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A degenerate diffusion problem with dynamical boundary conditions

Diffusion problem with dynamical boundary conditions

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Abstract. The purpose of this paper is to study the existence, the uniqueness and the limit in $L^1(\Omega)$, as $t \to \infty$, of solutions of general initial-boundary-value problems of the form $\partial_t u - \Delta w = 0$ and $u \in \beta(w)$ in a bounded domain Ω with dynamical boundary conditions of the form $\partial_t \rho(w) + \partial_\eta w = 0$.

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1. Introduction

Consider the equation

$$\partial_t u - \Delta w = 0, \quad u \in \beta(w) \quad \text{in } Q = (0, \infty) \times \Omega, \tag{1}$$

where Ω is a bounded domain of $\mathbb{R}^N$ with smooth boundary Γ and the nonlinearity β is a maximal monotone graph in $\mathbb{R}$ (see [7]). In particular β may be multivalued, so that (1) appears in various phenomena with changes of states, like multiphase Stefan problem (cf. [11]). On the other hand, β may be a continuous function in $\mathbb{R}$, so that (1) is the filtration equation which includes the flow of liquids or gases through porous media, the heat propagation in plasmas, population dynamics, spread of thin viscous films and others (cf. [3]).

Equation (1) needs to be completed by boundary conditions on w and initial data. Inspired by physical considerations, different sorts of boundary conditions exist in the literature. In this paper, we consider dynamical ones, that is

$$\partial_t z + \partial_\eta w = 0, \quad z = \rho(w) \quad \text{on } \Sigma = (0, \infty) \times \Gamma \tag{2}$$

where $\partial_\eta w$ is the normal derivative of w and $\rho : \mathbb{R} \to \mathbb{R}$ is a continuous nondecreasing function. This kind of boundary conditions appears when the boundary material has a large thermal conductivity and sufficiently small thickness. Hence, the boundary material is regarded as the boundary of the domain. For instance,

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one considers an iron ball in which water and ice coexists. For more details about above physical considerations one can see for instance [20] and [1]. Another interesting application we have in mind concerns the filtration equation with boundary conditions of the form (2) (see for instance [22]). It appears for example in the study of rainfall infiltration through the soil, when the accumulation of the water on the ground surfaces caused by the saturation of the surface layer is taken into account. Notice that ρ may be such that $\mathrm{Im}(\rho) \neq \mathbb{R}$, so that we can cover the case where the boundary conditions are dynamical only on a part of the boundary. For instance, one can think about the situation where the saturation happens only for values of w in a subinterval of $\mathbb{R}$.

Completed with initial data

$$u(0) = u_0 \quad \text{in } \Omega \quad \text{and} \quad z(0) = z_0 \quad \text{on } \Gamma, \tag{3}$$

Problem (1)-(2)-(3) was studied by many authors, for different particular cases of β and ρ. Interesting results may be found in [9], [17],[16] and [14]. We notice also that there exists a series of papers by Aiki where different methods of existence and uniqueness were used (see [1] and references therein).

In this paper, we study existence, uniqueness and asymptotic behavior of a weak solution (u, z) of (1)-(2)-(3) with general nonlinearities β and ρ, assumed, respectively, to be only a maximal monotone graph everywhere defined and a continuous function. Assuming $\beta(r) = c_1(r - b)^+ - c_2(r)^-$, Aiki proves in [1] that the abstract theory of nonlinear evolution equations governed by time-dependent subdifferentials in Hilbert space hands up very well this kind of problems. Our approach is completely different, we will treat (1), (2) and (3) in the context of nonlinear semigroup theory in Banach spaces. We prove that for any $(u_0, z_0) \in L^1(\Omega) \times L^1(\Gamma)$, such that $z_0(x) \in \mathrm{Im}(\rho)$ a.e. $x \in \Gamma$, the initial-boundary-value problem (1)-(2)-(3) has a unique mild-solution $(u, z) \in L^1(\Omega) \times L^1(\Gamma)$ (in the sense of Crandall-Ligget exponential formula). Moreover, if $(u_0, z_0) \in L^\infty(\Omega) \times L^\infty(\Gamma)$, we prove (u, z) is the unique weak solution, i.e. there exists $w \in L^2_{loc}([0, \infty); H^1(\Omega))$ and (u, w, z) solves Eqs. (1)-(2)-(3) in a weak sense.

The second part of this paper deals with the asymptotic behavior, as $t \to \infty$, of the solution (u, z). In [18] (see also [19]), the first author studied the asymptotic behavior of the Eq. (1) with general static boundary conditions of the form

$$\partial_\eta w + \gamma(w) \ni 0 \quad \text{on } \Gamma \tag{4}$$

where γ is assumed to be a maximal monotone graph in $\mathbb{R}$. So, assuming that $\rho \equiv 0$, Problem (1)-(2)-(3) is a particular case of [18]. Actually, if $\gamma \equiv 0$, we know (cf. Theorem 2 of [18]) that a solution u stabilizes, as $t \to \infty$, by converging in $L^1(\Omega)$, to a stationary solution $\underline{u}_0$, which satisfies $\displaystyle\int_\Omega \underline{u}_0 = \int_\Omega u_0$.

Moreover, $\underline{u}_0 \equiv \frac{1}{|\Omega|}\int_\Omega u_0$ if $\frac{1}{|\Omega|}\int_\Omega u_0$ is a continuous point of β. Otherwise, the characterization of the limit $\underline{u}_0$ is a difficult problem in general that the author solved only if additional assumptions on u_0 are fulfilled (cf. Theorem 4 of [18]). In this work, we generalize a part of these results to the case where the boundary conditions are of type (2), with ρ a continuous nondecreasing function. We prove that a solution (u, z) stabilizes, as $t \to \infty$, by converging in $L^1(\Omega) \times L^1(\Gamma)$ to $(\underline{u}_0, \rho(c)) \in L^1(\Omega) \times \mathbb{R}$, with $c \in \mathbb{R}$. The characterization of $(\underline{u}_0, \rho(c))$ depends on the quantities $m_0 := \Big(\int_\Omega u_0 + \int_\Gamma z_0\Big)/|\Omega|$ and $\Phi(m_0)$, with $\Phi(r) := \beta^{-1}(r) + \rho(r)|\Gamma|/|\Omega|$, for any $r \in \mathbb{R}$. Indeed, we will prove that $c \in \Phi^{-1}(m_0)$ and $\underline{u}_0(x) \in \beta(c)$, a.e. $x \in \Omega$; so that if m_0 is a continuous point of β, then $(\underline{u}_0, \rho(c))$ is uniquely given by $c = \Phi^{-1}(m_0)$ and $\underline{u}_0 \equiv \beta(c)$.

The paper is organized as follows. In the next section, we state assumptions that will hold throughout the paper and give our main results concerning existence, uniqueness and asymptotic behavior. In Sect. 3, we recall some basic tools from the nonlinear semigroup theory in Banach spaces and prove the existence and uniqueness results. Finally, in Sect. 4, we prove the stabilization result.

2. Main results

Throughout this paper, Ω is a bounded domain of $\mathbb{R}^N$ with smooth boundary Γ, ρ is a continuous nondecreasing function in $\mathbb{R}$ and b a maximal monotone graph in $\mathbb{R}$. We assume that

$$(H_1) \qquad \mathrm{Im}(b) = \mathcal{D}(b) = \mathbb{R},$$

and

$$(H_2) \qquad 0 \in b(0) \cap \rho(0).$$

Hereafter, we begin by announcing our existence and uniqueness results concerning the following evolution problem

$$E(u_0, z_0) \qquad \begin{cases} \partial_t u - \Delta w = 0, \quad w \in b(u) \quad \text{in } Q = (0,\infty) \times \Omega \\ \partial_t z + \partial_\eta w = 0, \quad z = \rho(w) \quad \text{on } \Sigma = (0,\infty) \times \Gamma \\ u(0) = u_0 \quad \text{in } \Omega, \quad z(0) = z_0 \quad \text{on } \Gamma. \end{cases}$$

Theorem 1. *For any $u_0 \in L^\infty(\Omega)$ and $z_0 \in L^\infty(\Gamma)$ such that*

$$z_0(x) \in Im(\rho), \quad a.e.\ x \in \Gamma, \tag{1}$$

there exists a unique (u, z) solution of $E(u_0, z_0)$ in the following sense

$$\begin{cases} u \in L^\infty(Q),\ z \in L^\infty(\Sigma),\ \exists\, w \in L^2_{loc}\left([0,\infty); H^1(\Omega)\right), \\ \\ w \in b(u) \text{ a.e. in } Q,\ z = \rho(w) \text{ a.e. on } \Sigma \text{ and} \\ \\ \displaystyle\int_0^\tau\int_\Omega \xi_t u + \int_0^\tau\int_\Gamma z\xi_t = \int_0^\tau\int_\Omega Dw.D\xi - \int_\Gamma z_0\xi(0) - \int_\Omega \xi(0)u_0 \\ \\ \forall\, \xi \in \mathcal{C}^1([0,\tau]\times\overline{\Omega}) \text{ with } \tau > 0 \text{ and } \xi(\tau) \equiv 0. \end{cases} \tag{2}$$

Moreover, $u \in \mathcal{C}([0,\infty); L^1(\Omega))$, $z \in \mathcal{C}([0,\infty); L^1(\Gamma))$,

$$\int_\Omega u(t) + \int_\Gamma z(t) = \int_\Omega u_0 + \int_\Gamma z_0 \quad \textit{for any } t \geq 0, \tag{3}$$

and if (u_i, z_i) is the solution of $E(u_{0i}, z_{0i})$, assuming $u_{0i} \in L^\infty(\Omega)$ and $z_{0i} \in L^\infty(\Gamma)$ satisfying (1), for $i = 1,\ 2$, then

$$\begin{aligned} &\left\| \left(u_1(t) - u_2(t)\right)^+ \right\|_{L^1(\Omega)} + \left\| \left(z_1(t) - z_2(t)\right)^+ \right\|_{L^1(\Gamma)} \\ &\qquad \leq \left\| \left(u_{01} - u_{02}\right)^+ \right\|_{L^1(\Omega)} + \left\| \left(z_{01} - z_{02}\right)^+ \right\|_{L^1(\Gamma)}. \end{aligned} \tag{4}$$

Now, in order to study the asymptotic behavior, as $t \to \infty$, of the solution (u, z), we introduce the maximal monotone graph in $\mathbb{R}$, defined by

$$\phi_{b\rho}(r) = \left\{ s + \frac{|\Gamma|}{|\Omega|}\rho(r)\ ;\ s \in b^{-1}(r) \right\}$$

where b^{-1} denotes the inverse of b in $\mathbb{R}$, defined by $r \in b^{-1}(s)$ if and only if $s \in b(r)$, for any $r \in \mathbb{R}$. We also define the set

$$\mathcal{E} = \left\{ r \in \mathbb{R}\ ;\ r \text{ is a point of discontinuity of } b_0^{-1} \right\},$$

where $b_0^{-1}(r) = \inf\ b^{-1}(r)$, for any $r \in \mathbb{R}$. On the other hand, for any $(u_0, z_0) \in L^1(\Omega) \times L^1(\Gamma)$, we set

$$m_0 = ⨍_\Omega u_0 + \frac{1}{|\Omega|}\int_\Gamma z_0$$

where $⨍_\Omega u_0 = \dfrac{1}{|\Omega|}\displaystyle\int_\Omega u_0$.

Theorem 2. *Let $u_0 \in L^\infty(\Omega)$, $z_0 \in L^\infty(\Gamma)$ satisfying (1) and let (u, z) be the solution of $E(u_0, z_0)$. Then, there exists a unique $c \in {\phi_{b\rho}}^{-1}(m_0)$, such that*

$$z(t) \to \rho(c) \quad \text{in } L^1(\Gamma), \text{ as } t \to \infty,$$

and there exists a unique $\underline{u} \in L^1(\Omega)$, such that $\underline{u}(x) \in b^{-1}(c)$ a.e. $x \in \Omega$, $⨍_\Omega \underline{u} = m_0 - \rho(c)|\Gamma|/|\Omega|$ and

$$u(t) \to \underline{u} \quad \text{in } L^1(\Omega), \text{ as } t \to \infty.$$

Corollary 1. *If $m_0 \notin \mathcal{E}$, then ${\phi_{b\rho}}^{-1}(m_0)$ is single valued,*

$$z(t) \to \rho\Big({\phi_{b\rho}}^{-1}(m_0)\Big) \quad \text{in } L^1(\Gamma)$$

and

$$u(t) \to b_0^{-1}\Big({\phi_{b\rho}}^{-1}(m_0)\Big) \quad \text{in } L^1(\Omega),$$

as $t \to \infty$.

In particular, if b is strictly increasing in a neighborhood of m_0, then $m_0 \notin \mathcal{E}$. The corollary gives the true value of the limit of the solution (u, z) as $t \to \infty$. But, in general we do not know exactly this value among the elements of the set $\mathcal{K}$, given by

$$\mathcal{K}(u_0, z_0) = \Big\{(\underline{u}, \rho(c)) \in L^1(\Omega) \times \mathbb{R}\ ;\ c \in {\phi_{b\rho}}^{-1}(m_0),$$
$$⨍_\Omega \underline{u} = m_0 - \rho(c)|\Gamma|/|\Omega| \quad \text{and } \underline{u}(x) \in b^{-1}(c)\Big\}.$$

In the next Theorem we give a description of this limit.

Theorem 3. *Let $u_0 \in L^\infty(\Omega)$, $z_0 \in L^\infty(\Gamma)$ satisfying (1), (u, z) the solution of $E(u_0, z_0)$ and consider $(\underline{u}, \rho(c)) \in \mathcal{K}(u_0, z_0)$ given by Theorem 2, such that $(\underline{u}, \rho(c)) = \lim_{t\to\infty} (u(t), z(t))$ in $L^1(\Omega) \times L^1(\Gamma)$. Setting $[l, L] = b^{-1}(c)$, we have*

$$l \le \underline{u} \le L \text{ a.e. in } \Omega, \tag{5}$$

and there exists $\underline{w} \in H^2(\Omega)$, such that

$$\begin{cases} \underline{u} = u_0 + \Delta \underline{w} \text{ a.e. in } \Omega \\ \\ \partial_\eta \underline{w} = z_0 - \rho(c) \text{ a.e. on } \Gamma \end{cases} \tag{6}$$

and, moreover,

$$\underline{w} = 0 \text{ a.e. in } \big\{\, x \in \Omega\ ;\ l < \underline{u}(x) < L \big\}. \tag{7}$$

Corollary 2. *Under the assumptions of Theorem 3, there exist disjoint subsets of Ω $A \subseteq [l \leq u_0 \leq L]$, A_1 and A_2 such that*

$$\underline{u} = u_0.\chi_A + l.\chi_{A_1} + L.\chi_{A_2}. \tag{8}$$

3. Preliminaries, existence and uniqueness

3.1. Preliminaries

As said in the introduction, we will treat $E(u_0, z_0)$ in the context of nonlinear semigroup theory in Banach spaces with a norm $\|.\|$. We refer the reader to [5], [10] and [15] for background materials on this theory. Nevertheless, we give a brief collection of materials that we need. Let X be a real Banach space. A mapping A from X into 2^X, the collection of all subsets of X, will be called an operator on X. The domain of A is denoted by $\mathcal{D}(A)$ and its range $\mathcal{R}(A)$. An operator A in X is *accretive* if

$$\|x - \hat{x}\| \leq \|x - \hat{x} + \lambda(y - \hat{y})\|, \quad \text{for } \lambda > 0,\ y \in Ax \text{ and } \hat{y} \in A\hat{x}. \tag{1}$$

From (1), it follows that for every $\lambda > 0$ the problem $x + \lambda Ax \ni z$ has at most one solution $x \in \mathcal{D}(A)$ for a given $z \in X$. Thus, we may define $\mathcal{J}_\lambda$, the resolvent of A, for each $\lambda > 0$ by $\mathcal{J}_\lambda = (I + \lambda A)^{-1}$ and $\mathcal{D}(\mathcal{J}_\lambda) = \mathcal{R}(I + \lambda A)$. From (1), it follows that $\mathcal{J}_\lambda$ is a nonexpansive mapping, i.e.,

$$\|\mathcal{J}_\lambda x - \mathcal{J}_\lambda \hat{x}\| \leq \|x - \hat{x}\| \quad \text{for } x,\ \hat{x} \in \mathcal{D}(\mathcal{J}_\lambda).$$

Let A be an accretive operator on X and consider the initial value problem

$$\begin{cases} u' + Au \ni 0 \quad \text{in } (0, T), \\ \\ u(0) = u_0 \ . \end{cases} \tag{2}$$

Discretizing the derivative in (2) and using an implicit difference scheme, we obtain for any partition $0 = t_0 < t_1 < ... < t_{n-1} < T \leq t_n$ a system of difference relations

$$\frac{u_i - u_{i-1}}{\varepsilon_{i-1}} + Au_i \ni 0,\ i = 1, 2, ..., n \tag{3}$$

where $\varepsilon_{i-1} = t_i - t_{i-1}$. Using the resolvent of A, the values u_i, in (3) are determined successively by

$$u_i = \mathcal{J}_{\varepsilon_{i-1}} u_{i-1},\ \ i = 1, 2, ..., n$$

and therefore (3) has a solution if and only if $u_i \in \mathcal{R}(I + \lambda A)$. The step function $v : [0, T] \to X$ defined by $v(0) = u_0$ and $v(t) = u_i$ for $t_{i-1} < t \leq t_i$

is considered to be an approximate solution of (2) and converge to a unique continuous function u on $[0, T]$. This function u is called the *mild-solution* of (2) on $[0, T]$. More concretely we have

Theorem. *Let A be an accretive operator in X such that $\mathcal{R}(I + \lambda A) \supseteq \mathcal{D}(A)$. Then, for any $u_0 \in \overline{\mathcal{D}}(A)$*

$$e^{-tA}u_0 = \lim_{n\to\infty} \mathcal{J}^n_{t/n}u_0 \tag{4}$$

on compact subsets of $[0, \infty[$. *Moreover, the family of operators e^{-tA}, $t > 0$, is a continuous semigroup of nonexpansive self-mappings of $\overline{\mathcal{D}(A)}$.*

Many partial differential equations that can be studied by means of nonlinear semigroup theory satisfy a "comparison principle". This fact is equivalent to the order preserving property of the semigroup $(e^{-tA}u_0)_{t\geq 0}$. The operators which generate order-preserving semigroups are the following : Let X be a Banach lattice and let A be an operator in X. A is called *T-accretive* if, for $\lambda > 0$,

$$\left\|\Big(x - \hat{x}\Big)^+\right\| \leq \left\|\Big(x - \hat{x} + \lambda(y - \hat{y})\Big)^+\right\| \text{ for } \ y \in Ax \text{ and } \hat{y} \in A\hat{x}.$$

It is clear that A is T-accretive if, and only if, its resolvents are T-contractions, i.e.,

$$\left\|(\mathcal{J}_\lambda x - \mathcal{J}_\lambda \hat{x})^+\right\| \leq \left\|(x - \hat{x})^+\right\| \quad \text{for } x,\ \hat{x} \in \mathcal{D}(\mathcal{J}_\lambda).$$

Now, since every T-contraction is order-preserving, we have that if A is T-accretive then each e^{-tA} is order-preserving. In general, T-accretivity does not implies accretivity, but in some Banach spaces T-accretivity implies accretivity, this is the case for the spaces $L^p(\Omega)$ for $1 \leq p \leq \infty$ (see for instance [4]).

3.2. Existence and uniqueness

Now, let us come back to $E(u_0, z_0)$ and consider its associate elliptic problem

$$S_\lambda(f, g, b, \rho) \qquad \begin{cases} v - \lambda\Delta w = f,\ w \in b(v) & \text{in } \Omega \\ \\ z + \lambda\partial_\eta w = g,\ z = \rho(w) & \text{on } \Gamma, \end{cases}$$

with $\lambda > 0$.

Definition 1. *For $f \in L^1(\Omega)$ and $g \in L^1(\Gamma)$, we say that (v, w, z) is a solution of $S_\lambda(f, g, b, \rho)$ if*

$$\begin{cases} v \in L^1(\Omega),\ w \in W^{1,1}(\Omega),\ w \in b(v) \text{ a.e. in } \Omega, \\ z \in L^1(\Gamma),\ z = \rho(w) \text{ a.e. on } \Gamma \text{ and} \\ \lambda \int_\Omega Dw.D\xi + \int_\Gamma z\xi = \int_\Omega (f - v)\xi + \int_\Gamma g\xi, \\ \forall\ \xi \in W^{1,\infty}(\Omega). \end{cases} \tag{5}$$

Proposition 1. *(cf. [6]) For any $f_1,\ f_2 \in L^1(\Omega)$ and $g_1,\ g_2 \in L^1(\Gamma)$, if (v_i, w_i, z_i) is a solution of $S_\lambda(f_i, g_i, b, \rho)$ for $i = 1,\ 2$, then*

$$\int_\Omega (v_1 - v_2)^+ + \int_\Gamma (z_1 - z_2)^+ \leq \int_\Omega (f_1 - f_2)^+ + \int_\Gamma (g_1 - g_2)^+$$

and

$$\int_\Omega |v_1 - v_2| + \int_\Gamma |z_1 - z_2| \leq \int_\Omega |f_1 - f_2| + \int_\Gamma |g_1 - g_2|\,.$$

Corollary 3. *For any $f \in L^1(\Omega)$ and $g \in L^1(\Gamma)$, $S_\lambda(f, g, b, \rho)$ has at most one solution.*

The existence of a solution of $S_\lambda(f, g, b, \rho)$ is well known by now in the case where b is a continuous increasing (strictly) function in $\mathbb{R}$ (cf. [8]) and also in the case where $g \equiv 0$ (cf. [6]). Next, we extend slightly part of results of [6] to the case $g \not\equiv 0$, that will be useful for the study of $E(u_0, z_0)$. We begin by giving a priori L^∞ estimates of solutions of $S_\lambda(f, g, b, \rho)$.

Proposition 2. *If $f \in L^\infty(\Omega)$, $g \in L^\infty(\Gamma)$ satisfies*

$$g(x) \in Im(\rho) \quad \text{a.e. } x \in \Gamma \tag{6}$$

and (v, w, z) is a solution of $S_\lambda(f, g, b, \rho)$, then $(u, w, z) \in L^\infty(\Omega) \times H^2(\Omega) \times L^\infty(\Gamma)$ and we have

$$\begin{aligned} \|v\|_{L^\infty(\Omega)} &\leq \max\Big(\|f\|_{L^\infty(\Omega)}, b_0^{-1} \circ \rho_0^{-1}(\|g\|_{L^\infty(\Gamma)}), \tilde{b}_0^{-1} \circ \tilde{\rho}_0^{-1}(\|g\|_{L^\infty(\Gamma)})\Big) \\ &=: M_1(f, g), \end{aligned} \tag{7}$$

$$\begin{aligned} \|z\|_{L^\infty(\Gamma)} &\leq \rho\ \max\Big(b(M_1(f, g)) \cup (-b_0(-M_1(f, g)))\Big) \\ &=: M_2(f, g), \end{aligned} \tag{8}$$

$$\begin{aligned}\|w\|_{L^\infty(\Omega)} \le \max\Big[b\Big(M_1(f,g)\Big) \cup \Big(-\tilde b(-M_1(f,g))\Big)\Big] \\ =: M_3(f,g),\end{aligned} \tag{9}$$

and

$$\|w\|_{H^1(\Omega)} \le C\Big(\|f\|_{L^1(\Omega)} + \|g\|_{L^1(\Gamma)}\Big) \tag{10}$$

where $\tilde b(r) = -b(-r)$ *and* $\tilde\rho(r) = -\rho(-r)$, *for any* $r \in \mathbb{R}$, *and* C *is a constant depending on* Ω, $M_1(f,g)$ *and* $M_2(f,g)$.

Now, we set our existence result for $S_\lambda(f,g,b,\rho)$, under the Assumption (6), sufficient for the study of $E(u_0,z_0)$.

Proposition 3. *For any* $f \in L^1(\Omega)$ *and* $g \in L^1(\Gamma)$ *satisfying (6), there exists a unique* (v,w,z) *solution of* $S_\lambda(f,g,b,\rho)$.

Remark 1. Notice that Condition (6) is not necessary for the existence of a solution of $S_\lambda(f,g,b,\rho)$ (see for instance Remark 2.12 of [4]). However, without this condition we do not know if L^∞ estimates of type (7), (8) and (9) remain true.

As a consequence of Proposition 1 and Proposition 2, one ses that the natural space where we can study $E(u_0,g)$ is $X = L^1(\Omega) \times L^1(\Gamma)$ provided with the natural norm

$$\|(f,g)\| = \|f\|_{L^1(\Omega)} + \|g\|_{L^1(\Gamma)}, \text{ for } (f,g) \in X.$$

In X, we define the operator (possibly multivalued) A by

$$\begin{aligned}A(v,z) = \Big\{ (f,g) \in X;\ \exists\, w \in W^{1,1}(\Omega), \\ (v,w,z) \quad \text{is a solution of } S_1(f+v, g+z, b, \rho)\Big\}\end{aligned}$$

and consider the evolution problem

$$\begin{cases} U_t + AU \ni 0 \quad \text{in } (0,\infty) \\ \\ U(0) = U_0\ . \end{cases} \tag{11}$$

As an immediate consequence of Proposition 1 and Proposition 2, we have

Corollary 4. *1.* A *is T-accretive in* X.
2. $R(I+\lambda A) \supseteq \Big\{(f,g) \in X\ ;\ g(x) \in Im(\rho)$ *a.e.* $x \in \Gamma\Big\}$.

Using the general theory of nonlinear semigroups, A generates a continuous nonlinear semigroup of contraction operators $S(t)$ in X. Moreover, we have

Proposition 4.

$$\overline{\mathcal{D}(A)} = \left\{(u,z) \in X \ ; \ z(x) \in \overline{Im(\rho)} \ a.e.\ x \in \Gamma\right\}$$

$$=: D_A.$$

So, for any $U_0 \in D_A$, $S(t)U_0$ is the unique mild solution of (11). By definition of $S(t)$,

$$S(t)U_0 = \lim_{\varepsilon\to 0} U_\varepsilon(t) \quad \text{in } X$$

uniformly for $t \in [0,\tau]$, where for $\varepsilon > 0$, U_ε is an $\varepsilon-$approximate solution corresponding to a subdivision $t_0 = 0 < t_1 < \ldots < t_{n-1} < \tau \leq t_n$, with $t_i - t_{i-1} = \varepsilon$ and defined by $U_\varepsilon(0) = U_0$, $U_\varepsilon(t) = U_i$ for $t \in]t_{i-1}, t_i]$ where $U_i \in X$ satisfies

$$\frac{U_i - U_{i-1}}{\varepsilon} + AU_i \ni 0.$$

Proposition 5. *If* $(u_0, z_0) \in D_A \cap L^\infty(\Omega) \times L^\infty(\Gamma)$, *then the curve* $(u(t), z(t)) := S(t)(u_0, z_0)$ *satisfies*

$$\begin{cases} u \in \mathcal{C}([0,\infty); L^1(\Omega)) \cap L^\infty(Q), \ z \in \mathcal{C}([0,\infty); L^1(\Gamma)) \cap L^\infty(\Sigma), \\ \exists\ w \in L^2_{loc}\left([0,\infty); H^1(\Omega)\right) \cap L^\infty(Q), \ w \in b(u) \ a.e.\ in\ Q, \\ z = \rho(w) \quad a.e.\ on \quad \Sigma \quad and \\ \displaystyle\int_0^\tau\int_\Omega \xi_t u + \int_\Omega \xi(0)u_0 + \int_0^\tau\int_\Gamma z\xi_t + \int_\Gamma z_0\xi(0) \\ \qquad\qquad \displaystyle = \int_0^\tau\int_\Omega Dw.D\xi + \int_\Omega \xi(\tau)u(\tau) + \int_\Gamma \xi(\tau)z(\tau), \\ \forall \quad \xi \in \mathcal{C}^1([0,\tau] \times \overline{\Omega}) \ with\ \tau > 0. \end{cases} \tag{12}$$

Moreover, for any $\tau \geq 0$,

$$\|u(\tau)\|_{L^\infty(\Omega)} \leq M_1(u_0, z_0), \tag{13}$$

$$\|z(\tau)\|_{L^\infty(\Gamma)} \leq M_2(u_0, z_0), \tag{14}$$

$$\|w\|_{L^\infty(Q)} \leq M_3(u_0, z_0), \tag{15}$$

$$\int_\Omega u(\tau) + \int_\Gamma z(\tau) = \int_\Omega u_0 + \int_\Gamma z_0 \tag{16}$$

and

$$\int_\Omega j(u(\tau)) + \int_\Gamma \psi(z(\tau)) + \int_0^\tau \int_\Omega |Dw|^2 \leq \int_\Omega j(u_0) + \int_\Gamma \psi(z_0) \tag{17}$$

where $j : \mathbb{R} \to [0, \infty]$ *is a proper convex l. s. c. function such that* $b = \partial j$ *and* $\psi(r) = \int_0^r \rho_0^{-1}(s)ds$, *for any* $r \in \mathbb{R}$.

Proof. By definition of mild solution $u \in \mathcal{C}([0, \infty); L^1(\Omega))$ and $z \in \mathcal{C}([0, \infty); L^1(\Gamma))$. Let $(u_\varepsilon, z_\varepsilon)$ be the $\varepsilon-$approximate solution with $\varepsilon = \tau/n$ and for $i = 1, \ ...., n$, let $w_i \in H^2(\Omega)$ such that

$$\begin{cases} u_i - \varepsilon \Delta w_i = u_{i-1}, \ w_i \in b(u_i) & \text{in } \Omega, \\ \\ z_i + \varepsilon \partial_\eta w_i = z_{i-1}, \ z_i = \rho(w_i) & \text{on } \Gamma. \end{cases} \tag{18}$$

Thanks to Propositions 1 and 2, it follows that $u_i \in L^\infty(\Omega)$, $z_i \in L^\infty(\Gamma)$, $\|u_i\|_{L^\infty(\Omega)} \leq M_1(u_0, z_0)$, $\|z_i\|_{L^\infty(\Gamma)} \leq M_2(u_0, z_0)$ and $\int_\Omega u_i + \int_\Gamma z_i = \int_\Omega u_0 + \int_\Gamma z_0$, so that

$$\|u_\varepsilon(\tau)\|_{L^\infty(\Omega)} \leq M_1(u_0, z_0), \tag{19}$$

$$\|z_\varepsilon(\tau)\|_{L^\infty(\Gamma)} \leq M_2(u_0, z_0)$$

and

$$\int_\Omega u_\varepsilon(\tau) + \int_\Gamma z_\varepsilon(\tau) = \int_\Omega u_0 + \int_\Gamma z_0.$$

Then, letting $\varepsilon \to 0$, and using the fact that $u_\varepsilon(t) \to u(t)$ in $L^1(\Omega)$ and $z_\varepsilon(t) \to z(t)$ in $L^1(\Gamma)$, for any $t \in [0, \tau)$, we deduce that u and z satisfy (13), (14) and (16). To prove that there exists w such that (u, w, z) satisfies (12), (15) and (17), we take w_i as a test function in (18). Using the fact that

$$\int_\Omega (u_{i-1} - u_i) w_i \leq \int_\Omega j(u_{i-1}) - \int_\Omega j(u_i)$$

and

$$\int_\Gamma (z_{i-1} - z_i) w_i \leq \int_\Gamma \psi(z_{i-1}) - \int_\Gamma \psi(z_i),$$

we conclude that

$$\int_\Omega j(u_i) + \varepsilon \int_\Omega |Dw_i|^2 + \int_\Gamma \psi(z_i) \leq \int_\Omega j(u_{i-1}) + \int_\Gamma \psi(z_{i-1}). \tag{20}$$

Adding (20) from $i = 1$ to n, we get

$$\int_\Omega j(u_\varepsilon(\tau)) + \int_0^\tau \int_\Omega |Dw_\varepsilon|^2 + \int_\Gamma \psi(z_\varepsilon(\tau)) \leq \int_\Omega j(u_0) + \int_\Gamma \psi(z_0) \tag{21}$$

where $w_\varepsilon : [0, \tau] \to H^1(\Omega)$ and $w_\varepsilon(t) = w_i$, for any $t \in]t_{i-1}, t_i]$, $i = 1, ...n$. Thanks to (H_1), $M_3(u_0, z_0) < \infty$, and (19) implies that

$$|w_\varepsilon| \leq M_3(u_0, z_0). \tag{22}$$

Since $j \geq 0$ and $\psi \geq 0$, we deduce from (21) and (22), that w_ε is bounded in $L^2(0, \tau; H^1(\Omega))$. There are a subsequence $\{\varepsilon_k\}$ and $w \in L^2(0, \tau; H^1(\Omega))$ such that $w_{\varepsilon_k} \to w$ weakly in $L^2(0, \tau; H^1(\Omega))$ and $w_{\varepsilon_k/\Gamma} \to w_{/\Gamma}$ weakly in $L^2(0, \tau; L^2(\Gamma))$. Clearly, $u_{\varepsilon_k} \to u$ in $L^2(0, \tau; L^2(\Omega))$ and $z_{\varepsilon_k} \to z$ $L^2(0, \tau;$ $L^2(\Gamma))$. Since b (resp. ρ) is a maximal monotone graph in $L^2(0, \tau; L^2(\Omega))$ (resp. $L^2(0, \tau; L^2(\Gamma))$), we obtain $w(t) \in b(u(t))$ a.e. in Ω (resp. $z(t) = \rho(w(t))$ a.e. on Γ), for any $t \in (0, \tau)$.

Finally, let $\tilde{u}_\varepsilon$ and $\tilde{z}_\varepsilon$ be the functions from $[0, \tau]$ into $L^1(\Omega)$, defined by $\tilde{u}_\varepsilon(t_i) = u_i$, $\tilde{z}_\varepsilon(t_i) = z_i$ and $\tilde{u}_\varepsilon$, $\tilde{z}_\varepsilon$ linear in $[t_{i-1}, t_i]$, then (18) implies that

$$\begin{aligned} &\int_0^\tau \int_\Omega \tilde{u}_\varepsilon \xi_t + \int_0^\tau \int_\Gamma \tilde{z}_\varepsilon \xi_t + \int_\Omega \xi(0) u_0 + \int_\Gamma z_0 \xi(0) \\ &\quad = \int_0^\tau \int_\Omega Dw_\varepsilon . D\xi + \int_\Gamma z_\varepsilon(\tau)\xi(\tau) + \int_\Omega \xi(\tau) u_\varepsilon(\tau) \end{aligned} \tag{23}$$

for any $\xi \in \mathcal{C}^1([0, \tau] \times \overline{\Omega})$. Letting $\varepsilon \to 0$ in (21), (22) and (23), we get (17), (15) and (12). □

Proof of Proposition 2. Using Proposition 1, we see that for any $a > 0$ and $c \in b(a)$, we have

$$\int_\Omega (u - a)^+ + \int_\Gamma (z - \rho(c))^+ \leq \int_\Omega (f - a)^+ + \int_\Gamma (g - \rho(c))^+. \tag{24}$$

It is clear that $(f - M_1)^+ = 0$ a.e. in Ω. On the other hand, we observe that either there exists $c \in b(M_1)$ such that $\|g\|_{L^\infty(\Gamma)} = \rho(c)$, or for any $c \in b(M_1)$ we have $\|g\|_{L^\infty(\Gamma)} \leq \rho(c)$. This implies that there exists $c \in b(M_1)$ such that $(g - \rho(c))^+ = 0$ a.e. on Γ. Then, taking $a = M_1$ in (24), we conclude that

$u \le M_1$ a.e. in Ω and $z \le M_2$ a.e. on Γ. Using the fact that $(-u, -w, -z)$ is a solution of $S_\lambda(-f, -g, \tilde{b}, \tilde{\rho})$, one can prove in the same way that $-u \le M_1$ a.e. in Ω and $-z \le M_2$. This ends the proof of (7) and (8). On the other hand, we see that (H_1) implies that $M_3(f, g) < \infty$ and, since $w \in b(u)$, then (7) implies (9). □

Lemma 1. *Let $f \in L^\infty(\Omega)$, $g \in L^\infty(\Gamma)$ satisfying (6), $\lambda > 0$ and let $(v, z) = \mathcal{J}_\lambda(f, g)$. For any $y \in \mathbb{R}^N$ and $\xi \in \mathcal{C}^2(\Omega)$ supported in $\{x \in \Omega\ ;\ distance\ (x, \Gamma) > |y|\ ,\}$ we have*

$$\int_\Omega \xi(x)\,|v(x+y) - v(x)|\,dx \le C\,|y|\,\|\Delta\xi\|_{L^\infty(\Omega)} \Big(\|f\|_{L^\infty(\Omega)} + \|g\|_{L^\infty(\Gamma)}\Big) + \int_\Omega \xi(x)\,|f(x+y) - f(x)|\,dx$$

where C is a constant depending only on Ω.

Proof. The proof follows exactly in the same way of Lemma 1 of [18]. Indeed, let $w \in H^1(\Omega)$, such that (v, w, z) is the solution of $S_\lambda(f, g, b, \rho)$. Using the results of [6] (cf. Step 3 of the proof of Theorem B'), for any $y \in \mathbb{R}^N$ and $\xi \in \mathcal{C}^2(\Omega)$ supported in $\{x \in \Omega\ ;\ \text{distance}(x, \Gamma) > |y|\}$, we have

$$\begin{aligned}\int_\Omega \xi(x)\,|v(x+y) - v(x)|\,dx &\le \lambda \int_\Omega |\Delta\xi|\,|w(x+y) - w(x)|\,dx \\ &\quad + \int_\Omega \xi(x)\,|f(x+y) - f(x)|\,dx \\ &\le \lambda\,|y|\,\|\Delta\xi\|_{L^\infty(\Omega)}\,|\Omega|^{\frac{1}{2}}\,\|Dw\|_{L^2(\Omega)} \\ &\quad + \int_\Omega \xi(x)\,|f(x+y) - f(x)|\,dx\end{aligned}$$

then, using (10), the result follows. □

Proof of Proposition 3. We begin by proving existence for $f \in L^\infty(\Omega)$ and $g \in L^\infty(\Gamma)$. For this, we consider b_n a sequence of continuous and increasing functions in $\mathbb{R}$ such that

$$b_n \to b \quad \text{in the sense of graph,}$$

i.e. $(I + b_n)^{-1} r \to (I + b)^{-1} r$, for any $r \in \mathbb{R}$. Using Corollary 21 of [8] and Proposition 2, for any $f \in L^\infty(\Omega)$ and $g \in L^\infty(\Gamma)$ there exists a unique $(u_n, w_n, z_n) \in L^\infty(\Omega) \times H^2(\Omega) \times L^\infty(\Gamma)$ solution of $S(f, g, b_n, \rho)$. Using (8), (9) and (10), it is not difficult to see that $\{u_n\}$ and $\{w_n\}$ (resp. $\{z_n\}$) are bounded in $L^\infty(\Omega)$ (resp. in $L^\infty(\Gamma)$). Thus, Lemma 1 implies that $\{u_n\}$ is relatively compact in $L^1(\Omega)$, and (10) implies that $\{w_n\}$ is bounded in $H^1(\Omega)$. Consider a subsequence that we denote again by n, such that w_n converges in $L^2(\Omega)$ and in $H^1(\Omega)-$weak, and $w_{n/\Gamma}$ converges in $L^2(\Gamma)$. Since ρ is continuous and the sequence $(z_n = \rho(w_{n/\Gamma}))$ is bounded in $L^\infty(\Gamma)$, then we deduce that (z_n) is also convergent in $L^1(\Gamma)$. Passing to the limit in the equation satisfied by (u_n, w_n, z_n), and using standard monotonicity and compactness arguments, we deduce that a limit of (u_n, w_n, z_n) is a solution of $S_\lambda(f, g, b, \rho)$.

For $f \in L^1(\Omega)$ and $g \in L^1(\Gamma)$, we consider $f_n \in L^\infty(\Omega)$ and $g_n \in L^\infty(\Gamma)$, such that as $n \to \infty$, $\|f_n\|_{L^1(\Omega)} \le \|f\|_{L^1(\Omega)}$, $\|g_n\|_{L^1(\Gamma)} \le \|g\|_{L^1(\Gamma)}$,

$$f_n \to f \text{ in } L^1(\Omega) \quad \text{and} \quad g_n \to g \text{ in } L^1(\Gamma)$$

and consider (u_n, w_n, z_n) the solution of $S_\lambda(f_n, g_n, b, \rho)$. Using Proposition 1, we have

$$\|u_n - u_m\|_{L^1(\Omega)} + \|z_n - z_m\|_{L^1(\Gamma)} \le \|f_n - f_m\|_{L^1(\Omega)} + \|g_n - g_m\|_{L^1(\Gamma)},$$

which implies that there exist $u \in L^1(\Omega)$ and $z \in L^1(\Gamma)$ such that $u_n \to u$ in $L^1(\Omega)$ and $z_n \to z$ in $L^1(\Gamma)$, as $n \to \infty$. To prove that there exists $w \in W^{1,1}(\Omega)$, such that (u, w, z) is a solution of $S_\lambda(f, g, b, \rho)$, it is enough to prove that (w_n) is bounded in $W^{1,1}(\Omega)$ and conclude by passing to the limit in the equation satisfied by (u_n, w_n, z_n), exactly in the same way of the first part of the proof. Using Proposition C of [6], we have

$$\|w_n - \fint w_n\|_{W^{1,q}(\Omega)} \le C\Big(\|f\|_{L^1(\Omega)} + \|g\|_{L^1(\Gamma)}\Big) \tag{25}$$

for $1 \le q < \frac{N-1}{N}$. On the other hand, following the same idea of [6], we see that $\fint w_n$ is bounded. Indeed if $\fint w_n \to \infty$ (resp. $\fint w_n \to -\infty$), then using (25) we will have $w_n(x) \to \infty$ (resp. $w_n(x) \to -\infty$) a.e. $x \in \Omega$, and since $w_n \in b(u_n)$, this contradicts the fact that (u_n) is convergent (through a subsequence) a.e. in Ω. This ends the proof of the Proposition. $\square$

Proof of Proposition 4. By definition of A, we see easily that $\overline{\mathcal{D}(A)} \subseteq D_A$. So, it is enough to prove that

$$\overline{\mathcal{D}(A)} \supseteq \Big\{(u, z) \in L^\infty(\Omega) \times L^\infty(\Gamma)\ ;\ z(x) \in \operatorname{Im}(\rho) \text{ a.e. } x \in \Gamma \Big\} =: K.$$

Let $(u, z) \in K$ and consider $(u_\varepsilon, w_\varepsilon, z_\varepsilon)$ the solution of $S_\varepsilon(u, z, b, \rho)$. By definition of A, it is clear that $(u_\varepsilon, z_\varepsilon) \in \mathcal{D}(A)$. Our aim now is to prove that $u_\varepsilon \to u$ in $L^1(\Omega)$ and $z_\varepsilon \to z$ in $L^1(\Gamma)$, as $\varepsilon \to 0$, which ends the proof of (3.2). Using Proposition 2 and Lemma 1 in the same way as for the proof of Proposition 3, we conclude that $\{u_\varepsilon\}$ (resp. $\{z_\varepsilon\}$) is relatively compact in $L^1(\Omega)$ (resp. in $L^1(\Gamma)$). Moreover, $\{w_\varepsilon\}$ is bounded in $L^\infty(\Omega)$, then $\varepsilon w_\varepsilon \to 0$ in $L^\infty(\Omega)$ and in $H^1(\Omega)-$weak. So, passing to the limit, as $\varepsilon \to 0$, in the integral equality satisfied by $(u_\varepsilon, w_\varepsilon, z_\varepsilon)$, we deduce that $u_\varepsilon \to u$ in $L^1(\Omega)$ and $z_\varepsilon \to z$ in $L^1(\Gamma)$. $\square$

Proof of Theorem 1. Obviously, the existence of a solution (u, z) in the sense of (2) follows by Proposition 5. Moreover, since $(u(t), z(t)) = S(t)U_0$, then $u \in \mathcal{C}([0,\infty), L^1(\Omega))$, $z \in \mathcal{C}([0,\infty), L^1(\Gamma))$, (3) and (4) are fulfilled. To prove uniqueness, let (u_i, z_i), for $i = 1,\ 2$, be two solutions of $E(u_0, z_0)$ and let w_i be such that (u_i, w_i, z_i) satisfies (2). Setting $U = u_1 - u_2$, $Z = z_1 - z_2$ and $W = w_1 - w_2$, we have

$$\int_0^T \int_\Omega U\xi_t + \int_0^T \int_\Gamma Z\xi_t = \int_0^T \int_\Omega \nabla W \cdot \nabla \xi \tag{26}$$

for any $\xi \in \mathcal{C}^1([0,T] \times \overline{\Omega})$ such that $\xi(T) \equiv 0$. By density, for an arbitrary $\tau > 0$, we can take ξ as follows

$$\xi(t) = \begin{cases} -\displaystyle\int_t^\tau W(s)ds & \text{if } t \le \tau \\ 0 & \text{if } t > \tau \end{cases}$$

as a test function in (26). Then

$$\begin{aligned} \int_0^\tau \int_\Omega UW + \int_0^\tau \int_\Gamma ZW &= -\int_0^\tau \int_\Omega \nabla W \cdot \nabla \left(\int_t^\tau W(s)\right) \\ &= \frac{1}{2}\int_0^\tau \int_\Omega \frac{\partial}{\partial t}\left|\nabla \int_t^\tau W(s)ds\right|^2 \\ &= -\frac{1}{2}\int_\Omega \left|\nabla \int_0^\tau W(s)ds\right|^2. \end{aligned} \tag{27}$$

Since $UW \ge 0$ a.e. $(0,T) \times \Omega$ and $ZW \ge 0$ a.e. on $(0,T) \times \Gamma$, then (27) implies that $\left|\nabla \int_0^\tau W(s)ds\right| \equiv 0$ in Ω for each $\tau > 0$, so that we deduce that $W(t)$ is a constante function in Ω for each $t > 0$. Then using (26) we get $U \equiv 0$ in $(0,T) \times \Omega$ and using again (27) and the fact that $ZW \ge 0$ a.e. on $(0,T) \times \Gamma$, we deduce that $ZW \equiv 0$ on $(0,T) \times \Gamma$ which implies that $Z \equiv 0$. This ends the proof of uniqueness. □

4. Stabilization results

Using Proposition 4, Theorem 2 is a particular case of the following result.

Theorem 4. *For any $U_0 \in D_A$, there exists a unique $\underline{U}_0 \in \mathcal{K}(U_0)$, such that*

$$S(t)U_0 \to \underline{U}_0 \quad \text{in } X,\ \text{as } t \to \infty.$$

In order to prove stabilization result, we need to know the orbits of the semigroup $S(t)$, i.e. $\{S(t)U_0\ ;\ t \geq 0\}$, are relatively compact in X. Among the results of [18], it is proved that this is true if $\rho \equiv 0$ (see also [19], [2,21]). The next Proposition is a generalization of these results.

Proposition 6. *For any $U_0 \in D_A$, $S(t)U_0$ is relatively compact in X.*

Proof. First, using Lemma 1 and (7), we see that for any $\lambda > 0$ fixed and B a bounded subset of $D_A \cap \Big(L^\infty(\Omega) \times L^\infty(\Gamma)\Big)$, $\mathcal{J}_\lambda B$ is a relatively compact subset of X. Indeed, for any $\{(f_n, g_n)\} \subseteq B$, if $(v_n, z_n) = \mathcal{J}_\lambda(f_n, g_n)$, then with an appropriate choice of ξ, we have

$$\lim_{|y|\to 0} \sup_n \int_{\Omega'} |v_n(x+y) - v_n(x)| = 0$$

for any $\Omega' \subset\subset \Omega$, which implies, with (7), that $\{v_n\}$ is relatively compact in $L^1(\Omega)$. On the other hand, since $z_n = \rho(w_n)$ where $w_n \in H^1(\Omega)$ is such that (v_n, w_n, z_n) is the solution of $S_\lambda(f_n, g_n, b, \rho)$, then using (8), (9), (10) and the continuity of ρ, we deduce that $\{z_n\}$ is relatively compact in $L^1(\Gamma)$. At last, the proof of the relative compactness of $S(t)U_0$, in X follows exactly in the same way as in the proof of Theorem 2.2 in [21] (see also [12], Theorem 3). In fact, one proves, firstly, that $\Big\{S(t)U_0\Big\}$ is relatively compact for any $U_0 \in D_A \cap \Big(L^\infty(\Omega) \times L^\infty(\Gamma)\Big)$ by using the inequality

$$\|S(t)U_0 - \mathcal{J}_\lambda S(t)U_0\| \leq \lambda \inf \Big\{\|U\|\ ;\ U \in AU_0\Big\}.$$

Then, for $U_0 \in D_A$, the compactness of a subsequence of $S(t)U_0$ follows by approximating U_0 and the fact that

$$\sup_{t\geq 0} \inf_{s\geq 0} \|S(t)U_0 - S(s)V_0\| \leq \|U_0 - V_0\|, \quad \text{for any } V_0 \in D_A. \tag{1}$$

□

Now, for any $u_0 \in L^1(\Omega)$ and $z_0 \in L^1(\Gamma)$, we define the $\omega-$limit set of $E(u_0, z_0)$ by

$$\omega(u_0, z_0) = \Big\{(\underline{u}, \underline{z}) \in X\ ;\ (\underline{u}, \underline{z}) = \lim_{t_n\to\infty} S(t_n)(u_0, z_0) \text{ in } X$$
$$\text{for some sequence } t_n \to \infty\Big\}.$$

This set is possibly empty. Now, it is well known (see [13]) that, if $S(t)U_0$ is relatively compact, then $\omega(U_0)$ is a non empty compact and connected subset of X. Furthermore $\omega(U_0)$ is invariant under $S(t)$, i.e., $S(t)\omega(U_0) \subseteq \omega(U_0)$ for any $t \geq 0$. An equilibrium or stationary solution is any $V \in X$ such that $\omega(V) = \{S(t)V\} = \{V\}$. We denote by $\mathcal{E}$ the set of equilibrium solutions. As a consequence of Proposition 6, we have the following result.

Remark 2. For any $U_0 \in D_A$, $\mathcal{K}(U_0) \subseteq \mathcal{E}$. Indeed, if $(\underline{u}, \rho(c)) \in \mathcal{K}(U_0)$, then it is not difficult to see that $(\underline{u}, c, \rho(c))$ is a solution of $S_\lambda(\underline{u}, \rho(c), b, \rho)$, for any $\lambda > 0$, so that $(I + \frac{t}{n}A)^{-n}(\underline{u}, \rho(c)) = (\underline{u}, \rho(c))$, for any $t > 0$ and $n = 1, 2, \ldots$. Then, by definition of mild solutions we deduce that $S(t)(\underline{u}, \rho(c)) = (\underline{u}, \rho(c))$.

Corollary 5. *For any* $U_0 \in D_A$, $\omega(U_0) \neq \emptyset$.

Proof of Theorem 4. Using the fact that $\mathcal{K}(U_0)$ is a closed subset of X and Inequality (1), one sees that it is sufficient to prove the Theorem for any U_0 in a dense subset of D_A. So, assume that $U_0 =: (u_0, z_0) \in D_A \cap \left(L^\infty(\Omega) \times L^\infty(\Gamma)\right)$ and let $(u(t), z(t)) = S(t)(u_0, z_0)$ and $w(t) \in H^1(\Omega)$, for each $t > 0$, such that (u, w, z) satisfies (12). Thanks to (17) and since $j \geq 0$ and $\psi \geq 0$, there exists a sequence (t_n), $t_n \to \infty$, such that

$$\lim_{t_n \to \infty} \int_\Omega |Dw(t_n)|^2 = 0 \quad , \tag{2}$$

then, using (15) and the Poincaré inequality, we deduce that $\{w(t_n)\}$ is bounded in $H^1(\Omega)$. Thanks to Proposition 6, let $(\underline{u}, \underline{z}) \in \omega(u_0, z_0)$ and (t_{nk}), $t_{nk} \to \infty$, such that $u(t_{nk}) \to \underline{u}$ in $L^1(\Omega)$, $z(t_{nk}) \to \underline{z}$ in $L^1(\Gamma)$ and, let $\underline{w} \in H^1(\Omega)$ be such that $w(t_{nk}) \to \underline{w}$ in $L^2(\Omega)$ and in $H^1(\Omega)$–weak, and $w(t_{nk})_{/\Gamma} \to \underline{w}_{/\Gamma}$ in $L^2(\Gamma)$. Then, as in the proof of Proposition 5, using standard compactness and monotonicity arguments, we obtain

$$\underline{w} \in b(\underline{u}) \text{ a.e. } \Omega \text{ and } \underline{z} = \rho(\underline{w}) \text{ a.e. } \Gamma. \tag{3}$$

Passing to the limit in (2), as $t_{nk} \to \infty$, we get

$$\int_\Omega \left|D\underline{w}\right|^2 = 0,$$

which implies that there exists $c \in \mathbb{R}$ such that

$$\underline{w} \equiv c \text{ a.e. in } \Omega. \tag{4}$$

Now, passing to the limit in (3), we obtain

$$⨍_\Omega \underline{u} + \rho(c)\frac{|\Gamma|}{|\Omega|} = m_0 \quad . \tag{5}$$

On the other hand, since $\underline{u} \in b^{-1}(c)$ and $b^{-1}(c)$ is a subinterval of $\mathbb{R}$, then $⨍_\Omega \underline{u} \in b^{-1}(c)$ and $⨍_\Omega \underline{u} + \rho(c)\frac{|\Gamma|}{|\Omega|} \in \phi_{b\rho}(c)$, which with (5) implies that $c \in {\phi_{b\rho}}^{-1}(m_0)$. From this, we deduce that $(\underline{u}, \underline{z}) \in \mathcal{K}(U_0)$, which implies by Remark 2 that $S(t)(\underline{u}, \underline{z}) = (\underline{u}, \underline{z})$. Then, the convergence of $S(t)U_0$, as $t \to \infty$, follows immediately by the contraction property of $S(t)$ in X. $\square$

For any $U_0 \in D_A$ and $r \in \mathcal{E}$, setting $(u(t), z(t)) = S(t)U_0$, we define

$$M_r(U_0) = \left\{(t,x) \in Q \; ; \; u(t,x) \in \mathrm{int}(b^{-1}(r))\right\},$$

for any $r \in \mathcal{E}$, and

$$M_r(U_0, t_0) = \left\{ x \in \Omega \; ; \; (t_0, x) \in M_r(U_0) \right\}.$$

Remark 3. In terms of the Stefan problem, $M_r(U_0, t_0)$ is called the "Mushy region", the set which separates two different phases.

Proposition 7. *For any $r \in \mathcal{E}$ and $U_0 \in D_A$,*

$$M_r(U_0, t_2) \subseteq M_r(U_0, t_1) \quad \textit{for any} \;\; t_2 > t_1, \tag{6}$$

in the sense of mes$(M_r(U_0, t_2) \setminus M_r(U_0, t_1)) = 0$.

Proof. The proof follows exactly the argument of Proposition 4 of [18]. We omit the details of the proof here to avoid to repeat unnecessarily the same arguments. □

We also recall the following Lemma that will be useful for the proof of Theorem 3.

Lemma 2. *(see for instance [18]) Let (f_n) be a sequence of $L^1(\Omega)$, $f \in L^1(\Omega)$, such that $f_n \to f$ in $L^1(\Omega)$. If $x_0 \in \Omega$ is a Lebesgue point of f such that $\theta_1 < f(x_0) < \theta_2$, for $\theta_1, \; \theta_2 \in \mathbb{R}$, then, for any $\delta > 0$,*

$$\textit{mes}\,\{x \in B(x_0, \delta) \; ; \; \theta_1 < f(x) < \theta_2\} > 0$$

and, there exists $n_0 = n_0(\theta_1, \theta_2, \delta) > 0$, such that

$$\textit{mes}\,\{x \in B(x_0, \delta) \; ; \; \theta_1 < f_n(x) < \theta_2\} > 0 \quad \textit{for any } n \geq n_0.$$

Proof of Theorem 3. The proof of this Theorem follows the ideas of Proposition 4 of [18]. First, we notice that (5) is a consequence of Theorem 4. Now, in order to prove (6) and (7), we consider $(u(t), z(t)) = S(t)(u_0, z_0)$ and $w \in L^2_{loc}([0,\infty); H^1(\Omega))$ given by Proposition 5, such that $w \in b(u)$ a.e. in Q and (u, w, z) satisfies (12). Obviously, for any $t \geq 0$, $W(t) = \displaystyle\int_0^t w(s)ds \; \in H^1(\Omega)$ and, by an appropriate choice of ξ in (12), we observe that $W(t)$ is a weak solution of

$$\begin{cases} -\Delta W(t) = u_0 - u(t) & \text{in } \Omega \\ \\ \partial_\eta W(t) = z_0 - z(t) & \text{on } \Gamma. \end{cases} \tag{7}$$

Thanks to (13) and (14), we have $W(t) \in H^2(\Omega)$ and $\tilde{W}(t) := W(t) - \fint W(t)$ is bounded in $H^1(\Omega)$. On the other hand, applying Theorem 2, we have $u(t) \to \underline{u}$ in $L^1(\Omega)$ and $z(t) \to \rho(c)$ in $L^1(\Gamma)$, as $t \to \infty$, with $c \in \phi_{b\rho}^{-1}(m_0)$ and $\underline{u} \in b^{-1}(c)$ a.e. in Ω. So, there exists $\underline{W} \in H^2(\Omega)$ and a sequence (t_k), $t_k \to \infty$, such that

$$\tilde{W}(t_k) \to \underline{W} \text{ weakly in } H^1(\Omega) \text{ and strongly in } L^2(\Omega),$$

$$\tilde{W}(t_k)_{/\Gamma} \to \underline{W}_{/\Gamma} \text{ strongly in } L^2(\Gamma),$$

and $\underline{W}$ satisfies

$$\begin{cases} -\Delta \underline{W} = u_0 - \underline{u} & \text{a.e. in } \Omega, \\ \\ \partial_\eta \underline{W} = z_0 - \rho(c) & \text{a.e. on } \Gamma. \end{cases}$$

Now, let $x_0 \in \Omega$ be a Lebesgue point of $\underline{u}$, such that $l < \underline{u}(x_0) < L$. Using Lemma 2, for any $\delta > 0$, there exists $t_0 = t_0(l, L, \delta) > 0$, such that

$$\text{mes}\,\{x \in B(x_0, \delta)\ ;\ l < u(t, x) < L\} > 0 \quad \text{for any } t \geq t_0\ ;$$

then, thanks to Proposition 7,

$$\text{mes}\,\{x \in B(x_0, \delta)\ ;\ l < u(t, x) < L\} > 0 \quad \text{for any } t \geq 0\ .$$

This implies that

$$\text{mes}\,\{x \in B(x_0, \delta)\ ;\ w(t, x) = c\} > 0 \quad \text{for any } t \geq 0$$

so that

$$\text{mes}\left\{x \in B(x_0, \delta)\ ;\ \tilde{W}(t, x) = tc - \fint W(t)\right\} > 0 \text{ for any } t \geq 0 \tag{8}$$

and $\underline{W}(x) = \lim\limits_{t_k \to \infty} \left(ct_k - \fint W(t_k))\right) =: k$ a.e. $x \in B(x_0, \delta)$, where $k \in \mathbb{R}$ does not depend on x_0. Taking $\underline{w} = \underline{W} - k$, we see that $\underline{w}$ satisfies (6) and

$$\underline{w} = 0 \quad \text{a.e. in } B(x_0, \delta)\ . \tag{9}$$

Since $\underline{w} \in \mathcal{C}(\Omega)$ by standard theory for elliptic problem and satisfies (9) for any $\delta > 0$, then $\underline{w}(x_0) = 0$, which ends the proof of the Theorem. $\square$

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