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The mesa problem for Neumann boundary value problem

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Abstract

In this paper, we study the singular limit of the Porous Medium equation $u_t = \Delta u^m + g(x,u)$, as $m \to \infty$, in a bounded domain with Neumann boundary condition.

Keywords: Singular limit; Mesa problem; Nonlinear semi group theory; Accretive operator; Mild solution; Parbaolic problem; Elliptic problem

1. Introduction

The aim of this paper is to study the effects of a lower-order nonlinearity and Neumann boundary condition on the limit of the Porous Medium equation $u_t = \Delta u^m$, when the parameter m goes to ∞. This is a particular case of an overall program of studying the so-called singular limit for nonlinear pdes, i.e., a perturbation problem where the perturbed problem is of totally different character than the unperturbed one. Recently, in light of Monge Kantorovich mass transfer theory, Evans et al. proved in [9] that the related problem of taking the limit $p \to \infty$, for the pde $u_t = \Delta_p u$ has turned out to be interesting. Our approach is different, it is based on the ideas we introduced in [6] (see also [10]) for the similar problem with Dirichlet boundary condition. However, in our case, i.e. Neumann boundary condition, the description of the limit is more delicate.

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✠ Philippe Bénilan sadly passed away last year.
[1] The main results of this work were obtained when the second author was a Ph.D. student of Bénilan in Besançon (c.f. [10]).

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Let Ω be a bounded open set in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$. For $m \geqslant 1$, we consider the problem

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u^m + g(u) & \text{on } Q = (0,T) \times \Omega, \\ \dfrac{\partial u^m}{\partial n} = 0 & \text{on } \Sigma = (0,T) \times \partial\Omega, \\ u(0,.) = u_0(.) & \text{on } \Omega, \end{cases} \tag{1}$$

where $g : \mathbb{R}_+ \to \mathbb{R}$ is continuous with

$$g(0) \geqslant 0, \quad \frac{dg}{dr} \leqslant K \text{ in } \mathscr{D}'(0,\infty), \; K \in \mathscr{C}(\mathbb{R}_+) \tag{2}$$

and $u_0 \in L^\infty(\Omega)$ with

$$0 \leqslant u_0 \leqslant M_0 \quad \text{a.e. on } \Omega. \tag{3}$$

According to (2), for any $r \in \mathbb{R}_+$ there exists a unique maximal solution $q(r,t)$ defined on the maximal interval $[0, T(r))$ of the o.d.e.

$$\frac{dq}{dt} = g(q) \text{ on } (0, T(r)), \quad q(0) = r. \tag{4}$$

Choosing

$$0 < T < T(M_0) \tag{5}$$

it is easy to show that there exists a unique bounded weak solution u of (1) in the sense:

$$\begin{cases} u \in \mathscr{C}([0,T]; L^1(\Omega)) \cap L^\infty(Q), \\ u \geqslant 0, \quad u^m \in L^2(0,T; H^1(\Omega)), \\ \int\int u\xi_t + \int\int g(u)\xi = \int\int Du^m \, D\xi + \int u_0 \xi(0,.), \\ \forall \xi \in W^{1,1}(0,T; L^1(\Omega)) \cap L^2(0,T; H^1(\Omega)), \quad \xi(T,.) \equiv 0. \end{cases} \tag{6}$$

We denote by u_m this solution. By maximum principle, it is clear that

$$0 \leqslant u_m(t,x) \leqslant q(M_0, t) \quad \text{a.e. } (t,x) \in Q. \tag{7}$$

This paper describes the limit of u_m as m goes to ∞. In the case $g \equiv 0$, it has been proved in [3] (c.f. Theorem 3) that $u_m(t) \to \underline{u}_0$ in $L^1(\Omega)$ for $t \in]0,T]$, where

$$\underline{u}_0 = \begin{cases} ⨍ u_0 \left(:= \dfrac{1}{|\Omega|} \int_\Omega u_0 \right) & \text{if } ⨍ u_0 \geqslant 1, \\ u_0 \chi_{[w=0]} + \chi_{[w>0]} & \text{if } ⨍ u_0 < 1 \end{cases} \tag{8}$$

with $w \in H^1(\Omega)$ the unique solution of the "mesa problem"

$$w \in H^2(\Omega), \quad w \geqslant 0, \; 0 \leqslant \Delta w + u_0 \leqslant 1,$$

$$w(\Delta w + u_0 - 1) = 0 \text{ a.e. } \Omega \quad \text{and} \quad \frac{\partial w}{\partial n} = 0 \text{ on } \Sigma.$$

Following the same approach as in [6] for the similar problem, where the Neumann boundary condition was replaced by the Dirichlet boundary condition, we prove for a general g satisfying (2) that

$$u_m \to u_\infty \quad \text{in } \mathscr{C}((0,T); L^1(\Omega)).$$

But the description of the limit u_∞ is more delicate. Indeed, we have the following cases:

Case 1: If $⨍ u_0 \geqslant 1$, then

$$u_\infty(t,x) = q\left(⨍ u_0, t\right) \quad \text{for a.a. } (t,x) \in Q;$$

Case 2: If $⨍ u_0 < 1$ and $g(1) \leqslant 0$, then

$$u_\infty(t,x) = q(\underline{u}_0(x), t) \quad \text{for a.a. } (t,x) \in Q;$$

Case 3: If $⨍ u_0 < 1$ and $g(1) > 0$, then there exists $T_0 \in (0,T]$ such that

(a) u_∞ is the unique solution on $(0,T_0) \times \Omega$ of

$$\begin{cases} u_\infty \in L^\infty((0,T_0) \times \Omega), \quad 0 \leqslant u_\infty \leqslant 1 \text{ a.e. on } (0,T_0) \times \Omega \\ \quad \text{there exists } w_\infty \in L^2_{\text{loc}}([0,T_0); H^1(\Omega)) \text{ such that} \\ \quad w_\infty \geqslant 0, \; w_\infty(u_\infty - 1) = 0 \text{ a.e. on } (0,T_0) \times \Omega \text{ and} \\ \quad \int_0^{T_0} \int_\Omega \xi_t u_\infty + g(u_\infty)\xi + \int_\Omega \xi(0,.)\underline{u}_0 = \int_0^{T_0} \int_\Omega D\xi \, Dw_\infty \\ \quad \forall \xi \in \mathscr{C}^1([0,T_0) \times \bar{\Omega}), \; \xi \text{ compactly supported;} \end{cases}$$

(b) $u_\infty(t,x) = q(1, t - T_0)$ for a.a. $x \in \Omega$, for any $t \in [T_0, T[$;

Actually we will consider problem (1) with a reaction term $g(u) = g(t,x,u)$ depending on (t,x); the exact assumptions and results will be precised in Section 3. In Section 2, we will prepare the results by studying problem (1) and its limit as $m \to \infty$, with $g(u)$ replaced by a function $h(t,x)$ independent of u.

2. The problem with reaction term independent of u

To apply abstract arguments of the nonlinear semigroups theory, we first consider the elliptic problem

$$v = \Delta v^m + f \ \text{ on } \Omega, \quad \frac{\partial v^m}{\partial n} = 0 \ \text{ on } \partial\Omega$$

with $f \in L^1(\Omega)$. Applying Theorem 20 in [7], for any $m > 0$, there exists a unique solution v of

$$\begin{cases} v \in L^1(\Omega), \quad v^m := |v|^{m-1} v \in W^{1,1}(\Omega), \\ \int_\Omega Dv^m \, D\xi = \int_\Omega (f - v)\xi, \quad \forall \xi \in W^{1,\infty}(\Omega). \end{cases} \tag{9}$$

If $v, \ \hat{v}$ are the solutions corresponding to $f, \ \hat{f} \in L^1(\Omega)$ then

$$\int_\Omega (v - \hat{v})^+ \leqslant \int_\Omega (f - \hat{f})^+. \tag{10}$$

One has the following result as $m \to \infty$:

Proposition 1. *Let $f \in L^1(\Omega)$ and for $m > 0$, v_m be the unique solution of* (9).

(1) (*c.f.* [5]). *If $|⨍ f| < 1$, there exists a unique solution (v, w) of*

$$\begin{cases} v \in L^\infty(\Omega), \quad w \in W^{1,1}(\Omega), \ v \in sign(w) \ a.e. \ on \ \Omega, \\ \int Dw \, D\xi = \int (f - v)\xi, \quad \forall \xi \in \mathscr{C}^1(\bar{\Omega}) \end{cases} \tag{11}$$

and $(v_m, (v_m)^m) \to (v, w)$ in $L^1(\Omega) \times W^{1,1}(\Omega)$ as $m \to \infty$.

(2) *If $|⨍ f| \geqslant 1$, then $v_m \to ⨍ f$ in $L^1(\Omega)$ as $m \to \infty$.*

Proof. Part (1) is a particular case of Theorem B in [5]. Let us prove part (2). Thanks to (10), it is enough to prove it for $|⨍ f| > 1$. Since the problem is odd, let us assume without loss of generality that $⨍ f > 1$. According to [5], we have

$$\{v_m\}_{m \geqslant 1} \quad \text{is relatively compact in } L^1(\Omega),$$

$$\{(v_m)^m - C_m\}_{m \geqslant 1} \quad \text{is relatively compact in } W^{1,1}(\Omega),$$

where $C_m = ⨍ (v_m)^m$. Let $m_k \to \infty$ such that $v_k := v_{m_k} \to v$ in $L^1(\Omega)$ and $\tilde{w}_k := (v_{m_k})^{m_k} - C_{m_k} \to \tilde{w}_\infty$ in $W^{1,1}(\Omega)$ and a.e. on Ω. Using $⨍ v_k = ⨍ f > 1$, one has

$$⨍ (v_k^+)^{m_k} \geqslant \left(⨍ v_k^+\right)^{m_k} \geqslant \left(⨍ f\right)^{m_k} \to \infty.$$

Since

$$C_{m_k}\frac{|\{v_k>0\}|}{|\Omega|}\geqslant ⨍(v_k^+)^{m_k}-⨍|\tilde{w}_k|$$

we have $C_{m_k}\to\infty$. Then $\frac{\tilde{w}_k}{C_{m_k}}\to 0$ a.e. and $(\frac{v_k}{C_{m_k}})^{\frac{1}{m_k}}=(1+\frac{\tilde{w}_k}{C_{m_k}})^{\frac{1}{m_k}}\to 1$ a.e. So $v=\lim_{m_k\to\infty}C_{m_k}^{\frac{1}{m_k}}$ a.e. is constant on Ω and equal to $⨍v=⨍f$. □

Those results may be restated in terms of operators in $L^1(\Omega)$. For $m\geqslant 1$, let A_m be the operator defined by

$$A_m v=-\Delta v^m \quad \text{with}$$

$$\mathscr{D}(A_m)=\Big\{v\in L^m(\Omega);\ v^m\in W^{1,1}(\Omega),\ h=-\Delta v^m\in L^1(\Omega)$$
$$\text{and}\ \int Dv^m\,D\xi=\int h\xi\ \ \forall\xi\in\mathscr{C}^1(\bar{\Omega})\Big\}. \tag{12}$$

Then A_m is m-accretive in $L^1(\Omega)$ and $A_m\to A_\infty$ in the sense of graph, where A_∞ is the multivalued m-accretive operator in $L^1(\Omega)$ defined by

$$z\in A_\infty v \iff \begin{cases} v,z\in L^1(\Omega),\ ⨍z=0 \text{ and}\\ \quad\text{either } v=\mu \text{ a.e. on } \Omega \text{ with } \mu\in\mathbb{R},\ |\mu|\geqslant 1\\ \quad\text{or there exists } w\in W^{1,1}(\Omega) \text{ such that}\\ \quad v\in sign(w) \text{ a.e. on } \Omega \text{ and}\\ \quad\int Dw\,D\xi=\int z\xi\ \ \forall\xi\in\mathscr{C}^1(\bar{\Omega}).\end{cases} \tag{13}$$

Indeed, A_∞ being defined as above, for $f\in L^1(\Omega)$, one has

$$v+A_\infty v\ni f \iff \begin{cases} v\in L^1(\Omega)\int v=\int f \text{ and}\\ \quad\text{either } v=\mu \text{ a.e. on } \Omega \text{ with } \mu\in\mathbb{R},\ |\mu|\geqslant 1\\ \quad\text{or there exists } w \text{ such that } (v,w)\\ \quad\text{is the solution of (11),}\end{cases}$$

so that according to Proposition 1, there exists a unique solution v of $v+A_\infty v\ni f$ and

$$v=\lim_{m\to\infty}(I+A_m)^{-1}f.$$

Let $T>0$ be fixed; set $Q=[0,T)\times\Omega$ and let $u_0\in L^1(\Omega)$ and $h\in L^1(Q)$ be given. Using the general theory of evolution equation, for any $m\geqslant 1$ there exists a unique mild solution (see [2,4,8]) $u_m \in \mathscr{C}([0,T);L^1(\Omega))$ of

$$\frac{du_m}{dt} + A_m u_m \ni h \;\text{ on }\; (0,T) \quad u_m(0) = u_0. \tag{14}$$

Assume $u_0 \geqslant 0$ a.e. on Ω. Using [3, c.f. Theorem 3] and [6, c.f. Theorem 1], $u_m \to u_\infty$ in $\mathscr{C}((0,T);L^1(\Omega))$ where u_∞ is the unique mild solution of

$$\frac{du_\infty}{dt} + A_\infty u_\infty \ni h \;\text{ on }\; (0,T) \quad u_\infty(0) = \underline{u}_0, \tag{15}$$

and $\underline{u}_0$ defined by (8) is $(I + A_\infty)^{-1}u_0$ (and then $e^{-tA_\infty}\underline{u}_0 = \underline{u}_0$). To translate this result in terms of p.d.e. we characterize the mild solutions of (14) and (15). First, one has the following result for (14):

Proposition 2. *Let $u_0 \in L^\infty(\Omega)$ and $h \in L^1(Q)$ with*

$$\int_0^T ||h(t,.)||_\infty \, dt < \infty. \tag{16}$$

For any $m \geqslant 1$, there exists a unique solution u of the problem

$$\begin{cases} u \in L^\infty(Q), \quad u^m \in L^2(0,T;H^1(\Omega)) \\ \int\int \xi_t u + \int\int \xi h + \int \xi(0,.)u_0 = \int\int Du^m \, D\xi \\ \forall \xi \in \mathscr{C}^1(\bar{Q}), \quad \xi(T,.) \equiv 0. \end{cases} \tag{17}$$

Moreover u is the mild solution u_m of (14).

Proof. This is a quite standard result (c.f. [2]). For completeness let us give the arguments. We first show that the mild solution u of (14) satisfies (17). By definition of a mild solution, $u(t) = L^1 - \lim u_\varepsilon(t)$ uniformly for $t \in [0,T)$, where for $\varepsilon > 0$, u_ε is an ε-approximate solution corresponding to a subdivision $t_0 = 0 < t_1 < \cdots < t_{n-1} < T \leqslant t_n$, with $t_i - t_{i-1} < \varepsilon$ and $h_1, \ldots h_n \in L^1(\Omega)$ with $\sum_{i=1}^n \int_{t_{i-1}}^{t_i} ||h(t) - h_i||_{L^1} \, dt \leqslant \varepsilon$, defined by $u_\varepsilon(0) = u_0 u_\varepsilon(t) = u_i$ for $t \in]t_{i-1}, t_i]$, where $u_i \in L^1(\Omega)$ satisfies

$$\frac{u_i - u_{i-1}}{t_i - t_{i-1}} + A_m u_i \ni h_i;$$

that is

$$\begin{cases} u_i = (t_i - t_{i-1})\Delta(u_i)^m + (t_i - t_{i-1})h_i + u_{i-1} \quad \text{on } \Omega \\ \dfrac{\partial (u_i)^m}{\partial n} = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{18}$$

We may choose $h_i \in L^\infty(\Omega)$, with

$$\sum_{i=1}^{n} (t_i - t_{i-1}) ||h_i||_\infty \leqslant \int_0^T ||h(t,.)||_{L^\infty} \, dt.$$

It follows that $u_i \in L^\infty(\Omega)$ and

$$||u_i||_\infty \leqslant ||u_0||_\infty + \sum_{j=1}^{i} (t_j - t_{j-1}) ||h_j||_\infty,$$

so

$$||u_\varepsilon||_{L^\infty(Q)} \leqslant M_1 := ||u_0||_\infty + \int_0^T ||h(t,.)||_{L^\infty} \, dt.$$

Then multiplying (18) by $(u_i)^m$, one gets

$$\frac{1}{m+1} \int |u_i|^{m+1} + (t_i - t_{i-1}) \int |D(u_i)^m|^2 \leqslant (t_i - t_{i-1}) M_1 \int |h_i| + \frac{1}{m+1} \int |u_{i-1}|^{m-1}$$

so

$$||Du_\varepsilon^m||^2_{L^2(Q)} \leqslant \frac{1}{m+1} \int |u_0|^{m+1} + M_1 ||h||_{L^1(Q)}. \tag{19}$$

Let $\tilde{u}_\varepsilon$ be the function from $[0, t_n]$ into $L^1(\Omega)$ defined by $\tilde{u}_\varepsilon(t_i) = u_i$, $\tilde{u}_\varepsilon$ is linear in $[t_{i-1}, t_i]$ and h_ε be defined by $h_\varepsilon(t) = h_i$ on $]t_{i-1}, t_i[$; for $\xi \in W^{1,1}(0, T; L^1(\Omega)) \cap L^2(0, T; H^1(\Omega))$ with $\xi(T,.) \equiv 0$

$$\int\int \tilde{u}_\varepsilon \xi_t + h_\varepsilon \xi + \int u_0 \xi(0,.) = \int\int D(u_\varepsilon)^m \, D\xi. \tag{20}$$

Passing to the limit in (19) and (20) one gets that u is a solution of (17).

At last, we show uniqueness of the solution to (17). It follows from Lemma A in the appendix: if u_1, u_2 are two solutions of (17), apply with $H = L^2(\Omega)$, $V = H^1(\Omega)$, $a(u,v) = \int Du \, Dv$ $u = u_1 - u_2$, $v = (u_1)^m - (u_2)^m$. $\square$

We consider now problem (15).

Proposition 3. *Let $u_0 \in L^1(\Omega)$ and $h \in L^2(Q)$. Set*

$$\mu(t) = ⨍ u_0 + \int_0^t \left(⨍_\Omega h(s) \right) ds \tag{21}$$

and

$$I = \{ t \in (0,T); \; \mu(t) < 1 \}. \tag{22}$$

Assume that the mild solution u_∞ *of* (15) *is nonnegative. Then* $u = u_\infty$ *is the unique solution of the following problem*:

$$\begin{cases} \text{(i) } u \in \mathscr{C}([0,T); L^1(\Omega)), \quad u(0) = \underline{u}_0, \\ \text{(ii) } u(t) \equiv \mu(t) \quad \textit{a.e. on } \Omega \textit{ for any } t \in (0,T)\backslash I \\ \text{(iii) } \textit{there exists } w \in L^\infty_{\rm loc}(I; H^1(\Omega)) \textit{ such that } u \in \textit{ sign}(w) \\ \qquad \textit{a.e. on } \Omega \textit{ and } \int\int \xi_t u + \xi h = \int\int Dw\, D\xi,\ \forall \xi \in \mathscr{C}^1(I \times \bar{\Omega}), \\ \qquad \textit{compactly supported.} \end{cases} \tag{23}$$

To prove this proposition we will use the following lemma:

Lemma 1. *Let* $\varepsilon > 0$, u, $\hat{u}$, $h \in L^1(\Omega)$ *and* $w \in H^1(\Omega)$ *such that* $u \in sign(w)$ *a.e. on* Ω, $|\hat{u}| \leqslant 1$ *and*

$$\int (Dw\, D\xi + h\xi) = \int \frac{u - \hat{u}}{\varepsilon} \xi, \quad \forall \xi \in \mathscr{C}^1(\bar{\Omega}).$$

If $⨍|u| < 1$, *then*

$$||w||_{L^1} \leqslant \frac{C}{1 - ⨍|u|} ||h||_{L^1},$$

where C *is a constant depending only on* Ω.

Proof. First, by Kato inequality (c.f. [1, Theorem 2.4]), for any $\xi \in W^{2,1}(\Omega)$ with $\xi \geqslant 0$, $\frac{\partial \xi}{\partial n} = 0$ on $\partial\Omega$, one has

$$\begin{aligned} \int |w|(-\Delta\xi) &\leqslant \int_{w \neq 0} \xi \left(h - \frac{u - \hat{u}}{\varepsilon} \right) sign(w) \\ &\leqslant \int_{w \neq 0} \xi h\, sign(w) \\ &\leqslant ||\xi||_{L^\infty} ||h||_{L^1}. \end{aligned}$$

Let ξ_0 be the solution of

$$\begin{cases} -\Delta\xi_0 = |u| - ⨍|u| & \text{in } \Omega, \\ \dfrac{\partial \xi_0}{\partial n} = 0 & \text{on } \partial\Omega, \\ ⨍\xi_0 = 0; \end{cases}$$

one has $\xi_0 \in W^{2,p}(\Omega)$ for any $1 < p < \infty$ and

$$\begin{aligned} ||\xi_0||_{L^\infty} &\leqslant C \big|\big| |u| - ⨍|u| \big|\big|_{L^\infty} \\ &\leqslant C, \end{aligned}$$

where C is a constant depending only on Ω. Set $\xi = \xi_0 + C$, one has $\xi \geqslant 0$ and

$$\begin{aligned}\int |w|\big(|u| - ⨍|u|\big) &= \int |w|(-\Delta\xi) \\ &\leqslant \int \xi|h| \\ &\leqslant 2C||h||_{L^1}\end{aligned}$$

and since $|uw| = |w|$ a.a. Ω, one has

$$||w||_{L^1} \leqslant \frac{2C}{1 - ⨍|u|}||h||_{L^1}. \qquad \square$$

Firstly, we prove a particular case of Proposition 3 stated in the following lemma:

Lemma 2. *Let u_0 and h be as in Proposition* 3. *Assume that $\mu(t)$ defined by* (21) *satisfies*

$$\mu(t) < 1 \quad \textit{for all } t \in [0, T] \tag{24}$$

and that the mild solution u_∞ of (15) *is nonnegative. Then u_∞ is the unique solution u of*

$$\begin{cases} u \in L^\infty(Q), \text{ there exists } w \in L^2(0,T; H^1(\Omega)) \\ \quad \text{such that } u \in sign(w) \text{ a.e. } \Omega \text{ and} \\ \quad \int\int \xi_t u + \int\int \xi h + \int \xi(0,.)u_0 = \int\int Dw\, D\xi \\ \quad \forall \xi \in \mathscr{C}^1(\bar{Q}),\ \xi(T,.) \equiv 0. \end{cases} \tag{25}$$

Proof. For uniqueness of a solution u of (25), apply Lemma A in the appendix in the same way as in the proof of Proposition 2. To prove that the mild solution $u = u_\infty$ of (15) satisfies (25), consider as in the proof of Proposition 2, an ε-approximate solution u_ε corresponding to a subdivision $t_0 < t_1 < \cdots < t_{n-1} < T \leqslant t_n$ and $h_1, \ldots, h_n \in L^2(\Omega)$ with $\sum_{i=1}^n \int_{t_{i-1}}^{t_i} ||h(t) - h_i||_{L^2}^2\, dt \leqslant \varepsilon$. One has $u_\varepsilon(t) = u_i$ on $]t_{i-1}, t_i]$ with $(u_i, w_i) \in L^\infty(\Omega) \times H^2(\Omega)$ solution of

$$\begin{cases} u_i = u_{i-1} + (t_i - t_{i-1})(\Delta w_i + h_i) \quad \text{on } \Omega, \\ u_i \in sign(w_i) \quad \text{on } \Omega, \\ \dfrac{\partial w_i}{\partial n} = 0 \quad \text{on } \partial\Omega \end{cases} \tag{26}$$

(using the convention for $i = 1,\ u_{i-1} = \underline{u}_0$).

Since $u_\varepsilon(t) \to u_\infty(t)$ in $L^1(\Omega)$ as $\varepsilon \to 0$ uniformly for $t \in [0, T]$, according to (24) for $\varepsilon > 0$ small enough, one has $⨍|u_i| \leqslant \theta$ for $i = 1, \ldots, n$ with $\theta < 1$ independent of ε.

Using Lemma 1,

$$\|w_i\|_{L^1} \leqslant C_1 \|h_i\|_{L^1} \quad \text{for } i = 1, \dots, n \tag{27}$$

with C_1 independent of ε.

Multiplying (26) by w_i, one gets

$$\int |Dw_i|^2 = \int w_i h_i - \int \frac{|w_i| - w_i u_{i-1}}{t_i - t_{i-1}} \leqslant \|w_i\|_{L^2} \|h_i\|_{L^2}.$$

Then, by Poincaré inequality and (27), one obtains

$$\|Dw_i\|_{L^2} \leqslant C_2 \|h_i\|_{L^2} \tag{28}$$

with C_2 independent of ε.

It follows from (27) and (28) that the function w_ε defined by $w_\varepsilon(t) = w_i$ on $]t_{i-1}, t_i[$, is bounded in $L^2(0, T; H^1(\Omega))$ as $\varepsilon \to 0$. Let $\varepsilon_k \to 0$ such that $w_{\varepsilon_k} \rightharpoonup w$ in $L^2(0, T; H^1(\Omega))$. Since $u_\varepsilon \to u_\infty$ in $L^1(Q)$ and $u_\varepsilon \in sign(w_\varepsilon)$ a.e. on Q, at the limit $u_\infty \in sign(w)$ a.e. on Q. Using the function $\tilde{u}_\varepsilon$ as in the proof of Proposition 2, one ends up the proof of $u = u_\infty$ satisfies (25). □

Proof of Proposition 3. Firstly, we prove uniqueness of a solution u of (23). By definition, a solution $u(t)$ of (23) is defined on $((0, T) \setminus I) \cup \{0\}$. Let (a, b) be a component of I. A solution $u(t)$ of (23) is defined for $t = a$. Applying Lemma 2, for $a < \alpha < \beta < b$, $u = u_\alpha$ on $(\alpha, \beta) \times \Omega$ where u_α is the mild solution of $\frac{du_\alpha}{dt} + A_\infty u_\alpha \ni h$ on (α, β), $u_\alpha(\alpha) = u(\alpha)$. If u_1, u_2 are two solutions of (15), by the contraction property for mild solutions,

$$\|u_1(t) - u_2(t)\|_{L^1} \leqslant \|u_1(\alpha) - u_2(\alpha)\|_{L^1}, \quad \forall a < \alpha \leqslant t < b.$$

Since $u_1(\alpha) - u_2(\alpha) \to 0$ in $L^1(\Omega)$ as $\alpha \to a$, $u_1 = u_2$ on $(a, b) \times \Omega$.

Now let $u = u_\infty$ be the mild solutions of (15). By assumption, u satisfies (23i) and $u \geqslant 0$. Being a mild solution it is clear that $u(t) \leqslant 1$ and $⨍ u(t) = \mu(t)$; then u satisfies (23ii). At last by Lemma 2, u satisfies (23iii). □

Summing up the results of Propositions 1–3, according to the results of [6,3], one has:

Corollary 1. *Let $u_0 \in L^\infty(\Omega)$, $u_0 \geqslant 0$ and $h \in L^\infty(0, T; L^1(\Omega))$ satisfying* (16). *For any $m \geqslant 1$, there exists a unique solution u_m of* (17) *and*

$$u_m \to u \quad in\, \mathscr{C}((0, T); L^1(\Omega)) \text{ as } m \to \infty.$$

If $u \geqslant 0$, then u is the unique solution of (23).

3. The general reaction–diffusion problem

We consider problem (1) with g depending on (t,x). We assume $g: Q \times \mathbb{R}_+ \to \mathbb{R}$ satisfies

$$\begin{cases} \text{(i) for any } r \in \mathbb{R}_+,\ g(.,r) \in L^\infty(0,T;L^1(\Omega)) \text{ and} \\ \quad \int_0^T \|g(t,.,r)\|_{L^\infty}\, dt < \infty, \\ \text{(ii) for a.a. } (t,x) \in Q,\ g(t,x,.) \text{ is continuous on } \mathbb{R}_+ \text{ and} \\ \quad \dfrac{\partial g}{\partial r}(t,x,.) \leqslant K(\cdot) \text{ in } \mathscr{D}'(0,\infty) \end{cases} \tag{29}$$

with $K: \mathbb{R}^+ \to \mathbb{R}^+$ continuous. Consequently, for any $u \in L^\infty(Q)$ with $u \geqslant 0$, the function $h = g(.,u)$ is in $L^\infty(0,\infty;L^1(\Omega))$ and satisfies (16); indeed

$$g(.,\|u\|_\infty) - \int_0^{\|u\|_\infty} K(r)\, dr \leqslant g(.,u) \leqslant g(.,0) + \int_0^{\|u\|_\infty} K(r)\, dr.$$

In this section, we fix $u_0 \in L^\infty(\Omega)$ satisfying (3). We assume there is $M \in W^{1,1}(0,T)$, so

$$M'(t) \geqslant g(t,x,M(t)) \quad \text{for a.a. } (t,x) \in Q,\ M(0) \geqslant M_0. \tag{30}$$

Applying Section 2, we have the following result:

Theorem 1. *Under the above assumption, for any $m \geqslant 1$, there exists a unique u_m solution of*

$$\begin{cases} u_m \in L^\infty(Q), \quad u_m \geqslant 0,\ (u_m)^m \in L^2(0,T;H^1(\Omega)) \\ \int\int u_m \xi_t + g(.,u_m)\xi + \int u_0 \xi(0,.) = \int\int D\xi\, D(u_m)^m \\ \forall \xi \in \mathscr{C}^1(\bar{Q}), \quad \xi(T,.) = 0. \end{cases} \tag{31}$$

Moreover $u_m \in \mathscr{C}([0,T);L^1(\Omega))$, $u_m(t,x) \leqslant M(t)$ for a.a. $(t,x) \in Q$; $u_m \to u$ in $\mathscr{C}((0,T);L^1(\Omega))$ as $m \to \infty$ and u is the unique function in $L^\infty(Q)$ with $u \geqslant 0$, satisfying (23) *with $h = g(.,u)$.*

Proof. For $R > 0$, let F_R be the map from $[0,T) \times L^1(\Omega)$ into $L^1(\Omega)$ defined by

$$F_R(t,u) = g(t,.,u^+ \wedge R).$$

With (29), F_R is integrable in $t \in (0,T)$ uniformly for any $u \in L^1(\Omega)$ and continuous in $u \in L^1(\Omega)$ for a.a. $t \in (0,T)$; moreover $(\max_{[0,R]} K)I - F_R(t,.)$ is accretive in $L^1(\Omega)$. Then (see for instance [6, Lemma 1]) there exists a unique mild solution of

$$\frac{du}{dt} + A_m u \ni F_R(.,u) \text{ on } (0,T), \quad u(0) = u_0. \tag{32}$$

Let first u_m be a solution of (31) and fix $R \geqslant \|u_m\|_\infty$.; Since $h := g(., u_m) = F_R(., u_m)$, applying Proposition 2, u_m is a mild solution of (32). From uniqueness of a solution to (32), follows uniqueness of a solution to (31). Conversely, let $R = \max_{[0,T]} M$ and consider the mild solution u_m of (32). By Proposition 2, u_m is solution of (17) with $h = g(., u_m^+ \wedge R)$. We will prove that

$$0 \leqslant u_m(t,x) \leqslant M(t) \quad \text{for a.a. } (t,x) \in Q \tag{33}$$

it will follow that $h = g(., u_m)$ and then u_m is solution of (32). To prove (33), we use the fact that, according to (10), the operator A_m is T-accretive in $L^1(\Omega)$ (c.f. [2,4]). If u_1, u_2 are mild solutions of (15) corresponding to (h_1, u_{01}), (h_2, u_{02}) in $L^1(Q) \times L^1(\Omega)$ respectively, one has for all $t \in [0, T)$

$$\int (u_1(t) - u_2(t))^+ \leqslant \int (u_{01} - u_{02})^+ + \int_0^t \int_{[u_1 \geqslant u_2]} (h_1 - h_2)^+, \tag{34}$$

Apply with $u_2 = u_m$, $h_2 = F_R(., u_m)$, $u_{02} = u_0$, $u_1 = 0$, $h_1 = 0$, $u_{01} = 0$. Since $u_m \geqslant 0$ and $F_R(., u_m)\chi_{[u_m \leqslant 0]} = g(., 0) \geqslant 0$, one first obtains $u_m \geqslant 0$. Secondly, notice that $u_2(t,x) = M(t)$ is strong solution, and then mild solution of (15) with $h_2(t,x) = M'(t)$, as $u_{02} = M(0)$. Using (29) and (30), one has

$$\begin{aligned} F_R(., u_m)\chi_{[u_m \geqslant M]} &= g(., u_m \wedge R)\chi_{[u_m \geqslant M]} \\ &\leqslant g(., M)\chi_{[u_m \geqslant M]} + \chi_{[u_m \geqslant M]} \int_M^{u_m \wedge R} k(r)\, dr \\ &\leqslant M' \chi_{[u_m \geqslant M]} + \left(\max_{[0,R]} K \right) (u_m - M)^+ \end{aligned}$$

and then, using (34), $u_m \leqslant M$. This proves first part of the theorem and u_m is the mild solution of (32) with $R = \max_{[0,T]} M$. Using Theorem 1 in [6], with Proposition 1, $u_m \to u$ in $\mathscr{C}((0,T); L^1(\Omega))$ where u is the unique mild solution of

$$\frac{du}{dt} + A_\infty u \ni F_R(., u) \text{ on } (0,T) \quad u(0) = (I + A_\infty)^{-1} u_0.$$

Since $0 \leqslant u \leqslant M$, with the above arguments, thanks to Proposition 3, u is the unique function in $L^\infty(Q)$ with $u \geqslant 0$ is solution of (23) with $h = g(., u)$. $\square$

Now we will make more explicit the limit solution u in the case $g(t,x,u) = g(u)$ (independent of $(t,x) \in Q$). Throughout the end of this section $g : \mathbb{R}_+ \to \mathbb{R}$ is defined by (2) and we assume (5), so $M'(t) = q(t, M_0)$ satisfies (30). Then we have the following characterization of the limit solution u.

Corollary 2. *If $g(t,x,u) = g(u)$ with $g : \mathbb{R}_+ \to \mathbb{R}$ satisfies* (2), *then the limit u of u_m is defined as it is claimed in the introduction*

Case 1: *If* $\fint u_0 \geqslant 1$, *then*

$$u(t,x) = q\big(\fint u_0, t\big) \quad \text{for a.a. } (t,x) \in Q.$$

Case 2: *If* $\fint u_0 < 1$ *and* $g(1) \leqslant 0$, *then*

$$u(t,x) = q(\underline{u}_0(x), t) \quad \text{for a.a. } (t,x) \in Q.$$

Case 3: *If* $\fint u_0 < 1$ *and* $g(1) > 0$, *then there exists* $T_0 \in (0, T]$ *such that*

(a) *u is the unique solution on* $(0, T_0) \times \Omega$ *of*

$$\begin{cases} u \in L^\infty((0,T_0) \times \Omega), \ 0 \leqslant u \leqslant 1 \ \text{a.e. on } (0,T_0) \times \Omega \\ \quad \text{there exists } w_\infty \in L^2_{\text{loc}}([0,T_0); H^1(\Omega)) \text{ such that} \\ \quad w_\infty \geqslant 0, \ w_\infty(u-1) = 0 \ \text{a.e. on } (0,T_0) \times \Omega \text{ and} \\ \quad \int_0^{T_0} \int_\Omega \xi_t u + g(u)\xi + \int_\Omega \quad \xi(0,.)\underline{u}_0 = \int_0^{T_0} \int_\Omega D\xi \, Dw_\infty \\ \quad \forall \xi \in \mathscr{C}^1([0,T_0) \times \bar{\Omega}), \ \xi \text{ compactly supported} \end{cases}$$

(b) $u(t,x) = q(1, t - T_0)$ *for a.a.* $x \in \Omega$, *for any* $t \in [T_0, T[$;

Proof. Recall that u is the unique function in $L^\infty(Q)$ with $u \geqslant 0$ satisfying (23) with $h = g(u)$. In the case $\fint u_0 \geqslant 1$, $\underline{u}_0 = \fint u_0$; the function $u(t,x) = q(\fint u_0, t)$ is clearly the solution of (23) with $h(t,x) = g(q(\fint u_0, t)) = u_t(t,x)$.

In the case $\fint u_0 < 1$, $\underline{u}_0 \leqslant 1$. If $g(1) \leqslant 0$, one has $u(t,x) = q(\underline{u}_0(x), t) \in [0,1]$ for a.a. $(t,x) \in Q$ and then u is the solution of (23) with $h(t,x) = u_t(t,x)$, $I = (0,T)$, $w \equiv 0$.

At last consider the case $g(1) > 0$. If $]a,b[$ is a component of

$$\big\{t \in (0,T); \fint u(t) > 1\big\},$$

one has $a > 0$, $\fint u(a) = 1$ and $u(t) \equiv \fint u(t)$ on $[a,b]$. Further $u(t) \equiv q(1, t-a)$ on $[a,b]$. Since $g(1) > 0$, one has $q(1, b-a) > 0$ and then $b = T$. So $I = (0, T_0)$ with $T_0 \in (0,T]$ and the result follows. $\square$

Remarques.

(i) In Case 3, if $M_0 < 1$, setting

$$T_1 = \max\{t \in [0,T]; \ q(u_0, t) \leqslant 1 \text{ a.e. on } \Omega\}$$

one has

$$T_0 \geqslant T_1 \quad \text{and} \quad u_\infty(t,x) = q(u_0(x), t) \ \text{for a.a. on } (0, T_1) \times \Omega.$$

In particular, if $g(M_0) \leqslant 0$ then $T_0 = T_1 = T$.

(ii) Still in case 3, define

$$T_2 = \sup\{t;\ q(⨍u_0, t) < 1\}.$$

If g is concave (resp. convex) on $[0, 1]$, then

$$\frac{d}{dt}⨍u(t) \leqslant (\text{resp.} \geqslant) g(⨍u(t)) \quad \text{for } t \in (0, T_0).$$

Further $⨍u(t) \leqslant$ (resp. $\geqslant$) $q(⨍u_0, t)$ for $t \in (0, T_0)$ so $T_0 \geqslant$ (resp. $\leqslant$) T_2.

Appendix

We give here a general lemma used to prove uniqueness. While this method is classical, we did not find such statement in the literature.

Lemma A. *Let $V \subseteq H$ be Hilbert spaces with continuous injection and $a : V \times V \to \mathbb{R}$ be continuous bilinear symmetric and nonnegative ($a(v, v) \geqslant 0$)). Let $u \in L^2(0, T; H)$, $w \in L^2(0, T; V)$ satisfying*

$$\int (u(t), \xi'(t))_H \, dt = \int a(w(t), \xi(t))$$

$$\forall \xi \in W^{1,2}(0, T; H) \cap L^2(0, T; V) \quad \textit{with } \xi(T) = 0 \tag{A.1}$$

and

$$(u(t), w(t))_H \geqslant 0 \quad \textit{a.e. } t \in (0, T) \tag{A.2}$$

then $u \equiv 0$.

Proof. Let $0 \leqslant \tau \leqslant T$ and apply (A.1) with $\xi(t) = \int_{t \wedge \tau}^{\tau} w(s)\, ds$. One gets

$$\begin{aligned}\int_0^{\tau} (u(t), w(t))_H \, dt &= \int_0^T a(\xi'(t), \xi(t)) \, dt \\ &= -\frac{1}{2} a(\xi(0), \xi(0)) \\ &= -\frac{1}{2} a\left(\int_0^{\tau} w(s)\, ds, \int_0^{\tau} w(s)\, ds\right).\end{aligned}$$

Using (A.2), $a(\int_0^{\tau} w(s)\, ds, \int_0^{\tau} w(s)\, ds) = 0$ for any $\tau \in [0, T)$ and then $a(w(t), v) = 0$ for any $v \in V$ and a.a. $t \in (0, T)$. Using (A.1) again, $u \equiv 0$. $\square$

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