Elliptic Problem Involving Diffuse Measure Data

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Abstract

In this paper, we study a suitable notion of solution for which a nonlinear elliptic problem governed by a general Leray-Lions operator is well posed for any diffuse measure data. In terms of the paper [12], we study the notion of solution for which any diffuse measure is "good measure".

Keywords : Nonlinear elliptic, diffuse measure, biting lemma of Chacon, maximal monotone graphe, Radon measure data, weak solution, entropic solution, Leray-Lions operator.

1 Introduction and main results

Our aim is to study existence and uniqueness of a solution for the nonlinear boundary value problem of the form

$$P(\beta,\mu)\begin{cases} -\nabla\cdot a(x,\nabla u)+\beta(u)\ni\mu & \text{in } \Omega,\\ \\ u=0 & \text{on } \partial\Omega,\end{cases}$$

where β is a maximal monotone graph on $\mathbb{R}$ such that $0\in\beta(0)$, a is a Leray-Lions operator, μ is a diffuse measure and $\Omega\subset\mathbb{R}^N$ is a smooth bounded domain ($N\geq 1$).

Recall that a Leray-Lions operator is a Carathéodory function $a:\Omega\times\mathbb{R}^N\to\mathbb{R}^N$ (i.e. $a(x,\xi)$ is measurable in $x\in\Omega$ for every $\xi\in\mathbb{R}^N$ and continuous in $\xi\in\mathbb{R}^N$ for almost every $x\in\Omega$) such that

- there exists $\lambda>0$ such that $\forall\xi\in\mathbb{R}^N$ and a.e. $x\in\Omega$,

$$a(x,\xi).\xi\geq\lambda|\xi|^p; \tag{1.1}$$

- for any $(\xi,\eta)\in\mathbb{R}^N\times\mathbb{R}^N$ with $\xi\neq\eta$ and a.e. $x\in\Omega$,

$$\Big(a(x,\xi)-a(x,\eta)\Big).\Big(\xi-\eta\Big)>0; \tag{1.2}$$

- there exists $\Lambda>0$ such that for a.e. $x\in\Omega$ and for any $\xi\in\mathbb{R}^N$,

$$\Big|a(x,\xi)\Big|\leq\Lambda\Big(j_1(x)+|\xi|^{p-1}\Big) \tag{1.3}$$

where j_1 is a nonnegative function in $L^{p'}(\Omega)$ with $p'=\dfrac{p}{p-1}$.

A Radon measure μ is said to be diffuse with respect to the capacity $W_0^{1,p}(\Omega)$ ($p-$capacity for

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short) if $\mu(E)=0$ for every set E such that $\text{cap}_p(E,\Omega)=0$. The $p-$capacity of every subset E with respect to Ω is defined as :

$$\text{cap}_p(E,\Omega)=\inf\Big\{\int_\Omega |\nabla u|^p dx\Big\}$$

and the infimum is taken on all functions $u\in W_0^{1,p}(\Omega)\cap C_0(\Omega)$ such that $u=1$ almost everywhere on $E,\ u\geq 0$ almost everywhere on Ω. The set of diffuse measures is denoted by $\mathcal{M}_b^p(\Omega)$.

In the case where $\mu\in W^{-1,p'}(\Omega)$, it is known (cf. [22], [15] and [4]) that $P(\beta,\mu)$ has a unique solution in the standard sense, the so called weak solution. That is a couple $(u,w)\in W_0^{1,p}(\Omega)\times L^1(\Omega)$ such that $w\in\beta(u),\ \mathcal{L}^N-$ a.e. in Ω, and

$$\int_\Omega a(x,\nabla u).\nabla\xi dx+\int_\Omega w\xi dx=\int_\Omega \xi d\mu, \forall\xi\in W_0^{1,p}(\Omega)\cap L^\infty(\Omega).$$

For diffuse measure, existence and uniqueness of a renormalized and/or entropic solution (some extension of the results of [4]) is treated in [9] for the case of continuous β defined in all $\mathbb{R}$ (we refer the reader to the paper [18] for an extensive references concerning existence and uniqueness of solutions for $P(\beta,\mu)$). But, in general $P(\beta,\mu)$ has no solution (see [3], [17], [6] and the references therein). This nonexistence mechanism is connected with the domain of the nonlinearity β and also with the regularity of the measure μ. This phenomena was analyzed and studied in [17] for the case

$$\begin{cases} -\Delta u+\beta(u)=\mu & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega. \end{cases} \tag{1.4}$$

In particular, the authors of [17] introduced the concept of "good measure" which is a Radon measure μ such that $P(\beta,\mu)$ has a weak solution. Moreover, they have introduced the notion of "reduced measure" denoted by μ^* associated with μ. It corresponds to the right measure that we can associate with μ such that (1.4) with μ replaced by μ^* has a unique weak solution. Indeed, by using natural approximation scheme (keep μ fixed and approximate β or keep β fixed and approximate μ) and passing to the limit in the equation they have characterized the right part of μ for which the problem is well posed. This approach was deeply analyzed and studied in the literature for the Laplacian (see [12], [13], [14] and [17]).

Our approach here is different, indeed as a consequence of the preceding arguments it is clear that the standard notion of weak solution neither standard renormalized/entropic solution is not the natural one for $P(\beta,\mu)$ when μ is a Radon measure. Indeed, the singular part of μ with respect to Lebesgue measure creates an obstruction to the existence of such kind of solutions. This is related to the fact that passing to the limit in the approximation scheme, singular parts may appear in the equation and need to be treated. In this paper we analyze and study the main feature of these quantities in the case of diffuse measure and maximal monotone graph β. Handling these parts gives the right notion of solutions for $P(\beta,\mu)$ when μ is diffuse with respect to the $p-$capacity. This notion of solution is such that any diffuse measure with respect to the $p-$capacity is a good measure for $P(\beta,\mu)$. Recall that, taking the nonlinearity β continuous and satisfies

$$\lim_{t\uparrow 1}\beta(t)=+\infty, \tag{1.5}$$

the authors of [17] shows that, there exists a diffuse measure μ with respect to the capacity $H^1(\Omega)$ such that the problem (1.4) has no weak solution. So, in general diffuse measure are not good measure for (1.4) with respect to the standard notion of weak solution. But, it will be good measure for (1.4) with respect to our notion of solution.

To give our notion of solution and main results, we set

$$\mathrm{int}(\mathrm{dom}\beta) = (m, M) \quad \text{with } -\infty \le m \le 0 \le M \le +\infty.$$

For any $r \in \mathbb{R}$ and any measurable function u on Ω, $[u = r]$, $[u \le r]$ and $[u \ge r]$ denote respectively the sets $\{x \in \Omega : u(x) = r\}$, $\{x \in \Omega : u(x) \le r\}$ and $\{x \in \Omega : u(x) \ge r\}$.

The main results in this work are the following theorems.

Theorem 1.1 *For any $\mu \in \mathcal{M}_b^p(\Omega)$, the problem $P(\beta, \mu)$ has at least one solution (u, w) in the sense that $w \in L^1(\Omega)$, u is measurable, $u \in dom(\beta)$ $\mathcal{L}^N - a.e.$ in Ω, $T_k(u) \in W_0^{1,p}(\Omega)$ $\forall k > 0$, $w \in \beta(u)$ $\mathcal{L}^N - a.e.$ in Ω, there exists a measure $\nu \in \mathcal{M}_b(\Omega)$ such that $\nu \perp \mathcal{L}^N$, for any $h \in \mathcal{C}_c(I\!R)$, $h(u) \in L^\infty(\Omega, d|\nu|)$, $h(u)\nu \in M_b^p(\Omega)$,*

$$\nu^+ \text{ is concentred on } [u = M] \cap [u \ne +\infty], \ \nu^- \text{ is concentred on } [u = m] \cap [u \ne -\infty],$$

$$\int_\Omega a(x, \nabla u).\nabla(h(u)\xi)\, dx + \int_\Omega w h(u)\xi dx + \int_\Omega h(u)\xi d\nu = \int_\Omega h(u)\xi d\mu, \tag{1.6}$$

for any $\xi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ and

$$\lim_{n\to+\infty} \int_{[n\le|u|\le n+1]} |\nabla u|^p \, dx = 0. \tag{1.7}$$

Here, since $T_k(u) \in W_0^{1,p}(\Omega)$, without abusing we are identifying the function u with its quasi-continuous representative. So, since the measures μ and ν are diffuse, all the terms of Theorem 1.1 have a sense.

See that, if $M = +\infty$ (resp. $m = -\infty$), then $\nu^+ \equiv 0$ (resp. $\nu^- \equiv 0$). In the particular case where the domain of β is equal to $I\!R$, Theorem 1.1 implies the existence of a renormalized solution in the standard sense. More precisely, we have

Corollary 1.1 *Assume that*

$$\mathcal{D}(\beta) = I\!R,$$

for any $\mu \in \mathcal{M}_b^p(\Omega)$, the problem $P(\beta, \mu)$ has at least one solution (u, w) in the sense that $w \in L^1(\Omega)$, u is measurable, $T_k(u) \in W_0^{1,p}(\Omega)$ $\forall k > 0$, $w \in \beta(u)$ $\mathcal{L}^N - a.e.$ in Ω, for any $h \in \mathcal{C}_c(I\!R)$, we have

$$\int_\Omega a(x, \nabla u).\nabla(h(u)\xi)\, dx + \int_\Omega w h(u)\xi dx = \int_\Omega h(u)\xi d\mu, \tag{1.8}$$

for any $\xi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ and

$$\lim_{n\to+\infty} \int_{[n\le|u|\le n+1]} |\nabla u|^p \, dx = 0. \tag{1.9}$$

The uniqueness of a solution in the sense of Theorem 1.1 and Corollary 1.1 is not clear in general. Thanks to [18], the uniqueness of a solution in the sense of Corollary 1.1 holds to be true for the so called comparable solutions. That is any two solutions (u_1, w_1) and (u_2, w_2) such that the difference

$u_1 - u_2$ is bounded. We'll not abort this question in this paper and refer the reader to the papers [18] and [19] for more details in this direction.

The connexion between our notion of solution and the entropic formulation of the solution is given in the following Theorem. In particular, this equivalent formulation is very useful for the proof of the uniqueness of solution for $P(\beta, \mu)$ in the case where the domain of β is bounded (see Theorem 1.3). We believe that it could be also useful for the proof of uniqueness in a more general setting like those of [18] and [19].

Theorem 1.2 *If (u, w) is a solution of $P(\beta, \mu)$, then (u, w) is a solution in the following sense : for any $\xi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ such that $\xi \in dom\beta$,*

$$\int_\Omega a(x, \nabla u).\nabla T_k(u - \xi)dx + \int_\Omega wT_k(u - \xi)dx \leq \int_\Omega T_k(u - \xi)d\mu, \quad \text{for any } k > 0. \tag{1.10}$$

In the case where the domain of β is bounded, the renormalization with the function h is not necessary in Theorem 1.1. We can take $h \equiv 1$. Moreover, by using Theorem 1.2 we have uniqueness. This is summarize in the following theorem.

Theorem 1.3 *If $-\infty < m \leq 0 \leq M < \infty$, then, for any $\mu \in \mathcal{M}_b^p(\Omega)$, the problem $P(\beta, \mu)$ has a unique solution (u, w) in the sense that $(u, w) \in W_0^{1,p}(\Omega) \times L^1(\Omega)$, $u \in dom(\beta)$ $\mathcal{L}^N -$ a.e. in Ω, $w \in \beta(u)$ $\mathcal{L}^N -$ a.e. in Ω, there exists $\nu \in \mathcal{M}_b^p(\Omega)$ such that $\nu \perp \mathcal{L}^N$,*

$$\nu^+ \text{ is concentred on } [u = M], \ \nu^- \text{ is concentred on } [u = m]$$

and

$$\int_\Omega a(x, \nabla u).\nabla \xi dx + \int_\Omega w\xi dx + \int_\Omega \xi d\nu = \int_\Omega \xi d\mu, \tag{1.11}$$

for any $\xi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$. Moreover

$$\nu^+ \leq \mu_s \lfloor \, [u = M] \tag{1.12}$$

and

$$\nu^- \leq -\mu_s \lfloor \, [u = m]. \tag{1.13}$$

Remark 1.1

1. *If $-\infty < m \leq 0 \leq M < \infty$, for any $\mu \in \mathcal{M}_b^p(\Omega)$, the "good measure" with respect to the notion of weak solution associated with μ is given by*

$$\mu^* = \mu - \nu.$$

2. *Assuming that $M < \infty$ (resp. $-\infty < m$) and $\overline{\mathcal{D}(\beta)} = (-\infty, M]$ (resp. $\overline{\mathcal{D}(\beta)} = [m, \infty)$), we can prove also that, if (u, w) is a solution in the sense of Theorem 1.1, then*

$$\nu^+ \leq \mu_s \lfloor \, [u = M] \quad \text{and } \nu^- \equiv 0$$

(resp.

$$\nu^- \leq \mu_s \lfloor \, [u = m] \quad \text{and } \nu^+ \equiv 0).$$

3. *If the measure μ is regular (i.e. absolutely continuous with respect to Lebesgue measure), Theorem 1.3 and the previous remark shows that $\nu = 0$ and a solution in the sense of Theorem 1.1 coincides with the usual renormalized solution for $P(\beta, \mu)$, which corresponds to the unique weak solution in the case where $\mathcal{D}(\beta)$ is bounded.*

Notice that this kind of formulation for the solution has already appeared in previous work, for instance in [10], [23] and [1] to deal with nonlinearities β depending on x. It appeared also in [13] to treat the obstacle problem associated with (1.4) ; i.e. the case where $\mathcal{D}(\beta) = [m, M]$. Our results here are some kind of generalization of the last result to general nonlinearity β and Leray-Lions operator a.

The paper is organized as follows. In section 2, we state some technical results, in section 3, we deal with the proof of Theorem 1.1 and in section 4, we deal with the proof of Theorem 1.2 and Theorem 1.3.

2 Preliminaries

If (u, w) is a solution of $P(\beta, \mu)$, choosing $\xi = T_k(u),\ k > 0$ in (1.11) , we get the following estimate:

$$\forall k > 0, \quad \frac{1}{k}\int_{[|u|<k]} |\nabla u|^p dx \leq K, \tag{2.1}$$

with $0 < K < +\infty$.
We denote by $\mathcal{T}_0^{1,p}(\Omega)$, the space of measurable functions $u : \Omega \to \mathbb{R}$ such that for any $k > 0,\ T_k(u) \in W_0^{1,p}(\Omega)$. The proof of the following two lemmas may be found in [4].

Lemma 2.1 *Let $1 < p < N$, Ω be as above and let $u \in \mathcal{T}_0^{1,p}(\Omega)$ be such that (2.1) holds. Then there exists $C = C(N, p) > 0$ such that*

$$meas\Big([|u| > k]\Big) \leq CK^{\frac{N}{N-p}} k^{-p_1} \tag{2.2}$$

with $p_1 = \dfrac{N(p-1)}{N-p}$.

Lemma 2.2 *Let $1 < p < N$ and assume that $u \in \mathcal{T}_0^{1,p}(\Omega)$ satisfies (2.1) for every $k > 0$. Then for every $h > 0$ we have*

$$meas\Big([|\nabla u| > h]\Big) \leq C(N, p)K^{\frac{N}{N-1}} h^{-p_2} \tag{2.3}$$

with $p_2 = \frac{N(p-1)}{N-1}$.

Now, let us prove the following result which will be useful in the sequel.

Lemma 2.3 *Let $(\beta_n)_{n\geq 1}$ be a sequence of maximal monotone graphs such that $\beta_n \longrightarrow \beta$ in the sense of graphs. We consider $(z_n)_{n\geq 1}$ and $(w_n)_{n\geq 1}$ two sequences of $L^1(\Omega)$, such that $w_n \in \beta_n(z_n)$, $\mathcal{L}^N-$a.e. in Ω, for any $n \in \mathbb{N}^*$. If*

$$(w_n)_{n\geq 1} \text{ is bounded in } L^1(\Omega) \text{ and } z_n \longrightarrow z \text{ in } L^1(\Omega),$$

then

$$z \in dom(\beta) \quad \mathcal{L}^N - a.e. \text{ in } \Omega.$$

The main tool for the proof of Lemma 2.3 is the "biting lemma of Chacon" (see [16]).
Let us recall it.

Lemma 2.4 ***The "biting lemma of Chacon"*** *Let Ω be an open bounded subset of $\mathbb{R}^N$ and $(f_n)_n$ a bounded sequence in $L^1(\Omega)$. Then, there exist $f \in L^1(\Omega)$, a subsequence $(f_{n_k})_k$ and a sequence of measurable sets $(E_j)_j$, $E_j \subset \Omega, \forall j \in \mathbb{N}$ with $E_{j+1} \subset E_j$ and $\lim_{j\to+\infty} |E_j| = 0$, such that for any $j \in \mathbb{N}, f_{n_k} \rightharpoonup f$ in $L^1(\Omega \backslash E_j)$.*

Proof of Lemma 2.3 Since the sequence $(w_n)_{n\geq 1}$ is bounded in $L^1(\Omega)$, using the "biting lemma of Chacon" there exist $w \in L^1(\Omega)$, a subsequence $(w_{n_k})_{k\geq 1}$ and a sequence of mesurable sets $(E_j)_{j\in\mathbb{N}}$ in Ω such that $E_{j+1} \subset E_j, \forall j \in \mathbb{N}$, $\lim_{j\to+\infty} |E_j| = 0$ and $\forall j \in \mathbb{N}, w_{n_k} \rightharpoonup w$ in $L^1(\Omega\backslash E_j)$. Since $z_{n_k} \longrightarrow z$ in $L^1(\Omega)$ and so in $L^1(\Omega\backslash E_j), \forall j \in \mathbb{N}$ and $\beta_{n_k} \longrightarrow \beta$ in the sense of graphs, we have $w \in \beta(z)$ a.e. in $\Omega\backslash E_j$. Thus $z \in \text{dom}(\beta)$ a.e. in $\Omega\backslash E_j$. Finally we obtain $z \in \text{dom}(\beta)$ a.e. in Ω $\square$

3 Proof of Theorem 1.1

For every $\epsilon > 0$, we consider the Yosida regularization β_ϵ of β given by

$$\beta_\epsilon = \frac{1}{\epsilon}\left(I - (I + \epsilon\beta)^{-1}\right).$$

Thanks to [11], there exists j a non negative, convex and l.s.c. function defined on $\mathbb{R}$, such that

$$\beta = \partial j.$$

To regularize β, we consider

$$j_\epsilon(s) = \min_{r\in\mathbb{R}} \left\{ \frac{1}{2\epsilon}|s-r|^2 + j(r) \right\}, \ \forall s \in \mathbb{R}, \ \forall \epsilon > 0.$$

By Proposition 2.11 in [11] we have

$$\begin{cases} \text{dom}(\beta) \subset \text{dom}(j) \subset \overline{\text{dom}(j)} = \overline{\text{dom}(\beta)}, \\ j_\epsilon(s) = \frac{\epsilon}{2}|\beta_\epsilon(s)|^2 + j(J_\epsilon(s)) \text{ where } J_\epsilon := (I + \epsilon\beta)^{-1}, \\ j_\epsilon \text{ is a convex, Frechet-differentiable function and } \beta_\epsilon = \partial j_\epsilon, \\ j_\epsilon \uparrow j \text{ as } \epsilon \downarrow 0. \end{cases}$$

Moreover, for any $\epsilon > 0$, β_ϵ is a nondecreasing and Lipschitz-continuous function.

For any measure μ assumed diffuse with respect to the capacity $W_0^{1,p}(\Omega)$, a well known result in [9] allows us to write

$$\mu = f - \nabla \cdot F \tag{3.1}$$

where $f \in L^1(\Omega)$ and $F \in \left(L^{p'}(\Omega)\right)^N$. To regularize μ, for any $\epsilon > 0$, we define the functions

$$f_\epsilon(x) = T_{\frac{1}{\epsilon}}(f(x)) \quad \text{for any } x \in \Omega$$

and

$$\mu_\epsilon = f_\epsilon - \nabla \cdot F \quad \text{for any } \epsilon > 0.$$

Then, we consider the following approximating scheme problem

$$P_\epsilon(\beta_\epsilon, \mu_\epsilon) \begin{cases} -\nabla \cdot a(x, \nabla u_\epsilon) + \beta_\epsilon(u_\epsilon) = \mu_\epsilon \text{ in } \Omega, \\ \\ u_\epsilon = 0 \qquad\qquad \text{on } \partial\Omega. \end{cases}$$

Thanks to [8], we know that $P_\epsilon(\beta_\epsilon, \mu_\epsilon)$ admits a unique weak solution u_ϵ in the sense that $u_\epsilon \in W_0^{1,p}(\Omega)$, $\beta_\epsilon(u_\epsilon) \in L^1(\Omega)$ and $\forall \varphi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$,

$$\int_\Omega a(x, \nabla u_\epsilon).\nabla \varphi dx + \int_\Omega \beta_\epsilon(u_\epsilon)\varphi dx = \int_\Omega f_\epsilon \varphi dx + \int_\Omega F.\nabla \varphi dx. \quad (3.2)$$

Let us prove the following result.

Proposition 3.1
(*i*) *There exists* $0 < C < +\infty$ *such that for any* $k > 0$,

$$\int_{[|u_\epsilon| \le k]} |\nabla u_\epsilon|^p dx \le Ck. \quad (3.3)$$

(*ii*) *The sequence* $(\beta_\epsilon(u_\epsilon))_{\epsilon > 0}$ *is uniformly bounded in* $L^1(\Omega)$.
(*iii*) *For any* $k > 0$, *the sequence* $(\beta_\epsilon(T_k(u_\epsilon)))_{\epsilon > 0}$ *is uniformly bounded in* $L^1(\Omega)$.
(*iv*) *There exists* $u \in \mathcal{T}_0^{1,p}(\Omega)$ *such that* $u \in dom(\beta)$ *a.e. in* Ω *and*

$$u_\epsilon \longrightarrow u \text{ in measure and a.e. in } \Omega, \qquad \text{as } \epsilon \to 0. \quad (3.4)$$

Proof (*i*) For any $k > 0$, we take $\varphi = T_k(u_\epsilon)$ as test function in (3.2). We get

$$\int_\Omega a(x, \nabla u_\epsilon).\nabla T_k(u_\epsilon) dx + \int_\Omega \beta_\epsilon(u_\epsilon) T_k(u_\epsilon) dx = \int_\Omega f_\epsilon T_k(u_\epsilon) dx + \int_\Omega F.\nabla T_k(u_\epsilon) dx. \quad (3.5)$$

Since

$$\left| \int_\Omega f_\epsilon T_k(u_\epsilon) dx + \int_\Omega F.\nabla T_k(u_\epsilon) dx \right| = \left| \int_\Omega T_k(u_\epsilon) d\mu_\epsilon \right| \le k|\mu|(\Omega) \le Ck$$

and

$$\int_\Omega \beta_\epsilon(u_\epsilon) T_k(u_\epsilon) dx \ge 0,$$

we deduce that

$$\int_\Omega a(x, \nabla u_\epsilon).\nabla T_k(u_\epsilon) dx \le Ck.$$

Using (1.1), we obtain $\lambda \int_\Omega |\nabla T_k(u_\epsilon)|^p dx \le Ck$ and (*i*) follows.

(*ii*) For any $k > 0$, the first term of (3.5) is non negative, then it follows that

$$\int_\Omega \beta_\epsilon(u_\epsilon) T_k(u_\epsilon) dx \le k|\mu|(\Omega) \le Ck.$$

Dividing by k, we get

$$\frac{1}{k} \int_\Omega \beta_\epsilon(u_\epsilon) T_k(u_\epsilon) dx \le C.$$

Letting k goes to 0, we deduce from the inequality above

$$\int_\Omega \beta_\epsilon(u_\epsilon) \text{sign}_0(u_\epsilon) dx \le C,$$

which implies $\int_\Omega |\beta_\epsilon(u_\epsilon)| dx \le C$ and so $(\beta_\epsilon(u_\epsilon))_\epsilon$ is bounded in $L^1(\Omega)$.

(*iii*) Since

$$\int_\Omega |\beta_\epsilon(T_k(u_\epsilon))| dx \le \int_\Omega |\beta_\epsilon(u_\epsilon)| dx,$$

(iii) follows obviously from (ii).

(iv) Using (i) we can assert that for all $k>0$, the sequence $\Big(\nabla T_k(u_\epsilon)\Big)_{\epsilon>0}$ is bounded in $L^p(\Omega)$, thus the sequence $(T_k(u_\epsilon))_{\epsilon>0}$ is bounded in $W_0^{1,p}(\Omega)$. Then up to a subsequence, we can assume that as $\epsilon\to 0$, $(T_k(u_\epsilon))_{\epsilon>0}$ converges strongly to some function σ_k in $L^q(\Omega)$ and a.e. in Ω for any $1\leq q<p^*=\dfrac{Np}{N-p}$. Let us see that the sequence $(u_\epsilon)_{\epsilon>0}$ is Cauchy in measure.
Indeed, let $s>0$ and define

$$E_1 := [|u_{\epsilon_1}|>k], \quad E_2 := [|u_{\epsilon_2}|>k] \quad \text{and} \quad E_3 := [|T_k(u_{\epsilon_1})-T_k(u_{\epsilon_2})|>s]$$

where $k>0$ is to be fixed. We note that

$$[|u_{\epsilon_1}-u_{\epsilon_2}|>s]\subset E_1\cup E_2\cup E_3$$

and hence

$$\text{meas}\Big([|u_{\epsilon_1}-u_{\epsilon_2}|>s]\Big)\leq \text{meas}(E_1)+\text{meas}(E_2)+\text{meas}(E_3). \tag{3.6}$$

Let $\theta>0$. Using Lemma 2.1, we choose $k=k(\theta)$ such that

$$\text{meas}(E_1)\leq\theta/3 \quad \text{and } \text{meas}(E_2)\leq\theta/3. \tag{3.7}$$

Since $(T_k(u_\epsilon))_{\epsilon>0}$ converges strongly in $L^q(\Omega)$ then it is a Cauchy sequence in $L^q(\Omega)$.
Thus

$$\text{meas}(E_3)\leq\frac{1}{s^q}\int_\Omega |T_k(u_{\epsilon_1})-T_k(u_{\epsilon_2})|^q dx\leq\frac{\theta}{3}, \tag{3.8}$$

for all $\epsilon_1,\epsilon_2\geq n_0(s,\theta)$. Finally from (3.6), (3.7) and (3.8) we obtain

$$\text{meas}\Big([|u_{\epsilon_1}-u_{\epsilon_2}|>s]\Big)\leq\theta \text{ for all } \epsilon_1,\epsilon_2\geq n_0(s,\theta). \tag{3.9}$$

Relation (3.9) means that the sequence $(u_\epsilon)_{\epsilon>0}$ is Cauchy in measure, so $u_\epsilon\longrightarrow u$ in measure and up to a subsequence, we have $u_\epsilon\longrightarrow u$ a.e. in Ω. Hence $\sigma_k=T_k(u)$ a.e. in Ω and so $u\in\mathcal{T}_0^{1,p}(\Omega)$. Using Lemma 2.3, we have $T_k(u)\in$ domβ a.e. in Ω for any $k>0$. This implies that $u\in$ domβ a.e. in Ω □

Proposition 3.2 *For any $k>0$, as ϵ tends to 0, we have*
(i) $a(x,\nabla T_k(u_\epsilon))\rightharpoonup a(x,\nabla T_k(u))$ weakly in $\Big(L^{p'}(\Omega)\Big)^N$.
(ii) $\nabla T_k(u_\epsilon)\longrightarrow\nabla T_k(u)$ a.e. in Ω.
(iii) $a(x,\nabla T_k(u_\epsilon)).\nabla T_k(u_\epsilon)\longrightarrow a(x,\nabla T_k(u)).\nabla T_k(u)$ a.e. in Ω and strongly in $L^1(\Omega)$.
(iv) $\nabla T_k(u_\epsilon)\longrightarrow\nabla T_k(u)$ strongly in $(L^p(\Omega))^N$.

Proof : (i) Using (1.3) we see that the sequence $(a(x,\nabla T_k(u_\epsilon)))_{\epsilon>0}$ is bounded in $\Big(L^{p'}(\Omega)\Big)^N$, then up to a subsequence

$$a(x,\nabla T_k(u_\epsilon))\rightharpoonup H_k \quad \text{in } \Big(L^{p'}(\Omega)\Big)^N.$$

Let us prove that $H_k=a(x,\nabla T_k(u))$ a.e. in Ω. The proof consists in four steps.
Step1 : For every function $h\in W^{1,+\infty}(\mathbb{R}), h\geq 0$ with supp(h) compact,

$$\limsup_{\epsilon\to 0}\int_\Omega a(x,\nabla u_\epsilon).\nabla\Big[h(u_\epsilon)(T_k(u_\epsilon)-T_k(u))\Big]dx\leq 0. \tag{3.10}$$

Taking $h(u_\epsilon)(T_k(u_\epsilon)-T_k(u))$ as test function in (3.2), we have

$$\begin{aligned}&\int_\Omega a(x,\nabla u_\epsilon).\nabla\Big[h(u_\epsilon)(T_k(u_\epsilon)-T_k(u))\Big]dx+\int_\Omega \beta_\epsilon(u_\epsilon)h(u_\epsilon)(T_k(u_\epsilon)-T_k(u))dx\\&=\int_\Omega f_\epsilon h(u_\epsilon)(T_k(u_\epsilon)-T_k(u))dx+\int_\Omega F.\nabla\Big[h(u_\epsilon)(T_k(u_\epsilon)-T_k(u))\Big]dx.\end{aligned}\tag{3.11}$$

In addition, we see that $h(u_\epsilon)(T_k(u_\epsilon)-T_k(u))\rightharpoonup 0$ weakly in $W_0^{1,p}(\Omega)$. Indeed, the sequence $(h(u_\epsilon)(T_k(u_\epsilon)-T_k(u)))_{\epsilon>0}$ is bounded in $W_0^{1,p}(\Omega)$ and converges to zero almost everywhere in Ω. This implies that

$$\lim_{\epsilon\to 0}\int_\Omega F.\nabla\Big[h(u_\epsilon)(T_k(u_\epsilon)-T_k(u))\Big]dx=0.$$

Note also that, by the generalized dominated convergence Theorem, we have

$$\lim_{\epsilon\to 0}\int_\Omega f_\epsilon h(u_\epsilon)(T_k(u_\epsilon)-T_k(u))dx=0.$$

Let us now prove that

$$\limsup_{\epsilon\to 0}\int_\Omega \beta_\epsilon(u_\epsilon)h(u_\epsilon)(T_k(u_\epsilon)-T_k(u))dx\geq 0.\tag{3.12}$$

For any $0<r$ sufficiently small we consider

$$u_r=(T_l(u)\wedge(M-r))\vee(m+r),$$

where l is such that $supp(h)\subset[-l,+l]$. For any $k>0$, $T_k(u_r)\in W_0^{1,p}(\Omega)$. Furthermore, since

$$\int_\Omega h(u_\epsilon)(\beta_\epsilon(u_\epsilon)-\beta_\epsilon(u_r))(T_k(u_\epsilon)-T_k(u_r))dx\geq 0,$$

we have

$$\begin{aligned}\int_\Omega \beta_\epsilon(u_\epsilon)h(u_\epsilon)(T_k(u_\epsilon)-T_k(u))dx\quad&\geq\quad\int_\Omega h(u_\epsilon)\beta_\epsilon(u_r)(T_k(u_\epsilon)-T_k(u_r))dx\\&\quad+\int_\Omega h(u_\epsilon)\beta_\epsilon(u_\epsilon)(T_k(u_r)-T_k(u))dx.\end{aligned}$$

See that

$$\max(m+r,-l)\leq u_r\leq\min(M-r,l),$$

so that

$$\beta_\epsilon(\max(m+r,-l))\leq\beta_\epsilon(u_r)\leq\beta_\epsilon(\min(M-r,l)).$$

Using Lebesgue dominated convergence Theorem, we get that

$$\limsup_{\epsilon\to 0}\int_\Omega h(u_\epsilon)\beta_\epsilon(u_r)(T_k(u_\epsilon)-T_k(u_r))dx=\int_\Omega h(u)\beta_0(u_r)(T_k(u)-T_k(u_r))dx.$$

As to the last term

$$I:=\int_\Omega h(u_\epsilon)\beta_\epsilon(u_\epsilon)(T_k(u_r)-T_k(u))dx,$$

see that

$$\begin{aligned} I &= \int_\Omega f_\epsilon h(u_\epsilon)(T_k(u_r)-T_k(u))dx + \int_\Omega F.\nabla\Big[h(u_\epsilon)(T_k(u_r)-T_k(u))\Big]dx \\ &\quad -\int_\Omega a(x,\nabla u_\epsilon).\nabla\Big[h(u_\epsilon)(T_k(u_r)-T_k(u))\Big]dx \\ &= \int_\Omega f_\epsilon h(u_\epsilon)(T_k(u_r)-T_k(u))dx + \int_\Omega F.\nabla\Big[h(u_\epsilon)(T_k(u_r)-T_k(u))\Big]dx \\ &\quad -\int_\Omega h(u_\epsilon)a(x,\nabla u_\epsilon).\nabla(T_k(u_r)-T_k(u))dx - \int_\Omega h'(u_\varepsilon)(T_k(u_r)-T_k(u))a(x,\nabla u_\epsilon).\nabla u_\epsilon dx. \end{aligned}$$

We need to let first $\varepsilon \to 0$ and next $r \to 0$. The three first terms are obvious. As to the last term, see that

$$\begin{aligned} |T_k(u_r)-T_k(u)| &\leq |(T_k(u)-T_k(M-r))\chi_{[M-r\leq u\leq M]}| + |(T_k(u)-T_k(m+r))\chi_{[m\leq u\leq m+r]}| \\ &\leq 2r, \end{aligned}$$

which implies that

$$\begin{aligned} \lim_{r\to 0}\limsup_{\epsilon\to 0}\left|\int_\Omega h'(u_\epsilon)(T_k(u_r)-T_k(u))a(x,\nabla u_\epsilon).\nabla u_\epsilon dx\right| &\leq \lim_{r\to 0}\limsup_{\epsilon\to 0} 2r\int_\Omega \Big|h'(u_\epsilon)a(x,\nabla u_\epsilon).\nabla u_\epsilon\Big|\,dx \\ &= 0, \end{aligned}$$

where we use the fact that $\int_\Omega \Big|h'(u_\varepsilon)a(x,\nabla u_\epsilon).\nabla u_\epsilon\Big|\,dx$ is bounded.
Now, see that

$$h(u)\beta_0(u_r)(T_k(u)-T_k(u_r)) \geq 0.$$

Indeed,

$$\begin{aligned} h(u)\beta_0(u_r)(T_k(u)-T_k(u_r)) &= h(u)\beta_0(M-r)(T_k(u)-T_k(M-r))\chi_{[M-r\leq u\leq M]} \\ &\quad + h(u_\epsilon)\beta_0(m+r)(T_k(u)-T_k(m+r))\chi_{[m\leq u\leq m+r]} \quad \geq 0, \end{aligned}$$

where we use the fact that $\beta_0(M-r)\geq 0$ and $\beta_0(m+r)\leq 0$ (since $0\in\beta(0)$ and $m+r\leq 0\leq M-r$). This gives (3.12).
Passing to the limit in (3.11) and using the above results we obtain the inequality (3.10).

Step2 : We prove that

$$\limsup_{l\to+\infty}\limsup_{\epsilon\to 0}\int_{[l<|u_\epsilon|<l+1]} a(x,\nabla u_\epsilon).\nabla u_\epsilon dx \leq 0. \tag{3.13}$$

Taking $w_l(u_\epsilon)$ as test function in (3.2), where $w_l(r)=T_1(r-T_l(r))$, we get

$$\int_\Omega a(x,\nabla u_\epsilon).\nabla T_1(u_\epsilon - T_l(u_\epsilon))dx + \int_\Omega \beta_\epsilon(u_\epsilon)T_1(u_\epsilon - T_l(u_\epsilon))dx$$

$$= \int_\Omega f_\epsilon T_1(u_\epsilon - T_l(u_\epsilon))dx + \int_\Omega F.\nabla T_1(u_\epsilon - T_l(u_\epsilon))dx.$$

Since

$$\int_\Omega \beta_\epsilon(u_\epsilon) T_1(u_\epsilon - T_l(u_\epsilon))dx \geq 0$$

and

$$\int_\Omega a(x, \nabla u_\epsilon).\nabla T_1(u_\epsilon - T_l(u_\epsilon))dx = \int_{[l<|u_\epsilon|<l+1]} a(x, \nabla u_\epsilon).\nabla u_\epsilon dx,$$

we get

$$\int_{[l<|u_\epsilon|<l+1]} a(x, \nabla u_\epsilon).\nabla u_\epsilon dx \leq \int_\Omega f_\epsilon T_1(u_\epsilon - T_l(u_\epsilon))dx + \int_\Omega F.\nabla T_1(u_\epsilon - T_l(u_\epsilon))dx. \tag{3.14}$$

Recall that by step1, we have

$$\lim_{\epsilon\to 0} \int_\Omega f_\epsilon T_1(u_\epsilon - T_l(u_\epsilon))dx = \int_\Omega f T_1(u - T_l(u))dx.$$

So, using the fact that $T_1(u - T_l(u)) \longrightarrow 0$ a.e. in Ω as $l \to +\infty$ and the Lebesgue dominated convergence Theorem, we obtain

$$\lim_{l\to+\infty} \lim_{\epsilon\to 0} \int_\Omega f_\epsilon T_1(u_\epsilon - T_l(u_\epsilon))dx = 0.$$

Now, let us see that $\lim_{l\to+\infty} \lim_{\epsilon\to 0} \int_\Omega F.\nabla T_1(u_\epsilon - T_l(u_\epsilon))dx = 0$. Indeed, we begin by proving that

$$\lim_{l\to+\infty} \lim_{\epsilon\to 0} \int_{[l<|u_\epsilon|<l+1]} |\nabla u_\epsilon|^p dx = 0.$$

Thanks to (1.1), we have

$$\begin{aligned} \lambda \int_{[l<|u_\epsilon|<l+1]} |\nabla u_\epsilon|^p dx &\leq \int_{[l<|u_\epsilon|<l+1]} a(x, \nabla u_\epsilon).\nabla u_\epsilon dx \\ &\leq \int_\Omega f_\epsilon T_1(u_\epsilon - T_l(u_\epsilon))dx + \int_{[l<|u_\epsilon|<l+1]} F.\nabla u_\epsilon dx. \end{aligned}$$

Using Young's inequality, for every $\tilde{\epsilon} > 0$, we get

$$\int_{[l<|u_\epsilon|<l+1]} F.\nabla u_\epsilon dx \leq \frac{(\tilde{\epsilon})^{1-p'}}{p'} \int_{[l<|u_\epsilon|<l+1]} |F|^{p'} dx + \frac{\tilde{\epsilon}}{p} \int_{[l<|u_\epsilon|<l+1]} |\nabla u_\epsilon|^p dx.$$

Taking $\tilde{\epsilon} = \frac{p}{2}\lambda$ we obtain

$$\frac{\lambda}{2} \int_{[l<|u_\epsilon|<l+1]} |\nabla u_\epsilon|^p dx \leq \int_\Omega f_\epsilon T_1(u_\epsilon - T_l(u_\epsilon))dx + C(\lambda, p) \int_{[l<|u_\epsilon|<l+1]} |F|^{p'} dx.$$

Furthermore

$$\int_{[l<|u_\epsilon|<l+1]} |F|^{p'} dx \leq \int_{[|u_\epsilon|>l]} |F|^{p'} dx$$

and

$$\lim_{\epsilon\to 0} \int_{[|u_\epsilon|>l]} |F|^{p'} dx \leq \int_{[|u|\geq l]} |F|^{p'} dx.$$

Since

$$\text{meas}([|u| \geq l]) \to 0, \text{ as } l \to +\infty \text{ by (2.2)}$$

we have

$$\lim_{l \to +\infty} \lim_{\epsilon \to 0} \int_{[l<|u_\epsilon|<l+1]} |F|^{p'} dx = 0.$$

Hence, $\lim_{l \to +\infty} \lim_{\epsilon \to 0} \int_{[l<|u_\epsilon|<l+1]} |\nabla u_\epsilon|^p dx = 0$. Now, using the above results we obtain

$$\lim_{l \to +\infty} \lim_{\epsilon \to 0} \int_\Omega F.\nabla T_1(u_\epsilon - T_l(u_\epsilon))dx = 0.$$

Then passing to the limit as $\epsilon \to 0$ and as $l \to +\infty$ in (3.14) we get (3.13).

Step3: We prove that for every $k > 0$,

$$\limsup_{\epsilon \to 0} \int_\Omega a(x, \nabla T_k(u_\epsilon)).\Big[\nabla T_k(u_\epsilon) - \nabla T_k(u)\Big] dx \leq 0. \tag{3.15}$$

Indeed, for all $l > 0$ we define the function h_l by $h_l(r) = \inf\{1, (l + 1 - |r|)^+\}$. For $l > k$, we have

$$\begin{aligned}
&\int_\Omega a(x, \nabla u_\epsilon).\nabla\Big[h_l(u_\epsilon)(T_k(u_\epsilon) - T_k(u))\Big] dx \\
&= \int_{[|u_\epsilon|\leq k]} h_l(u_\epsilon) a(x, \nabla T_k(u_\epsilon)).\Big[\nabla T_k(u_\epsilon) - \nabla T_k(u)\Big] dx \\
&\quad + \int_{[|u_\epsilon|>k]} h_l(u_\epsilon) a(x, \nabla u_\epsilon)(-\nabla T_k(u)) dx \\
&\quad + \int_\Omega h_l'(u_\epsilon)(T_k(u_\epsilon) - T_k(u)) a(x, \nabla u_\epsilon).\nabla u_\epsilon dx \\
&:= (E_1) + (E_2) + (E_3).
\end{aligned}$$

Since $l > k$, on the set $[|u_\epsilon| \leq k]$ we have $h_l(u_\epsilon) = 1$ so that we can write (E_1) as

$$\begin{aligned}
(E_1) &= \int_{[|u_\epsilon|\leq k]} a(x, \nabla T_k(u_\epsilon)).\Big[\nabla T_k(u_\epsilon) - \nabla T_k(u)\Big] dx \\
&= \int_\Omega a(x, \nabla T_k(u_\epsilon)).\Big[\nabla T_k(u_\epsilon) - \nabla T_k(u)\Big] dx.
\end{aligned}$$

Hence we obtain

$$\limsup_{\epsilon \to 0} (E_1) = \limsup_{\epsilon \to 0} \int_\Omega a(x, \nabla T_k(u_\epsilon)).\Big[\nabla T_k(u_\epsilon) - \nabla T_k(u)\Big] dx.$$

Let us write the term (E_2) as

$$(E_2) = -\int_{[|u_\epsilon|>k]} h_l(u_\epsilon) a(x, \nabla T_{l+1}(u_\epsilon)).\nabla T_k(u) dx.$$

Using Lebesgue dominated convergence theorem, we get

$$\lim_{\epsilon \to 0} (E_2) = -\int_{[|u|\geq k]} h_l(u) H_{l+1}.\nabla T_k(u) dx = 0.$$

For the term (E_3), we have

$$\begin{aligned}\left(-\int_\Omega h_l^{'}(u_\epsilon)(T_k(u_\epsilon)-T_k(u))a(x,\nabla u_\epsilon).\nabla u_\epsilon dx\right) &\le \left|\int_\Omega h_l^{'}(u_\epsilon)(T_k(u_\epsilon)-T_k(u))a(x,\nabla u_\epsilon).\nabla u_\epsilon dx\right|\\ &\le 2k\int_{[l<|u_\epsilon|<l+1]} a(x,\nabla u_\epsilon).\nabla u_\epsilon dx.\end{aligned}$$

Using the result of the step2 we deduce that

$$\limsup_{l\to+\infty}\limsup_{\epsilon\to 0}\left(-\int_\Omega h_l^{'}(u_\epsilon)(T_k(u_\epsilon)-T_k(u))a(x,\nabla u_\epsilon).\nabla u_\epsilon dx\right)\le 0.$$

Applying (3.10) with h replaced by $h_l,\ l>k$ we get

$$\begin{aligned}&\limsup_{\epsilon\to 0}\int_\Omega a(x,\nabla T_k(u_\epsilon)).\Big[\nabla(T_k(u_\epsilon)-\nabla T_k(u))\Big]dx\\ &\qquad\le \limsup_{\epsilon\to 0}\left(-\int_\Omega h_l^{'}(u_\epsilon)(T_k(u_\epsilon)-T_k(u))a(x,\nabla u_\epsilon).\nabla u_\epsilon dx\right),\end{aligned}$$

so that letting $l\to+\infty$ yields the inequality (3.15).

Step4: In this step we prove by standard monotonicity arguments that for all $k>0,\ H_k = a(x,\nabla T_k(u))$ a.e. in Ω. Let $\varphi\in\mathcal{D}(\Omega)$ and $\tilde\alpha\in\mathbb{R}^*$. Using (3.15), we have

$$\begin{aligned}&\tilde\alpha\lim_{\epsilon\to 0}\int_\Omega a(x,\nabla T_k(u_\epsilon)).\nabla\varphi dx\\ &\quad\ge \limsup_{\epsilon\to 0}\int_\Omega a(x,\nabla T_k(u_\epsilon)).\Big[\nabla T_k(u_\epsilon)-\nabla T_k(u)+\nabla(\tilde\alpha\varphi)\Big]dx\\ &\quad\ge \limsup_{\epsilon\to 0}\int_\Omega a(x,\nabla(T_k(u)-\tilde\alpha\varphi)).\Big[\nabla T_k(u_\epsilon)-\nabla T_k(u)+\nabla(\tilde\alpha\varphi)\Big]dx\\ &\quad\ge \limsup_{\epsilon\to 0}\int_\Omega a(x,\nabla(T_k(u)-\tilde\alpha\varphi)).\nabla(\tilde\alpha\varphi)dx\\ &\quad\ge \tilde\alpha\int_\Omega a(x,\nabla(T_k(u)-\tilde\alpha\varphi)).\nabla\varphi dx.\end{aligned}$$

Dividing by $\tilde\alpha>0$ and by $\tilde\alpha<0$, passing the limit with $\tilde\alpha\to 0$ we obtain

$$\lim_{\epsilon\to 0}\int_\Omega a(x,\nabla T_k(u_\epsilon)).\nabla\varphi dx = \int_\Omega a(x,\nabla T_k(u)).\nabla\varphi dx.$$

This means that $\forall k>0,\ \int_\Omega H_k.\nabla\varphi dx = \int_\Omega a(x,\nabla T_k(u)).\nabla\varphi dx$ and then

$$H_k = a(x,\nabla T_k(u)) \text{ in } \mathcal{D}^{'}(\Omega)$$

for all $k>0$. Hence $H_k = a(x,\nabla T_k(u))$ a.e. in Ω and so $a(x,\nabla T_k(u_\epsilon))\rightharpoonup a(x,\nabla T_k(u))$ weakly in $\left(L^{p'}(\Omega)\right)^N$.

(ii) Thanks to (3.15), we deduce that that for all $k>0$

$$\lim_{\epsilon\to 0}\int_\Omega\Big[a(x,\nabla T_k(u_\epsilon))-a(x,\nabla T_k(u))\Big].\Big[\nabla T_k(u_\epsilon)-\nabla T_k(u)\Big]dx = 0.$$

Since

$$g_\epsilon(.) := \Big[a(.,\nabla T_k(u_\epsilon)) - a(.,\nabla T_k(u))\Big].\Big[\nabla T_k(u_\epsilon) - \nabla T_k(u)\Big] \geq 0,$$

up to a subsequence we have

$$g_\epsilon(.) \longrightarrow 0 \quad \text{ a.e. in } \Omega.$$

This implies that, there exists $Z \subset \Omega$ such that meas$(Z) = 0$ and $g_\epsilon(.) \longrightarrow 0$ a.e. in $\Omega \setminus Z$. Let $x \in \Omega \setminus Z$. Using the assumptions (1.1) and (1.3), it follows that the sequence $(\nabla T_k(u_\epsilon(x)))_{\epsilon>0}$ is bounded in $\mathbb{R}^N$ so we can extract a subsequence which converges to some $\tilde{\xi}$ in $\mathbb{R}^N$. Passing to the limit in the expression of $g_\epsilon(x)$, we get

$$0 = \Big[a(x,\tilde{\xi}) - a(x,\nabla T_k(u(x)))\Big].\Big[\tilde{\xi} - \nabla T_k(u(x))\Big].$$

This yields $\tilde{\xi} = \nabla T_k(u(x)) \ \ \forall x \in \Omega \setminus Z$. As the limit does not depend on the subsequence, the whole sequence $(\nabla T_k(u_\epsilon(x)))_{\epsilon>0}$ converges to $\tilde{\xi}$ in $\mathbb{R}^N$. This means that

$$\nabla T_k(u_\epsilon) \longrightarrow \nabla T_k(u) \quad \text{ a.e. in } \Omega.$$

(iii) The continuity of $a(x,\xi)$ with respect to $\xi \in \mathbb{R}^N$ gives us

$$a(x,\nabla T_k(u_\epsilon)) \longrightarrow a(x,\nabla T_k(u)) \text{ a.e. in } \Omega$$

and then we obtain

$$a(x,\nabla T_k(u_\epsilon)).\nabla T_k(u_\epsilon) \longrightarrow a(x,\nabla T_k(u)).\nabla T_k(u) \text{ a.e. in } \Omega.$$

Setting $y_\epsilon = a(x,\nabla T_k(u_\epsilon)).\nabla T_k(u_\epsilon)$ and $y = a(x,\nabla T_k(u)).\nabla T_k(u)$, we have

$$\begin{cases} y_\epsilon \geq 0, \ y_\epsilon \longrightarrow y \text{ a.e. in } \Omega, \ y \in L^1(\Omega), \\ \displaystyle\int_\Omega y_\epsilon dx \longrightarrow \int_\Omega y dx. \end{cases}$$

Since

$$\int_\Omega |y_\epsilon - y| dx = 2\int_\Omega (y - y_\epsilon)^+ dx + \int_\Omega (y_\epsilon - y) dx$$

and $(y - y_\epsilon)^+ \leq y$ it follows by the Lebesgue dominated convergence theorem that

$$\lim_{\epsilon \to 0} \int_\Omega |y_\epsilon - y| dx = 0,$$

which means that

$$a(x,\nabla T_k(u_\epsilon)).\nabla T_k(u_\epsilon) \longrightarrow a(x,\nabla T_k(u)).\nabla T_k(u) \text{ strongly in } L^1(\Omega).$$

(iv) By (1.1), we have $|\nabla T_k(u_\epsilon)|^p \leq \dfrac{1}{\lambda} a(x,\nabla T_k(u_\epsilon)).\nabla T_k(u_\epsilon)$. Using the L^1-convergence of (iii) and the generalized dominated convergence Theorem, we obtain the result of (iv). □

Lemma 3.1 *For any $h \in C_c^1(\mathbb{R})$ and $\xi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$,*

$$\nabla[h(u_\epsilon)\xi] \longrightarrow \nabla[h(u)\xi] \textit{ strongly in } L^p(\Omega) \textit{ as } \epsilon \to 0.$$

Proof : For any $h \in C_c^1(\mathbb{R})$ and $\xi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, we have

$$\begin{aligned}\nabla[h(u_\epsilon)\xi] &= h(u_\epsilon)\nabla\xi + h'(u_\epsilon)\xi\nabla u_\epsilon \\ &= h(u_\epsilon)\nabla\xi + h'(u_\epsilon)\xi\nabla T_l(u_\epsilon) \text{ for } l > 0 \text{ such that } \mathrm{supp}(h) \subset]-l,+l[.\end{aligned}$$

Using the Lebesgue dominated convergence Theorem ,we get

$$h(u_\epsilon)\nabla\xi \longrightarrow h(u)\nabla\xi \text{ strongly in } L^p(\Omega) \text{ as } \epsilon \to 0.$$

Moreover, since $|h'(u_\epsilon)\xi\nabla T_l(u_\epsilon)| \leq C|\nabla T_l(u_\epsilon)|$, then using the generalized convergence Theorem and Proposition 3.2-(iv) we deduce that

$$h'(u_\epsilon)\xi\nabla T_l(u_\epsilon) \longrightarrow h'(u)\xi\nabla T_l(u) = h'(u)\xi\nabla u \text{ strongly in } L^p(\Omega) \text{ as } \epsilon \to 0.$$

So Lemma 3.1 follows □

Now, to pass to the limit in $\beta_\varepsilon(u_\varepsilon)$, let us consider the function $h_0 = h_{l_0},\ l_0 > 0$ to be fixed later such that

$$\begin{cases} h_0 \in C_c^1(\mathbb{R}),\ h_0(r) \geq 0, \forall r \in \mathbb{R}, \\ \\ h_0(r) = 1 \text{ if } |r| \leq l_0 \text{ and } h_0(r) = 0 \text{ if } |r| \geq l_0 + 1. \end{cases}$$

Since, for any $k > 0$, $(h_k(u_\varepsilon)z_\varepsilon)_{\epsilon>0}$ is bounded in $L^1(\Omega)$, there exists $z_k \in \mathcal{M}_b(\Omega)$, such that

$$h_k(u_\epsilon)\beta_\epsilon(u_\epsilon) \overset{*}{\rightharpoonup} z_k \text{ in } \mathcal{M}_b(\Omega) \text{ as } \epsilon \to 0.$$

Moreover, for any $\xi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, we have

$$\int_\Omega \xi\, dz_k = \int_\Omega \xi h_k(u)\, d\mu - \int_\Omega a(x,\nabla u) \cdot \nabla(h_k(u)\xi)dx,$$

which implies that $z_k \in \mathcal{M}_b^p(\Omega)$ and, for any $k \leq l$,

$$z_k = z_l \quad \text{on } [|T_k(u)| < k].$$

Let us consider the Radon measure z defined by

$$\begin{cases} z = z_k, \quad \text{on } [|T_k(u)| < k] \text{ for } k \in \mathbb{N}^*, \\ \\ z = 0 \quad \text{on } \bigcap_{k\in\mathbb{N}^*} [|T_k(u)| = k]. \end{cases} \tag{3.16}$$

For any $h \in \mathcal{C}_c(\mathbb{R})$, $h(u) \in L^\infty(\Omega, d|z|)$ and

$$\int_\Omega h(u)\xi\, dz = -\int_\Omega a(x,\nabla u) \cdot \nabla(h(u)\xi)dx + \int_\Omega h(u)\xi d\mu,$$

for any $\xi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$. Indeed, let $k_0 > 0$ be such that $\mathrm{supp}(h) \subseteq [-k_0, k_0]$,

$$\begin{aligned}\int_\Omega h(u)\xi\, dz &= \int_\Omega h(u)\xi\, dz_{k_0} \\ &= -\lim_{\varepsilon\to 0}\int_\Omega a(x,\nabla u_\varepsilon) \cdot \nabla(h(u_\varepsilon)\xi)dx + \lim_{\varepsilon\to 0}\int_\Omega h(u_\varepsilon)\xi d\mu_\varepsilon \\ &= -\lim_{\varepsilon\to 0}\int_\Omega a(x,\nabla T_{k_0}(u_\varepsilon)) \cdot \nabla(h(u_\varepsilon)\xi)dx + \lim_{\varepsilon\to 0}\int_\Omega h(u_\varepsilon)\xi d\mu_\varepsilon \\ &= -\int_\Omega a(x,\nabla u) \cdot \nabla(h(u)\xi)dx + \int_\Omega h(u)\xi d\mu.\end{aligned}$$

Moreover, we have

Lemma 3.2 *The Radon-Nikodym decomposition of the measure z given by (3.16) with respect to $\mathcal{L}^N$,*

$$z = w\,\mathcal{L}^N + \nu, \qquad \textit{with } \nu \perp \mathcal{L}^N,$$

satisfies the following properties

$$\begin{cases} w \in \beta(u)\ \mathcal{L}^N - a.e.\ \ in\ \Omega,\ \ w \in L^1(\Omega),\ \ \nu \in \mathcal{M}_b^p(\Omega), \\ \\ \nu^+\ \ is\ concentrated\ on\ [u = M] \cap [u \neq +\infty] \\ \\ \ and\ \nu^-\ \ is\ concentrated\ on\ [u = m] \cap [u \neq -\infty]. \end{cases}$$

Proof : Since, for any $\epsilon > 0$, $z_\epsilon \in \partial j_\epsilon(u_\epsilon)$, we have

$$j(t) \geq j_\epsilon(t) \geq j_\epsilon(u_\epsilon) + (t - u_\epsilon) z_\epsilon \ \ \mathcal{L}^N - \text{a.e. in } \Omega, \quad \forall t \in \mathbb{R}.$$

Then for any $h \in \mathcal{C}_c(\mathbb{R})$, $h \geq 0$ and $k > 0$ such that $\text{supp}(h) \subseteq [-k, k]$, we have

$$\xi h(u_\epsilon) j(t) \geq \xi h(u_\epsilon) j_\epsilon(u_\epsilon) + (t - u_\epsilon)\xi h(u_\epsilon) h_k(u_\varepsilon) z_\epsilon.$$

In addition, for any $0 < \epsilon < \tilde{\epsilon}$, we have

$$\xi h(u_\epsilon) j(t) \geq \xi h(u_\epsilon) j_{\tilde{\epsilon}}(u_\epsilon) + (t - u_\epsilon)\xi h(u_\epsilon) h_k(u_\varepsilon) z_\epsilon$$

and, integrating over Ω yields

$$\int_\Omega \xi h(u_\epsilon) j(t) dx \geq \int_\Omega \xi h(u_\epsilon) j_{\tilde{\epsilon}}(u_\epsilon) dx + \int_\Omega (t - u_\epsilon)\xi h(u_\epsilon) h_k(u_\varepsilon) z_\epsilon dx.$$

As $\epsilon \to 0$, we get by using Fatou's Lemma

$$\int_\Omega \xi h(u) j(t) dx \quad \geq \quad \int_\Omega \xi h(u) j_{\tilde{\epsilon}}(u) dx + \liminf_{\epsilon \to 0} \int_\Omega (t - u_\epsilon)\xi h(u_\epsilon) h_k(u_\varepsilon) z_\epsilon dx.$$

Now, for any $\xi \in C_c^1(\Omega)$ and $t \in \mathbb{R}$, setting

$$\tilde{h}(r) = (t - r) h(r),$$

we have

$$\begin{aligned} \lim_{\epsilon \to 0} \int_\Omega (t - u_\epsilon) h(u_\epsilon) \xi h_k(u_\varepsilon) z_\epsilon dx \quad &= \quad \lim_{\epsilon \to 0} \int_\Omega \tilde{h}(u_\epsilon) \xi h_k(u_\varepsilon) z_\epsilon dx \\ &= \quad \int_\Omega (t - u) h(u) \xi dz_k \\ &= \quad \int_\Omega (t - u) h(u) \xi dz. \end{aligned}$$

So,

$$\int_\Omega \xi h(u) j(t) dx \geq \int_\Omega \xi h(u) j_{\tilde{\epsilon}}(u) dx + \int_\Omega \xi (t - u) h(u) dz.$$

As $\tilde{\epsilon} \to 0$, we get by using again Fatou's Lemma

$$\int_\Omega \xi h(u) j(t) dx \geq \int_\Omega \xi h(u) j(u) dx + \int_\Omega \xi (t - u) h(u) dz.$$

From the inequality above we have

$$h(u)j(t) \geq h(u)j(u) + (t-u)h(u)z, \quad \text{ in } \mathcal{M}_b(\Omega), \ \forall t \in \mathbb{R}. \tag{3.17}$$

Using the Radon-Nikodym decomposition of z we have $z = w\mathcal{L}^N + \nu$ with $\nu \perp \mathcal{L}^N, w \in L^1(\Omega)$, then comparing the regular part and the singular part of (3.17), for any $h \in \mathcal{C}_c(\mathbb{R})$, we obtain

$$h(u)j(t) \geq h(u)j(u) + (t-u)h(u)w \ \ \mathcal{L}^N - \text{a.e. in } \Omega, \ \forall t \in \mathbb{R} \tag{3.18}$$

and

$$(t-u)h(u)\nu \leq 0 \text{ in } \mathcal{M}_b(\Omega), \ \forall \ t \in \overline{\text{dom}(j)}. \tag{3.19}$$

From (3.18) we get

$$j(t) \geq j(u) + (t-u)w \ \ \mathcal{L}^N - \text{a.e. in } \Omega, \ \forall t \in \mathbb{R},$$

so that $w \in \partial j(u) \ \mathcal{L}^N -$ a.e in Ω. As to (3.19), this implies that for any $t \in \overline{\text{dom}(j)}$,

$$\nu \geq 0 \text{ in } [u \in (t, \infty) \cap \text{supp}(h)] \tag{3.20}$$

and

$$\nu \leq 0 \text{ in } [u \in (-\infty, t) \cap \text{supp}(h)]. \tag{3.21}$$

In particular, this implies that

$$\nu([m < u < M]) = 0.$$

Moreover, if $m \neq -\infty$ (resp. $M \neq +\infty$), then (3.21) (resp. (3.20)) implies that

$$\nu^- \text{ is concentrated on } [u = m] \quad (\text{resp. } \nu^+ \text{ is concentrated on } [u = M] \ ).$$

By construction of z, we see that

$$\nu([u = \pm\infty]) = 0$$

and the proof of the Lemma 3.2 is finished □

To finish the proof of Theorem 1.1, we consider $\xi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ and $h \in C_c^1(\mathbb{R})$. Then, we take $h(u_\epsilon)\xi$ as test function in (3.2). We get

$$\int_\Omega a(x, \nabla u_\epsilon).\nabla[h(u_\epsilon)\xi]dx + \int_\Omega \beta_\epsilon(u_\epsilon)h(u_\epsilon)\xi dx = \int_\Omega h(u_\epsilon)\xi f_\epsilon dx + \int_\Omega F.\nabla[h(u_\epsilon)\xi]dx. \tag{3.22}$$

Using Lemma 3.1, it is not difficult to see that

$$\lim_{\epsilon \to 0} \left(\int_\Omega h(u_\epsilon)\xi f_\epsilon dx + \int_\Omega F.\nabla[h(u_\epsilon)\xi]dx \right) = \int_\Omega h(u)\xi d\mu.$$

The first term of (3.22) can be written as

$$\int_\Omega a(x, \nabla u_\epsilon).\nabla[h(u_\epsilon)\xi]dx = \int_\Omega a(x, \nabla T_{l_0+1}(u_\epsilon)).\nabla[h_0(u_\epsilon)\xi]dx,$$

for some $l_0 > 0$ so that, by Proposition 3.2-(i) and Lemma 3.1, we have

$$\begin{aligned}
\lim_{\epsilon \to 0} \int_\Omega a(x, \nabla u_\epsilon).\nabla[h(u_\epsilon)\xi]dx &= \lim_{\epsilon \to 0} \int_\Omega a(x, \nabla T_{l_0+1}(u_\epsilon)).\nabla[h_0(u_\epsilon)\xi]dx \\
&= \int_\Omega a(x, \nabla T_{l_0+1}(u)).\nabla[h_0(u)\xi]dx \\
&= \int_\Omega a(x, \nabla u).\nabla[h(u)\xi]dx.
\end{aligned}$$

Thanks to the convergence of Lemma 3.1 and Proposition 3.2-(i) we have from (3.22)

$$\begin{aligned}\lim_{\epsilon\to 0}\int_\Omega \beta_\epsilon(u_\epsilon)h(u_\epsilon)\xi dx &= \int_\Omega h(u)\xi d\mu - \int_\Omega a(x,\nabla u).\nabla[h(u)\xi]dx.\\ &= \int_\Omega h(u)\xi dz\\ &= \int_\Omega h(u)w\xi dx + \int_\Omega h(u)\xi d\nu.\end{aligned}$$

Letting ϵ goes to 0 in (3.22) it yields that (u,w) is a solution of the problem $P(\beta,\mu)$.
To end the proof of Theorem 1.1, we prove (1.9). We take $\varphi = T_1(u_\epsilon - T_n(u_\epsilon))$ as test function in (3.2) to get

$$\int_\Omega a(x,\nabla u_\epsilon).\nabla[T_1(u_\epsilon - T_n(u_\epsilon))]dx + \int_\Omega \beta_\epsilon(u_\epsilon)T_1(u_\epsilon - T_n(u_\epsilon))dx = \int_\Omega T_1(u_\epsilon - T_n(u_\epsilon))f_\epsilon dx$$
$$+\int_\Omega F.\nabla[T_1(u_\epsilon - T_n(u_\epsilon))]dx. \tag{3.23}$$

Since $\int_\Omega \beta_\epsilon(u_\epsilon)T_1(u_\epsilon - T_n(u_\epsilon))dx \geq 0$ and $\nabla[T_1(u_\epsilon - T_n(u_\epsilon))] = \nabla u_\epsilon \chi_{[n<|u_\epsilon|<n+1]}$, we have from equality (3.23)

$$\int_{[n<|u_\epsilon|<n+1]} a(x,\nabla u_\epsilon).\nabla u_\epsilon dx \leq \int_\Omega T_1(u_\epsilon - T_n(u_\epsilon))f_\epsilon dx + \int_\Omega F.\nabla[T_1(u_\epsilon - T_n(u_\epsilon))]dx. \tag{3.24}$$

As $\epsilon \to 0$ in (3.24), we get

$$\int_{[n\leq|u|\leq n+1]} a(x,\nabla u).\nabla u dx \leq \int_\Omega T_1(u - T_n(u))f dx + \int_\Omega F.\nabla[T_1(u - T_n(u))]dx. \tag{3.25}$$

Using assumption (1.1), it follows

$$\lambda\int_{[n\leq|u|\leq n+1]} |\nabla u|^p dx \leq \int_\Omega T_1(u - T_n(u))f dx + \int_\Omega F.\nabla[T_1(u - T_n(u))]dx. \tag{3.26}$$

Using the proof of Proposition 3.2 (*i) Step 2*), one sees that $\lim_{n\to+\infty}\int_\Omega T_1(u-T_n(u))f dx = 0$ and $\lim_{n\to+\infty}\int_\Omega F.\nabla[T_1(u-T_n(u))]dx = 0$ so that we get (1.9).

4 Proof of Theorem 1.2 and Theorem 1.3

The existence part of Theorem 1.3 follows by Theorem 1.1 and the fact that u in this case is bounded. As we said in the introduction, the proof of the uniqueness follows by the using entropic formulation of the solution. So let us first prove Theorem 1.2.

Proof of Theorem 1.2 : Let us consider the function h_n, $n > 0$ defined on $\mathbb{R}$ by $h_n(r) = \inf\{1, (n+1-|r|)^+\}$. If (u,w) is a solution of $P(\beta,\mu)$, we take $h = h_n$ in (1.8). We have for any $\varphi \in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$

$$\int_\Omega a(x,\nabla u).\nabla(h_n(u)\varphi)\,dx + \int_\Omega w h_n(u)\varphi dx + \int_\Omega h_n(u)\varphi d\nu = \int_\Omega h_n(u)\varphi d\mu. \tag{4.1}$$

Since $\nabla(h_n(u)\varphi) = h_n(u)\nabla\varphi + h_n'(u)\varphi\nabla u$, it follows from (4.1)

$$\int_\Omega h_n(u)a(x,\nabla u).\nabla\varphi dx + \int_\Omega h_n'(u)\varphi a(x,\nabla u).\nabla u dx + \int_\Omega w h_n(u)\varphi dx + \int_\Omega h_n(u)\varphi d\nu = \int_\Omega h_n(u)\varphi d\mu. \tag{4.2}$$

We have $h_n(u) \longrightarrow 1$ a.e. in $\mathbb{R}$ as $n \to +\infty$. Excepted the second term, all the terms in (4.2) pass to the limit by Lebesgue dominated convergence Theorem and when $n \to +\infty$, we get

$$\int_\Omega a(x,\nabla u).\nabla\varphi dx + \limsup_{n\to+\infty}\int_\Omega h_n'(u)\varphi a(x,\nabla u).\nabla u dx + \int_\Omega w\varphi dx + \int_\Omega \varphi d\nu = \int_\Omega \varphi d\mu. \tag{4.3}$$

For the second term in (4.3), we have

$$\begin{aligned}
\Big|\int_\Omega h_n'(u)\varphi a(x,\nabla u).\nabla u dx\Big| &\leq \int_{[n\leq|u|\leq n+1]} \Big|h_n'(u)\varphi a(x,\nabla u).\nabla u\Big| dx \\
&\leq \|\varphi\|_\infty \int_{[n\leq|u|\leq n+1]} \Big|a(x,\nabla u).\nabla u\Big| dx \\
&\leq \Lambda\|\varphi\|_\infty \int_{[n\leq|u|\leq n+1]} \Big(j_1(x)|\nabla u| + |\nabla u|^p\Big) dx \\
&\leq \Lambda\|\varphi\|_\infty \left(\|j_1\|_{L^{p'}(\Omega)}\left(\int_{[n\leq|u|\leq n+1]}|\nabla u|^p dx\right)^{\frac{1}{p}} + \int_{[n\leq|u|\leq n+1]}|\nabla u|^p dx\right) \\
&\longrightarrow 0 \text{ as } n\to+\infty \ \text{ (thanks to (1.9))}.
\end{aligned}$$

Now, we replace φ by $T_k(u-\xi)$ in (4.3) to get

$$\int_\Omega a(x,\nabla u).\nabla T_k(u-\xi)dx + \int_\Omega wT_k(u-\xi)dx + \int_\Omega T_k(u-\xi)d\nu = \int_\Omega T_k(u-\xi)d\mu. \tag{4.4}$$

Note that, since $\xi \in \mathrm{dom}\beta$,

$$\begin{aligned}
\int_\Omega T_k(u-\xi)d\nu &= \int_\Omega T_k(u-\xi)d\nu^+ - \int_\Omega T_k(u-\xi)d\nu^- \\
&= \int_{[u=M]} T_k(u-\xi)d\nu^+ - \int_{[u=m]} T_k(u-\xi)d\nu^- \\
&\geq 0;
\end{aligned}$$

then (1.10) follows from (4.4)□

Proof of Theorem 1.3: Now, let us prove the uniqueness of the solution for $P(\beta,\mu)$ when $-\infty < m \leq 0 \leq M < \infty$. Suppose that $(u_1,w_1),(u_2,w_2)$ are two solutions of $P(\beta,\mu)$. For u_1, we choose $\xi = u_2$ as test function in (1.10), we have

$$\int_\Omega a(x,\nabla u_1).\nabla T_k(u_1-u_2)dx + \int_\Omega w_1T_k(u_1-u_2)dx \leq \int_\Omega T_k(u_1-u_2)d\mu.$$

Similarly we get for u_2

$$\int_\Omega a(x,\nabla u_2).\nabla T_k(u_2-u_1)dx + \int_\Omega w_2T_k(u_2-u_1)dx \leq \int_\Omega T_k(u_2-u_1)d\mu.$$

Adding these two last inequalities yields

$$\int_\Omega \Big(a(x,\nabla u_1)-a(x,\nabla u_2)\Big).\nabla T_k(u_1-u_2)dx+\int_\Omega (w_1-w_2)\,T_k(u_1-u_2)dx\le 0. \qquad (4.5)$$

For any $k>0$, from (4.5) it yields

$$\int_\Omega \Big(a(x,\nabla u_1)-a(x,\nabla u_2)\Big).\nabla T_k(u_1-u_2)dx=0. \qquad (4.6)$$

From (4.6), it follows that there exists a constant c such that $u_1-u_2=c$ a.e. in Ω. Using the fact that $u_1=u_2=0$ on $\partial\Omega$ we get $c=0$. Thus, $u_1=u_2$ a.e. in Ω. At last, let us see that $w_1=w_2$ a.e. in Ω and $\nu_1=\nu_2$. Indeed for any $\varphi\in\mathcal{D}(\Omega)$, taking φ as test function in (1.11) for the solutions (u_1,w_1) and (u_1,w_2), after substraction of these equalities we get

$$\int_\Omega (w_1-w_2)\varphi dx+\int_\Omega \varphi d(\nu_1-\nu_2)=0.$$

Hence

$$\int_\Omega w_1\varphi dx+\int_\Omega \varphi d\nu_1=\int_\Omega w_2\varphi dx+\int_\Omega \varphi d\nu_2.$$

Therefore

$$w_1\mathcal{L}^N+\nu_1=w_2\mathcal{L}^N+\nu_2.$$

Since the Radon-Nikodym decomposition of a measure is unique, we get $w_1=w_2$ a.e. in Ω and $\nu_1=\nu_2$.

To complete the proof of Theorem 1.3, it remains to show that (1.12) and (1.13) hold. To this aim, let us prove the following result.

Lemma 4.1 *Let $\eta\in W_0^{1,p}(\Omega)$, $Z\in\mathcal{M}_b^p(\Omega)$ and $\lambda\in\mathbb{R}$ be such that*

$$\begin{cases}\eta\le\lambda \text{ a.e. in } \Omega \quad (\text{resp. } \eta\ge\lambda),\\[2ex] Z=-div\, a(x,\nabla\eta) \text{ in } \mathcal{D}'(\Omega).\end{cases} \qquad (4.7)$$

Then

$$\int_{[\eta=\lambda]}\xi dZ\ge 0 \qquad (4.8)$$

(resp.)

$$\int_{[\eta=\lambda]}\xi dZ\le 0, \qquad (4.9)$$

for any $\xi\in C_c^1(\Omega)$, $\xi\ge 0$.

Proof : The proof of this lemma follows the same steps of [2]. For seek of completeness, let us give the arguments. For any $n\ge 1$, we set $\varphi_n(r)=\inf\{1,(nr-n\lambda+1)^+\}$ $\forall r\in\mathbb{R}$. Since $Z\in\mathcal{M}_b^p(\Omega)$, $\varphi_n(\eta)\longrightarrow\chi_{\{\eta=\lambda\}}$ quasi everywhere, and since Z is diffuse the convergence is also Z-a.e. Then for any $\xi\in C_c^1(\Omega)$, $\xi\ge 0$, we have

$$\begin{aligned}\int_{[\eta=\lambda]}\xi dZ &= \lim_{n\to+\infty}\int_\Omega \xi\varphi_n(\eta)dZ\\ &= \lim_{n\to+\infty}\int_\Omega a(x,\nabla\eta).\nabla[\xi\varphi_n(\eta)]dx\\ &\ge \lim_{n\to+\infty}\int_\Omega \varphi_n(\eta)a(x,\nabla\eta).\nabla\xi dx.\end{aligned}$$

Furthermore

$$\left|\int_\Omega \varphi_n(\eta)a(x,\nabla\eta).\nabla\xi dx\right| \;\leq\; \|\nabla\xi\|_\infty \int_{[\lambda-\frac{1}{n}\leq\eta\leq\lambda]} |a(x,\nabla\eta)|dx$$
$$\longrightarrow 0 \text{ as } n\to+\infty.$$

This give (4.8). The proof of (4.9) follows the same way by letting $\tilde\eta=-\eta,\ \tilde\lambda=-\lambda,\ \tilde Z=-Z$ and $\tilde a(x,z)=-a(x,-z)$ $\square$

Coming back to the proof of (1.12) and (1.13) , we see that since

$$\nu=\text{div } a(x,\nabla u)-w\mathcal{L}^N+\mu,$$

we have

$$\mu-\nu-w\mathcal{L}^N=-\text{div } a(x,\nabla u).$$

By Lemma 4.1, for any $\xi\in C_c^1(\Omega),\ \xi\geq 0$, we have

$$\int_{[u=M]}\xi d\nu^+\leq\int_{[u=M]}\xi d\mu-\int_{[u=M]}\xi w dx$$

and

$$\int_{[u=m]}\xi d\nu^-\geq\int_{[u=m]}\xi d\mu-\int_{[u=m]}\xi w dx.$$

The first inequality implies that

$$\int_\Omega\xi d\nu^+\leq\int_\Omega\xi d\mu\lfloor[u=M]-\int_\Omega\xi w\chi_{[u=M]}dx.$$

Consequently (1.12) holds. Similarly we get (1.13).

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