Metric Character for the Sub-Hamilton-Jacobi Obstacle Equation

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Abstract

For a given nonnegative continuous function $\mathfrak{g}$, we establish an new explicit formula of the distance related to the sub-Hamilton-Jacobi obstacle equation : $u \geq \mathfrak{g}$ and $H(x, \nabla u) = 0$ in the set $[u > \mathfrak{g}]$. We introduce a new inf-sup integral formula involving the trajectories joining two given points and the obstacle $\mathfrak{g}$. This defines a new length quasi-metric $\mathcal{I}_{\mathfrak{g}}$ in $I\!R^N$ which handle the obstacle and enters in a representation formulae for a viscosity solution of the sub-Hamilton-Jacobi obstacle equation.

1 Introduction

Let $\Omega \subset I\!R^N$ be a regular connected open set. Our aim is to establish an explicit formula of the distance in Ω related to the sub-Hamilton-Jacobi obstacle equation (SHJO equation for short)

$$(1) \qquad \begin{cases} u \geq \mathfrak{g} & \text{in } \Omega, \\ H(x, \nabla u) = 0 & \text{in } [u > \mathfrak{g}], \end{cases}$$

where $\mathfrak{g} : \overline{\Omega} \to I\!R^+$ is a given continuous function and $H : \overline{\Omega} \times I\!R^N \to I\!R$ is a continuous Hamiltonian (satisfying additional assumptions that will be precise in Section 2). Here we denote by $[u > \mathfrak{g}]$ the subset given by $\{x \in \Omega \; ; \; u(x) > \mathfrak{g}(x)\}$. The metric character of the Hamilton-Jacobi equation $H(x, \nabla u) = 0$ is well recognized and investigated by now (see [16], [15], [10], [6] and [14]) under the standard assumptions of convexity and compactness of the level sets $Z(x) := \{p \in I\!R^N \; ; \; H(x,p) \leq 0\}$, for any $x \in \Omega$. In [19], it is shown that the metric character is preserved even if the convexity assumption is removed. In fact, a class of fundamental (viscosity) subsolutions can be identified by using a distance function associated with the equation, the so called optical length function.

The main achievement of the present paper is to show that the metric character is preserved when the Hamilton-Jacobi equation $H(x, \nabla u) = 0$ is subject to unilateral constraint of the type $u \geq \mathfrak{g}$, where $\mathfrak{g}$ is a given continuous function not necessary a subsolution of $H(x, \nabla u) = 0$.

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Notice here, that even if we are able to rewrite (1) into the general form $F(x,u,\nabla u)=0$, the common convexity assumption of the level set of F (see (see [16] and [15]) does not necessarily holds to be true here.

To our knowledge there is no study addressing the question of metric character and explicit formula for the SHJO problem. The converse problem (the super-Hamilton-Jacobi obstacle equation) where the obstacle is above, the problem reads

$$\begin{cases} u \le \mathfrak{g} & \text{in } \Omega, \\ \\ H(x,\nabla u)=0 & \text{in } [u<\mathfrak{g}]. \end{cases} \tag{2}$$

In this case the problem falls into the scope of the form $F(x,u,\nabla u)=0$, with a convex Hamiltonian $F(x,t,p)$ with respect to (t,p). We refer the reader to the paper [15] and [16] for explicit formulation for this type of problems. One can see also [5], where the authors establish an new explicit formula for the solution of (7) in a bounded domain with a boundary condition. The connection with the evolution problem is studied in [17] in the case of degenerate viscous Hamilton-Jacobi equation.

In the present paper, in order to establish an explicit formula of the distance related to the SHJO equation we use again the level set $\left\{p \in I\!R^N \; ; \; H(x,p)\le 0\right\}$. We introduce a new inf-sup integral formula involving the trajectories joining two given points and the obstacle $\mathfrak{g}$, which defines a quasi-metric $\mathcal{I}_{\mathfrak{g}}$ in Ω. Roughly speaking, our representation formula is of game theory type. Indeed, recall that the solution of the Hamilton-Jacobi equation $H(x,\nabla u)=0$ is the maximal subsolution. So, assuming that $u\ge \mathfrak{g}$ and $\mathfrak{g}$ is not a subsolution is a real conflict situation for the Hamilton-Jacobi equation and our inf-sup expression shows that the equation (1) sorts out some kind of least worst strategy. Then, we show that for any fixed $y\in\overline{\Omega}$, the function $S_{\mathfrak{g}}(y,.) := \mathcal{I}_{\mathfrak{g}}(y,.)+\mathfrak{g}(y)$ is a viscosity solution of the Hamilton-Jacobi equation (1) taking the value $\mathfrak{g}(y)$ on y. Moreover, for the boundary value problem in a bounded domain Ω, we use $S_{\mathfrak{g}}$ to establish a Hopf-Lax type formula for the representation of the solution of (1) subject to boundary condition $u=\mathfrak{g}$, on $\partial\Omega$.

The Hamilton Jacobi equation occurs in a large field of applications including optimal control, image processing, fluid dynamics, robotics and geophysics. In particular, the SHJO equation appears with the Eikonal equation in the study of the equilibrium of a growing sandpile over a uneven table under the action of a given vertical source (cf. [18]). Another peculiar application deals with an heuristic metaphor for the formation of lakes (and rivers) along landscape (cf. [18] and [9]). For more details in these directions one can see the forthcoming paper [12]. The notion of viscosity solution is the powerful and flexible notion of solution for problem of Hamilton Jacobi type. It has been generalized in many different directions. We refer the reader to the book of [2] for a more complete presentation of the notion of viscosity solution including applications to deterministic optimal control problems, and to the users guide [7] for extensions to second-order equations.

The paper is organized as follows. In the next section, we recall some preliminaries concerning the metric character of the Hamilton-Jacobi equation $H(x,\nabla u)=0$. We give an overview on the optical length function associated with the Hamiltonian H; that we denote by $S(y,x)$, for any $x, y \in \Omega$. We recall its connexion with the Hamilton-Jacobi equation $H(x, \nabla u) = 0$ in Ω. In Section 3, we establish a new inf-sup integral formula and we prove that it defines a new length quasi-distance in Ω; that we denote by $\mathcal{I}_{\mathfrak{g}}(y, x)$, for any $x, y \in \Omega$. In Section 4, we prove the connection between the distance $\mathcal{I}_{\mathfrak{g}}$ and the Hamilton-Jacobi equation (1). We prove that, for any $y \in \Omega$, $S_{\mathfrak{g}}(y, .) := \mathcal{I}_{\mathfrak{g}}(y, x) + \mathfrak{g}(y)$ is both a maximal subsolution of (1) in Ω and a supersolution of (1) in $\Omega \setminus \{y\}$ satisfying $S_{\mathfrak{g}}(y, y) = \mathfrak{g}(y)$. Then, we establish a representation formula of Hopf-Lax type to solve the boundary value SHJO problem. For the uniqueness, we prove the comparaison principle. At last, Section 6 is devoted to some remarks, comments on extensions to the critical case and the description of qualitative properties of the solution related to the regularity of the free boundary problems. Some qualitative properties of the solution along the geodesics are also given.

2 Preliminaries

Let $\Omega \subset I\!R^N$ be a regular connected open set and $\mathfrak{g} \in \mathcal{C}(\overline{\Omega})$ that we assume, with no loss of generality, to be such that

$$\mathfrak{g} \geq 0 \quad \text{ in } \overline{\Omega}.$$

The Hamiltonian $H : \overline{\Omega} \times I\!R^N \to I\!R$ is assumed to be a continuous function satisfying, for any $x \in \overline{\Omega}$, the following assumptions :

$$(H1) \qquad Z(x) := \Big\{ p \in I\!R^N \; ; \; H(x, p) \leq 0 \Big\} \text{ is convex,}$$

$$(H2) \qquad Z(x) \text{ is compact,}$$

and

$$(H3) \qquad H(x, 0) < 0.$$

Our aim here is to study the Sub-Hamilton-Jacobi obstacle equation (1) that we can write in the following form :

$$(3) \qquad F(x, u, \nabla u) = 0, \quad \text{ in } \Omega,$$

where $F : \overline{\Omega} \times I\!R \times I\!R^N \to I\!R$ is the Hamiltonian given by

$$(4) \qquad F(x, r, p) = \min\Big(H(x, p), r - \mathfrak{g}(x) \Big), \quad \text{ for any } (x, r, p) \in \overline{\Omega} \times I\!R \times I\!R^N.$$

A continuous function $u : \Omega \to I\!R$ is a viscosity solution of the Hamilton-Jacobi equation (3) if

- u is a viscosity subsolution; that is whenever $\phi \in \mathcal{C}^1(\Omega)$, $u - \phi$ attains a local maximum at $x \in \Omega$, then $F(x, u(x), \nabla \phi(x)) \leq 0$.

- u is a viscosity supersolution; that is whenever $\phi \in \mathcal{C}^1(\Omega)$, $u - \phi$ attains a local minimum at $x \in \Omega$, then $F(x, u(x), \nabla\phi(x)) \geq 0$.

To simplify the notation throughout the paper, we use the notation $H[u] = 0$ and $H_{\mathfrak{g}}[u] = 0$ to express $H(x, \nabla u(x)) = 0$ and $\min\Big(H(x, \nabla u(x)), u(x) - \mathfrak{g}(x)\Big) = 0$, respectively.

To define the metric associated with the Hamilton-Jacobi equation $H[u] = 0$, we consider the support function of $Z(x)$ given by

$$\sigma(x, q) = \sup\Big\{p \cdot q \,;\, q \in Z(x)\Big\}, \qquad \text{for any x } \in \overline{\Omega} \text{ and } q \in I\!R^N.$$

Thanks to the assumptions (H1-H3), σ is a continuous nonnegative function in $\Omega \times I\!R^N$, which is convex and positive homogeneous with respect to q, for any $x \in \Omega$. For any $x, y \in \overline{\Omega}$, we denote

$$\Gamma(x, y) = \Big\{\varphi \in \mathrm{Lip}_\Omega;\ \varphi(0) = x \text{ and } \varphi(1) = y\Big\},$$

where

$$\mathrm{Lip}_\Omega = \Big\{\varphi \,:\, [0,1] \to \overline{\Omega}\,;\ \|\varphi(s) - \varphi(t)\| \leq c|s - t|, \text{ for any } s, t \in [0,1]\Big\},$$

$\|.\|$ denote the Euclidean norm of $I\!R^N$ and c is an arbitrary constant in $I\!R$. Then, we define the so called optical distance from x to y by

$$S(x, y) = \inf\left\{\int_0^1 \sigma(\xi(s), \xi'(s))\, ds\,;\ \xi \in \Gamma(x, y)\right\}, \text{ for any } x, y \in \overline{\Omega}.$$

In general, the term $\sigma(\varphi(t), \varphi'(t))$ is called the running cost, and the term $\displaystyle\int_0^1 \sigma(\varphi(t), \varphi'(t))\, dt$ is called the action functional. Under the assumptions (H1-H3), S is a quasi-distance; i.e. a distance which is not necessary symmetric (see for instance [10]). In other words, S satisfies

- $S(x, y) = 0$ if and only if $x = y$.
- $S(x, y) \leq S(x, z) + S(z, y)$, for any $x, y, z \in \overline{\Omega}$.

Moreover, thanks to (H2) there exists $M > 0$ such that

$$S(x, y) \leq M\, \|x - y\|, \qquad \text{for any } x, y \in \overline{\Omega}.$$

The metric character of Hamilton-Jacobi equation follows from the connection between S and the viscosity solutions of $H[u] = 0$. This connection is summarized in the following proposition.

Proposition 1. *(cf. [10]) Under the assumptions (H1-H3), we have*

1. *For any $y \in \overline{\Omega}$, $S(y, .)$ is a viscosity subsolution in Ω and a viscosity supersolution in $\Omega \setminus \{y\}$ of $H[u] = 0$.*

2. *v is a viscosity subsolution of $H[u] = 0$ in Ω if and only if*

$$v(x) - v(y) \le S(y, x), \qquad \text{for any } x, y \in \Omega. \tag{5}$$

For the uniqueness of the viscosity solution, it is well known by now that this is connected to the existence of a strict subsolution. The collection of points around which no subsolution is strict is a set of bad points for the uniqueness. It is called the Aubry set and is usually denoted by A. Counterexamples showing the existence of infinite viscosity solution if one doesn't assigned datum in the Aubry set can be found in in [11] and [13]. Thanks to the assumption (H3), it is clear that the Aubry set is empty in our situation. So, under the assumption $(H3)$ the uniqueness of a viscosity solution follows by the assignment of a boundary value on $\partial\Omega$.

Proposition 2. *(cf. [10]) Under the assumptions (H1-H3), we assume moreover that Ω is bounded. We have*

1. *Let u and v be a subsolution and a supersolution of $H[u] = 0$ in Ω, respectively. If $u \le v$ on $\partial\Omega$, then $u \le v$ in Ω.*
2. *If $h \in \mathcal{C}(\partial\Omega)$ and satisfies the compatibility condition*

$$h(x) - h(y) \le S(y, x), \qquad \text{for any } x, y \in \partial\Omega, \tag{6}$$

then $u(x) := \min\Big\{ S(y, x) + h(y) \ ; \ y \in \partial\Omega \Big\}$, is the unique viscosity solution of $H[u] = 0$ in Ω satisfying $u(x) = h(x)$, for any $x \in \partial\Omega$.

Coming back to the formulation (4), a representation formula of the viscosity solution of (3) is given in [16] by using the conjugate of F (see also further developments on this case in [15]). However, the convexity of the sub-level set $\Big\{ (t, p) \in I\!R \times I\!R^N \ ; \ F(x, t, p) \le 0 \Big\}$ seems to be again a key ingredient among others in this case. See here, that under the assumption (H1)-(H3), the Hamiltonian F given by (4) does not falls into the scope of this theory. For instance, assume $H(x, \xi) = |\xi| - 1$. In this situation, we have

$$F(x, t, p) = \min(H(x, p), t - \mathfrak{g}(x)), \qquad \text{for any } (x, t, p) \in \overline{\Omega} \times I\!R \times I\!R^N.$$

Then, for any $x \in I\!R^N$, the sub-level set $Z(x)$ is given by

$$Z(x) = (-\infty, \mathfrak{g}(x)] \times I\!R^N \cup [\mathfrak{g}(x), \infty) \times B(0, 1)\,;$$

which is clearly non-convex in general. Here $B(0, 1)$ denotes the unit ball of $I\!R^N$.

To end up the preliminaries, we notice that in the case where the obstacle is above, the unilateral obstacle problem reads

$$\begin{cases} u \le \psi \\ \\ H(x, \nabla u) = 0 \qquad \text{in } [u > \psi]. \end{cases} \tag{7}$$

Again this problem falls into the scope of the formulation (3) by taking

$$F(x,t,p) = \max(H(x,p), t - \mathfrak{g}(x)), \qquad \text{for any } (x,t,p) \in \overline{\Omega} \times I\!R \times I\!R^N.$$

In the contrast of the problem (1), in this case the corresponding sub-level sets $Z(x)$ are convex and the problem (7) may falls into the scope of [16] and [15]. Indeed,

$$Z(x) = \Big\{ (t,p) \in I\!R \times I\!R^N \ ; \ t \le \mathfrak{g}(x) \text{ and } H(x,p) \le 0 \Big\},$$

which is convex for any $x \in I\!R^N$. The problem (7) with Dirichlet boundary condition was studied in [5] and the authors shows the following results

Theorem 1. *(cf. [5]) Let $h : \partial\Omega \to I\!R$ and $\mathfrak{g} : \overline{\Omega} \to I\!R$ be continuous functions such that*

$$-S(y,x) \le h(x) - h(y) \le S(y,x), \qquad \text{for any } x, y \in \partial\Omega$$

and

$$-S(y,x) \le \mathfrak{g}(x) - \mathfrak{g}(y) \le S(y,x), \qquad \text{for any } x, y \in \overline{\Omega}.$$

The function

$$u(x) = \min \Big\{ \min_{y \in \partial\Omega} (h(y) + S(y,x), \min_{y \in \overline{\Omega}} (\mathfrak{g}(y) + S(y,x) \Big\},$$

is the unique solution of (7) satisfying $u(x) = \min(\mathfrak{g}(x), h(x))$, for any $x \in \partial\Omega$.

3 Representation formula with respect to the obstacle

For any path $\varphi \in Lip_\Omega$ and $0 \le t_1 \le t_2 \le 1$, we denote by $\Lambda_\varphi(t_1, t_2)$, the quantity given by

$$\Lambda_\varphi(t_1,t_2) := \int_{t_1}^{t_2} \sigma(\varphi(t), \varphi'(t))\, dt.$$

Then, for any given path $\varphi \in \mathrm{Lip}_\Omega$, we introduce the *action of φ with respect to the obstacle* $\mathfrak{g}$ given by

$$A_{\mathfrak{g}}(\varphi) := \max_{t \in [0,1]} \Big\{ \mathfrak{g}(\varphi(t)) + \Lambda_\varphi(t,1) \Big\}.$$

We call $A_{\mathfrak{g}}(\varphi)$ the $\mathfrak{g}-$action of φ. See that, when $\mathfrak{g} \equiv 0$, $A_{\mathfrak{g}}(\varphi)$ coincides with the standard action given by $\Lambda_\varphi(0,1)$. At last, we introduce the minimum $\mathfrak{g}-$action, for any $x, y \in \overline{\Omega}$:

$$S_{\mathfrak{g}}(x,y) = \inf \Big\{ A_{\mathfrak{g}}(\varphi) \ ; \ \varphi \in \Gamma(x,y) \Big\}.$$

And, for any $x, y \in \overline{\Omega}$, we denote by

$$\mathcal{I}_{\mathfrak{g}}(x,y) = S_{\mathfrak{g}}(x,y) - \mathfrak{g}(x).$$

First, let us see that $S_{\mathfrak{g}}$ is well defined in Ω and can be expressed in a simple way when the obstacle $\mathfrak{g}$ is a subsolution of $H[u] = 0$.

Proposition 3. *Under the assumptions (H1)-(H3), $S_{\mathfrak{g}}$ is well defined in $\overline{\Omega} \times \overline{\Omega}$. Moreover,*

i) For any $x, y \in \overline{\Omega}$, we have

$$\max(S(y,x), \mathfrak{g}(x) - \mathfrak{g}(y)) \leq \mathcal{I}_{\mathfrak{g}}(y,x) \leq S(y,x) + \max_{x \in \overline{\Omega}} \mathfrak{g}(x) - \mathfrak{g}(y). \tag{8}$$

ii) For any $y \in \overline{\Omega}$, $\mathcal{I}_{\mathfrak{g}}(y,y) = 0$.

iii) If $\mathfrak{g}$ is such that

$$\mathfrak{g}(x) - \mathfrak{g}(y) \leq S(y,x), \quad \text{for any } x, y \in \overline{\Omega}, \tag{9}$$

then

$$S_{\mathfrak{g}}(y,x) = \mathfrak{g}(y) + S(y,x), \quad \text{for any } x, y \in \overline{\Omega}$$

and $\mathcal{I}_{\mathfrak{g}} = S$.

Proof : It is not difficult to see that, for any $x, y \in \overline{\Omega}$ and $\varphi \in \Gamma(y,x)$, we have

$$\max(\mathfrak{g}(y) + \Lambda_\varphi(0,1), \mathfrak{g}(x)) \leq A_{\mathfrak{g}}(\varphi) \leq \max_{t \in [0,1]} \mathfrak{g}(\varphi(t)) + \int_0^1 \sigma(\varphi(t), \varphi'(t))\, dt.$$

Taking the infimum over $\varphi \in \Gamma(y,x)$, we deduce that

$$\max(S(y,x) + \mathfrak{g}(y), \mathfrak{g}(x)) \leq S_{\mathfrak{g}}(y,x) \leq \max_{t \in [0,1]} \mathfrak{g}(\varphi(t)) + \int_0^1 \sigma(\varphi(t), \varphi'(t))\, dt, \tag{10}$$

for any $\varphi \in \Gamma(y,x)$. Thus

$$\max(S(y,x) + \mathfrak{g}(y), \mathfrak{g}(x)) \leq S_{\mathfrak{g}}(y,x) \leq S(y,x) + \max_{x \in \overline{\Omega}} \mathfrak{g}(x).$$

and (8) follows by subtracting $g(y)$. The proof of $ii)$ follows by using (10) and taking $\varphi \in \Gamma(y,y)$ given by $\varphi(t) = y$, for any $t \in [0,1]$, in (8). At last, let us prove $iii)$. Thanks to (9), we see that for any $\varphi \in \Gamma(y,x)$ and $t \in [0,1]$, we have

$$\begin{aligned} \mathfrak{g}(\varphi(t)) &\leq \mathfrak{g}(y) + S(y, \varphi(t)) \\ &\leq \mathfrak{g}(y) + \int_0^t \sigma(\varphi(s), \varphi'(s))\, ds. \end{aligned}$$

This implies that

$$\mathfrak{g}(\varphi(t)) + \int_t^1 \sigma(\varphi(t), \varphi'(t))\, dt \leq \mathfrak{g}(y) + \int_0^1 \sigma(\varphi(t), \varphi'(t))\, dt, \quad \text{for any } t \in [0,1],$$

so that

$$A_{\mathfrak{g}}(\varphi) = \mathfrak{g}(y) + \int_0^1 \sigma(\varphi(s), \varphi'(s))\, ds.$$

Taking the infimum over $\varphi \in \Gamma(y, x)$, we deduce iii).

$\square$

Example : For instance, assume that $N = 1$, $\Omega = (a, b)$ and $H(x, \xi) = |\xi| - 1$, for any $\xi \in I\!R$. In this situation, we have $\sigma(x, y) = |y|$, for any $(x, y) \in (a, b) \times I\!R$. Here,

$$S(y, x) = |y - x|, \qquad \text{for any } y, x \in (a, b).$$

The following picture shows the graphic of the function $z \in (a, b) \to S_{\mathfrak{g}}(x, z)$ for $x = x_1$ (in blue color) and $x = x_2$ (in red color), where x_1, x_2 are two given real values. The obstacle $\mathfrak{g}$ is given by the shaded area where the gradient is bigger than 1 and outside this region $\mathfrak{g}$ is equals to 0 :

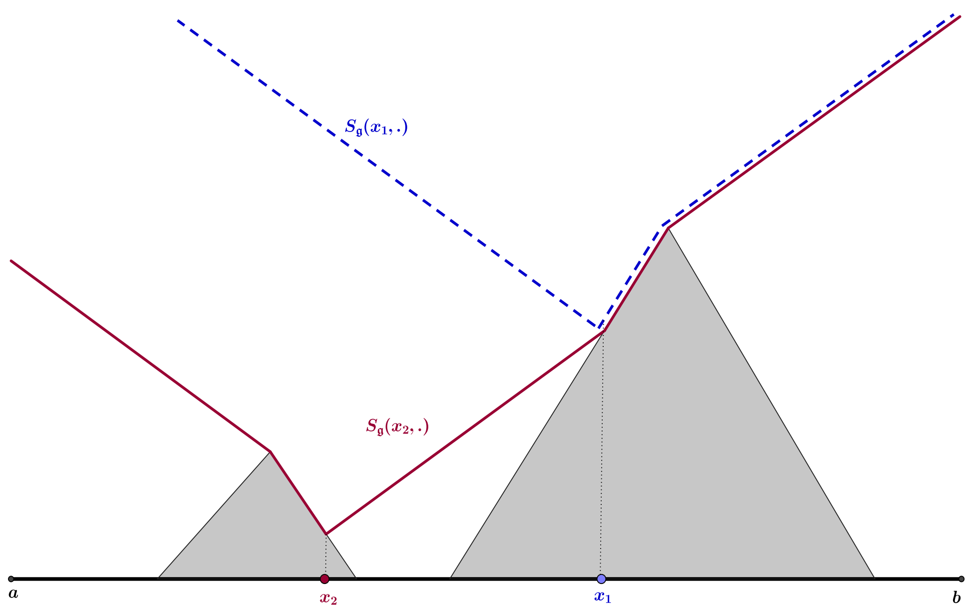


Figure 1: Representation of the functions $z \to S_{\mathfrak{g}}(x_1, z)$ and $z \to S_{\mathfrak{g}}(x_2, z)$

Remark 1. *1. It is possible to rewrite the $\mathfrak{g}$-action as follows*

$$\mathfrak{g}(\varphi(t)) + \Lambda_\varphi(t, 1) = \int_t^1 \Big(\sigma(\varphi(s), \varphi'(s)) - \nabla \mathfrak{g}(\varphi(s)) \cdot \varphi'(s)\Big)\, ds + \mathfrak{g}(\varphi(1)), \qquad \textit{for any } t \in [0, 1].$$

So, the $\mathfrak{g}-$action of φ may be re-written as

$$A_{\mathfrak{g}}(\varphi) = \max_{t\in[0,1]} \Big\{ \int_t^1 \sigma_{\mathfrak{g}}(\varphi(s), \varphi'(s))\, ds \Big\} + \mathfrak{g}(\varphi(1)),$$

where $\sigma_{\mathfrak{g}}(x,p) = \sigma(x,p) - \nabla \mathfrak{g}(x)\cdot p$, for any $(x,p) \in \Omega \times I\!R^N$. So, for any $x, y \in I\!R^N$,

$$S_{\mathfrak{g}}(x,y) = \inf \left\{ \max_{t\in[0,1]} \Big\{ \int_t^1 \sigma_{\mathfrak{g}}(\varphi(s), \varphi'(s))\, ds \Big\} + \mathfrak{g}(y)\ ;\ \xi \in \Gamma(x,y) \right\}$$

Here $\sigma_{\mathfrak{g}}(\varphi(t), \varphi'(t))$ appears as the running cost. See that $\sigma_{\mathfrak{g}}$ is not necessary nonnegative, so that the $\mathfrak{g}-$action $A_{\mathfrak{g}}$ is not necessary of Finsler type.

2. *In general $S_{\mathfrak{g}}$ is not symmetric even if S is so. Indeed, it is not difficult to see from Figure 1 on page 8, that there exists $x, y \in \Omega$ such that $S_{\mathfrak{g}}(y,x) \neq S_{\mathfrak{g}}(x,y)$.*

The following result shows that the application $(x,y) \to \mathcal{I}_{\mathfrak{g}}(x,y)$ inherit the metric character property of the application $(x,y) \to S(x,y)$.

Theorem 2. *Under the assumptions (H1)-(H2), $\mathcal{I}_{\mathfrak{g}}$ is a quasi-metric in Ω.*

The proof of Theorem 2 follows as a consequence of the following lemmas. To simplify the presentation, let us introduce some notations. For any $\varphi_1 \in Lip_\Omega$, $\varphi_2 \in Lip_\Omega$ and $\tau \in (0,1)$, such that $\varphi_1(1) = \varphi_2(0)$, we denote by $\varphi_1 \cup_\tau \varphi_2$, the juxtaposition of φ_1 and φ_2 up to the following reparametrization

$$\varphi(t) = \begin{cases} \varphi_1\left(\dfrac{t}{\tau}\right) & \text{for } t \in [0,\tau] \\ \\ \varphi_2\left(\dfrac{\tau - t}{\tau - 1}\right), & \text{for any } t \in [\tau, 1]. \end{cases}$$

Moreover, for any $\varphi \in Lip_\Omega$, and $0 \le t_1 \le t_2 \le 1$, we denote by $\varphi_{(t_1,t_2)}$, the path $\varphi \in \Gamma(\varphi(t_1), \varphi(t_2)))$ given by

$$\varphi_{(t_1,t_2)}(t) = \varphi\left((t_2 - t_1)t + t_1\right), \quad \text{for any } t \in [0,1].$$

Thanks to the homogeneity of $\sigma(x,p)$ with respect to p, we see that

$$\Lambda_{\varphi_{(t_1,t_2)}}(0,1) = \Lambda_\varphi(t_1,t_2).$$

Moreover, we have

Lemma 1. *Let $x_1, x_2, x_3 \in \overline{\Omega}$, $\varphi_1 \in \Gamma(x_1, x_2)$ and $\varphi_2 \in \Gamma(x_2, x_3)$. For any $\tau \in [0,1]$, we have*

$$A_{\mathfrak{g}}(\varphi_1 \cup_\tau \varphi_2) = \max\Big(A_{\mathfrak{g}}(\varphi_1) + \Lambda_{\varphi_2}(0,1), A_{\mathfrak{g}}(\varphi_2)\Big). \tag{11}$$

In particular, for any $\varphi \in Lip_\Omega$ and $\tau \in [0,1]$, we have

$$A_{\mathfrak{g}}(\varphi) = \max\Big(A_{\mathfrak{g}}(\varphi_{(\tau,1)}), A_{\mathfrak{g}}(\varphi_{(0,\tau)}) + \Lambda_\varphi(\tau,1)\Big). \tag{12}$$

Proof : We set $\varphi = \varphi_1 \cup_\tau \varphi_2$, for an arbitrary $\tau \in [0,1]$ (see for instance Figure 2 on page 10). It is not difficult to see that

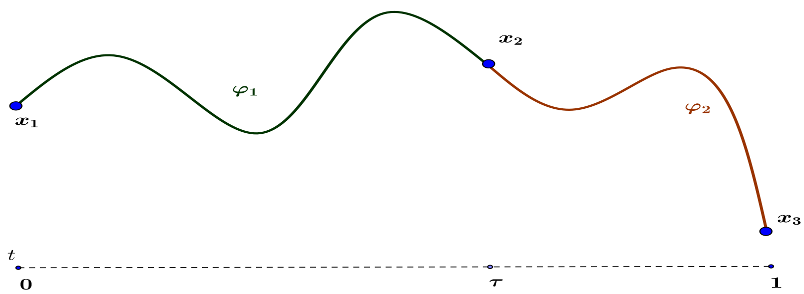


Figure 2: Juxtaposition of φ_1 and φ_2

$$\max_{t\in[0,\tau]}(\mathfrak{g}(\varphi(t)) + \Lambda_\varphi(t,\tau)) = A_{\mathfrak{g}}(\varphi_1) \quad \text{and} \quad \max_{t\in[\tau,1]}(\mathfrak{g}(\varphi(t)) + \Lambda_\varphi(t,1) = A_{\mathfrak{g}}(\varphi_2).$$

So,

$$\begin{aligned}
A_{\mathfrak{g}}(\varphi) &= \max_{t\in[0,1]} \mathfrak{g}(\varphi(t)) + \Lambda_\varphi(t,1)) \\
&= \max\Big(\max_{t\in[0,\tau]}(\mathfrak{g}(\varphi(t)) + \Lambda_\varphi(t,1)), \max_{t\in[\tau,1]}(\mathfrak{g}(\varphi(t)) + \Lambda_\varphi(t,1)\Big) \\
&= \max\Big(\max_{t\in[0,\tau]}(\mathfrak{g}(\varphi(t)) + \Lambda_\varphi(t,\tau)) + \Lambda_\varphi(\tau,1), \max_{t\in[\tau,1]}(\mathfrak{g}(\varphi(t)) + \Lambda_\varphi(t,1)\Big) \\
&= \max\Big(A_{\mathfrak{g}}(\varphi_1) + \Lambda_{\varphi_2}(0,1), A_{\mathfrak{g}}(\varphi_2)\Big).
\end{aligned}$$

This finish the proof of (11). As to (12), it follows by taking $\varphi_1 = \varphi_{(0,\tau)}$ and $\varphi_2 = \varphi_{(\tau,1)}$ in (11) for a given $\varphi \in Lip_\Omega$. □

Proof of Theorem 2 : For any $x, y, z \in \overline{\Omega}$, let us prove that

$$S_{\mathfrak{g}}(x,z) \leq S_{\mathfrak{g}}(x,y) + S_{\mathfrak{g}}(y,z) - \mathfrak{g}(y). \tag{13}$$

Thanks to Lemma 1, for any $\varphi_1 \in \Gamma(x,y)$ and $\varphi_2 \in \Gamma(y,z)$, we have

$$\begin{aligned} S_{\mathfrak{g}}(x,z) &\leq \max\Big(A_{\mathfrak{g}}(\varphi_1) + \Lambda_{\varphi_2}(0,1), A_{\mathfrak{g}}(\varphi_2)\Big) \\ &= \max\Big(A_{\mathfrak{g}}(\varphi_1) + \Lambda_{\varphi_2}(0,1) + \mathfrak{g}(\varphi_2(0)) - \mathfrak{g}(\varphi_2(0)), A_{\mathfrak{g}}(\varphi_2)\Big) \\ &\leq \max\Big(A_{\mathfrak{g}}(\varphi_1) + A_{\mathfrak{g}}(\varphi_2) - \mathfrak{g}(y), A_{\mathfrak{g}}(\varphi_2)\Big) \end{aligned}$$

where we use the fact that $\Lambda_{\varphi_2}(0,1) + \mathfrak{g}(\varphi_2(0)) \leq A_{\mathfrak{g}}(\varphi_2)$. On the other hand, we see that

$$\mathfrak{g}(y) = \mathfrak{g}_{\varphi_1}(1) \leq A_{\mathfrak{g}}(\varphi_1).$$

so that $S_{\mathfrak{g}}(x,z) \leq A_{\mathfrak{g}}(\varphi_1) + A_{\mathfrak{g}}(\varphi_2) - \mathfrak{g}(y)$. Taking the infimum with respect to φ_1 and φ_2, we deduce (13). At last, subtracting $\mathfrak{g}(y)$ in both sides of (13) we get $\mathcal{I}_{\mathfrak{g}}(x,z) \leq \mathcal{I}_{\mathfrak{g}}(x,y) + \mathcal{I}_{\mathfrak{g}}(y,z)$. To finish the proof, we see by Proposition 3, that $\mathcal{I}_{\mathfrak{g}}(x,x) = 0$, and we have $S(x,y) \leq \mathcal{I}_{\mathfrak{g}}(x,y)$, for any $x, y \in \Omega$. Thus, $\mathcal{I}_{\mathfrak{g}}(x,y) = 0$ if and only if $x = y$.

□

To end up this section, we prove the local continuous 1−Lipschitz regularity result of $S_{\mathfrak{g}}(y,.)$ with respect to the distance S on the set where $S_{\mathfrak{g}}(y,.) \neq \mathfrak{g}$. This will be very useful for the study of the connection between $S_{\mathfrak{g}}(y,.)$ and the SHJO equation $H_{\mathfrak{g}}[u] = 0$ in Ω.

Theorem 3. *Let $y \in \overline{\Omega}$ be fixed and C_y be a connected component of the set $[S_{\mathfrak{g}}(y,.) > \mathfrak{g}(.)]$. Under the assumptions (H1)-(H3), for any $x_0 \in C_y$, there exists $R > 0$, such that*

$$S_{\mathfrak{g}}(y,x) - S_{\mathfrak{g}}(y,z) \leq S(z,x), \qquad \textit{for any } x, z \in B(x_0, R) \tag{14}$$

and

$$\mathcal{I}_{\mathfrak{g}}(y,x) - \mathcal{I}_{\mathfrak{g}}(y,z) \leq S(z,x) \qquad \textit{for any } x, z \in B(x_0, R). \tag{15}$$

To prove this theorem, we begin with the following lemmas.

Lemma 2. *Let $x, y \in \overline{\Omega}$ and $\varphi \in \Gamma(y,x)$. For any $\overline{t} \in [0,1]$ such that*

$$A_{\mathfrak{g}}(\varphi) = \mathfrak{g}(\varphi(\overline{t})) + \Lambda_{\varphi}(\overline{t}, 1),$$

we have

$$S_{\mathfrak{g}}(y, \varphi(\overline{t})) = \mathfrak{g}(\varphi(\overline{t})).$$

Proof : Let us denote by $\overline{x} := \varphi(\bar{t})$ and $\overline{\varphi} := \varphi_{(0,\bar{t})} \in \Gamma(y,\overline{x})$. Thanks to Proposition 3, we have

$$\begin{aligned}
\mathfrak{g}(\overline{x}) \leq S_{\mathfrak{g}}(y,\overline{x}) \leq A_{\mathfrak{g}}(\overline{\varphi}) &= \max_{t\in[0,1]} \Big(\mathfrak{g}(\varphi(\bar{t}\,t)) + \Lambda_{\varphi}(\bar{t}\,t,\bar{t})\Big) \\
&= \max_{t\in[0,1]} \Big(\mathfrak{g}(\varphi(\bar{t}\,t)) + \Lambda_{\varphi}(\bar{t}\,t,1)\Big) - \Lambda_{\varphi}(\bar{t},1) \\
&= \max_{t\in[0,\bar{t}]} \Big(\mathfrak{g}(\varphi(t)) + \Lambda_{\varphi}(t,1)\Big) - \Lambda_{\varphi}(\bar{t},1) \\
&= A_{\mathfrak{g}}(\varphi) - \Lambda_{\varphi}(\bar{t},1) \\
&= \mathfrak{g}(\overline{x}).
\end{aligned}$$

□

Lemma 3. *Let $y \in \overline{\Omega}$ be fixed and $x, z \in C_y$. For any $\varphi_2 \in \Gamma(z,x)$, such that $\varphi_2([0,1]) \subset C_y$, we have*

$$S_{\mathfrak{g}}(y,x) \leq S_{\mathfrak{g}}(y,z) + \Lambda_{\varphi_2}(0,1). \tag{16}$$

Proof : For a given $\varphi_1 \in \Gamma(y,z)$, we fix an arbitrary $\tau \in [0,1]$, and we consider $\varphi := \varphi_1 \cup_\tau \varphi_2 \in \Gamma(y,x)$ (see for instance Figure 3 on page 13). Then, let

$$\bar{t} = \max\Big\{t \in [0,1]\ ;\ \mathfrak{g}(\varphi(t)) + \Lambda_{\varphi}(t,1) = A_{\mathfrak{g}}(\varphi)\Big\}.$$

Since $x \in C_y$, thanks to Lemma 3, we have $\varphi(\bar{t}) \notin C_y$ and $\bar{t} < \tau$. This implies that and we have

$$\begin{aligned}
S_{\mathfrak{g}}(y,x) &\leq A_{\mathfrak{g}}(\varphi) \\
&\leq \max_{t\in[0,\tau]} \mathfrak{g}(\varphi(t)) + \int_t^1 \sigma(\varphi(s),\varphi'(s))\,ds \\
&\leq A_{\mathfrak{g}}(\varphi_1) + \Lambda_{\varphi_2}(0,1).
\end{aligned}$$

Then, by taking the infimum over $\varphi_1 \in \Gamma(y,z)$, we deduce the result. □

The next lemma which is a slight modification of Lemma 5.5 of [10] is useful for the proof of Theorem 3 .

Lemma 4. *Under the assumptions (H1)-(H3), let $x_0 \in \Omega$. Then, for any $\epsilon > 0$, there exists $\delta_\epsilon \in (0,\epsilon)$ such that*

$$S(x,z) = \inf\Big\{A(\varphi)\ ;\ \varphi \in \Gamma(x,z) \cap Lip_{B(x_0,\epsilon)}\Big\}, \quad \textit{for any } x,z \in B(x_0,\delta_\epsilon). \tag{17}$$

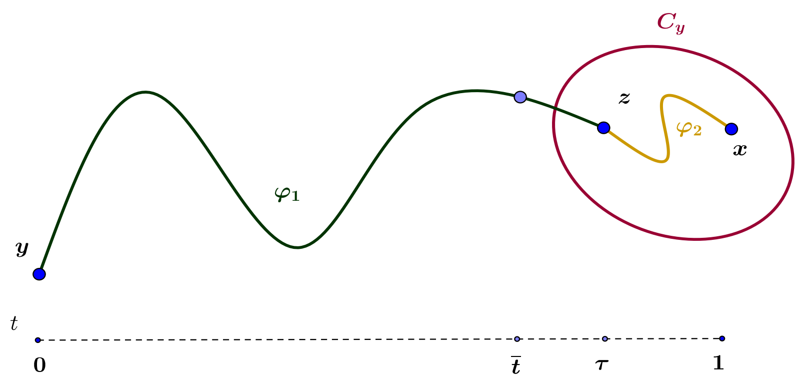


Figure 3: Jusxtaposition inside C_y

Proof : Assume (17) is not true. Then, there exists $\epsilon_0 > 0$, a sequence x_n, y_n and $\varphi_n \in \Gamma(x_n, y_n)$ such that $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = x_0$, $l(\varphi_n) > \epsilon_0$ and $\lim_{n\to\infty} \int_0^1 \sigma(\varphi_n(t), \varphi_n'(t))\, dt = 0$. Taking $\xi_n^1 \in \Gamma(x_0, x_n)$ and $\xi_n^2 \in \Gamma(y_n, x_0)$ the Euclidian geodesics and juxtaposing ξ_n^1, φ_n and ξ_n^2, we construct a sequence of curves $\tilde{\varphi}_n$ such that $\tilde{\varphi}_n \in \Gamma(x_0, x_0)$, $l(\tilde{\varphi}_n) \geq l(\varphi_n) > \epsilon_0$ and

$$\lim_{n\to\infty} \int_0^1 \sigma(\tilde{\varphi}_n(t), \tilde{\varphi}_n'(t))\, dt = 0.$$

This is in contradiction with the fact that the Aubry set is empty.

□

Proof of Theorem 3 : Thanks to Proposition 3 and Theorem 2, we know that $S(y,.) \geq \mathfrak{g}$ in Ω. The continuity of $S_{\mathfrak{g}}(y,.)$ and $\mathfrak{g}$ implies that C_y is an open domain. Let $x_0 \in C_y$. Thanks to Lemma 4, there exists $0 < \delta < \epsilon$, such that $B(x_0, \epsilon) \subset C_y$ and

$$(18) \quad S(x,z) = \inf \left\{ \int_0^1 \sigma(\varphi(t), \varphi'(t))\, dt \ ;\ \varphi \in \Gamma(x,z) \cap Lip_{B(x_0,\epsilon)} \right\} \quad \text{for any } x, z \in B(x_0, \delta).$$

We take $R = \delta$. Thanks to Lemma 3, by taking the infimum over $\varphi \in \Gamma(x,z)$ such that $\varphi([0,1]) \subset B(x_0, \epsilon)$, we deduce the result. □

4 Viscosity solution for the SHJO equation

The aim of this section is to establish the connection between the explicit formula $S_{\mathfrak{g}}$ and the SHJO equation $H_{\mathfrak{g}}[u] = 0$ in Ω. We summarize this connection in the following theorems.

Theorem 4. *Under the assumptions (H1-H3), for any $y \in \overline{\Omega}$, we have*

1. *$S_{\mathfrak{g}}(y,.)$ is a subsolution of $H_{\mathfrak{g}}[u] = 0$ in Ω.*
2. *For any $x \in \Omega$,*

$$(19) \qquad S_{\mathfrak{g}}(y,x) = \max\Big\{ u(x) \ ; \ u \text{ is a subsolution of } H_{\mathfrak{g}}[u] = 0 \text{ in } \Omega \text{ with } u(y) = \mathfrak{g}(y) \Big\}.$$

3. *$S_{\mathfrak{g}}(y,.)$ is a solution of $H_{\mathfrak{g}}[u] = 0$ in $\Omega \setminus \{y\}$.*

In order to deal with the boundary value problem, we introduce as usual for Hamilton-Jacobi equation, the intrinsic Hopf-Lax-formula. To this aim, let $D \subset \overline{\Omega}$ be a given closed subset. Following the same idea of the Hamilton-Jacobi equation, the Hopf-Lax formula is given by

$$(20) \qquad \delta_{\mathfrak{g}}(x) := \inf\Big\{ S_{\mathfrak{g}}(y,x) \ ; \ y \in D \Big\}, \qquad \text{for any } x \in \overline{\Omega}.$$

We have

Theorem 5. *Under the assumptions (H1)-(H3), $\delta_{\mathfrak{g}}$ is a viscosity solution of $H_{\mathfrak{g}}[u] = 0$ in Ω, satisfying $\delta_{\mathfrak{g}}(x) = \mathfrak{g}(x)$, for any $x \in D$.*

Then, for the boundary value SHJO problem, we have the following existence and uniqueness result.

Theorem 6. *Assume that Ω is a bounded domain. Taking $D = \partial\Omega$ in (20), δ_g is the unique viscosity solution of $H_{\mathfrak{g}}[u] = 0$ in Ω, satisfying $\delta_{\mathfrak{g}}(x) = \mathfrak{g}(x)$, for any $x \in \partial\Omega$.*

Example : Coming back to the example of Figure 1 on page 8, the picture on Figure 4 on page 15 shows moreover the graphic of the function $x \in I\!R \to \delta_{\mathfrak{g}}(x)$.

The proof of Theorem 4, Theorem 5 and Theorem 6 follows from the sequence of lemmas we will present. Throughout this section, we use the notation χ_B to denote the indicator function of the subset $B \subseteq \Omega$, given by

$$\chi_B(x) = \begin{cases} 1 & \text{if } x \in B \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 5. *1. If u is a supersolution of $H_{\mathfrak{g}}[u] = 0$ in Ω, then u and $\max(u,\mathfrak{g})$ are supersolutions of $H[u] = 0$ in Ω.*

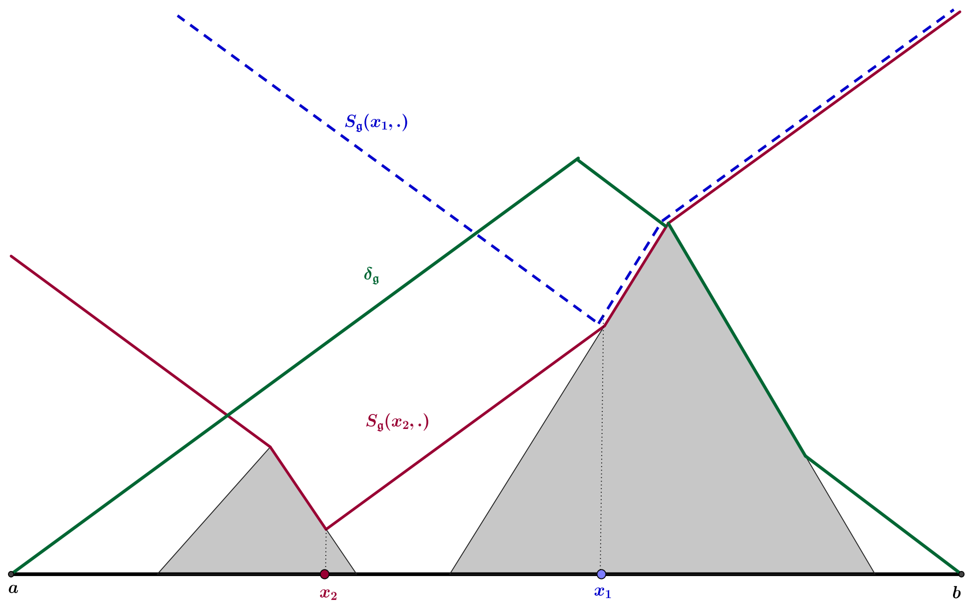


Figure 4: Representation of the functions $z \to S_{\mathfrak{g}}(x_1, z)$, $z \to S_{\mathfrak{g}}(x_2, z)$ and $z \to \delta_{\mathfrak{g}}(z)$

2. *Any subsolution of $H[u] = 0$ in Ω is a subsolution of $H_{\mathfrak{g}}[u] = 0$ in Ω.*

3. *Let $u \in \mathcal{C}(\Omega)$ be such that $u \geq \mathfrak{g}$ in Ω. Then, u is a subsolution of $H_{\mathfrak{g}}[u] = 0$ in Ω if and only if u is a subsolution of $H[u] = 0$ in the set $[u > \mathfrak{g}]$.*

4. *Let $u \in \mathcal{C}(\Omega)$ be a given function and let C be a connected component of the set $[u > \mathfrak{g}]$. Then, u is subsolution (resp. supersolution) of $H_{\mathfrak{g}}[u] = 0$ in C if and only if u is subsolution (resp. supersolution) of $H[u] = 0$ in C.*

5. *Let $A \subseteq \Omega$ be a given open set. If u is a subsolution of $H[u] = 0$ in A and $u = \mathfrak{g}$ on ∂A, then the function $\tilde{u} = u\chi_A + \mathfrak{g}\chi_{\Omega \setminus A}$ is a subsolution of $H_{\mathfrak{g}}[u] = 0$ in Ω.*

Proof : The proof of this lemma use in an obvious way the definition of viscosity solution. We omit to give the details of the proof and let it as an exercise for the reader. □

Lemma 6. *For any $y \in \overline{\Omega}$, $S_{\mathfrak{g}}(y, .)$ is a subsolution of $H_{\mathfrak{g}}[u] = 0$ in Ω.*

Proof : First, let C_y be a connected component of the set $[S_{\mathfrak{g}}(y, .) > \mathfrak{g}]$. Thanks to Theorem 3 and Proposition 1 we know that, for any $x \in C_y$, there exists $R > 0$ such that $S_{\mathfrak{g}}(y, .)$ is a subsolution of $H[u] = 0$ in $B(x, R)$. This implies that $S_{\mathfrak{g}}(y, .)$ is a subsolution of $H[u] = 0$ in C_y. Then, by using Lemma 5, we deduce that $S_{\mathfrak{g}}(y, .)$ is a subsolution of $H_{\mathfrak{g}}[u] = 0$ in Ω.

□

Lemma 7. *Let $y \in \overline{\Omega}$ be fixed. For every $x \in \Omega$,*

$$S_{\mathfrak{g}}(y,x) = \max\Big\{u(x)\ ;\ u \text{ subsolution of } H_{\mathfrak{g}}[u]=0 \text{ with } u(y)=\mathfrak{g}(y)\Big\}. \tag{21}$$

Proof : Thanks to Lemma 6, $S_{\mathfrak{g}}(y,.)$ is a subsolution of $H_{\mathfrak{g}}[u]=0$ and $S_{\mathfrak{g}}(y,y)=g(y)$. This implies that

$$S_{\mathfrak{g}}(y,x) \le \max\Big\{u(x)\ ;\ u \text{ subsolution of } H_{\mathfrak{g}}[u]=0 \text{ with } u(y)=\mathfrak{g}(y)\Big\}.$$

Now, let u be a subsolution of $H_{\mathfrak{g}}[u]=0$ such that $u(y)=\mathfrak{g}(y)$. Let us prove that

$$S_{\mathfrak{g}}(y,x) \ge u(x), \qquad \text{for any } x \in \Omega. \tag{22}$$

Assume by contradiction that, there exists $x \in \Omega$, such that $S_{\mathfrak{g}}(y,x) < u(x)$. This implies that there exists a curve $\xi \in \Gamma(y,x)$ such that

$$u(x) > A_{\mathfrak{g}}(\xi) \ge g(x).$$

Since $u(y)=\mathfrak{g}(y)$ and $u(x)>\mathfrak{g}(x)$, there exists $\tau \in [0,1]$ such that

$$u(\xi(\tau)) = \mathfrak{g}(\xi(\tau) \text{ and } u(\xi(t)) > \mathfrak{g}(\xi(t)), \text{ for any } \tau < t \le 1.$$

Taking C a connected component of the set $[u>\mathfrak{g}]$ such that $x \in C$, we have that $x_0 := \xi(\tau) \in \partial C$, $\xi([\tau,1] \subseteq C$ and

$$u(x) > A_{\mathfrak{g}}(\xi) \ge g(x_0) + \Lambda_{\xi}(\tau,1).$$

This implies that

$$u(x) - g(x_0) > \inf\Big\{\Lambda_{\xi}(0,1)\ ;\ \varphi \in \Gamma(x_0,y) \text{ and } \varphi([0,1] \subseteq C\Big\},$$

which is impossible because u is a subsolution of $H(x,\nabla u)=0$ in C satisfying $u(x_0 = \mathfrak{g}(x_0)$ (see (5) in Proposition 1). This ends up the proof of the lemma. □

Lemma 8. *Let $y \in \overline{\Omega}$ be fixed. The function $S_{\mathfrak{g}}(y,.)$ is a supersolution of $H_{\mathfrak{g}}[u]=0$, in $\Omega \setminus \{y\}$.*

Proof : Using the formulation (3) and (4), the proof follows the same ideas of Proposition 3.2 of [10]. Indeed, assume that $S_{\mathfrak{g}}(y,.)$ is not a supersolution. This implies that there exists $z \neq y$ and $w \in \mathcal{C}^1(\Omega)$ such that $S_{\mathfrak{g}}(y,.) - w(.)$ has a local minimum at z and $H(z,\nabla w(z)) < 0$. Using the continuity of the function $x \to F(x,w(x),\nabla w(x))$ and the function $x \to S_{\mathfrak{g}}(y,x) - w(x)$, there exists $\varepsilon > 0$ and $r > 0$ such that

$$F(.,w(.),\nabla w(.)) < 0, \qquad \text{in } B(z,r)$$

and

$$S_{\mathfrak{g}}(y,x) - w(x) > \varepsilon, \qquad \text{in } \partial B(z,r).$$

Then, choosing

$$\chi(x) = \begin{cases} \max(w(x)+\varepsilon, S_{\mathfrak{g}}(y,x)) & \text{for } x \in B(z,r) \\ \\ S_{\mathfrak{g}}(y,x) & \text{otherwise,} \end{cases}$$

yields a subsolution with $\chi(y) = \mathfrak{g}(y)$. This is impossible since $\chi(z) > S_{\mathfrak{g}}(y,z)$. □

Proof of Theorem 4 : The proof of this theorem follows by Lemma 6, Lemma 7 and Lemma 8. □

Proof of Theorem 5: Since, for any $y \in \overline{\Omega}$, $S_{\mathfrak{g}}(y,.)$ is a supersolution of $H_{\mathfrak{g}}[u] = 0$ in Ω, it is not difficult to see that $\delta_{\mathfrak{g}}$ is a supersolution of $H_{\mathfrak{g}}[u] = 0$ in Ω satisfying $\delta_{\mathfrak{g}}(y) = \mathfrak{g}(y)$, for any $y \in D$. Now, let us prove that $\delta_{\mathfrak{g}}$ is a subsolution $H_{\mathfrak{g}}[u] = 0$ in Ω. It is clear that

$$\mathfrak{g}(x) \le \delta_{\mathfrak{g}}(x) \le S_{\mathfrak{g}}(y,x), \quad \text{for any } x \in \overline{\Omega} \text{ and } y \in D. \tag{23}$$

Let C be a connected component of the set $[\delta_{\mathfrak{g}} > \mathfrak{g}]$. Since $S_{\mathfrak{g}}$ is continuous, for any $z \in C$, there exists $y_z \in D$ such that $\delta_{\mathfrak{g}}(z) = S_{\mathfrak{g}}(y_z, z)$, and we have

$$\delta_{\mathfrak{g}}(x) - \delta_{\mathfrak{g}}(z) \le S_{\mathfrak{g}}(y_z, x) - S_{\mathfrak{g}}(y_z, z), \quad \text{for any } x, z \in C.$$

Moreover, since $C \subset [S_{\mathfrak{g}}(y_z,.) > \mathfrak{g}]$, by using Theorem 3 and the same arguments of the proof of Lemma 6, we deduce that $\delta_{\mathfrak{g}}$ is a subsolution of $H_{\mathfrak{g}}[u] = 0$ in Ω. □

For the proof of Theorem 6, we prove the comparison principle for the SHJO equation in a bounded domain.

Lemma 9. *Assume that Ω is a bounded domain and let u and v be a subsolution and a supersolution of $H_{\mathfrak{g}}[u] = 0$ in Ω, respectively. If $u \le v$ on $\partial\Omega$, then*

$$u(x) \le v(x), \quad \textit{for any } x \in \Omega. \tag{24}$$

Proof : Let u and v be a subsolution and a supersolution of $H_{\mathfrak{g}}[u] = 0$ in Ω, respectively. Recall that $u \ge \mathfrak{g}$ and $v \ge \mathfrak{g}$, in Ω. If $u(x) = \mathfrak{g}(x)$, then obviously $u(x) \le v(x)$. Now, assume that $u(x) > \mathfrak{g}(x)$ and let us prove that $u(x) \le v(x)$. Let us consider C the connected component of $[u > \mathfrak{g}]$, such that $x \in C$. Thanks to Lemma 5, u and v are a subsolution and a supersolution of $H[u] = 0$ in C, respectively. Moreover, we have $u = \mathfrak{g}$ and $v \ge \mathfrak{g}$ on ∂C. Using Proposition 2, we deduce that $u \le v$ in C and the proof of the first part of the theorem is finished. □

Proof of Theorem 6 : The proof of this theorem is a simple consequence of Theorem 5 and Lemma 9. □

5 Remarks, extensions and comments

5.1 The function $S_{\mathfrak{g}}(y,.)$ along the geodesic

Now, assume that $x, y \in \Omega$ are geodesically connected; that is there exists $\varphi \in \Gamma(y,x)$, that we denote by $\varphi_{y,x}$, such that

$$S_{\mathfrak{g}}(y,x) = A_{\mathfrak{g}}(\varphi_{y,x}).$$

Denoting by $t_{y,x} \in [0,1]$, the value given by

$$t_{y,x} = \max\Big\{ t \in [0,1]\ ;\ A_{\mathfrak{g}}(\varphi_{y,x}) = \mathfrak{g}(\varphi_{y,x}(t) + \int_t^A \sigma(\varphi_{y,x}(s), \varphi'_{y,x}(s)))\, ds \Big\},$$

we know by Lemma 3 that

$$S_{\mathfrak{g}}(y, \varphi_{y,x}(t_{y,x})) = \mathfrak{g}(\varphi_{y,x}(t_{y,x})).$$

In particular, this implies that $S_{\mathfrak{g}}(y,x) > \mathfrak{g}(x)$, if and only if $t_{y,x} < 1$. Moreover, $t_{y,x} = 0$ if and only if $S_{\mathfrak{g}}(y,x) = S(y,x) + \mathfrak{g}(y)$. Indeed, in one hand, we have

$$\begin{aligned} S_{\mathfrak{g}}(y,x) &\geq \Lambda_{\varphi_{y,x}}(0,1) + \mathfrak{g}(y) \\ &\geq S(y,x) + \mathfrak{g}(y). \end{aligned} \tag{25}$$

On the other hand, by definition of $t_{y,x}$, we know that $S_{\mathfrak{g}}(y,x) = \mathfrak{g}(y) + \Lambda_{\varphi_{y,x}}(0,1)$. This implies that $S(y,x) \leq \Lambda_{\varphi_{y,x}}(0,1)$, and, by (25), we get $S(y,x) = \Lambda_{\varphi_{y,x}}(0,1)$.

Theorem 7. *Let $x, y \in \Omega$, be such that $S_{\mathfrak{g}}(y,x) > \mathfrak{g}(x)$. We have*

1. *For any $t \in (t_{y,x}, 1]$, we have*

$$S_{\mathfrak{g}}(y, \varphi_{y,x}(t)) > \mathfrak{g}(\varphi_{y,x}(t)).$$

2. *For any $t_{y,x} < t_1 \leq t_2 \leq 1$, we have*

$$S_{\mathfrak{g}}(\varphi_{y,x}(t_2)) - S_{\mathfrak{g}}(\varphi_{y,x}(t_1)) = \Lambda_{\varphi_{y,x}}(t_1, t_2). \tag{26}$$

Moreover, for any $\tau \in (t_{y,x}, 1)$, there exists $\epsilon > 0$, such that for any $t_1, t_2 \in (\tau - \epsilon, \tau + \epsilon)$, we have

$$S_{\mathfrak{g}}(\varphi_{y,x}(t_2)) - S_{\mathfrak{g}}(\varphi_{y,x}(t_1)) = S(\varphi_{y,x}(t_1), \varphi_{y,x}(t_2)). \tag{27}$$

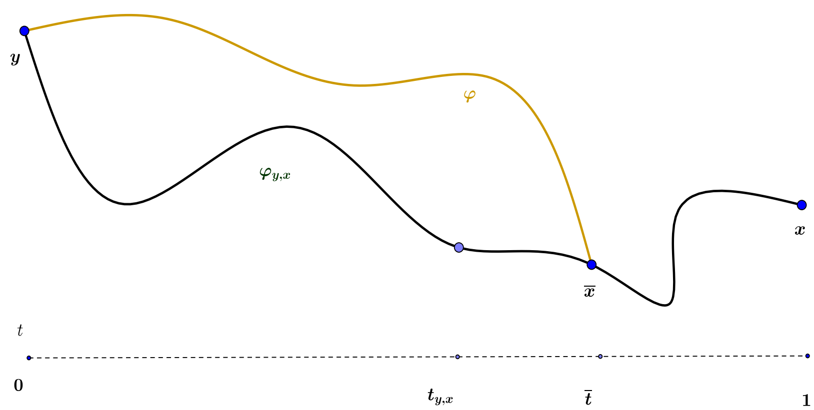
3. *Let $t \in (t_{y,x}, 1)$, if $\varphi_{y,x}$ is differentiable at t and $S(y,.)$ is differentiable at $\varphi_{y,x}(t)$, then*

$$\nabla_x S_{\mathfrak{g}}(y, \varphi_{y,x}(t)) \cdot \varphi'_{y,x}(t) = \sigma(\varphi_{y,x}(t), \varphi'_{y,x}(t)) = \nabla_x S(y, \varphi_{y,x}(t)) \cdot \varphi'_{y,x}(t). \tag{28}$$

Lemma 10. *Under the assumption of Theorem 7, for any $\bar{t} \in (t_{y,x}, 1)$, we have*

$$S_{\mathfrak{g}}(y, \varphi_{y,x}(\bar{t})) = A_{\mathfrak{g}}\left((\varphi_{y,x})_{(0,\bar{t})}\right) = \mathfrak{g}(\varphi_{y,x}(t_{y,x})) + \Lambda_{\varphi_{y,x}}(t_{y,x}, \bar{t}) \tag{29}$$

Proof : Let us denote by $\overline{x} = \varphi_{y,x}(\bar{t})$. For a given $\varphi \in \Gamma(y, \overline{x})$, we consider the path $\overline{\varphi} := \varphi \cup_{\bar{t}} (\varphi_{y,x})_{(\bar{t},1)} \in \Gamma(y, x)$.



By definition of $\varphi_{y,x}$ and $t_{y,x}$, we have

$$\mathfrak{g}(\varphi_{y,x}(t_{y,x})) + \Lambda_{\varphi_{y,x}}(t_{y,x}, 1) \leq \max\left(A_{\mathfrak{g}}((\varphi_{y,x})_{(\bar{t},1)}), A_{\mathfrak{g}}(\varphi) + \Lambda_{(\varphi_{y,x})_{(\bar{t},1)}}(0, 1)\right) \tag{30}$$

Since, $\bar{t} > t_{y,x}$, by definition of $t_{y,x}$, we have

$$A_{\mathfrak{g}}((\varphi_{y,x})_{(\bar{t},1)}) < \mathfrak{g}(\varphi_{y,x}(t_{y,x})) + \Lambda_{\varphi_{y,x}}(t_{y,x}, 1).$$

So, (30) implies that

$$\max\left(A_{\mathfrak{g}}((\varphi_{y,x})_{(\bar{t},1)}), A_{\mathfrak{g}}(\varphi) + \Lambda_{(\varphi_{y,x})_{(\bar{t},1)}}(0, 1)\right) = A_{\mathfrak{g}}(\varphi) + \Lambda_{(\varphi_{y,x})_{(\bar{t},1)}}(0, 1),$$

and

$$\mathfrak{g}(\varphi_{y,x}(t_{y,x})) + \Lambda_{\varphi_{y,x}}(t_{y,x}, 1) \leq A_{\mathfrak{g}}(\varphi) + \Lambda_{(\varphi_{y,x})_{(\bar{t},1)}}(0, 1).$$

This implies that

$$A_{\mathfrak{g}}(\varphi) \geq A_{\mathfrak{g}}\left((\varphi_{y,x})_{(0,\bar{t})}\right).$$

Since $\varphi \in \Gamma(y, \varphi_{y,x}(\bar{t}))$ is arbitrary and $(\varphi_{y,x})_{(0,\bar{t})} \in \Gamma(y, \varphi_{y,x}(\bar{t}))$, we deduce that

$$S_{\mathfrak{g}}(y, \varphi_{y,x}(\bar{t})) = A_{\mathfrak{g}}\left((\varphi_{y,x})_{(0,\bar{t})}\right).$$

Then, using the fact that $t > t-y,x$, it not difficult to see that $A_{\mathfrak{g}}\Big((\varphi_{y,x})_{(0,\bar{t})}\Big) = \mathfrak{g}(\varphi_{y,x}(t_{y,x})) + \Lambda_{\varphi_{y,x}}(t_{y,x},\bar{t})$, and the proof is complete. □

Proof of Theorem 7 :

1. This follows from Lemma 10 and the definition of $t_{y,x}$. Indeed, for any $t > t_{y,x}$, we have

$$\begin{aligned} g(\varphi_{y,x}(t)) &= g(\varphi_{y,x}(t_{y,x})) + \Lambda_{\varphi_{y,x}}(t_{y,x},t) \\ &= g(\varphi_{y,x}(t_{y,x})) + \Lambda_{\varphi_{y,x}}(t_{y,x},1) - \Lambda_{\varphi_{y,x}}(t,1) \\ &> g(\varphi_{y,x}(t)) + \Lambda_{\varphi_{y,x}}(t,1) - \Lambda_{\varphi_{y,x}}(t,1) = g(\varphi_{y,x}(t)). \end{aligned}$$

2. Thanks again to Lemma 10, we have (26). Now, for the proof of (27), we see that combining (26) with Theorem 3, we have

$$\Lambda_{\varphi_{y,x}}(t_1,t_2) \leq S(\varphi_{y,x}(t_1),\varphi_{y,x}(t_2)),$$

for any t_1 and t_2 in the neighbor of τ (because $S_{\mathfrak{g}}(y,\varphi_{y,x}(\tau)) > \mathfrak{g}(\varphi_{y,x}(\tau))$. Thus $\Lambda_{\varphi_{y,x}}(t_1,t_2) = S(\varphi_{y,x}(t_1),\varphi_{y,x}(t_2))$. This finish the proof of the second part of the theorem.

3. Thanks to the second part of the theorem, for any $t \in (t_{y,x},1)$, and $h > 0$ small enough, we have

$$\begin{aligned} \frac{1}{h}\Big(S_{\mathfrak{g}}(y,\varphi_{y,x}(t+h)) - S_{\mathfrak{g}}(y,\varphi_{y,x}(t)\Big) &= \frac{1}{h}\Big(S(y,\varphi_{y,x}(t+h)) - S(y,\varphi_{y,x}(t)\Big) \\ &= \frac{1}{h}\int_t^{t+h}\sigma(\varphi_{y,x}(s),\varphi'_{y,x}(s))\,ds\,ds. \end{aligned}$$

Letting $h \to 0$, we obtain

$$\begin{aligned} \nabla S_{\mathfrak{g}}(y,\varphi_{y,x}(t)) \cdot \varphi'_{y,x}(t) &= \nabla S(y,\varphi_{y,x}(t)) \cdot \varphi'_{y,x}(t) \\ &= \sigma(\varphi_{y,x}(t),\varphi'_{y,x}(t)). \end{aligned}$$

□

5.2 The critical case

At last, let us mention that most of the results of this paper holds to be true if we replace the assumptions (H3) by the general one

$$(H'3) \qquad H(x,0) \leq 0, \quad \text{for any } x \in \Omega.$$

However, even if we do believe that the condition that the Aubry set is empty is not necessary to get (18), it is not clear for us if the result of Theorem 3 holds to be true or not under the assumption $(H'3)$. Actually, the proof of Theorem 3 is based on the estimate (16) and the property (18). So, it is possible to weaken the assumption (H3) to new conditions which enters into (18). We do not abort these particular cases in this paper and avoid to splint the result with respect to the partitions of $\mathcal{A}$. This corresponds to the critical case for the Hamilton-Jacobi equation $H[u] = 0$ as well as for the SHJO equation $H_{\mathfrak{g}}[u] = 0$; for which both metrics S and $I_{\mathfrak{g}}$ would degenerate. We are planning to give a complete study to the critical case in a forthcoming paper.

5.3 Comments and remarks on the free boundary formulation

As usual for the obstacle problem, the domain Ω is divided into two regions separated by the so called free boundary. The region where the solution u coincides with the obstacle is called the contact set ; that we denoted by Ω_c. In the remaining region $\Omega \setminus \Omega_c$, u is a viscosity solution of the Hamilton-Jacobi equation $H[u] = 0$. Interesting questions about the regularity of the free boundary and the "geodesic convexity" of the set $\Omega\setminus\Omega_c$ remain to be open. Indeed, knowing these kind of behaviors one can rewrite the solution of the SHJO problem using local intrinsic metrics corresponding to connected component of the set $\Omega \setminus \Omega_c$, and establish qualitative behavior of the solution in these regions. To name just a few, these kind of questions appears in many other different context : equilibrium position of a membrane, fluid filtration in porous media, optimal control or financial mathematics, etc.

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