ON THE UNIQUENESS AND NUMERICAL APPROXIMATIONS FOR A MATCHING PROBLEM[*]

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Abstract. The paper deals with some theoretical and numerical aspects for an optimal matching problem with constraints. It is known that the uniqueness of optimal matching measure does not hold even with L^p sources and targets. In this paper, the uniqueness is proven under geometric conditions. On the other hand, we also introduce a dual formulation with a linear cost functional on convex set and show that its Fenchel–Rockafellar dual formulation gives right solution to the optimal matching problem. Basing on our formulations, a numerical approximation is given via an augmented Lagrangian method. We compute at the same time the optimal matching measure, optimal flows and Kantorovich potentials. The convergence of the discretization is also studied in detail.

Key words. Optimal matching, optimal transport, Fenchel–Rockafellar duality, augmented Lagrangian methods

AMS subject classifications. 49Q20, 65M60, 65K10, 60K30

1. Introduction. *Optimal matching* problem (see [11, 9, 8] and the references therein) deals with the problem to transport two measures of commodities into a prescribed location and to match them there in such a way to minimize the total cost of both transportations. The problem with uniformly convex costs is studied in [7, 11, 9, 8] with applications in economic theory. The case where costs are governed by the Euclidean distance is studied in [19] with connection to $p-$Laplacian type equations.

Optimal constrained matching problem (see [3, 20]), which is a variant from the optimal matching problem and the partial transport problem (see for instance [14, 18]), consists in transporting two kinds of goods and matching them into a target set with constraints on the amount of matter at the target. For example, the target represents the capacities of some companies, the amount of goods matching at each company should have a predetermined bound from above. In mathematical language, the optimal matching problem with constraints for the Euclidean costs can be modeled as follows: Let Ω be a bounded, convex set of $\mathbb{R}^N$ and $f_1, f_2 \in \mathcal{M}_b^+(\Omega)$ represent source measures of the same mass, i.e., $f_1(\Omega) = f_2(\Omega)$. The constraint on the target set is represented by a measure $\Theta \in \mathcal{M}_b^+(\Omega)$, which must satisfy

$$f_1(\Omega) = f_2(\Omega) < \Theta(\Omega).$$

The optimal matching problem reads

$$(1.1) \qquad W(f_1, f_2; \Theta) := \inf_{(\gamma_1, \gamma_2) \in \pi(f_1, f_2; \Theta)} \left(\int_{\Omega \times \Omega} |x - y| \mathrm{d}\gamma_1 + \int_{\Omega \times \Omega} |x - y| \mathrm{d}\gamma_2 \right)$$

with

$$\pi(f_1, f_2; \Theta) := \left\{ (\gamma_1, \gamma_2) \in \mathcal{M}_b^+(\Omega \times \Omega)^2 : \pi_y \# \gamma_1 = \pi_y \# \gamma_2 \le \Theta, \pi_x \# \gamma_i = f_i, i = 1, 2 \right\},$$

where $\pi_x \# \gamma$ and $\pi_y \# \gamma$ are the first and second marginals of γ, respectively. An optimal solution (γ_1, γ_2) is called a *couple of optimal plans* and $\rho := \pi_y \# \gamma_1 = \pi_y \# \gamma_2$ is called an *optimal matching measure*. Obviously, we can write (1.1) as follows

$$\text{(OM)} \quad W(f_1, f_2; \Theta) = \inf_{\rho \in \mathcal{M}_b^+(\Omega)} \Big\{ W_1(f_1, \rho) + W_1(f_2, \rho) \, : \, \rho \leq \Theta, \ \rho(\Omega) = f_1(\Omega) \Big\},$$

where $W_1(.,.)$ is the 1-Wasserstein distance (see e.g. [21, Chapter 6]). By definition, given two non-negative Radon measures μ_1 and μ_2 with the same mass on Ω,

$$(1.2) \qquad\qquad W_1(\mu_1, \mu_2) := \min_{\gamma \in \pi(\mu_1, \mu_2)} \Big\{ \int_{\Omega \times \Omega} |x - y| \mathrm{d}\gamma \Big\},$$

where $\pi(\mu_1, \mu_2)$ is the set of *transport plans*, i.e.

$$\pi(\mu_1, \mu_2) := \Big\{ \gamma \in \mathcal{M}_b^+(\Omega \times \Omega) : \pi_x \# \gamma = \mu_1, \ \pi_y \# \gamma = \mu_2 \Big\}.$$

The problem (OM) can be also reformulated by saying that masses moving from f_1 to f_2 are forced to pass through an unknown (optimal) distribution less than Θ and the transportation cost should be optimal. In applications, f_1 and f_2 can be distributions of consumers while Θ would be a distribution of commodities.

Using the direct method in Calculus of Variations, it is not difficult to prove the existence of an optimal matching measure. Our main interest lies in the uniqueness and numerical approximation of the solution. As we will see, the uniqueness of optimal matching measure does not hold even with regular f_1, f_2, Θ. An additional geometrical condition, as well as the absolute continuity of the measure Θ, is needed for the uniqueness. Concerning the numerical computation, we develop the variational study of the problem.

The optimal constrained matching problem (1.1) is recently studied theoretically in [20] in connection with p–Laplacian type systems by using PDE techniques. Inspiring from the work of Evans and Gangbo [13] on the optimal transport theory, the authors in [20] show that an optimal matching measure and associated Kantorovich potentials can be obtained from limits in p–Laplacian type equations as $p \to +\infty$. In [3], Barrett and Prigozhin give a numerical approximation to the problem (1.1) in the case where $\Theta = C\mathcal{L}^N \llcorner D$, i.e., Θ is a constant C on the destination set D. Their approach based on nonlinear approximated PDEs. They use Raviart–Thomas finite elements for discretization and solve nonlinear resulting equations.

In this paper, we focus on more variational aspects and the uniqueness of optimal matching measure. We introduce some equivalent formulations for the problem (OM). We give a sufficient condition to ensure the uniqueness of optimal matching measure and show that a solution of the Fenchel–Rockafellar dual formulation is the right solution to the optimal matching problem under a suitable geometric condition. Numerical aspects are also studied with the help of the equivalent formulations. We show the convergence of our approximation and give the details for solving the discretized problems consisting in solving linear equations (with a symmetric positive definite coefficient matrix) or updating an explicit formula.

The paper is organized as follows: In the following section we present our main results such as the uniqueness of optimal matching measure, dual maximization problem, connection between minimal matching flow problem and (OM), the convergence of the discretization and a numerical example illustrating our approach. The details

and proofs are discussed in the next sections. More precisely, section 3 is devoted to the duality issue while the uniqueness is discussed in section 4. Numerical analysis of the problem is given in section 5 with a study of the convergence of the discretization. We end up the article by giving some numerical examples which validate the approach.

2. Main results. Throughout the paper, $\Omega \subset \mathbb{R}^N$ is a bounded convex domain, and f_1, f_2, $\Theta \in \mathcal{M}_b^+(\Omega)$ are nonnegative Radon measures such that

$$f_1(\Omega) = f_2(\Omega) < \Theta(\Omega).$$

It is not difficult to see that the feasible set $\pi(f_1, f_2; \Theta)$ is closed under the weak convergence of Radons measures. This observation gives easily the existence of a couple of optimal plans (γ_1, γ_2) and thus an optimal matching measure $\rho := \pi_y \# \gamma_1 = \pi_y \# \gamma_2$. However, in general the uniqueness of optimal matching measures does not hold. For instance, let $f_1 = \mathcal{L} \llcorner [0, 1], f_2 = \mathcal{L} \llcorner [5, 6]$ and $\Theta = \mathcal{L} \llcorner [2, 4]$, where $\mathcal{L}$ is the Lebesgue measure on $\mathbb{R}$. We see that there are infinitely many optimal matching measures with the total cost $W(f_1, f_2; \Theta) = 5$ (one can verify this by using the duality in Theorem 2.3 below).

Here, we prove that under additional conditions on the supports of Θ, f_i, for $i = 1, 2$ and the absolute continuity of Θ, there is a unique optimal matching measure. Let us fix the assumption

(H) $$S(f_1, f_2) \cap \text{supp}(\Theta) = \emptyset,$$

where $S(f_1, f_2) := \Big\{ z = (1-t)x + ty : x \in \text{supp}(f_1),\, y \in \text{supp}(f_2) \text{ and } t \in [0, 1] \Big\}$.

THEOREM 2.1. *Assume that $\Theta \in L^1$ and that* (H) *holds. There exists a unique optimal matching measure ρ.*

Notice that the absolute continuity of Θ is necessary for the uniqueness. Indeed, taking $f_1 = \delta_{(0,-1)}, f_2 = \delta_{(0,1)}$ and $\Theta = \delta_{(-1,0)} + \delta_{(1,0)}$ in $\mathbb{R}^2$, then $S(f_1, f_2) \cap \text{supp}(\Theta) = \emptyset$ and there are again infinitely many optimal matching measures of form $\rho = \alpha \delta_{(-1,0)} + \beta \delta_{(1,0)}$ with $\alpha \geq 0, \beta \geq 0, \alpha + \beta = 1$. So, the conditions in Theorem 2.1 are somehow optimal for the uniqueness.

Now, to build a numerical computation of the solution to the optimal matching problem (OM), our main objective is to prove rigorously all the necessary materials to use the augmented Lagrangian method. Our approach is variational. To this aim, we introduce a suitable dual formulation to (OM) which moves the problem into the scope of the general formulation

(2.1) $$\inf_{u \in V} \mathcal{F}(u) + \mathcal{G}(\Lambda u),$$

where V and Z are two Hilbert spaces, $\mathcal{F} : V \longrightarrow (-\infty, +\infty]$ and $\mathcal{G} : Z \longrightarrow (-\infty, +\infty]$ are convex and l.s.c., and $\Lambda \in \mathcal{L}(V, Z)$ the space of continuous linear operators. Once such a dual formulation is given, the ALG2 method (see section 5) can be applied to give numerical solutions to both the problem (2.1) and the Fenchel–Rockafellar dual problem of (2.1):

(2.2) $$\sup_{\sigma \in Z^*} \left(-\mathcal{F}^*(-\Lambda^*\sigma) - \mathcal{G}^*(\sigma) \right).$$

Recall that the necessary and sufficient condition for optimality of (2.1) and (2.2) reads as

$$(2.3) \qquad -\Lambda^*\sigma \in \partial\mathcal{F}(u) \quad \text{and} \quad \sigma \in \partial\mathcal{G}(\Lambda u).$$

It is expected that the Fenchel–Rockafellar dual form (2.2) will give informations on the original matching problem. We will see that this is again true under the necessary geometrical condition (H).

We come back to the duality issue for (OM). As usual, let us denote by $Lip_1(\Omega)$ the set of 1-Lipschitz functions on Ω. By extension, we usually identify $Lip_1(\Omega)$ with $Lip_1(\overline{\Omega})$. Let us recall that the duality issue was already studied in [20] with the following result:

THEOREM 2.2 ([20]). *Assume that $f_1, f_2 \in L^\infty(\Omega)$. One has*
$$(2.4)$$
$$W(f_1, f_2; \Theta) = \max\left\{ -\int u_1 \mathrm{d}f_1 - \int u_2 \mathrm{d}f_2 - \int (u_1 + u_2)^- \mathrm{d}\Theta : u_1, u_2 \in Lip_1(\Omega) \right\}.$$

However, even if the maximization problem in (2.4) falls into the scope of (2.1), unfortunately the corresponding $\mathcal{F}$ is nonlinear on its variable $u := (u_1, u_2)$ and (2.4) is not very useful for the numerical computation. Here, we introduce a new dual formulation with the following linear cost functional:

$$(2.5) \qquad \max\left\{ \int (u_1 + u_2) \mathrm{d}\Theta - \int u_1 \mathrm{d}f_1 - \int u_2 \mathrm{d}f_2 : (u_1, u_2) \in K \right\},$$

where

$$K := \left\{ (u_1, u_2) \in Lip_1(\Omega) \times Lip_1(\Omega) \; : \; u_1 + u_2 \leq 0 \right\}.$$

Using the Fenchel–Rockafellar dual theory for the maximization problem (2.5), we also introduce the *minimal matching flow* (MMF) problem:

$$(\text{MMF}) \qquad \min\left\{ |\Phi_1|(\overline{\Omega}) + |\Phi_2|(\overline{\Omega}) : (\Phi_1, \Phi_2, \nu) \in \Psi(f_1, f_2; \Theta) \right\},$$

where

$$\Psi(f_1, f_2; \Theta)$$
$$:= \left\{ (\Phi_1, \Phi_2, \nu) \in \mathcal{M}_b(\overline{\Omega})^N \times \mathcal{M}_b(\overline{\Omega})^N \times \mathcal{M}_b^+(\overline{\Omega}) : -\nabla \cdot \Phi_i = \Theta - \nu - f_i \text{ in } \mathbb{R}^N, i = 1, 2 \right\}.$$

Here, the divergence constraint is understood in the sense of distribution, i.e.,

$$\langle \nabla\phi, \Phi_i \rangle_{(C(\overline{\Omega})^N, \mathcal{M}_b(\overline{\Omega})^N)} = \int_{\overline{\Omega}} \nabla\phi \cdot \frac{\Phi_i}{|\Phi_i|} \mathrm{d}|\Phi_i| = \int_{\overline{\Omega}} \phi \mathrm{d}(\Theta - \nu - f_i),$$

for any smooth compactly supported function $\phi \in C_c^\infty(\mathbb{R}^N)$. We denote by $\frac{\Phi}{|\Phi|}$ the density of Φ w.r.t. the measure $|\Phi|$ and by $\nabla_{|\Phi|}u$ the tangential gradient w.r.t. $|\Phi|$, which is well-defined for any Lipschitz function u (see e.g. [6]).

Our main result concerning duality and quivalent formulations is summarized in the following theorem.

THEOREM 2.3. *Let f_1, f_2, Θ be Radon measures. We have*

$$(2.6) \qquad \begin{aligned} W(f_1, f_2; \Theta) &= \max\left\{ \int (u_1 + u_2) \mathrm{d}\Theta - \int u_1 \mathrm{d}f_1 - \int u_2 \mathrm{d}f_2 : (u_1, u_2) \in K \right\} \\ &= \min\left\{ |\Phi_1|(\overline{\Omega}) + |\Phi_2|(\overline{\Omega}) : (\Phi_1, \Phi_2, \nu) \in \Psi(f_1, f_2; \Theta) \right\}. \end{aligned}$$

Moreover, we have that

- $(\gamma_1, \gamma_2) \in \pi(f_1, f_2; \Theta)$ *and* $(u_1, u_2) \in K$ *are optimal for the optimal constrained matching problem* (1.1) *and the maximization problem* (2.5), *respectively, if and only if*

$$(2.7) \qquad \begin{cases} u_1 + u_2 = 0, \ (\Theta - \rho)\text{-}a.e., \ with \ \rho := \pi_y \# \gamma_1 = \pi_y \# \gamma_2 \\ u_1(y) - u_1(x) = |y - x| \ for \ all \ (x, y) \in \operatorname{supp}(\gamma_1) \\ u_2(y) - u_2(x) = |y - x| \ for \ all \ (x, y) \in \operatorname{supp}(\gamma_2). \end{cases}$$

- $(u_1, u_2) \in K$ *and* $(\Phi_1, \Phi_2, \nu) \in \Psi(f_1, f_2; \Theta)$ *are optimal for* (2.5) *and* (MMF), *respectively, if and only if the following optimality condition holds*

$$(2.8) \qquad \begin{cases} -\nabla \cdot \Phi_i = \Theta - \nu - f_i \ \ in \ \mathbb{R}^N, \ i = 1, 2 \\ \frac{\Phi_i}{|\Phi_i|} = \nabla_{|\Phi_i|} u_i \ \ |\Phi_i|\text{-}a.e. \ in \ \overline{\Omega}, \ i = 1, 2 \\ u_1 + u_2 = 0 \ \ \ \nu - a.e. \ in \ \overline{\Omega}. \end{cases}$$

Remark 2.4. If Θ is absolutely continuous, the optimality condition (2.8) can be simplified by using the usual gradient instead of the tangential gradient. In fact, in this case, it is known that Φ_i is also absolutely continuous (see for instance [1]) and that u is then differentiable $|\Phi_i|$-a.e.. By regularization via convolution, we can use u as test function in the first equation of (2.8), and using the duality (2.6), we get $\frac{\Phi_i}{|\Phi_i|} = \nabla u_i$ for $|\Phi_i|$-a.e. in $\overline{\Omega}$.

Roughly speaking, the dual maximization formulation (2.5), the problem (MMF) and the system (2.8) correspond to (2.1), (2.2) and the optimality condition (2.3), respectively. In the optimal mass transportation problem, these three formulations contain all the informations concerning the optimal transportation. This is extensively used to give numerical approximations for some variants of the optimal mass transport problem (see for instance [4, 5, 17]). For the optimal matching problem, we need to compute moreover the optimal matching measure. As an immediate consequence of the duality equalities in Theorem 2.3, the following result shows how this can be carried out.

COROLLARY 2.5. *Let ρ be an optimal matching measure and Φ_i be optimal flows for transporting f_i onto ρ, $i = 1, 2$. Then $(\Phi_1, \Phi_2, \nu) := (\Phi_1, \Phi_2, \Theta - \rho)$ is an optimal solution for the associated problem* (MMF). *Conversely, if (Φ_1, Φ_2, ν) is an optimal solution for the problem* (MMF) *and $\nu \leq \Theta$, then $\rho := \Theta - \nu$ is an optimal matching measure and Φ_i is an optimal flow of transporting f_i onto ρ, $i = 1, 2$.*

This result shows that the connection between (MMF) and (OM) lies in the condition $\nu \leq \Theta$ for an optimal solution (Φ_1, Φ_2, ν) of (MMF). Unfortunately, this does not hold in general as shown in the following example.
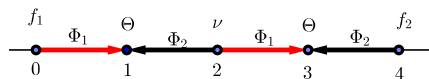

Fig. 1: Example of $\nu \nleq \Theta$

Example 2.6. On $\mathbb{R}$, taking $f_1 = \delta_0, f_2 = \delta_4, \Theta = \delta_1 + \delta_3$, where δ_i is the Dirac mass at i on $\mathbb{R}$ (see Figure 1). Let $\nu = \delta_2$ and Φ_1 be the optimal flow of transporting

$f_1 + \nu$ onto Θ (the corresponding plan is described as follows: $f_1 = \delta_0 \to \delta_1, \nu = \delta_2 \to \delta_3$) and Φ_2 be the optimal flow of transporting $f_2 + \nu$ onto Θ (the corresponding plan is described as follows: $f_2 = \delta_4 \to \delta_3, \nu = \delta_2 \to \delta_1$). Then (Φ_1, Φ_2, ν) is an optimal solution of the problem (MMF). Indeed, it is not difficult to see that the total cost of matching f_1 and f_2 into Θ is 4. The cost of the problem (MMF) corresponding to this choice of (Φ_1, Φ_2, ν) is also 4. From our duality results, we have the optimality of (Φ_1, Φ_2, ν), but $\nu \nleq \Theta$.

However, under the assumption (H), we prove that the constraint $\nu \leq \Theta$ is fulfilled. More precisely, we have

THEOREM 2.7. *Let $f_1, f_2, \Theta \in \mathcal{M}_b^+(\Omega)$ be Radon measures. Assume that* (H) *holds. Let $(\Phi_1, \Phi_2, \nu) \in \Psi(f_1, f_2; \Theta)$ be an optimal solution for the problem* (MMF) *and set $\rho := \Theta - \nu$. Then $\rho \geq 0$ and it is an optimal matching measure.*

Before ending up this section let us show how we use the ALG2 method to solve numerically the optimal matching problem (OM). For any $u = (u_1, u_2) \in V := C^1(\overline{\Omega}) \times C^1(\overline{\Omega})$, we set

$$\mathcal{F}(u) := \int u_1 \mathrm{d}f_1 + \int u_2 \mathrm{d}f_2 - \int (u_1 + u_2) \mathrm{d}\Theta$$

$$\Lambda(u) := (\nabla u_1, \nabla u_2, u_1 + u_2),$$

and, for any $(p, q, s) \in Z := C(\overline{\Omega})^N \times C(\overline{\Omega})^N \times C(\overline{\Omega})$, we set

$$\mathcal{G}(p, q, s) := \begin{cases} 0 & \text{if } |p(x)| \leq 1, |q(x)| \leq 1, s(x) \leq 0 \;\; \forall x \in \overline{\Omega} \\ +\infty & \text{otherwise.} \end{cases}$$

Then the problem

$$(2.9) \qquad\qquad \inf_{u \in V} \mathcal{F}(u) + \mathcal{G}(\Lambda u)$$

provides all information on the optimal matching problem. Indeed, u_1, u_2 give Kantorovich potentials and dual variables Φ_1, Φ_2, ν give information on optimal flows and optimal matching measure. To solve numerically the problem (2.9) and its Fenchel-Rockafellar dual formulation (MMF), we consider a regular triangulation $\mathcal{T}_h$ of $\overline{\Omega}$. We fix an integer $k \geq 1$, and we consider P_k the set of polynomials of degree less or equal than k. Let $E_h \subset H^1(\Omega)$ be the space of continuous functions on $\overline{\Omega}$ and belonging to P_k on each triangle of $\mathcal{T}_h$. We denote by Y_h the space of vectorial functions such that their restrictions belong to $(P_{k-1})^N$ on each triangle of $\mathcal{T}_h$. Let $f_{1,h}, f_{2,h}, \Theta_h \in E_h$ such that $f_{1,h}(\Omega) = f_{2,h}(\Omega) < \Theta_h(\Omega)$ and $f_{1,h} \rightharpoonup f_1, f_{2,h} \rightharpoonup f_2, \Theta_h \rightharpoonup \Theta$ weakly* in $\mathcal{M}_b(\overline{\Omega})$. Set $V_h := E_h \times E_h$ and $Z_h := Y_h \times Y_h \times E_h$. We approximate the problem (2.9) by the following finite-dimensional problem: For any $(u_1, u_2) \in V_h$, we set

$$\Lambda_h(u_1, u_2) := (\nabla u_1, \nabla u_2, u_1 + u_2) \in Z_h,$$

$$\mathcal{F}_h(u_1, u_2) := \langle u_1, f_{1,h} \rangle + \langle u_2, f_{2,h} \rangle - \langle u_1 + u_2, \Theta_h \rangle,$$

and for any $(p, q, s) \in Z_h$,

$$\mathcal{G}_h(p, q, s) := \begin{cases} 0 & \text{if } |p(x)| \leq 1, |q(x)| \leq 1, s(x) \leq 0 \text{ a.e. } x \in \Omega \\ +\infty & \text{otherwise.} \end{cases}$$

The finite-dimensional approximation of (2.9) is given by

$$(2.10) \qquad \inf_{(u_1, u_2) \in V_h} \mathcal{F}_h(u_1, u_2) + \mathcal{G}_h(\Lambda_h(u_1, u_2)).$$

Note that the cost functional does not change under the translation $\tilde{u}_1 := u_1 + C$, $\tilde{u}_2 := u_2 - C$, for $C \in \mathbb{R}$. In particular, the new couple

$$\left(\tilde{u}_1 := u_1 - \frac{|\Omega|}{2} \int_\Omega (u_1 - u_2), \ \tilde{u}_2 := u_2 + \frac{|\Omega|}{2} \int_\Omega (u_1 - u_2) \right)$$

satisfies $\int_\Omega \tilde{u}_1 = \int_\Omega \tilde{u}_2$ and is optimal if (u_1, u_2) is optimal.

The next theorem shows that (2.10) is a suitable approximation of (2.9) in the sense that primal and dual solutions converge to a solution of (2.9) (i.e. a solution of the maximization problem (2.5)) and a solution of (MMF).

THEOREM 2.8. *Let $(u_{1,h}, u_{2,h}) \in V_h$ be an optimal solution to the approximated problem (2.10) such that $\int_\Omega u_{1,h} = \int_\Omega u_{2,h}$ and let $(\Phi_{1,h}, \Phi_{2,h}, \nu_h)$ be an optimal dual solution to (2.10). Then, up to a subsequence, $(u_{1,h}, u_{2,h})$ converges uniformly to (u_1^*, u_2^*) an optimal solution of the dual maximization problem (2.5) and $(\Phi_{1,h}, \Phi_{2,h}, \nu_h)$ converges weakly* to (Φ_1, Φ_2, ν) an optimal solution of (MMF).*

At last, we solve the finite-dimensional problem (2.10) by using the ALG2 method. The details of the method are given in section 5. Here, we just give an illustration of our numerical results on the following example (see Figure 2): In $\mathbb{R}^2$, we take $\Omega = [0,1] \times [0,1]$,

$$f_1 = 4\chi_{[(x-0.2)^2 + (y-0.8)^2 < 0.01]}, \ f_2 = 4\chi_{[(x-0.2)^2 + (y-0.2)^2 < 0.01]},$$

and

$$\Theta = 4\chi_{[(x-0.8)^2 + (y-0.2)^2 < 0.04]},$$

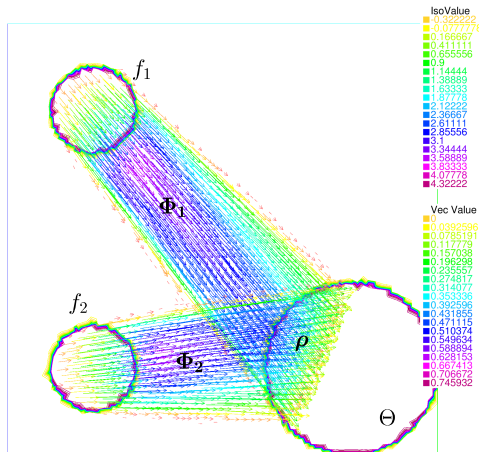where $\chi_A(x)$ is the characteristic function of A.

Fig. 2: Optimal matching measure ρ and optimal flows Φ_1 and Φ_2

3. Proofs of the equivalent formulations. The present section deals with the proofs of dual formulations as well as the connection between the minimal matching flow and the optimal constrained matching problem. To begin with, let us recall some well-known results in optimal transport theory with the Euclidean cost. It is well-known that the optimal transport problem (1.2) has the dual formulation

$$
(3.1) \qquad \max\left\{\int_\Omega u\mathrm{d}(\mu_2 - \mu_1) : u \in Lip_1(\Omega)\right\},
$$

i.e.,

$$
(3.2) \qquad W_1(\mu_1,\mu_2) = \min_{\gamma \in \Pi(\mu_1,\mu_2)}\left\{\int_{\Omega\times\Omega} |x-y|\mathrm{d}\gamma\right\} = \max_{u\in Lip_1(\Omega)}\left\{\int_\Omega u\mathrm{d}(\mu_2 - \mu_1)\right\}.
$$

It follows immediately from (3.2) that the 1-Wasserstein distance depends only on the difference of its arguments. On the other hand, the Fenchel–Rockafellar dual formulation of (3.1), called *minimal flow* problem, reads as

$$
(3.3) \qquad W_1(\mu_1,\mu_2) = \min_{\Phi\in\mathcal{M}_b(\overline{\Omega})^N}\left\{|\Phi|(\overline{\Omega}) : -\nabla\cdot\Phi = \mu_2 - \mu_1 \text{ in } \mathbb{R}^N\right\}.
$$

Optimizers of (3.1) and (3.3) are called *Kantorovich potentials* and *optimal flows*, respectively. These optimizers are completely characterized by the following PDE

$$
(3.4) \qquad
\begin{cases}
-\nabla\cdot\Phi = \mu_2 - \mu_1 \text{ in } \mathbb{R}^N \\
\frac{\Phi}{|\Phi|} = \nabla_{|\Phi|}u \quad |\Phi|\text{-a.e. in } \overline{\Omega} \\
u \in Lip_1(\Omega).
\end{cases}
$$

This fact can be done by using the integration by parts formula (see for instance [6]):

$$
(3.5) \qquad \forall \psi \in Lip_1(\Omega), \quad \langle \nabla\cdot\Phi, \psi\rangle = \int \nabla_{|\Phi|}\psi \frac{\Phi}{|\Phi|}\mathrm{d}|\Phi|.
$$

Coming back to the optimal constrained matching problem, we start with the Fenchel–Rockafellar duality between (2.5) and (MMF).

LEMMA 3.1. *Let $f_1, f_2, \Theta \in \mathcal{M}_b^+(\Omega)$ be Radon measures. We have*

$$
\max\left\{\int(u_1+u_2)\mathrm{d}\Theta - \int u_1\mathrm{d}f_1 - \int u_2\mathrm{d}f_2 : (u_1,u_2)\in K\right\}
$$
$$
= \min\left\{|\Phi_1|(\overline{\Omega}) + |\Phi_2|(\overline{\Omega}) : (\Phi_1,\Phi_2,\nu)\in\Psi(f_1,f_2;\Theta)\right\}.
$$

Keeping in mind the use of augmented Lagrangian method for numerical approximations, we use the Fenchel–Rockafellar duality technique to prove Lemma 3.1.

Proof. We observe that, using the standard smooth approximation by convolution,

$$
\max\left\{\int(u_1+u_2)\mathrm{d}\Theta - \int u_1\mathrm{d}f_1 - \int u_2\mathrm{d}f_2 : (u_1,u_2)\in K\right\}
$$
$$
= \sup\left\{\int(u_1+u_2)\mathrm{d}\Theta - \int u_1\mathrm{d}f_1 - \int u_2\mathrm{d}f_2 : (u_1,u_2)\in K,\, u_1,u_2\in C^1(\overline{\Omega})\right\}
$$
$$
= -\inf_{(u_1,u_2)\in C^1(\overline{\Omega})\times C^1(\overline{\Omega})}\mathcal{F}(u_1,u_2) + \mathcal{G}(\Lambda(u_1,u_2)),
$$

where $\mathcal{F}$, $\mathcal{G}$ and Λ are given in section 2. Now, using the Fenchel–Rockafellar dual theory (see for instance [12, Chapter III, Remark 4.2]), we have

(3.6)
$$\inf_{(u_1, u_2) \in C^1(\overline{\Omega}) \times C^1(\overline{\Omega})} \mathcal{F}(u_1, u_2) + \mathcal{G}(\Lambda(u_1, u_2))$$
$$= \max_{(\Phi_1, \Phi_2, \nu) \in \mathcal{M}_b(\overline{\Omega})^N \times \mathcal{M}_b(\overline{\Omega})^N \times \mathcal{M}_b(\overline{\Omega})} \left\{ -\mathcal{F}^*(-\Lambda^*(\Phi_1, \Phi_2, \nu)) - \mathcal{G}^*(\Phi_1, \Phi_2, \nu) \right\}.$$

So, it is enough to compute explicitly the above quantities. Since $\mathcal{F}$ is linear, the quantity $\mathcal{F}^*(-\Lambda^*(\Phi_1, \Phi_2, \nu))$ is finite (and is thus equal to 0) if and only if

$$\langle -\Lambda^*(\Phi_1, \Phi_2, \nu), (u_1, u_2) \rangle = \mathcal{F}(u_1, u_2) = \int u_1 \mathrm{d}f_1 + \int u_2 \mathrm{d}f_2 - \int (u_1 + u_2) \mathrm{d}\Theta$$

for all $(u_1, u_2) \in C^1(\overline{\Omega}) \times C^1(\overline{\Omega})$, or

$$-\langle \Phi_1, \nabla u_1 \rangle - \langle \Phi_2, \nabla u_2 \rangle - \langle \nu, u_1 + u_2 \rangle = \int u_1 \mathrm{d}f_1 + \int u_2 \mathrm{d}f_2 - \int (u_1 + u_2) \mathrm{d}\Theta$$

for all $(u_1, u_2) \in C^1(\overline{\Omega}) \times C^1(\overline{\Omega})$. This implies that (by taking $(u_1, u_2) = (u_1, 0)$ and $(u_1, u_2) = (0, u_2)$ as test functions)

$$-\nabla \cdot \Phi_i = \Theta - \nu - f_i \ \text{ in } \mathbb{R}^N, i = 1, 2.$$

Next, it is easy to see that

$$\mathcal{G}^*(\Phi_1, \Phi_2, \nu) = \begin{cases} |\Phi_1|(\overline{\Omega}) + |\Phi_2|(\overline{\Omega}) & \text{if } \nu \geq 0, \\ +\infty, & \text{otherwise.} \end{cases}$$

Therefore the proof is completed by substituting $\mathcal{F}^*$ and $\mathcal{G}^*$ into (3.6). $\square$

Following immediately from (3.3), we see that

(3.7)
$$\min \left\{ |\Phi_1|(\overline{\Omega}) + |\Phi_2|(\overline{\Omega}) : (\Phi_1, \Phi_2, \nu) \in \Psi(f_1, f_2; \Theta) \right\}$$
$$= \min_{\nu \in \mathcal{M}_b^+(\overline{\Omega})} \left\{ W_1(f_1 + \nu, \Theta) + W_1(f_2 + \nu, \Theta) : \nu(\overline{\Omega}) = \Theta(\Omega) - f_1(\Omega) \right\}.$$

This proposes an alternative formulation of (OM) that we prove directly in the following lemma.

LEMMA 3.2. *Assume that $f_1, f_2, \Theta \in \mathcal{M}_b^+(\Omega)$ are Radon measures. We have*

(3.8)
$$\min_{\rho \in \mathcal{M}_b^+(\Omega)} \left\{ W_1(f_1, \rho) + W_1(f_2, \rho) \, : \, \rho \leq \Theta, \ \rho(\Omega) = f_1(\Omega) \right\}$$
$$= \min_{\nu \in \mathcal{M}_b^+(\overline{\Omega})} \left\{ W_1(f_1 + \nu, \Theta) + W_1(f_2 + \nu, \Theta) : \nu(\overline{\Omega}) = \Theta(\Omega) - f_1(\Omega) \right\}.$$

Moreover, if ν is optimal for the right hand side of (3.8) then there exist $0 \leq \theta_1, \theta_2 \leq \Theta$, $\theta_1(\Omega) = \theta_2(\Omega) = f_1(\Omega)$ such that

(3.9)
$$W_1(f_1, \theta_2) = W_1(f_1, \theta_1) + W_1(\theta_1, \theta_2),$$

(3.10)
$$W_1(f_2, \theta_1) = W_1(f_2, \theta_2) + W_1(\theta_1, \theta_2)$$

and

(3.11)
$$W_1(\nu, \Theta - \theta_1) + W_1(\nu, \Theta - \theta_2) = W_1(\theta_1, \theta_2).$$

Proof. The existence of minimizers follows easily from the direct method. Now, fix any $\rho \in \mathcal{M}_b^+(\Omega)$ with $\rho \leq \Theta$, $\rho(\Omega) = f_1(\Omega)$ and set $\nu := \Theta - \rho$. We have

$$W_1(f_1, \rho) + W_1(f_2, \rho) = W_1(f_1 + \nu, \Theta) + W_1(f_2 + \nu, \Theta).$$

This shows that the left hand side of (3.8) is greater than or equal to the right hand side. Conversely, take $\nu \in \mathcal{M}_b^+(\overline{\Omega})$ with $\nu(\overline{\Omega}) = \Theta(\Omega) - f_1(\Omega)$. Consider the optimal plan γ_i between $f_i + \nu$ and Θ. It sends f_i to some $\theta_i \leq \Theta$, $i = 1, 2$ such that

$$W_1(f_1+\nu, \Theta) = W_1(f_1, \theta_1)+W_1(\nu, \Theta-\theta_1), \ W_1(f_2+\nu, \Theta) = W_1(f_2, \theta_2)+W_1(\nu, \Theta-\theta_2),$$

$$f_1(\Omega) = \theta_1(\Omega) = \theta_2(\Omega).$$

By triangular inequality and $W_1(\Theta - \theta_1, \Theta - \theta_2) = W_1(\theta_1, \theta_2)$, we get
(3.12)
$$\begin{aligned}
W_1(f_1 + \nu, \Theta) + W_1(f_2 + \nu, \Theta) &= W_1(f_1, \theta_1) + W_1(\nu, \Theta - \theta_1) + W_1(f_2, \theta_2) + W_1(\nu, \Theta - \theta_2) \\
&\geq W_1(f_1, \theta_1) + W_1(f_2, \theta_2) + W_1(\theta_1, \theta_2) \\
&\geq \max_{i=1,2} \{W_1(f_1, \theta_i) + W_1(f_2, \theta_i)\} \\
&\geq \min_{i=1,2} \{W_1(f_1, \theta_i) + W_1(f_2, \theta_i)\} \\
&\geq \min_{\rho \in \mathcal{M}_b^+(\Omega)} \left\{ W_1(f_1, \rho) + W_1(f_2, \rho) : \rho \leq \Theta, \ \rho(\Omega) = f_1(\Omega) \right\}.
\end{aligned}$$

Therefore, the proof of the equality (3.8) is done.
At last, if ν is optimal then all the inequalities in (3.12) become equalities. This implies the conclusions (3.9), (3.10) and (3.11). □

Proof of Theorem 2.3. The duality (2.6) follows from (3.7), Lemmas 3.1 and 3.2. It remains to show the optimality conditions (2.7) and (2.8). Let us begin with the proof of (2.7). For any admissible $(u_1, u_2) \in K$ and $(\gamma_1, \gamma_2) \in \pi(f_1, f_2; \Theta)$, taking $\rho := \pi_y \# \gamma_1 = \pi_y \# \gamma_2$, we have

(3.13)
$$\begin{aligned}
&\int (u_1 + u_2) \mathrm{d}\Theta - \int u_1 \mathrm{d}f_1 - \int u_2 \mathrm{d}f_2 \\
&\leq \int (u_1 + u_2) \mathrm{d}\rho - \int u_1 \mathrm{d}f_1 - \int u_2 \mathrm{d}f_2 \\
&= \int (u_1(y) - u_1(x)) \, \mathrm{d}\gamma_1 + \int (u_2(y) - u_2(x)) \, \mathrm{d}\gamma_2 \\
&\leq \int |x - y| \mathrm{d}\gamma_1 + \int |x - y| \mathrm{d}\gamma_2.
\end{aligned}$$

From the duality equalities (2.6), we deduce that (γ_1, γ_2) and (u_1, u_2) are optimal if and only if all the inequalities in (3.13) become equalities. The latter conditions read as

$$\begin{cases}
\int (u_1 + u_2) \, \mathrm{d}\Theta = \int (u_1 + u_2) \, \mathrm{d}\rho \\
\int (u_1(y) - u_1(x)) \, \mathrm{d}\gamma_1 = \int |x - y| \mathrm{d}\gamma_1 \\
\int (u_2(y) - u_2(x)) \, \mathrm{d}\gamma_2 = \int |x - y| \mathrm{d}\gamma_2.
\end{cases}$$

This condition is equivalent to (2.7).
For the proof of (2.8), we see that, for any admissible $(\Phi_1, \Phi_2, \nu) \in \Psi(f_1, f_2; \Theta)$, by

the integration by parts formula (3.5), we have
(3.14)
$$-\int u_1 \mathrm{d}f_1 - \int u_2 \mathrm{d}f_2 + \int (u_1 + u_2)\mathrm{d}\Theta \leq -\int u_1 \mathrm{d}f_1 - \int u_2 \mathrm{d}f_2 + \int (u_1 + u_2)\mathrm{d}(\Theta - \nu)$$

$$= \int u_1 \mathrm{d}(\Theta - \nu - f_1) + \int u_2 \mathrm{d}(\Theta - \nu - f_2)$$

$$= \int_{\overline{\Omega}} \frac{\Phi_1}{|\Phi_1|} \cdot \nabla_{|\Phi_1|} u_1 \mathrm{d}|\Phi_1| + \int_{\overline{\Omega}} \frac{\Phi_2}{|\Phi_2|} \cdot \nabla_{|\Phi_2|} u_2 \mathrm{d}|\Phi_2|$$

$$\leq |\Phi_1|(\overline{\Omega}) + |\Phi_2|(\overline{\Omega}).$$

Thanks to (2.6), (u_1, u_2) and (Φ_1, Φ_2, ν) are optimal if and only if all the inequalities in (3.14) become equalities. This is equivalent to the system (2.8).   □

We end up this section by giving the proof of Theorem 2.7 concerning the relation between (MMF) and (OM).

Proof of Theorem 2.7. Assume that (Φ_1, Φ_2, ν) is optimal for (MMF) which implies that ν is optimal for the alternative formulation of (OM) given in Lemma 3.2. Take θ_1 and θ_2 given by Lemma 3.2. Then (3.9) and (3.10) mean that θ_2 is on a geodesic joining θ_1 to f_2 and θ_1 is on a geodesic joining θ_2 to f_1. The assumption (H) imposes that $\theta_1 = \theta_2$. To convince the reader, take $\gamma_{f_1,1}, \gamma_{1,2}$ and γ_{2,f_2} the optimal plans from f_1 to θ_1, from θ_1 to θ_2 and from θ_2 to f_2. Using the gluing lemma (see e.g. [21, Lemma 7.6]), we build $\gamma_{f_1,1,2}$ obtained by gluing $\gamma_{f_1,1}$ to $\gamma_{1,2}$ and $\gamma_{1,2,f_2}$ obtained by gluing $\gamma_{1,2}$ to γ_{2,f_2}. Then, it holds

$$W_1(f_1, \theta_1) + W_1(\theta_1, \theta_2) = \int_{\overline{\Omega}^3} |x_1 - z_1| + |z_1 - z_2| \mathrm{d}\gamma_{f_1,1,2}(x_1, z_1, z_2)$$

$$= W_1(f_1, \theta_2) \leq \int_{\overline{\Omega}^3} |x_1 - z_2| \mathrm{d}\gamma_{f_1,1,2}(x_1, z_1, z_2).$$

By triangular inequality and the continuity of the integrands, we get
(3.15)
$$|x_1 - z_1| + |z_1 - z_2| = |x_1 - z_2|, \text{ i.e., } z_1 \in [x_1, z_2] \text{ for all } (x_1, z_1, z_2) \in \mathrm{supp}(\gamma_{f_1,1,2}).$$

In the same way,
(3.16)
$$|z_1 - z_2| + |z_2 - x_2| = |z_1 - x_2|, \text{ i.e., } z_2 \in [z_1, x_2] \text{ for all } (z_1, z_2, x_2) \in \mathrm{supp}(\gamma_{1,2,f_2}).$$

If there exists $(z_1, z_2) \in \mathrm{supp}(\gamma_{1,2})$ such that $z_1 \neq z_2$ then, using (3.15) and (3.16), there are $x_1 \in \mathrm{supp}(f_1), x_2 \in \mathrm{supp}(f_2)$ such that $z_1 \in [x_1, z_2], z_2 \in [z_1, x_2]$ and therefore $z_1, z_2 \in [x_1, x_2]$ (by $z_1 \neq z_2$), a contradiction with the assumption (H). This shows that $\theta_1 = \theta_2$. At last, using (3.11), we obtain $\nu = \Theta - \theta_i \leq \Theta$, $i = 1, 2$.   □

4. Uniqueness of optimal matching measure. This section concerns the proof for the uniqueness of optimal matching measure in Theorem 2.1. Let ρ be an optimal matching measure. By Corollary 2.5 and Theorem 2.3, setting $\nu := \Theta - \rho$, we have

(4.1)
$$\begin{cases} -\nabla \cdot \Phi_i = \Theta - \nu - f_i & \text{in } \mathbb{R}^N, \ i = 1, 2 \\ \frac{\Phi_i}{|\Phi_i|} = \nabla_{|\Phi_i|} u_i & |\Phi_i|\text{-a.e. in } \overline{\Omega}, \ i = 1, 2 \\ u_1 + u_2 = 0 & \nu\text{-a.e. in } \overline{\Omega}, \end{cases}$$

where Φ_i and u_i are optimal flows and Kantorovich potentials, respectively. To prove the uniqueness of optimal matching measure under the assumption (H), we establish precise expression of ν w.r.t. Θ and u_i, for $i = 1, 2$. More precisely, we have

PROPOSITION 4.1. *Assume that* $\Theta \in L^1(\Omega)^+$, $f_i \in \mathcal{M}_b^+(\Omega)$ $i = 1, 2$ *and that* $(\Phi_1, \Phi_2, \nu, u_1, u_2) \in \mathcal{M}_b(\overline{\Omega})^N \times \mathcal{M}_b(\overline{\Omega})^N \times \mathcal{M}_b^+(\overline{\Omega}) \times Lip_1(\Omega) \times Lip_1(\Omega)$ *satisfies the PDE* (4.1). *Under the assumption* (H), *we have* $\nu \leq \Theta$ *and*

$$\nu = \Theta \llcorner [u_1 + u_2 = 0],$$

where the set $[u_1 + u_2 = 0] := \{x \in \Omega : u_1(x) + u_2(x) = 0\}$.

The proof of this result follows as a consequence of the following lemmas.

LEMMA 4.2. *Let* u, v *be 1-Lipschitz functions on* Ω *such that* $u + v \leq 0$ *on* Ω. *Assume that* $u(y_1) - u(x_1) = |y_1 - x_1|$ *and that* $u(z) + v(z) = 0$ *for some* $z \in [x_1, y_1]$ *the segment joining* x_1 *to* y_1. *Then*

$$(4.2) \qquad u(s) + v(s) = 0 \quad \forall s \in [z, y_1].$$

Moreover, if $x_2 \in \Omega$ *is such that* $v(y_1) - v(x_2) = |y_1 - x_2|$, *then* z, y_1 *and* x_2 *are aligned.*

Proof. We prove first that

$$(4.3) \qquad v(s) = v(z) - |s - z| \quad \forall s \in [z, y_1].$$

Since u is 1-Lipschitz and $u(y_1) - u(x_1) = |y_1 - x_1|$, we have

$$(4.4) \qquad u(s) = u(z) + |s - z| \quad \forall s \in [z, y_1].$$

Using the fact that $u + v \leq 0$, we have

$$\begin{aligned} v(s) &\leq -u(s) \\ &= -u(z) - |s - z| \\ &= v(z) - |s - z| \quad \forall s \in [z, y_1]. \end{aligned}$$

Since v is 1-Lipschitz, we get the equality (4.3) and thus (4.2) (by $u(z) + v(z) = 0$ and (4.4)).
At last, following (4.3) with $s = y_1$,

$$v(y_1) = v(z) - |y_1 - z|,$$

we get, for x_2 as in the hypothesis,

$$\begin{aligned} |z - x_2| &\geq v(z) - v(x_2) \\ &= |z - y_1| + v(y_1) - v(x_2) \\ &= |z - y_1| + |y_1 - x_2|. \end{aligned}$$

This implies that z, y_1 and x_2 are aligned. □

Next, we need the following behaviors of f_i and Φ_i, $i = 1, 2$ on the set $[u_1 + u_2 = 0]$.

LEMMA 4.3. *Assume that* $f_1, f_2, \Theta \in \mathcal{M}_b^+(\Omega)$ *and that* (H) *holds. Suppose that* $(\Phi_1, \Phi_2, \nu, u_1, u_2)$ *satisfies the PDE* (4.1). *Then*

(i) $f_1 \llcorner [u_1 + u_2 = 0] = f_2 \llcorner [u_1 + u_2 = 0] = 0$;

(ii) $\mathcal{L}^N(\mathrm{supp}(\Phi_1) \cap [u_1 + u_2 = 0]) = \mathcal{L}^N(\mathrm{supp}(\Phi_2) \cap [u_1 + u_2 = 0]) = 0$, *where* $\mathcal{L}^N$ *is the Lebesgue measure on* $\mathbb{R}^N$.

Proof. (i) Thanks to Theorem 2.7, we have $\nu \leq \Theta$. Let us first show that $f_1 \llcorner [u_1 + u_2 = 0] = 0$. Assume on the contrary that the conclusion is not true. Then there exist $x_1 \in [u_1 + u_2 = 0]$ and $y_1 \in \mathrm{supp}(\Theta - \nu)$ such that $(x_1, y_1) \in \mathrm{supp}(\gamma_1)$, where γ_1 is the optimal plan from f_1 to $\Theta - \nu$. Since u_1 is a Kantorovich potential for $W_1(f_1, \Theta - \nu)$ (as explained in the characterization PDE (3.4)), we get

$$u_1(y_1) - u_1(x_1) = |x_1 - y_1|.$$

Similarly, since $y_1 \in \mathrm{supp}(\Theta - \nu)$, there is $x_2 \in \mathrm{supp}(f_2)$ such that $(x_2, y_1) \in \mathrm{supp}(\gamma_2)$ and

$$u_2(y_1) - u_2(x_2) = |x_2 - y_1|.$$

Using Lemma 4.2, we deduce that x_1, y_1, x_2 are aligned which contradicts with (H). In the same way, we get $f_2 \llcorner [u_1 + u_2 = 0] = 0$.

(ii) Now, we prove that

(4.5)
$$\mathcal{L}^N(\mathrm{supp}(\Phi_1) \cap [u_1 + u_2 = 0]) = 0.$$

Thanks to [1, Corollary 6.1] or [2, Theorem 6.2], we know that the set E of right endpoints of maximal transport rays w.r.t. the Kantorovich potential u_1 satisfies $\mathcal{L}^N(E) = 0$. To prove (4.5), it is enough to show that

$$\mathrm{supp}(\Phi_1) \cap [u_1 + u_2 = 0] \subset E.$$

Assume on the contrary that there exists $z \in \mathrm{supp}(\Phi_1) \cap [u_1 + u_2 = 0]$ such that $z \notin E$. Then there exists $(x_1, y_1) \in \mathrm{supp}(f_1) \times \mathrm{supp}(\Theta - \nu)$ such that $z \in [x_1, y_1[$ and $u_1(y_1) = u_1(x_1) + |y_1 - x_1|$. On the other hand, since $y_1 \in \mathrm{supp}(\Theta - \nu)$, there exists $x_2 \in \mathrm{supp}(f_2)$ such that

$$u_2(y_1) - u_2(x_2) = |y_1 - x_2|.$$

Since $u_1(z) + u_2(z) = 0$, using Lemma 4.2, we deduce that z, y_1 and x_2 are on a straight line. Thus x_1, y_1 and x_2 are aligned (by $z \in [x_1, y_1[$). This is again a contradiction with (H). □

Proof of Proposition 4.1. We use notations of the above lemmas. By Theorem 2.7, we have $\nu \leq \Theta$. Following directly from the PDE (4.1), we have

$$-\nabla \cdot (\Phi_1 + \Phi_2) = 2(\Theta - \nu) - (f_1 + f_2).$$

This implies that

the measure $2(\Theta - \nu) - (f_1 + f_2)$ is concentrated on $\mathrm{supp}(\Phi_1 + \Phi_2)$.

In particular, $2(\Theta - \nu) \llcorner [u_1 + u_2 = 0] - (f_1 + f_2) \llcorner [u_1 + u_2 = 0]$ is concentrated on $[u_1 + u_2 = 0] \cap \mathrm{supp}(\Phi_1 + \Phi_2)$. Thanks to Lemma 4.3 (i), we deduce that $2(\Theta - \nu) \llcorner [u_1 + u_2 = 0]$ is concentrated on $[u_1 + u_2 = 0] \cap \mathrm{supp}(\Phi_1 + \Phi_2)$. Since $(\Theta - \nu) \in L^1$, using Lemma 4.3 (ii) and the fact that $\mathrm{supp}(\Phi_1 + \Phi_2) \subset \mathrm{supp}(\Phi_1) \cup \mathrm{supp}(\Phi_2)$, we get

$$(\Theta - \nu) \llcorner [u_1 + u_2 = 0] = 0.$$

Since $u_1 + u_2 = 0$ ν–a.e. in $\overline{\Omega}$, we deduce that

$$\nu = \Theta \llcorner [u_1 + u_2 = 0].$$

□

We can now conclude the proof of the uniqueness of optimal matching measures with the following arguments.

Proof of Theorem 2.1. We fix a maximizer (u_1, u_2) of the maximization problem (2.5). Then if ρ_1 and ρ_2 are optimal matching measures then $\nu_i := \Theta - \rho_i$, $i = 1, 2$ satisfies the PDE (4.1). Thanks to Proposition 4.1, we get

□

$$\rho_1 = \Theta - \nu_1 = \Theta \llcorner [u_1 + u_2 < 0] = \Theta - \nu_2 = \rho_2.$$

Remark 4.4. Following from the proof, the unique optimal matching measure has the form

$$\rho = \Theta \llcorner [u_1 + u_2 < 0],$$

for any maximizer (u_1, u_2) of the dual problem (2.5).

5. Numerical analysis for the problem. The present section concerns on numerical aspects of the matching problem.

5.1. Convergence of the discretization.

Proof of Theorem 2.8. The optimality condition of (2.10) is (see for instance [12])

$$-\Lambda_h^*(\Phi_{1,h}, \Phi_{2,h}, \nu_h) = \partial \mathcal{F}_h(u_{1,h}, u_{2,h}) \text{ in } V_h^*$$

(or equivalently, $-\langle (\Phi_{1,h}, \Phi_{2,h}, \nu_h), \Lambda_h(u,v) \rangle = \mathcal{F}_h(u,v) \ \forall (u,v) \in V_h$), and

$$(\Phi_{1,h}, \Phi_{2,h}, \nu_h) \in \partial \mathcal{G}_h(\Lambda_h(u_{1,h}, u_{2,h})).$$

Writing these in details, we have
(5.1)
$$-\langle \Phi_{1,h}, \nabla u \rangle - \langle \Phi_{2,h}, \nabla v \rangle - \langle \nu_h, u+v \rangle = \langle f_{1,h}, u \rangle + \langle f_{2,h}, v \rangle - \langle \Theta_h, u+v \rangle \ \forall (u,v) \in V_h,$$

and

$$\begin{cases} \Phi_{1,h} \in \partial \mathbb{I}_{B_{(Y_h, \|\cdot\|_\infty)}}(\nabla u_{1,h}) \\ \Phi_{2,h} \in \partial \mathbb{I}_{B_{(Y_h, \|\cdot\|_\infty)}}(\nabla u_{2,h}) \\ \nu_h \in \partial \mathbb{I}_{\{z \in E_h : z \leq 0\}}(u_{1,h} + u_{2,h}). \end{cases}$$

Choosing test functions $u \equiv 1, v \equiv 0$ in (5.1) and using the fact that $f_{1,h}(\Omega) < \Theta_h(\Omega)$, we have that $\nu_h \neq 0$ and $\{\nu_h\}$ is bounded in $L^1(\Omega)$. Since $\nu_h \in \partial \mathbb{I}_{\{z \in E_h : z \leq 0\}}(u_{1,h} + u_{2,h})$, we get $\nu_h \geq 0$ and $\langle u_{1,h} + u_{2,h}, \nu_h \rangle = 0$. Since $\nu_h \neq 0$, there exists $x_h \in \overline{\Omega}$ such that $u_{1,h}(x_h) + u_{2,h}(x_h) = 0$. Combining this with the fact $\int u_{1,h} = \int u_{2,h}$, we imply that $\{u_{1,h}\}$ and $\{u_{2,h}\}$ are bounded in $C(\overline{\Omega})$. Since $u_{1,h}, u_{2,h}$ are 1-Lipschitz functions, up to a subsequence (using the Ascoli–Arzela Theorem),

$$u_{1,h} \rightrightarrows u_1^*, u_{2,h} \rightrightarrows u_2^* \text{ uniformly on } \overline{\Omega}.$$

It is clear that u_1^*, u_2^* are 1-Lipschitz and $u_1^* + u_2^* \leq 0$ on $\overline{\Omega}$.

On the other hand, using the optimality of $(u_{1,h}, u_{2,h})$, $(\Phi_{1,h}, \Phi_{2,h}, \nu_h)$ and the duality equality for (2.10), we have

$$\mathcal{F}_h(u_{1,h}, u_{2,h}) + \mathcal{G}_h(\Lambda_h(u_{1,h}, u_{2,h})) = -\mathcal{F}_h^*(-\Lambda_h^*(\Phi_{1,h}, \Phi_{2,h}, \nu_h)) - \mathcal{G}_h^*(\Phi_{1,h}, \Phi_{2,h}, \nu_h),$$

or more explicitly,

$$\langle f_{1,h}, u_{1,h} \rangle + \langle f_{1,h}, u_{2,h} \rangle - \langle \Theta_h, u_{1,h} + u_{2,h} \rangle$$

(5.2)
$$= -\sup\left\{ \langle \Phi_{1,h}, q \rangle : q \in Y_h, |q(x)| \le 1, \text{ a.e. } x \in \Omega \right\}$$

$$- \sup\left\{ \langle \Phi_{2,h}, q \rangle : q \in Y_h, |q(x)| \le 1, \text{ a.e. } x \in \Omega \right\}.$$

Using the boundedness of $(u_{1,h}, u_{2,h})$, we obtain that $\Phi_{1,h}$ and $\Phi_{2,h}$ are bounded in $L^1(\Omega)^N$. Thus, up to a subsequence,

$$(\Phi_{1,h}, \Phi_{2,h}, \nu_h) \rightharpoonup (\Phi_1, \Phi_2, \nu) \quad \text{weakly* in } \mathcal{M}_b(\overline{\Omega})^N \times \mathcal{M}_b(\overline{\Omega})^N \times \mathcal{M}_b(\overline{\Omega}).$$

Then (Φ_1, Φ_2, ν) is feasible for the problem (MMF). Indeed, thanks to (5.1) and the nonnegativity of ν_h, we obtain that

$$\langle \Phi_1, \nabla u \rangle + \langle \Phi_2, \nabla v \rangle + \langle \nu, u+v \rangle = -\langle f_1, u \rangle - \langle f_2, v \rangle + \langle \Theta, u+v \rangle \; \forall (u,v) \in V := C^1(\overline{\Omega}) \times C^1(\overline{\Omega}),$$

and

$$\nu \ge 0,$$

i.e., the feasibility of (Φ_1, Φ_2, ν). Now, we show the optimality of (u_1^*, u_2^*) and (Φ_1, Φ_2, ν). Thanks to Theorem 2.3, it is sufficient to show that

(5.3)
$$-\langle f_1, u_1^* \rangle - \langle f_2, u_2^* \rangle + \langle \Theta, u_1^* + u_2^* \rangle \ge |\Phi_1|(\overline{\Omega}) + |\Phi_2|(\overline{\Omega}).$$

To this aim, let $q_1, q_2 \in C(\overline{\Omega})^N$ be such that $|q_1(x)| \le 1, |q_2(x)| \le 1 \; \forall x \in \overline{\Omega}$ and $q_{1,h}, q_{2,h} \in Y_h$ be such that $\|q_{i,h} - q_i\|_{L^\infty(\Omega)} \to 0$ as $h \to 0, i = 1, 2$. By the fact that

$$|q_{i,h}(x)| = |q_i(x)| + |q_{i,h}(x)| - |q_i(x)| \le 1 + O(h) \text{ a.e. } x \in \Omega,$$

and, taking $\frac{q_{i,h}}{1+O(h)}$ if necessary, we can assume that

$$q_{i,h} \in Y_h, |q_{i,h}(x)| \le 1 \text{ a.e. } x \in \Omega \text{ and } \|q_{i,h} - q_i\|_{L^\infty(\Omega)} \to 0 \text{ as } h \to 0, i = 1, 2.$$

We see that

$$\langle \Phi_1, q_1 \rangle = \langle \Phi_{1,h}, q_{1,h} \rangle + \langle \Phi_1 - \Phi_{1,h}, q_1 \rangle + \langle \Phi_{1,h}, q_1 - q_{1,h} \rangle$$
$$\le \sup\left\{ \langle \Phi_{1,h}, q \rangle : q \in Y_h, |q(x)| \le 1 \text{ a.e. } x \in \Omega \right\} + O(h).$$

Similarly,

$$\langle \Phi_2, q_2 \rangle \le \sup\left\{ \langle \Phi_{2,h}, q \rangle : q \in Y_h, |q(x)| \le 1 \text{ a.e. } x \in \Omega \right\} + O(h).$$

Combining these with (5.2),

$$-\langle f_{1,h}, u_{1,h} \rangle - \langle f_{2,h}, u_{2,h} \rangle + \langle \Theta_h, u_{1,h} + u_{2,h} \rangle + O(h) \ge \langle \Phi_1, q_1 \rangle + \langle \Phi_2, q_2 \rangle.$$

Letting $h \to 0$ and taking supremum in q_1, q_2, we get the desired inequality (5.3). $\square$

5.2. Solving the discretized problem. Our task is now to solve the finite-dimensional problem (2.10). We recall first the ALG2 method. Let V and Z be two Hilbert spaces. Let us consider the problem

$$(5.4) \qquad \inf_{u \in V} \mathcal{F}(u) + \mathcal{G}(\Lambda u)$$

where $\mathcal{F} : V \longrightarrow (-\infty, +\infty]$ and $\mathcal{G} : Z \longrightarrow (-\infty, +\infty]$ are convex, l.s.c. and $\Lambda \in \mathcal{L}(V, Z)$. The Fenchel–Rockafellar dual problem is

$$\sup_{\sigma \in Z} \left(-\mathcal{F}^*(-\Lambda^* \sigma) - \mathcal{G}^*(\sigma) \right).$$

Introducing new variable $q \in Z$ to the primal problem (5.4), we rewrite it in the form

$$\inf_{(u,q) \in V \times Z \,:\, \Lambda u = q} \mathcal{F}(u) + \mathcal{G}(q).$$

Consider the augmented Lagrangian w.r.t. the quadratic term

$$L(u, q; \sigma) := \mathcal{F}(u) + \mathcal{G}(q) + \langle \sigma, \Lambda u - q \rangle + \frac{r}{2} |\Lambda u - q|^2, \; r > 0.$$

The ALG2 algorithm (see e.g. [10, 15]) is a primal-dual method (i.e., providing numerical solutions of both primal and dual problems) consisting of 3 steps. Known q_i, σ_i, the next step $(u_{i+1}, q_{i+1}, \sigma_{i+1})$ is computed as follows: Fix any parameter $r > 0$ (in practice $r = 1, 2$),

- Step 1: Minimizing $\inf_u L(u, q_i; \sigma_i)$, i.e.

$$u_{i+1} = \arg\min_{u \in V} \left\{ \mathcal{F}(u) + \langle \sigma_i, \Lambda u \rangle + \frac{r}{2} |\Lambda u - q_i|^2 \right\}.$$

- Step 2: Minimizing $\inf_{q \in Z} L(u_{i+1}, q; \sigma_i)$, i.e.

$$q_{i+1} = \arg\min_{q \in Z} \left\{ \mathcal{G}(q) - \langle \sigma_i, q \rangle + \frac{r}{2} |\Lambda u_{i+1} - q|^2 \right\}.$$

- Step 3: Update the multiplier σ,

$$\sigma_{i+1} = \sigma_i + r(\Lambda u_{i+1} - q_{i+1}).$$

Now, apply the ALG2 to our discretized problem (2.10). To simplify the notations, let us drop out the subscript h in $u_{1,h}, u_{2,h}, \Phi_{1,h}, \Phi_{2,h}, \nu_h$. We denote by $\Phi_1^i, \Phi_2^i, \nu^i, u_1^i, u_2^i, p^i, q^i, s^i$ the values at iteration i. Known $(p^i, q^i, s^i), (\Phi_1^i, \Phi_2^i, \nu^i)$,

- Step 1:

$$(u_1^{i+1}, u_2^{i+1}) = \arg\min_{(u_1, u_2) \in V_h} \mathcal{F}_h(u_1, u_2) + \langle (\Phi_1^i, \Phi_2^i, \nu^i), \Lambda_h(u_1, u_2) \rangle + \frac{r}{2} |\Lambda_h(u_1, u_2) - (p^i, q^i, s^i)|^2$$

$$= \arg\min_{(u_1, u_2) \in V_h} \langle u_1, f_{1,h} \rangle + \langle u_2, f_{2,h} \rangle - \langle u_1 + u_2, \Theta_h \rangle + \langle \Phi_1^i, \nabla u_1 \rangle + \langle \Phi_2^i, \nabla u_2 \rangle$$

$$+ \langle \nu^i, u_1 + u_2 \rangle + \frac{r}{2} |\nabla u_1 - p^i|^2 + \frac{r}{2} |\nabla u_2 - q^i|^2 + \frac{r}{2} |u_1 + u_2 - s^i|^2.$$

- Step 2:

$$(p^{i+1}, q^{i+1}, s^{i+1}) = \arg\min_{(p,q,s) \in Z_h} \mathcal{G}_h(p, q, s) - \langle (\Phi_1^i, \Phi_2^i, \nu^i), (p, q, s) \rangle + \frac{r}{2} |\Lambda_h(u_1^{i+1}, u_2^{i+1}) - (p, q, s)|^2$$

$$= \arg\min_{(p,q,s) \in Z_h} \mathbb{I}_{B_{(Y_h, \|\cdot\|_\infty)}}(p) + \mathbb{I}_{B_{(Y_h, \|\cdot\|_\infty)}}(q) + \mathbb{I}_{\{s \in E_h \,:\, s \le 0\}}(s) - \langle \Phi_1^i, p \rangle - \langle \Phi_2^i, q \rangle - \langle \nu^i, s \rangle$$

$$+ \frac{r}{2} |\nabla u_1^{i+1} - p|^2 + \frac{r}{2} |\nabla u_2^{i+1} - q|^2 + \frac{r}{2} |u_1^{i+1} + u_2^{i+1} - s|^2.$$

- Step 3:

$$(\Phi_1^{i+1}, \Phi_2^{i+1}, \nu^{i+1}) = (\Phi_1^i, \Phi_2^i, \nu^i) + r(\nabla u_1^{i+1} - p^{i+1}, \nabla u_2^{i+1} - q^{i+1}, u_1^{i+1} + u_2^{i+1} - s^{i+1}).$$

Let us give more details of the above iteration. Overall, Step 1 is a quadratic programming. Step 2 can be calculated explicitly and Step 3 updates obviously. We denote by $\mathrm{Proj}_C(.)$ the projection onto a closed convex subset C.

- In Step 1: We split variables u_1 and u_2, i.e., first minimizing w.r.t. u_1 and using u_1^{i+1} to calculate u_2^{i+1}.

1. For u_1^{i+1},

$$u_1^{i+1} = \arg\min_{u \in E_h} \langle u, f_{1,h} - \Theta_h \rangle + \langle \Phi_1^i, \nabla u \rangle + \langle \nu^i, u \rangle + \frac{r}{2} |\nabla u - p^i|^2 + \frac{r}{2} |u + u_2^i - s^i|^2.$$

This is a quadratic problem with the associated linear equation:

$$r \langle \nabla u_1^{i+1}, \nabla \phi \rangle + r \langle u_1^{i+1}, \phi \rangle = \langle \Theta_h - f_{1,h} - \nu^i, \phi \rangle + \langle r p^i - \Phi_1^i, \nabla \phi \rangle + r \langle s^i - u_2^i, \phi \rangle \ \forall \phi \in E_h.$$

2. Similarly for u_2^{i+1},

$$r \langle \nabla u_2^{i+1}, \nabla \phi \rangle + r \langle u_2^{i+1}, \phi \rangle = \langle \Theta_h - f_{2,h} - \nu^i, \phi \rangle + \langle r q^i - \Phi_2^i, \nabla \phi \rangle + r \langle s^i - u_1^{i+1}, \phi \rangle \ \forall \phi \in E_h.$$

- In Step 2: Since the function $\mathcal{G}(p, q, s)$ has the form of $\mathcal{G}_1(p) + \mathcal{G}_2(q) + \mathcal{G}_3(s)$, we solve them separately.

1. For s^{i+1}, if we choose P_2 finite element for s^{i+1},

$$s^{i+1} = \arg\min_{s \in P_2} \left\{ \mathbb{I}_{[s \leq 0]} - \langle \nu^i, s \rangle + \frac{r}{2} |u_1^{i+1} + u_2^{i+1} - s|^2 \right\}$$

$$= \mathrm{Proj}_{\{s \in P_2 : s \leq 0\}} \left(u_1^{i+1} + u_2^{i+1} + \frac{\nu^i}{r} \right).$$

This is computed in pointwise, i.e., at vertices x_k of a given grid,

$$s^{i+1}(x_k) = \mathrm{Proj}_{[r \in \mathbb{R}: \, r \leq 0]} \left(u_1^{i+1}(x_k) + u_2^{i+1}(x_k) + \frac{\nu^i(x_k)}{r} \right).$$

2. For p^{i+1} and q^{i+1}, similarly, at each vertice x_l,

$$p^{i+1}(x_l) = \mathrm{Proj}_{\overline{B}(0,1)} \left(\nabla u_1^{i+1}(x_l) + \frac{\Phi_1^i(x_l)}{r} \right)$$

and

$$q^{i+1}(x_l) = \mathrm{Proj}_{\overline{B}(0,1)} \left(\nabla u_2^{i+1}(x_l) + \frac{\Phi_2^i(x_l)}{r} \right).$$

6. Numerical experiments. We base on [4, 5, 17] and on FreeFem++ [16] to give some numerical examples. We use P_2 finite element for u_1^i, u_2^i, s^i, ν^i and P_1 finite element for $\Phi_1^i, \Phi_2^i, p^i, q^i$.

6.1. Stopping criterion. The measures f_1, f_2 and Θ are approximated by non-negative regular functions that we denote again by f_1, f_2 and Θ. We use the PDE of optimality condition as stopping criteria:

1. MIN $:= \min \left\{ \min_{\overline{\Omega}} \left(-u_1(x) - u_2(x) \right), \min_{\overline{\Omega}} \nu(x) \right\}$.

2. Lip $:= \max \left\{ \max_{\overline{\Omega}} |\nabla u_1(x)|, \max_{\overline{\Omega}} |\nabla u_2(x)| \right\}$.

3. DIV $:= \frac{Div_1 + Div_2}{2}$, where

$$Div_1 := \|\nabla \cdot \Phi_1 + \Theta - \nu - f_1\|_{L^2}, \ \ Div_2 := \|\nabla \cdot \Phi_2 + \Theta - \nu - f_2\|_{L^2}.$$

4. DUAL $:= \frac{Dual_1 + Dual_2}{2}$, with

$$Dual_1 := \||\Phi_1(x)| - \Phi_1(x) \cdot \nabla u_1\|_{L^2}, \ \ Dual_2 := \||\Phi_2(x)| - \Phi_2(x) \cdot \nabla u_2\|_{L^2}.$$

We expect that MIN ≥ 0, Lip ≤ 1; DIV and DUAL are small.

6.2. Some examples. In all the examples below, we take $\Omega = [0,1] \times [0,1]$ and work on a grid 60×60. Computation time for each example is about 8 minutes on a PC Mac OSX 10.9.

Example 6.1. We take

$$f_1 = 4\chi_{[(x-0.2)^2 + (y-0.8)^2 < 0.01]},$$

$$f_2 = 2\chi_{[(x-0.8)^2 + (y-0.8)^2 < 0.01]} + 2\chi_{[(x-0.2)^2 + (y-0.2)^2 < 0.01]},$$

$$\Theta = 4\chi_{[(x-0.5)^2 + (y-0.5)^2 < 0.04]}.$$

The optimal matching measure and optimal flows are given in Figure 3. Stopping criterion is given in Figure 4.

Fig. 3: Optimal matching measure and optimal flows

(a) log_{10} of DIV and DUAL errors
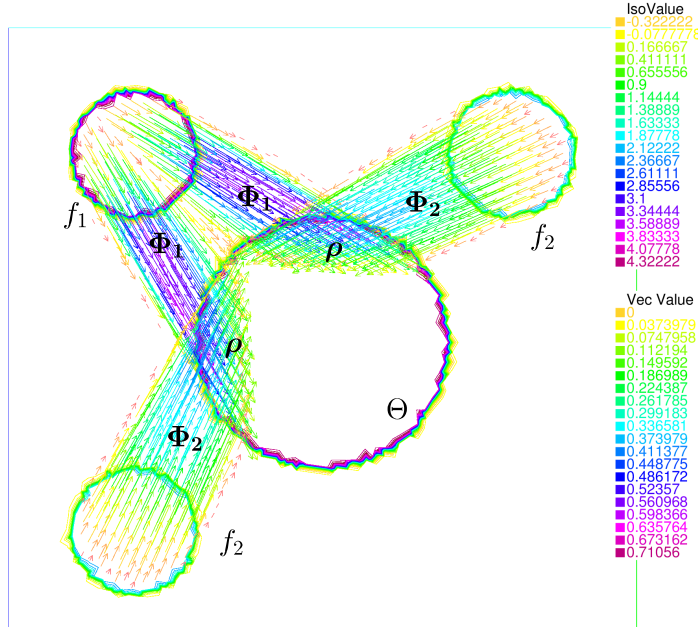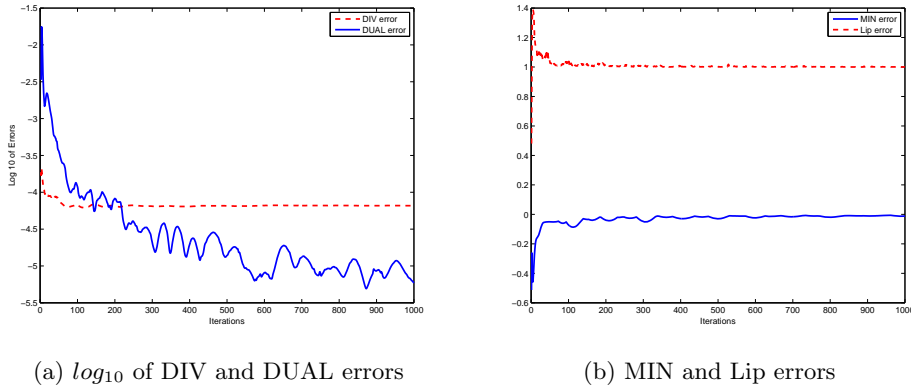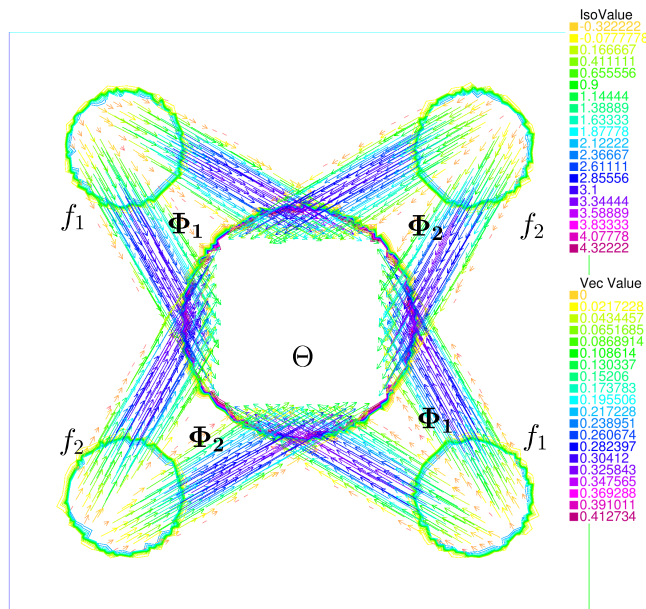
(b) MIN and Lip errors

Fig. 4: Stopping criterion

Example 6.2. The results are given in Figures 5 and 6 for

$$f_1 = 2\chi_{[(x-0.2)^2+(y-0.8)^2<0.01]} + 2\chi_{[(x-0.8)^2+(y-0.2)^2<0.01]},$$

$$f_2 = 2\chi_{[(x-0.8)^2+(y-0.8)^2<0.01]} + 2\chi_{[(x-0.2)^2+(y-0.2)^2<0.01]},$$

$$\Theta = 4\chi_{[(x-0.5)^2+(y-0.5)^2<0.04]}.$$

Fig. 5: Optimal matching measure and optimal flows

(a) log_{10} of DIV and DUAL errors   (b) MIN and Lip errors
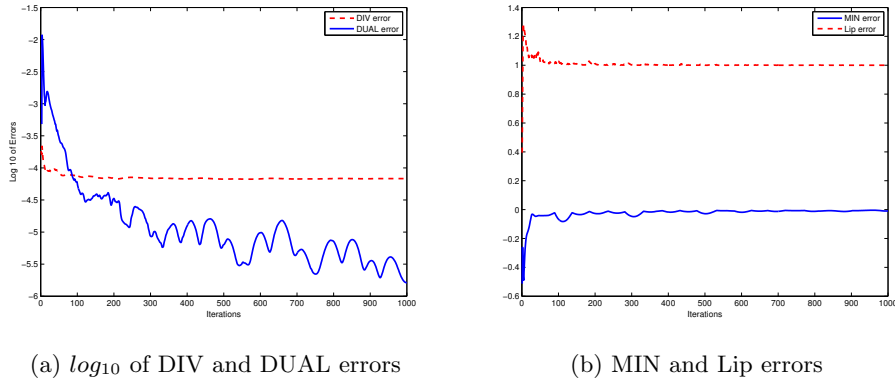
Fig. 6: Stopping criterion

Example 6.3. We take

$$f_1 = 4\chi_{[(x-0.1)^2+(y-0.9)^2<0.01]},$$

$$f_2 = 4\chi_{[(x-0.7)^2+(y-0.3)^2<0.01]},$$

$$\Theta = 4\chi_{[(x-0.2)^2+(y-0.2)^2<0.04]} + 4\chi_{[(x-0.6)^2+(y-0.6)^2<0.0064]}.$$
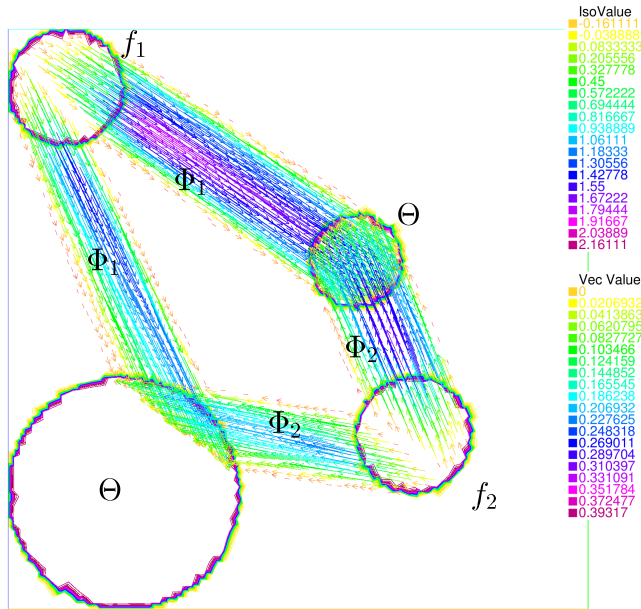
The results are given in Figures 7 and 8.

Fig. 7: Optimal matching measure and optimal flows
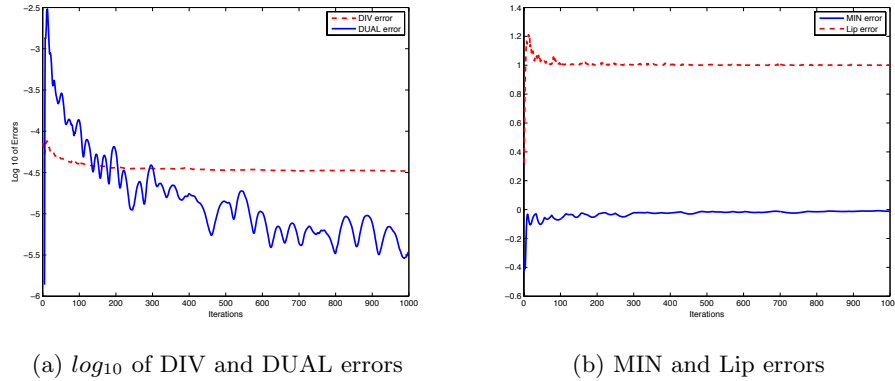
(a) log_{10} of DIV and DUAL errors   (b) MIN and Lip errors

Fig. 8: Stopping criterion

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