicated techniques (cf [14]). Our main result is the following.

Theorem 1.1. *Let $\mu \in \mathcal{M}_b(\Omega)$, $\nu \in \mathcal{M}_b(\Gamma)$ and $v \in K$. Then v is solution of* (1.1), *i.e.*

$$\int_\Omega (v-\xi)\, d\mu + \int_{\Gamma_N} (v-\xi)\, d\nu \geq 0 \;\; \text{for any } \xi \in K$$

if and only if there exists $\phi \in (\mathcal{M}_b(\Omega))^N$ such that (v,ϕ) satisfies (1.6). *Moreover, the formulations* $(1.6)-(1.8)$ *are equivalent.*

Notice also that the main interest in the study of (1.1) and the equivalent formulation $(1.6)-(1.8)$ besides their connection with mass transport problem as well as mass optimization is their connection with sand dunes problems. Indeed, in the dynamic of the formulation of sand dunes, we need a nonhomogeneous Neumann boundary condition on the part of the boundary exposed upon the arrival of grains of sand.
The rest of paper is organized as follows : in the next section we give some preliminaries and we recall some technical lemmas. Section 3 is devoted to the proof of the main theorem.

2. Preliminary

In this section we introduce some notations and lemmas that will be useful later on. Let Ω be a bounded open subset of $\mathbb{R}^N$ $(N \geq 2)$ equipped with the N-dimensional Lebesgue measure. The space of Radon measure and the set of continuous functions with compact support in Ω will be denoted by $\mathcal{M}(\Omega)$ and $\mathcal{C}_c(\Omega)$ respectively. We recall that each Radon measure μ can be interpreted as an element of the dual of the space $\mathcal{C}_c(\Omega)$. This result can be extended to the space $C\left(\overline{\Omega}\right)$ ie $\mathcal{M}(\Omega) = \left(\mathcal{C}\left(\overline{\Omega}\right)\right)^*$ in the sense that, every $\mu \in \mathcal{M}(\Omega)$ is equal to $\tilde{\mu} \in \left(\mathcal{C}\left(\overline{\Omega}\right)\right)^*$ with $\tilde{\mu}(\partial\Omega) = 0$. So, for any $\mu \in \mathcal{M}(\Omega)$ and $\xi \in \mathcal{C}\left(\overline{\Omega}\right)$, the notation $\displaystyle\int_\Omega \xi \mathrm{d}\mu$ is equivalent to $\langle \tilde{\mu}, \xi \rangle$.
$\mathcal{M}^+(\Omega)$ denote the space of all nonnegative Radon measure on Ω. The variation measure $|\mu|$ associated with $\mu \in \mathcal{M}(\Omega)$ is defined by

$$|\mu|(B) := \sup \left\{ \sum_{i=1}^{\infty} |\mu(B_i)| \,;\; B = \cup_{i=1}^{\infty} B_i, \;\; B_i \text{ a Borelean set} \right\}.$$

For $\mu \in \mathcal{M}(\Omega)$, $\mu^+ = \dfrac{1}{2}(|\mu| + \mu)$ and $\mu^- = \dfrac{1}{2}(|\mu| - \mu)$ are positives and bounded measures. We say that μ^+, μ^- is the positive, negative variation of μ respectively.
The space of Radon measures with bounded total variation $|\mu|(\Omega)$ will be denoted by $\mathcal{M}_b(\Omega)$. Recall that $\mathcal{M}_b(\Omega)$ equipped with the norm $|\mu|(\Omega)$ is a Banach space.
Let $(\mathcal{M}(\Omega))^N$ the space of $\mathbb{R}^N$-valued Radon measures of Ω. Then $\mu \in (\mathcal{M}(\Omega))^N$ if and only if $\mu = (\mu_1, \cdots, \mu_n)$ with $\mu_i \in \mathcal{M}(\Omega)$. We recall that the total variation measure associated with $\mu \in (\mathcal{M}(\Omega))^N$ is denoted again by $|\mu|$ and the subspace $(\mathcal{M}_b(\Omega))^N$ equipped with the norm $\|\mu\| = |\mu|(\Omega)$ is a Banach space. The space $(\mathcal{M}(\Omega))^N$ endowed with the norm $\|.\|$ is isometric to the dual space of $(\mathcal{C}_c(\Omega))^N$.
For any $\mu \in (\mathcal{M}_b(\Omega))^N$ and $\nu \in (\mathcal{M}_b(\Omega))^+$, the density of μ with respect to ν is the

unique bounded $\mathbb{R}^N-$ valued Radon measure denoted by $D_\nu\mu$ such that

$$\mu(A) = \int_A D_\nu\mu \mathrm{d}\nu \quad \text{for any} \quad A \subseteq \Omega.$$

We have $D_\nu\mu \in \mathcal{M}_b(\Omega)$ and

$$\nu(A) = 0 \Rightarrow |\mu|(A) = 0.$$

For any $\mu \in (\mathcal{M}_b(\Omega))^N$ and $\nu \in (\mathcal{M}_b(\Omega))^+$, μ is absolutely continuous with respect to ν; denoted by $\mu << \nu$, provided

$$\nu(A) = 0 \Rightarrow |\mu|(A) = 0, \text{ for any A} \in \Omega.$$

Let's recall that for $\mu \in (\mathcal{M}_b(\Omega))^N$ and $\nu \in (\mathcal{M}_b(\Omega))^+$ such that $\mu << \nu$, the previous splitting up of μ according to ν is always possible by using Radon-Nikodym Decomposition Theorem. Since $|\mu(A)| \leq |\mu|(A)$, for all $\mu \in (\mathcal{M}(\Omega))^N$, then we have $\mu << |\mu|$, and $\left|D_{|\mu|}\mu\right| = 1$, $|\mu|$-*a.e.* in Ω. In the litterature $D_{|\mu|}\mu$ is denoted by $\dfrac{\mu}{|\mu|}$. So, for any $\mu \in (\mathcal{M}(\Omega))^N$ we have

$$\mu(A) = \int_A \frac{\mu}{|\mu|} \mathrm{d}\mu \quad \text{for any borel set} \quad A \subseteq \Omega.$$

Hence, every $\mu \in (\mathcal{M}(\Omega))^N$ can be identified with the linear application

$$\xi \in (\mathcal{C}_c(\Omega))^N \mapsto \int_\Omega \frac{\mu}{|\mu|} . \xi \mathrm{d}|\mu|.$$

For any $\Phi \in (\mathcal{M}_b(\Omega))^N$ and $\nu \in \mathcal{M}_b(\Omega)$, we say that $-\nabla \cdot \Phi = \nu$ in $\mathcal{D}'(\Omega)$ provided that

$$\int_\Omega \nabla\xi . \frac{\Phi}{|\Phi|} \mathrm{d}\Phi = \int_\Omega \xi \mathrm{d}\nu \quad \text{for any} \quad \xi \in \mathcal{D}(\Omega).$$

In particular, this remains true for any $\xi \in \mathcal{C}_0^1(\Omega)$, where $\mathcal{C}_0^1(\Omega)$ is the subset of $\mathcal{C}^1$ function in Ω, such that ξ and $\nabla\xi$ are null on the boundary of Ω. In other words, $-\nabla \cdot \Phi = \nu$ in $\mathcal{D}'(\Omega)$ is equivalent to $-\nabla \cdot \left(\dfrac{\Phi}{|\Phi|}|\Phi|\right) = \nu$ in $\mathcal{D}'(\Omega)$.
We recall the following sets used in the definition of tangential gradient with respect to $\nu \in \mathcal{M}_b(\Omega)^+$ (see [4]).

$$\mathcal{N}_\nu := \left\{ \begin{array}{l} \xi \in (L^\infty_\nu(\Omega))^N \text{ ; } \exists u_n \in C^\infty(\Omega), \ \ u_n \to 0 \quad \text{in} \quad C(\Omega) \quad \text{and} \\ Du_n \to \xi \quad \text{in}\, \sigma\left((L^\infty_\nu(\Omega))^N, (L^1_\nu(\Omega))^N\right) \end{array} \right\}$$

and

$$\mathcal{N}_\nu^\perp := \left\{ \eta \in \left(L^1_\nu(\Omega)\right)^N \text{ ; } \int_\Omega \eta . \xi \mathrm{d}\nu = 0, \forall \xi \in \mathcal{N}_\nu \right\}.$$

For ν-a.e. $x \in \Omega$, we define the tangent space $T_\nu(x)$ to measure ν, as the subspace of $\mathbb{R}^N$:

$$T_\nu(x) = \left\{ A \in \mathbb{R}^N; \exists\, \xi \in \mathcal{N}_\nu^\perp, A = \xi(x) \right\}.$$

Then (cf. Proposition 3.2 of [6]) the operator $\nabla_\nu : Lip(\Omega) \to (L^\infty_\nu(\Omega))^N$ is the continuous operator such that for any $u \in \mathcal{C}^1(\Omega)$,

$$\nabla_\nu u(x) = P_{T_{\nu(x)}} \nabla u(x) \quad \nu - p.p. \quad x \in \Omega,$$

where $P_{T_{\nu(x)}}$ is the orthogonal projection on $T_\nu(x)$, $Lip(\Omega)$ is the set of Lipchitz continuous function equipped with the uniform convergence and $L^\infty_\nu(\Omega)$ is equipped with the weak star topology. A $\mathbb{R}^N$-valued Radon measure ϕ is said to be tangential measure on Ω provided there exist $\nu \in \mathcal{M}_b(\Omega)^+$ and $\sigma \in L^1_\nu(\Omega)^N$, such that $\sigma(x) \in T_\nu(x)$, ν-a.e. $x \in \Omega$ and $\phi = \sigma\nu$. Thanks to Proposition 3.5 of [6], we know that for any tangential measure $\phi = \sigma\nu$ on Ω, such that $-\nabla \cdot \phi = \mu \in \mathcal{M}_b(\Omega)$, we have the following integration by parts

$$\int_\Omega u d\mu = \int_\Omega \sigma.\nabla_\nu u d\nu,$$

for any $u \in Lip(\Omega)$ null on the boundary of Ω.
In the sequel, we need the following two lemmas.

Lemma 2.1. *For any $z \in K$, there exists $(z_\varepsilon)_{\varepsilon>0}$ a sequence in $C^1(\Omega) \cap K$ such that*

$$z_\varepsilon \to z \text{ uniformly in } \overline{\Omega}.$$

Proof. Let $z \in K$ and $(d_\varepsilon)_{\varepsilon>0}$ be a subsequence defined in Ω by

$$d_\varepsilon(x) = \begin{cases} 0 & \text{if} \quad |z(x)| \le \varepsilon \\ z(x) - \varepsilon & \text{otherwise.} \end{cases}$$

We have $d_\varepsilon \in K$ and converges uniformly to z in Ω.
For any $\varepsilon > 0$, we have

$$\text{supp } d_\varepsilon \subset \overline{\{x \in \Omega\,;\, |z(x)| \ge \varepsilon\}} \tag{2.1}$$

and we set

$$\Omega_\varepsilon = \overline{\{x \in \Omega\,;\, |z(x)| \ge \varepsilon\}}.$$

Since Ω is bounded, d_ε is compactly supported in Ω_ε.
Now we introduce the sequences $(\tilde{z}_\varepsilon)_{\varepsilon>0}$ by

$$\tilde{z}_\varepsilon(x) = \begin{cases} d_\varepsilon(x) & \text{if} \quad x \in \Omega_\varepsilon \\ 0 & \text{if} \quad x \in \mathbb{R}^N \backslash \Omega_\varepsilon. \end{cases} \tag{2.2}$$

For any $\varepsilon > 0$, we have

$$|\tilde{z}_\varepsilon(x)| \le |z(x)| \text{ and } |\nabla \tilde{z}_\varepsilon(x)| \le |\nabla z(x)| \le 1 \text{ for any } x \in \Omega. \tag{2.3}$$

Thus, $\tilde{z}_\varepsilon \in K$ and $\tilde{z}_\varepsilon$ is compactly supported in Ω_ε.
Moreover,

$$\sup_{x \in \overline{\Omega}} |\tilde{z}_\varepsilon(x) - z(x)| \le \varepsilon, \tag{2.4}$$

which implies that $\tilde{z}_\varepsilon$ converges uniformly to z in $\overline{\Omega}$.
Let $(\rho_n)_{n>0}$ be the standard sequence of mollifiers such that

$$\rho_n \in C^\infty_c(\mathbb{R}^N), \text{ Supp}\rho_n \subset B\left(0, \frac{1}{n}\right), \int_{\mathbb{R}^N} \rho_n = 1, \text{ and } \rho_n \ge 0. \tag{2.5}$$

Thanks to the propositions $IV.20$ and $IV.21$ in [7] we have

$$z_\varepsilon := \tilde{z}_\varepsilon * \rho_n \in C^1(\Omega) \tag{2.6}$$

and

$$\tilde{z}_\varepsilon * \rho_n \to \tilde{z}_\varepsilon \text{ uniformly in } \overline{\Omega} \text{ as } n \to +\infty. \tag{2.7}$$

Since $\tilde{z}_\varepsilon$ converges uniformly to z in $\overline{\Omega}$, it follows that z_ε converges uniformly to z in $\overline{\Omega}$. At last, we use Höelder inequality to obtain

$$\begin{aligned}
|\nabla z_\varepsilon(x)|^2 &= \sum_{k=1}^{N} \left| \frac{\partial z_\varepsilon}{\partial x_k}(x) \right|^2 \\
&= \sum_{k=1}^{N} \left| \int_{\mathbb{R}^N} \frac{\partial \tilde{z}_\varepsilon}{\partial x_k}(x-y)\,\rho_n(y)\,dy \right|^2 \\
&\le \sum_{k=1}^{N} \left| \left(\int_{\mathbb{R}^N} \left(\frac{\partial \tilde{z}_\varepsilon}{\partial x_k}(y)\,\rho_n(x-y)^{\frac{1}{2}} \right)^2 dy \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^N} \left(\rho_n(x-y)^{\frac{1}{2}} \right)^2 dy \right)^{\frac{1}{2}} \right|^2 \\
&\le \sum_{k=1}^{N} \left(\int_{\mathbb{R}^N} \left| \frac{\partial \tilde{z}_\varepsilon}{\partial x_k}(y) \right|^2 \rho_n(x-y)\,dy \right) \left(\int_{\mathbb{R}^N} \rho_n(x-y)\,dy \right) \\
&\le \int_{\mathbb{R}^N} \sum_{k=1}^{N} \left| \frac{\partial \tilde{z}_\varepsilon}{\partial x_k}(y) \right|^2 \rho_n(x-y)\,dy \\
&\le \int_{\mathbb{R}^N} |\nabla \tilde{z}_\varepsilon(y)|^2 \rho_n(x-y)\,dy \\
&\le \int_{\mathbb{R}^N} \rho_n(x-y)\,dy \le 1 \text{ a.e. } x \in \Omega
\end{aligned} \tag{2.8}$$

□

Then, similarly as in [14], the following result can be prouved.

Lemma 2.2. *For any $v \in K$ and $\nu \in \mathcal{M}_b(\Omega)^+$, we have*

$$|\nabla_\nu v| \le 1 \ \nu\text{-}a.e.in\ \Omega.$$

3. Proof of Theorem 1.1

To get the proof of the Theorem 1.1 , we introduce a set of lemmas

Lemma 3.1. *Let $\phi \in (\mathcal{M}_b(\Omega))^N$ and $v \in K$. If (v, ϕ) satisfies* (1.6) *then, v is solution of* (1.1).

Proof. Let $\xi \in K$, thanks to Lemma 2.1, there exists $\xi_\varepsilon \in C^1(\Omega) \cap K$ such that $\xi_\varepsilon \to \xi$, uniformly in Ω. Taking ξ_ε as a test function in (1.6), we have

$$\int_\Omega \nabla \xi_\varepsilon . \frac{\phi}{|\phi|} \mathrm{d}\,|\phi| = \int_\Omega \xi_\varepsilon \mathrm{d}\mu + \int_{\Gamma_N} \xi_\varepsilon \mathrm{d}\nu.$$

Using the fact that $\xi_\varepsilon \in K$, we get

$$\int_\Omega \nabla \xi_\varepsilon . \frac{\phi}{|\phi|} \mathrm{d}\,|\phi| \leq |\phi|\,(\Omega)$$

Thus,

$$\begin{aligned} \int_\Omega \xi \mathrm{d}\mu + \int_{\Gamma_N} \xi \mathrm{d}\nu &= \lim_{\varepsilon\to 0} \int_\Omega \xi_\varepsilon \mathrm{d}\mu + \int_{\Gamma_N} \xi_\varepsilon \mathrm{d}\nu \\ &= \lim_{\varepsilon\to 0} \int_\Omega \nabla \xi_\varepsilon . \frac{\phi}{|\phi|} \mathrm{d}\,|\phi| \\ &\leq |\phi|\,(\Omega) \end{aligned} \tag{3.1}$$

i.e.

$$\int_\Omega \xi \mathrm{d}\mu + \int_{\Gamma_N} \xi \mathrm{d}\nu \leq |\phi|\,(\Omega)\,. \tag{3.2}$$

Since

$$|\phi|\,(\Omega) = \int_\Omega v \mathrm{d}\mu + \int_{\Gamma_N} v \mathrm{d}\nu, \tag{3.3}$$

then

$$\int_\Omega v \mathrm{d}\mu + \int_{\Gamma_N} v \mathrm{d}\nu \geq \int_\Omega \xi \mathrm{d}\mu + \int_{\Gamma_N} \xi \mathrm{d}\nu \ \text{ for all } \xi \in K.$$

□

Lemma 3.2. *Let* $\phi \in (\mathcal{M}_b(\Omega))^N$ *and* $v \in K$. *Then* (v,ϕ) *satisfies* (1.6) *if and only if* (v,ϕ) *satisfies the formulations* (1.7) *and* (1.8)

Proof. Assume that (v,ϕ) satisfies (1.6) and taking $v_\varepsilon \in C^1(\Omega) \cap K$ the approximation of v given by Lemma 2.1; we have

$$\begin{aligned} |\phi|\,(\Omega) &= \int_\Omega v \mathrm{d}\mu + \int_{\Gamma_N} v \mathrm{d}\nu \\ &= \lim_{\varepsilon\to 0} \left(\int_\Omega v_\varepsilon \mathrm{d}\mu + \int_{\Gamma_N} v_\varepsilon \mathrm{d}\nu \right) \\ &= \lim_{\varepsilon\to 0} \int_\Omega \nabla_{|\phi|} v_\varepsilon . \frac{\phi}{|\phi|} \mathrm{d}\,|\phi| = \int_\Omega \nabla_{|\phi|} v . \frac{\phi}{|\phi|} \mathrm{d}\,|\phi|\,. \end{aligned}$$

So

$$\int_\Omega \left(1 - \nabla_{|\phi|} v . \frac{\phi}{|\phi|} \right) \mathrm{d}\,|\phi| = 0. \tag{3.4}$$

Since by Lemma 2.2, we have $\left| \nabla_{|\phi|} v . \frac{\phi}{|\phi|} \right| \leq \left| \nabla_{|\phi|} v \right| \leq 1$ $|\phi|$-a.e. in Ω, then by (3.4), we deduce that

$$\nabla_{|\phi|} v . \frac{\phi}{|\phi|} = 1 \quad |\phi|\text{-a.e. in } \Omega.$$

This implies that

$$\nabla_{|\phi|} v = \frac{\phi}{|\phi|} \quad |\phi|\text{-a.e. in } \Omega.$$

Therefore,

$$\phi = |\phi| \frac{\phi}{|\phi|} = |\phi| \nabla_{|\phi|} v \quad |\phi| \text{-a.e. in } \Omega. \tag{3.5}$$

Moreover if $\Phi \in (\mathcal{M}_b(\Omega))^N$ such that Φ satisfies the first equality of (1.6), we have

$$\begin{aligned} |\phi|(\Omega) &= \int_\Omega v \mathrm{d}\mu + \int_{\Gamma_N} v \mathrm{d}\nu \\ &= \lim_{\varepsilon \to 0} \int_\Omega v_\varepsilon \mathrm{d}\mu + \int_{\Gamma_N} v_\varepsilon \mathrm{d}\nu \\ &= \lim_{\varepsilon \to 0} \int_\Omega \nabla v_\varepsilon . \frac{\Phi}{|\Phi|} \mathrm{d}|\Phi| \\ &\leq \int_\Omega \mathrm{d}|\Phi| \end{aligned} \tag{3.6}$$

hence (1.6) implies (1.8). It's clear that (1.8) implies (1.6), now suppose that (v, ϕ) satisfies (1.7), we have

$$\begin{aligned} |\phi|(\Omega) &= \int_\Omega \nabla_{|\phi|} v . \frac{\phi}{|\phi|} \mathrm{d}|\phi| \\ &= \lim_{\varepsilon \to 0} \int_\Omega \nabla v_\varepsilon . \frac{\phi}{|\phi|} \mathrm{d}|\phi| \\ &= \lim_{\varepsilon \to 0} \int_\Omega v_\varepsilon \mathrm{d}\mu + \int_{\Gamma_N} v_\varepsilon \mathrm{d}\nu \\ &= \int_\Omega v \mathrm{d}\mu + \int_{\Gamma_N} v \mathrm{d}\nu. \end{aligned} \tag{3.7}$$

Thus,

$$|\phi|(\Omega) = \int_\Omega v \mathrm{d}\mu + \int_{\Gamma_N} v \mathrm{d}\nu.$$

$\square$ As a consequence of lemmas 3.1 and 3.2, we have (1.6) implies (1.1) and $(1.6) \Longleftrightarrow (1.7) \Longleftrightarrow (1.8)$.

To prove that (1.1) implies (1.6), we consider the following system.

$$(S_\varepsilon) \left\{ \begin{array}{rcll} -\nabla . \phi_\varepsilon (\nabla v_\varepsilon) & = & \mu & \text{in} \quad \Omega \\ v_\varepsilon & = & 0 & \text{on} \quad \Gamma_D \\ \phi_\varepsilon (\nabla v_\varepsilon) . \eta & = & \nu & \text{on} \quad \Gamma_N, \end{array} \right.$$

where η is the unit outward normal vecteur on $\partial\Omega$, for any $\varepsilon > 0$ and $x \in \Omega$, $\phi_\varepsilon : \mathbb{R}^N \to \mathbb{R}^N$ is given by

$$\phi_\varepsilon(r) = \frac{1}{\varepsilon} \left((|r| - 1)^+ \right)^{(p-1)} \frac{r}{|r|} \quad \text{for all} \quad r \in \mathbb{R}^N \quad \text{and} \quad x \in \Omega,$$

with $p > N$ fixed.

It is not difficult to see that ϕ_ε satisfies the following properties.

(i) For any $r_1, r_2 \in \mathbb{R}^N$ and $x \in \Omega$, $(\phi_\varepsilon(r_1) - \phi_\varepsilon(r_2)).(r_1 - r_2) \geq 0$.

(ii) There exist $\varepsilon_0 > 0$ and $C_0 > 1$ such that $\phi_\varepsilon(r).r \geq |r|^p$ for any $|r| \geq C_0$ and $\varepsilon < \varepsilon_0$.

(iii) For any $\varepsilon > 0$, $r \in \mathbb{R}^N$ and $x \in \Omega$, $|\phi_\varepsilon(r)| \leq \phi_\varepsilon(r).r$.

We define the following separable and reflexive Banach space for $W^{1,p}(\Omega)$-norm

$$W^{1,p}_{\Gamma_D}(\Omega) = \left\{ z \in W^{1,p}(\Omega) \,;\; z_{/\Gamma_D} = 0 \right\}.$$

Lemma 3.3. *For any $0 < \varepsilon < \varepsilon_0$, the problem (S_ε) has a unique solution v_ε in the sense that $v_\varepsilon \in W^{1,p}_{\Gamma_D}(\Omega)$ and*

$$\int_\Omega \phi_\varepsilon(\nabla v_\varepsilon).\nabla z\,dx = \int_\Omega z\,d\mu + \int_{\Gamma_N} z\,d\nu \tag{3.8}$$

for all $z \in W^{1,p}_{\Gamma_D}(\Omega)$.

Proof We define the operator $A_\varepsilon : W^{1,p}_{\Gamma_D}(\Omega) \to \left(W^{1,p}_{\Gamma_D}(\Omega)\right)'$ by,

$$\langle A_\varepsilon v, z\rangle = \int_\Omega \phi_\varepsilon(\nabla v).\nabla z\,\mathrm{d}x. \tag{3.9}$$

A_ε is monotone, coercive, hemicontinous and bounded. Indeed, the property (i) of ϕ_ε gives the monotonicity.
For any $v, z \in W^{1,p}_{\Gamma_D}(\Omega)$, we have

$$\begin{aligned}
|\langle A_\varepsilon(v), z\rangle| &\leq \frac{1}{\varepsilon}\int_\Omega \left|(|\nabla v| - 1)^+\right|^{p-1} |\nabla z|\,\mathrm{d}x \\
&\leq \frac{1}{\varepsilon}\int_\Omega \left|(|\nabla v| - 1)^+\right|^{p-1} |\nabla z|\,\mathrm{d}x \\
&\leq \frac{1}{\varepsilon}\int_\Omega |\nabla v|^{p-1} |\nabla z|\,\mathrm{d}x \\
&\leq \frac{1}{\varepsilon}\left(\int_\Omega |\nabla v|^p\,\mathrm{d}x\right)^{\frac{1}{p'}} \left(\int_\Omega |\nabla z|^p\,\mathrm{d}x\right)^{\frac{1}{p}} \\
&\leq \frac{1}{\varepsilon}\|v\|^{\frac{p}{p'}}_{W^{1,p}(\Omega)} \|z\|_{W^{1,p}(\Omega)},
\end{aligned} \tag{3.10}$$

which implies that

$$\|A_\varepsilon(v)\|_{\left(W^{1,p}_{\Gamma_D}(\Omega)\right)'} \leq \frac{1}{\varepsilon}\|v\|^{\frac{p}{p'}}_{W^{1,p}(\Omega)}. \tag{3.11}$$

Let B be a bounded set of $W^{1,p}_{\Gamma_D}(\Omega)$, there exists $M > 0$ such that

$$\|A_\varepsilon(v)\|_{\left(W^{1,p}_{\Gamma_D}(\Omega)\right)'} \leq \frac{1}{\varepsilon}M^{\frac{p}{p'}}, \forall\, v \in B. \tag{3.12}$$

Hence, A_ε is a bounded operator. Moreover, using the properties (ii) and (iii) of ϕ_ε, we obtain

$$\begin{aligned}\langle A_\varepsilon(v),v\rangle &= \int_\Omega \Phi_\varepsilon(\nabla v).\nabla v\mathrm{d}x\\ &= \int_{[|\nabla v|<C_0]}\Phi_\varepsilon(\nabla v).\nabla v\mathrm{d}x+\int_{[|\nabla v|\geq C_0]}\Phi_\varepsilon(\nabla v).\nabla v\mathrm{d}x\\ &\geq \int_{[|\nabla v|<C_0]}|\Phi_\varepsilon(\nabla v)|\,\mathrm{d}x+\int_{[|\nabla v|\geq C_0]}|\nabla v|^p\,\mathrm{d}x\\ &\geq \int_{[|\nabla v|\geq C_0]}|\nabla v|^p\,\mathrm{d}x. \end{aligned}\tag{3.13}$$

Consequently,

$$\langle A_\varepsilon(v),v\rangle+\int_{[|\nabla v|<C_0]}|\nabla v|^p\,\mathrm{d}x\geq\int_\Omega|\nabla v|^p\,\mathrm{d}x. \tag{3.14}$$

Thus,

$$\begin{aligned}\langle A_\varepsilon(v),v\rangle &\geq -\int_{[|\nabla v|<C_0]}|\nabla v|^p\,\mathrm{d}x+\int_\Omega|\nabla v|^p\,\mathrm{d}x\\ &\geq -\int_{[|\nabla v|<C_0]}C_0^p\mathrm{d}x+\int_\Omega|\nabla v|^p\,\mathrm{d}x\\ &\geq -C_0^p\mathrm{meas}\left([|\nabla v|<C_0]\right)+\int_\Omega|\nabla v|^p\,\mathrm{d}x\\ &\geq -C_0^p\mathrm{meas}\left(\Omega\right)+\|v\|^p_{W^{1,p}(\Omega)}. \end{aligned}\tag{3.15}$$

Hence,

$$\frac{\langle A_\varepsilon(v),v\rangle}{\|v\|_{W^{1,p}(\Omega)}}\geq-\frac{C_0^p\mathrm{meas}\left(\Omega\right)}{\|v\|_{W^{1,p}(\Omega)}}+\|v\|^{p-1}_{W^{1,p}(\Omega)}. \tag{3.16}$$

Since $p>1$, letting $\|v\|_{W^{1,p}(\Omega)}\to+\infty$ in (3.16), it follows that A_ε is coercive. Now consider the map $F:\mathbb{R}\to\mathbb{R}$ defined by

$$F\left(\lambda\right)=\langle A(u+\lambda v),w\rangle=\int_\Omega\phi_\varepsilon\left(\nabla u+\lambda\nabla v\right).\nabla w\mathrm{d}x, \tag{3.17}$$

with u,v,w in $W^{1,p}_{\Gamma_D}(\Omega)$. We will prove that F is continuous. The functions $x\mapsto\phi_\varepsilon\left(\nabla u+\lambda\nabla v\right).\nabla w$, $\lambda\mapsto\phi_\varepsilon\left(\nabla u+\lambda\nabla v\right).\nabla w$ are respectively mesurable $a.e.$ in Ω and continuous in $\mathbb{R}$. Let $(\lambda_n)_{n\in\mathbb{N}}$ be such that $\lambda_n\to\lambda$, so that there exists a constant $c>0$ with $|\lambda_n|\leq c$.
Therefore,

$$\begin{aligned}\left|\phi_\varepsilon\left(\nabla u+\lambda_n\nabla v\right).\nabla w\right| &\leq \left|\phi_\varepsilon\left(\nabla u+\lambda_n\nabla v\right)\right|\left|\nabla w\right|\\ &\leq \frac{1}{\varepsilon}\left(|\nabla u|+|\lambda_n|\,|\nabla v|+1\right)^{p-1}|\nabla w|\\ &\leq \frac{1}{\varepsilon}\left(|\nabla u|+c\,|\nabla v|+1\right)^{p-1}|\nabla w|. \end{aligned}\tag{3.18}$$

Letting $n \to +\infty$ in (3.18) and using the fact that the function $\lambda \mapsto |\phi_\varepsilon(\nabla u + \lambda\nabla v).\nabla w|$ is continuous we obtain

$$|\phi_\varepsilon(\nabla u + \lambda\nabla v).\nabla w| \leq \frac{1}{\varepsilon}(|\nabla u| + c|\nabla v| + 1)^{p-1}|\nabla w| \in L^1(\Omega). \tag{3.19}$$

Therefore thanks to the Lebesgue theorem, we can say that F is continuous, hence the operator A is hemicontinuous.
Since $p > N$, we have $W^{1,p}_{\Gamma_D}(\Omega) \subset C(\overline{\Omega})$ and the linear form $G : W^{1,p}_{\Gamma_D}(\Omega) \to \mathbb{R}$ define by

$$\langle G, v\rangle = \int_\Omega v d\mu + \int_{\Gamma_N} v d\nu \tag{3.20}$$

belongs to the dual space of $W^{1,p}_{\Gamma_D}(\Omega)$. So (see for instance [16]), for any $0 < \varepsilon < \varepsilon_0$, and $p > N$ there exists $v_\varepsilon \in W^{1,p}_{\Gamma_D}(\Omega)$ such that $A(v_\varepsilon) = G$, i.e

$$\int_\Omega \phi_\varepsilon(\nabla v_\varepsilon).\nabla z dx = \int_\Omega z d\mu + \int_{\Gamma_N} z d\nu \text{ for all } z \in W^{1,p}_{\Gamma_D}(\Omega). \tag{3.21}$$

Now, suppose that v_ε and $\tilde{v}_\varepsilon$ are two solutions of (S_ε). For v_ε and $\tilde{v}_\varepsilon$, we take $z = v_\varepsilon - \tilde{v}_\varepsilon$ in (3.21) to get

$$\int_\Omega (\phi_\varepsilon(\nabla v_\varepsilon) - \phi_\varepsilon(\nabla\tilde{v}_\varepsilon)).(\nabla v_\varepsilon - \nabla\tilde{v}_\varepsilon) dx = 0. \tag{3.22}$$

It follows that there exists a constant $\tilde{c}$ such that $v_\varepsilon - \tilde{v}_\varepsilon = \tilde{c}$ a.e. in Ω. Using the fact that $v_\varepsilon = \tilde{v}_\varepsilon = 0$ on Γ_D, we get $\tilde{c} = 0$. Thus $v_\varepsilon = \tilde{v}_\varepsilon$ a.e. in Ω.
□

Lemma 3.4. *Let $(v_\varepsilon)_{0<\varepsilon<\varepsilon_0}$ be the sequence of solutions of (S_ε). Then*

(1) $(v_\varepsilon)_{0<\varepsilon<\varepsilon_0}$ *is bounded in* $W^{1,p}_{\Gamma_D}(\Omega)$.

(2) $(\phi_\varepsilon(\nabla v_\varepsilon))_{0<\varepsilon<\varepsilon_0}$ *is bounded in* $(L^1(\Omega))^N$.

(3) *For any Borel set $B \subseteq \Omega$,*

$$\liminf_{\varepsilon\to 0}\left(\int_B |\nabla v_\varepsilon|^{p-1} dx\right)^{\frac{1}{p-1}} \leq |B|^{\frac{1}{p-1}}.$$

Proof. (1) Taking v_ε as test function in (3.22) and using the fact that $W^{1,p}(\Omega) \subset C(\overline{\Omega})$, we get following estimate.

$$\begin{aligned}
\frac{1}{\varepsilon}\int_\Omega (|\nabla v_\varepsilon| - 1)^{+(p-1)}|\nabla v_\varepsilon| \,\mathrm{d}x &= \int_\Omega \phi_\varepsilon(\nabla v_\varepsilon).\nabla v_\varepsilon \mathrm{d}x x \\
&= \int_\Omega v_\varepsilon \mathrm{d}\mu + \int_{\Gamma_N} v_\varepsilon \mathrm{d}\nu \\
&\leq (|\mu|(\Omega) + |\nu|(\Gamma_N))\|v_\varepsilon\|_\infty \\
&\leq C(|\mu|(\Omega) + |\nu|(\Gamma_N))\|v_\varepsilon\|_{W^{1,p}(\Omega)}.
\end{aligned} \tag{3.23}$$

Combining (3.23) and property (ii) of ϕ_ε, for any $0<\varepsilon<\varepsilon_0$, we get

$$\begin{aligned}
\int_\Omega |\nabla v_\varepsilon|^p \,\mathrm{d}x &\leq \int_{[|\nabla v_\varepsilon|\leq C_0]} |\nabla v_\varepsilon|^p \,\mathrm{d}x + \int_{[|\nabla v_\varepsilon|> C_0]} |\nabla v_\varepsilon|^p \,\mathrm{d}x \\
&\leq \int_{[|\nabla v_\varepsilon|\leq C_0]} |\nabla v_\varepsilon|^p \,\mathrm{d}x + \frac{1}{\varepsilon}\int_\Omega (|\nabla v_\varepsilon|-1)^{+(p-1)} |\nabla v_\varepsilon| \,\mathrm{d}x \\
&\leq \int_{[|\nabla v_\varepsilon|\leq C_0]} |\nabla v_\varepsilon|^p \,\mathrm{d}x \\
&\quad + C\left(|\mu|(\Omega)+|\nu|(\Gamma_N)\right)\|\nabla v_\varepsilon\|_{L^p(\Omega)} \\
&\leq C_0^p|\Omega| + C\left(|\mu|(\Omega)+|\nu|(\Gamma_N)\right)\|\nabla v_\varepsilon\|_{L^p(\Omega)}. \qquad (3.24)
\end{aligned}$$

Thus, according to Young inequality, we deduce that

$$\|\nabla v_\varepsilon\|^p_{L^p(\Omega)} \leq p' C_0^p |\Omega| + \left[C\left(|\mu|(\Omega)+|\nu|(\Gamma_N)\right)\right]^{p'}, \qquad (3.25)$$

which implies that $(\nabla v_\varepsilon)_{0<\varepsilon<\varepsilon_0}$ is bounded in $(L^p(\Omega))^N$. Hence $(v_\varepsilon)_{0<\varepsilon<\varepsilon_0}$ is bounded in $W^{1,p}_{\Gamma_D}(\Omega)$.

(2) Using (3.23) and the property (iii) of ϕ_ε we deduce that

$$\begin{aligned}
\int_\Omega |\phi_\varepsilon(\nabla v_\varepsilon)| \,\mathrm{d}x &\leq \int_\Omega \phi_\varepsilon(\nabla v_\varepsilon).\nabla v_\varepsilon \mathrm{d}x \\
&\leq \frac{1}{\varepsilon}\int_\Omega (|\nabla v_\varepsilon|-1)^{(p-1)} |\nabla v_\varepsilon| \,\mathrm{d}x \\
&\leq C\left(|\mu|(\Omega)+|\nu|(\Gamma_N)\right)\|\nabla v_\varepsilon\|_{L^p(\Omega)}. \qquad (3.26)
\end{aligned}$$

So by (3.25) we deduce that $\phi_\varepsilon(\nabla v_\varepsilon)$ is bounded in $\left(L^1(\Omega)\right)^N$.

(3) Now, let $B\subseteq\Omega$ be a fixed Borel set. We have

$$\begin{aligned}
\|\nabla v_\varepsilon\|_{L^{p-1}(B)} &\leq \left\|(\nabla v_\varepsilon-1)^+ + 1\right\|_{L^{p-1}(B)} \\
&\leq \left\|(\nabla v_\varepsilon-1)^+\right\|_{L^{p-1}(B)} + |B|^{\frac{1}{p-1}} \qquad (3.27)\\
&\leq \left(\int_B (\nabla v_\varepsilon-1)^{+(p-1)} |\nabla v_\varepsilon| \,\mathrm{d}x\right)^{\frac{1}{p-1}} + |B|^{\frac{1}{p-1}} \\
&\leq \left[\varepsilon C\left(|\mu|(\Omega)+|\nu|(\Gamma_N)\right)\|\nabla v_\varepsilon\|_{L^p(\Omega)}\right]^{\frac{1}{p-1}} + |B|^{\frac{1}{p-1}}. \qquad (3.28)
\end{aligned}$$

Letting $\varepsilon\to 0$ and using the fact that v_ε is bounded in $W^{1,p}(\Omega)$, we obtain

$$\liminf_{\varepsilon\to 0}\left(\int_\Omega |\nabla v_\varepsilon|^{p-1}\,\mathrm{d}x\right)^{\frac{1}{p-1}} \leq |B|^{\frac{1}{p-1}}. \qquad (3.29)$$

□

Lemma 3.5. *Under the assumptions of Lemma* 3.4*, there exists a subsequence that we denote again by* v_ε*, such that, as* $\varepsilon\to 0$*,*

$$v_\varepsilon \to \tilde{v} \text{ uniformly in } \overline{\Omega} \text{ and in } W^{1,\infty}(\Omega) - \text{weak}, \qquad (3.30)$$

$$\phi_\varepsilon(\nabla v_\varepsilon) \to \phi \text{ in } \left(\mathcal{M}_b(\Omega)\right)^N - \text{weak}^*, \qquad (3.31)$$

and
$$\int_{\Omega}|\phi_{\varepsilon}(\nabla v_{\varepsilon})|\,dx \to |\phi|(\Omega). \tag{3.32}$$
Moreover, $\tilde{v} \in K$ *and* $(\tilde{v}, \phi)$ *satisfies* (1.6).

Proof. Thanks to the Lemma 3.4, there exist $\tilde{v} \in W^{1,p}_{\Gamma_D}(\Omega)$, $\phi \in \mathcal{M}_b(\Omega)$ and a subsequence denoted again by v_ε, such that (3.30) and (3.31) are fulfilled.
For any $\xi \in \mathcal{C}^1(\Omega) \cap W^{1,\infty}_D(\Omega)$, we have
$$\int_{\Omega}\phi_{\varepsilon}(\nabla v_{\varepsilon}).\nabla\xi \mathrm{d}x = \int_{\Omega}\xi \mathrm{d}\mu + \int_{\Gamma_N}\xi \mathrm{d}\nu.$$
Thus, letting $\varepsilon \to 0$, we deduce that
$$\int_{\Omega}\nabla\xi.\frac{\phi}{|\phi|}\mathrm{d}|\phi| = \int_{\Omega}\xi \mathrm{d}\mu + \int_{\Gamma_N}\xi \mathrm{d}\nu. \tag{3.33}$$
To prove that $\tilde{v} \in K$, let us consider $A_\delta = [|\nabla\tilde{v}| \geq 1+\delta]$, with arbitrary $\delta > 0$. Since as $\varepsilon \to 0$, $\nabla v_\varepsilon \to \nabla\tilde{v}$ in $\left(L^1(\Omega)\right)^N$-weak then,
$$\begin{aligned}(1+\delta)|A_\delta| &\leq \int_{A_\delta}|\nabla\tilde{v}|\,\mathrm{d}x \\ &\leq \liminf_{\varepsilon\to 0}\int_{A_\delta}|\nabla v_\varepsilon|\,\mathrm{d}x \\ &\leq \liminf_{\varepsilon\to 0}\left(\int_{A_\delta}|\nabla v_\varepsilon|^{p-1}\,\mathrm{d}x\right)^{\frac{1}{p-1}}|A_\delta|^{\frac{p-2}{p-1}}. \end{aligned}\tag{3.34}$$
So that, by using the third part of Lemma 3.4, we deduce that
$$(1+\delta)|A_\delta| \leq |A_\delta|,$$
which implies that $|A_\delta| = 0$. Since δ is arbitrary, we deduce that $|\nabla\tilde{v}| \leq 1$ a.e. in Ω. Therefore,
$$\tilde{v} \in K.$$
To prove (3.32), we see that according to the property (iii) of ϕ_ε and (3.30), we have
$$\begin{aligned}\limsup_{\varepsilon\to 0}\int_{\Omega}|\phi_\varepsilon(\nabla v_\varepsilon)|\,\mathrm{d}x &\leq \limsup_{\varepsilon\to 0}\int_{\Omega}\phi_\varepsilon(\nabla v_\varepsilon)\cdot\nabla v_\varepsilon \mathrm{d}x \\ &\leq \limsup_{\varepsilon\to 0}\left(\int_{\Omega}v_\varepsilon \mathrm{d}\mu + \int_{\Gamma_N}v_\varepsilon \mathrm{d}\nu\right) \\ &\leq \int_{\Omega}\tilde{v}\mathrm{d}\mu + \int_{\Gamma_N}\tilde{v}\mathrm{d}\nu. \end{aligned}\tag{3.35}$$
In addition, we have
$$\begin{aligned}\int_{\Omega}\tilde{v}\mathrm{d}\mu + \int_{\Gamma_N}\tilde{v}\mathrm{d}\nu &= \lim_{\varepsilon\to 0}\int_{\Omega}\phi(\nabla v_\varepsilon)\cdot\nabla\tilde{v}\mathrm{d}x \\ &\leq \lim_{\varepsilon\to 0}\int_{\Omega}|\phi_\varepsilon(\nabla v_\varepsilon)|\,\mathrm{d}x. \end{aligned}\tag{3.36}$$

So, (3.35) and (3.36) implies that

$$\lim_{\varepsilon\to 0}\int_\Omega |\phi_\varepsilon(\nabla v_\varepsilon)|\,\mathrm{d}x = \int_\Omega \tilde{v}\mathrm{d}\mu + \int_{\Gamma_N}\tilde{v}\mathrm{d}\nu \tag{3.37}$$

and by (3.31) we get

$$|\phi|(\Omega) = \int_\Omega \mathrm{d}|\phi| \leq \liminf_{\varepsilon\to 0}\int_\Omega |\phi_\varepsilon(\nabla v_\varepsilon)|\,\mathrm{d}x = \int_\Omega \tilde{v}\mathrm{d}\mu + \int_{\Gamma_N}\tilde{v}\mathrm{d}\nu. \tag{3.38}$$

Using $\tilde{v}_\varepsilon$, the approximation of $\tilde{v}$ given by Lemma 2.1, we see that

$$\begin{aligned}\int_\Omega \tilde{v}\mathrm{d}\mu + \int_{\Gamma_N}\tilde{v}\mathrm{d}\nu &= \lim_{\varepsilon\to 0}\int_\Omega \nabla\tilde{v}_\varepsilon . \frac{\phi}{|\phi|}\mathrm{d}|\phi| \\ &\leq \int_\Omega \mathrm{d}|\phi| = |\phi|(\Omega).\end{aligned} \tag{3.39}$$

Combining the above inequality with (3.38) we obtain

$$|\phi|(\Omega) = \int_\Omega \tilde{v}\mathrm{d}\mu + \int_{\Gamma_N}\tilde{v}\mathrm{d}\nu \tag{3.40}$$

and

$$\lim_{\varepsilon\to 0}\int_\Omega |\phi(\nabla v_\varepsilon)|\,\mathrm{d}x = |\phi|(\Omega). \tag{3.41}$$

□

Lemma 3.6. *Let $v \in K$, be a solution of* (1.1), *then there exists $\phi \in (\mathcal{M}_b(\Omega))^N$ such that (v, ϕ) satisfies* (1.6).

Proof. Let $\tilde{v} = \lim_{\epsilon\to 0}\tilde{v}_\varepsilon$, where $\tilde{v}_\varepsilon$ is the solution of (S_ε). According to Lemma 3.5, there exists ϕ in $(\mathcal{M}_b(\Omega))^N$ such that

$$\int_\Omega \nabla\xi\cdot\frac{\phi}{|\phi|}\mathrm{d}|\phi| = \int_\Omega \xi\mathrm{d}\mu + \int_{\Gamma_N}\xi\mathrm{d}\nu \text{ for all } \xi\in\mathcal{C}^1(\Omega)\cap W_D^{1,\infty}(\Omega) \tag{3.42}$$

and

$$|\phi|(\Omega) = \int_\Omega \tilde{v}\mathrm{d}\mu + \int_{\Gamma_N}\tilde{v}\mathrm{d}\nu. \tag{3.43}$$

Let $v_\varepsilon \in \mathcal{C}^1(\Omega)\cap K$ be the approximation of v given by the Lemma 2.1, we have

$$\begin{aligned}\int_\Omega v\mathrm{d}\mu + \int_{\Gamma_N} v\mathrm{d}\nu &= \lim_{\varepsilon\to 0}\int_\Omega v_\varepsilon\mathrm{d}\mu + \int_{\Gamma_N} v_\varepsilon\mathrm{d}\nu \\ &= \lim_{\varepsilon\to 0}\int_\Omega \nabla v_\varepsilon\cdot\frac{\phi}{|\phi|}\mathrm{d}|\phi| \\ &\leq \int_\Omega \mathrm{d}|\phi| = |\phi|(\Omega).\end{aligned} \tag{3.44}$$

Since v is solution of (1.1), we have

$$\int_\Omega v\mathrm{d}\mu + \int_{\Gamma_N} v\mathrm{d}\nu \geq \int_\Omega z\mathrm{d}\mu + \int_{\Gamma_N} z\mathrm{d}\nu \text{ for all } z\in K. \tag{3.45}$$

In particular, taking $z = \tilde{v}$, we deduce that

$$\int_\Omega v \mathrm{d}\mu + \int_{\Gamma_N} v \mathrm{d}\nu \geq \int_\Omega \tilde{v} \mathrm{d}\mu + \int_{\Gamma_N} \tilde{v} \mathrm{d}\nu = |\phi| (\Omega) . \tag{3.46}$$

Consequently, (3.44) and (3.46) implies that

$$|\phi| (\Omega) = \int_\Omega v \mathrm{d}\mu + \int_{\Gamma_N} v \mathrm{d}\nu.$$

□

Proof of Theorem 1.1. Thanks to lemma 3.1 we have (1.6) implies (1.1). As a consequence of lemmas 3.6, we have that (1.1) implies (1.6) . The equivalence between (1.6) , (1.7) and (1.8)is given by Lemma 3.2 □

References

[1] Ambrosio, L. (2003) *Lecture notes on optimal transport, in mathematical aspect of evolving interfaces*, in : Lecture Notes in Mathematics, LNM 1812, Spinger, Berlin.

[2] Barrett, J. W., Prigozhin, L. (2007) *A mixed formulation for the Monge-Kantorovich equations.* M2AN, 41(6), 1041-1060.

[3] Bouchitté, G., Buttazzo, G. (2001) *Characterization of optimal Shapes and masses throught Monge-Kantorovich equation.* J. Eur. Math. Soc 3(2), 139-168.

[4] Bouchitté, G., Buttazzo, G., Seppecher, P. (1997) *Energies with respect to a measure and applications to low dimensional structures.* Calc. Var. Partial Differential Equations 5, 37-54.

[5] Bouchitté, G., Buttazzo, G., Seppecher, P. (1997) *Shape Optimization Solutions via Monge-Kantorovich.* C.R. Acad. Sci. Paris, t.324 Série I, 1185-1991.

[6] Bouchitté, G., Champion, Th., Jimenez, C. (2005) *Completion of the space of measures in the Kantorovich norm,* Riv.Mat.della Univ.Parma Serie 7(4)(2005)127-139.

[7] Brezis. H. (1987) ; *Analyse Fonctionnelle Théorie et application.* Masson.

[8] Buttazzo, G., De Pascal, L *Optimal shapes and mass, and optimal transportation problems*, in : Optimal Transportation and Applications, Martina Franca 2-8 september 2001, in : Lecture Notes in Mathematics, vol.1813, Springer-verlag, Berlin, 2003, pp. 11-52.

[9] Chipot, M. (2000) ; *Elements of Nonlinear Analysis,* ISBN 3-7643-6406-8 Birkhäuser Verlag, Basel-Boston-Berlin.

[10] Dumont, S., Igbida, N. (2009) *Back on a dual formulation for growing sandpile problem.* Euro. Jnl of Applied Mathematics(in press)

[11] Evans, L. C. (1997) *Partial differential equations and Monge-Kantorovich mass transfert,* in: Current Developments in Mathematics, International Press, Bostan, MA, pp. 65-126.

[12] Evans, L. C. (1998) *Partial differential equations.* Grad. stud. Math. 19 Amer. Math. Soc.

[13] Evans, L. C., Gangbo, W. (1999) *Differential equations methods for the Monge-Kantorovich mass transfert problem,* Mem. Am. Math. Soc., 137, 653.

[14] Igdida, N. *Equivalent formulations for Monge-Kantorovich equation.* Nonlinear Analysis 71(2009)3805-3813

[15] Igdida, N. *Evolution Monge-Kantorovich equation.* J.Differential Equations 255(2003)1383-1407.

[16] Lions, J. L. (1996) ; *Quelques methodes de résolution des problèmes aux limites non linéaires.* Dunod. Gauthier-Villars, Paris.

[17] Monge, G. (1781) *Memoire sur la théorie des délais et des remblais.* Histoire de L'Académie des Sciences de Paris(1781).

[18] Pratelli, A. (2005) *Equivalence between some definitions for the optimal mass transport problem and for density on manifolds.* Ann.Mat.Pura Appl. 184, 215-238.

[19] Prigozhin , L (1996) *Variational model of sandpile growth.* Euro. J. App. Math. 7, 225-236.

(NOUREDDINE. IGBIDA) Institut de recherche XLIM-DMI, UMR-CNRS 7252, Faculté des sciences et techniques, Université de Limoges, 123, Avenue Albert Thomas
E-mail address: noureddine.igbida@unilim.fr

(STANISLAS. OUARO) Laboratoire d'Analyse Mathématique des Equations (LAME), UFR, Sciences Exactes et Appliquées, Université de Ouagadougou, 03 BP 7021 Ouaga 03, Ouagadougou, Burkina Faso

(URBAIN. TRAORE) Laboratoire d'Analyse Mathématique des Equations (LAME), UFR, Sciences Exactes et Appliquées, Université de Ouagadougou, 03 BP 7021 Ouaga 03, Ouagadougou, Burkina Faso., Institut de recherche XLIM-DMI, UMR-CNRS 7252, Faculté des sciences et techniques, Université de Limoges, 123, Avenue Albert Thomas
E-mail address: urbain.traore@etu.unilim.fr and urbain.traore@yahoo.fr