Well-positioned convex sets and functions and applications

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Outline

- Setting

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- Some properties of the barrier cone to a convex set

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- Well-positioned sets

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- Closure of the difference of closed convex sets
- Continuous and Slice-continuous sets
- Separation of convex sets

Framework

Throughout this presentation, we suppose that X is a reflexive Banach space (unless otherwise stated) with continuous dual X^*.

An important object: the barrier cone

The recession cone to the closed convex set S is the closed convex cone S^∞ defined as

$$S_\infty = \{v \in X : \ \ \forall\, \lambda > 0,\, \forall\, x_0 \in S,\ x_0 + \lambda v \in S\}\,.$$

S is called linearly bounded whenever $S_\infty = \{0\}$.
Given a closed convex subset S of X, the domain of the support function given by

$$\sigma_S(f) := \ \sup_{x\in S} \langle f, x\rangle$$

is the barrier cone of S:

$$\mathcal{B}(S) = \ \{f \in X^* : \ \sigma_S(f) < +\infty\} = \mathrm{Dom}\,\sigma_S\,.$$

The closure of the barrier cone

Let C be a closed convex set in a normed linear space X.

When X is reflexive: By the Bipolar Theorem,

$$\overline{\mathcal{B}(C)} = \mathcal{B}(C)^{\circ\circ}$$

But

$$\mathcal{B}(C)^{\circ\circ} = (C_\infty)^\circ.$$

Thus, $\overline{\mathcal{B}(C)}$ is characterized in X^* by the formula:

$$\overline{\mathcal{B}(C)} = (C^\infty)^\circ.$$

Setting: Normed linear space

Set

$$C_f = \{x \in X : \langle f, x\rangle \geq \|x\|\}\,.$$

For every closed convex set C of a normed linear space X, the following two facts are equivalent:

- $\overline{\mathcal{B}(C)} = (C^\infty)^\circ$;
- $C \cap C_f$ is bounded for every $f \in (C^\infty)^\circ$.

When C is linearly bounded $\overline{\mathcal{B}(C)} = X^*$ if and only if $C \cap C_f$ is bounded for every $f \in X^*$. The second condition defines the family of conically bounded sets which is a subclass of the class of linearly bounded sets for which holds the property of density of the barrier cone in X^*.

The closure of the barrier cone

We define the temperate cone of C:

$$\mathcal{T}(C) = \left\{ f \in X^* : \lim_{r \to \infty} \left(\inf_{x \in C, \langle f, x \rangle \geq r} \frac{\|x\|}{r} \right) = \infty \right\}.$$

- As $\inf_{x \in \emptyset} \|x\| = \infty$, it follows that $\mathcal{B}(C) \subseteq \mathcal{T}(C)$.
- The temperate cone of a closed convex set is norm-closed.

FACT [AET, PAMS, 2003] : For every closed convex set C of a normed linear space X, the closure of the barrier cone is the temperate cone.

$$\overline{\mathcal{B}(C)} = \mathcal{T}(C).$$

Closure of the domain of the *Fenchel conjugate* of $\Psi \in \Gamma_0(X)$

Take

$$\|(x,\mu)\|_{X\times\mathbb{R}} := \sqrt{\|x\|^2+\mu^2},\ \forall (x,\mu) \in X\times\mathbb{R},$$

as norm on $X \times \mathbb{R}$ and

$$\|(f,\lambda)\|_{X^*\times\mathbb{R}} := \sqrt{\|f\|_*^2+\lambda^2},\ \forall (f,\lambda) \in X^*\times\mathbb{R}$$

as dual norm on $X^* \times \mathbb{R}$.

The duality pairing is given $\forall (f,\lambda) \in X^* \times \mathbb{R}, (x,\mu) \in X \times \mathbb{R}$ by

$$\langle (f,\lambda),(x,\mu)\rangle_{X^*\times\mathbb{R},X\times\mathbb{R}} = \langle f,x\rangle + \lambda\mu$$

Closure of the domain of the *Fenchel conjugate* of $\Psi \in \Gamma_0(X)$

Given an extended-real valued function $\Psi : X \to \mathbb{R} \cup \{\infty\}$, recall that the Fenchel conjugate of Ψ is the function $\Psi^* : X^* \to \mathbb{R} \cup \{+\infty\}$ given by

$$\Psi^*(f) := \sup_{x \in X}\{\langle f, x\rangle - \Psi(x)\}.$$

Obviously, the domain of Ψ^* is connected to the barrier cone $\mathcal{B}(\text{epi } \Psi)$ of the epigraph of Ψ through the following equivalence:

$$g \in \mathrm{Dom}\Psi^* \iff (g, -1) \in \mathcal{B}(\text{epi } \Psi).$$

This yields

$$\mathrm{Dom}\Psi^* \times \{-1\} = \mathcal{B}(\text{epi } \Psi) \cap \left(X^* \times \{-1\}\right).$$

Standard techniques from convex analysis allow us to prove that

$$\overline{\mathrm{Dom}\Psi^*} = \overline{\mathcal{B}(\mathrm{epi}\ \Psi)} \cap (X^* \times \{-1\}).$$

Precedent results allow us to find a new proof of the well-known characterization of $\overline{\mathrm{Dom}\Psi^*}$

Let $\Psi : X \to \mathbb{R} \cup \{+\infty\}$ be an extended-real valued proper convex and lower semicontinuous functional. Then $0 \in \overline{\mathrm{Dom}(\Psi^*)} \iff$ the map $x \mapsto (\Psi(x) + \varepsilon\|x\|)$ is bounded below for every $\varepsilon > 0$.

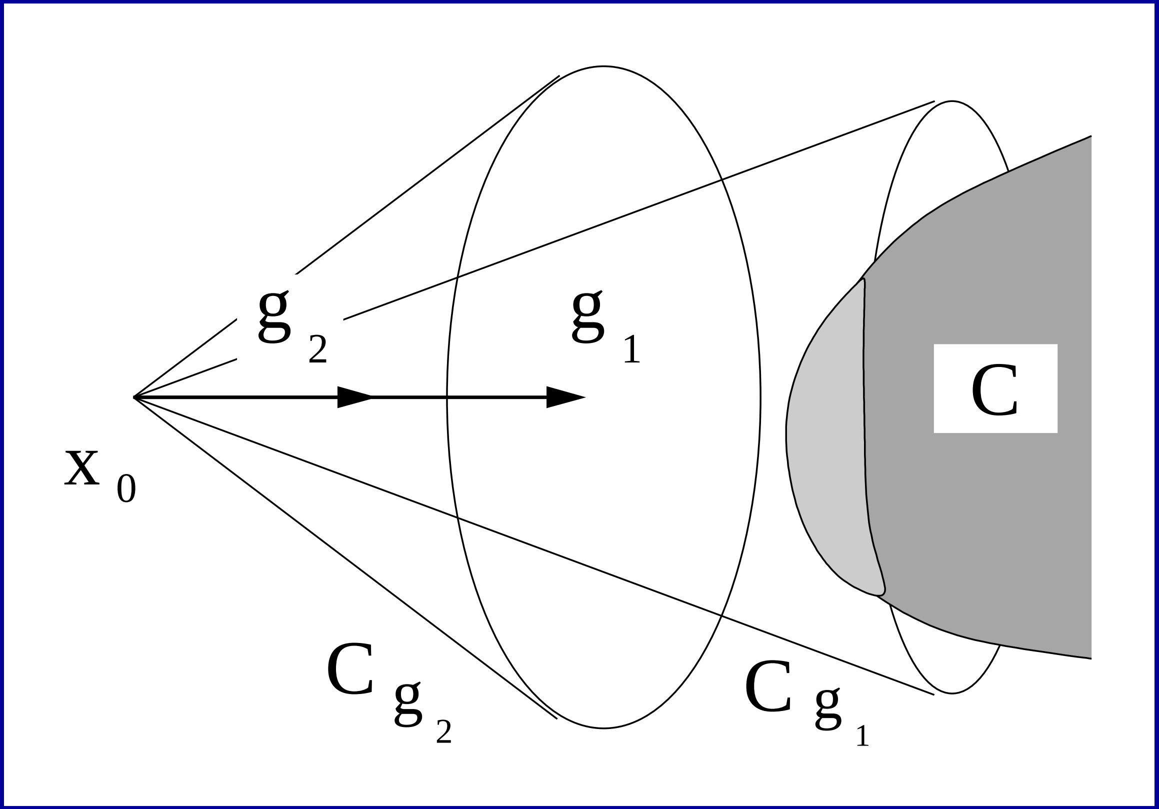
Well-Positioned Sets [AET, JOGO, 2004]

The concept of well-positioned closed convex set is a geometric notion equivalent, in the framework of reflexive Banach spaces, to the absence of lines and to weak local compactness. The necessity of well-positionedness in this separation problem was established by Adly et al., while sufficiency goes back to Dieudonné

A nonempty subset C of the normed vector space X is *well-positioned* if there exist $x_0 \in X$ and $g \in X^*$ such that:

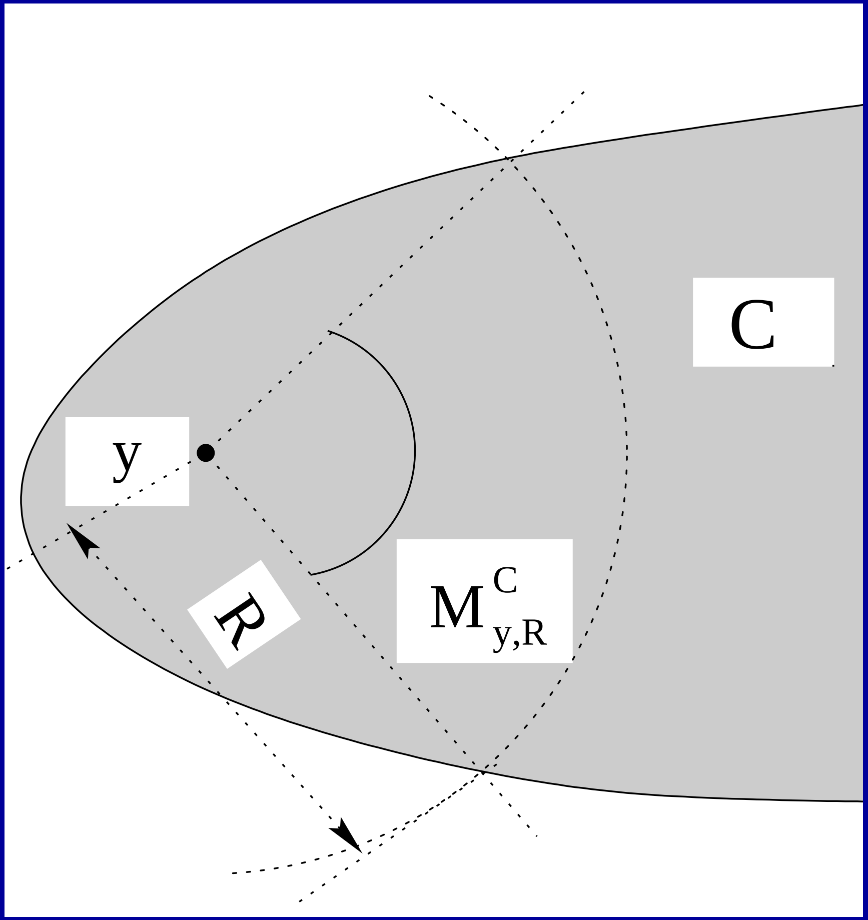
$$\langle g, x - x_0 \rangle \geq \|x - x_0\|, \quad \forall\, x \in C.$$

Equivalently, denoting by $C_g := \{x \in X : \langle g, x\rangle \geq \|x\|\}$, then C is well-positioned $\iff$ C is included in some translate of C_g.



A closed convex set C is well-positioned $\iff$ for every $\mathbf{y} \in C$ there exists $R > 0$ such that $0 \notin \overline{\mathrm{co}}(M^C_{y,R})$, where

$$M^C_{y,R} = \left\{ \frac{\mathbf{x} - \mathbf{y}}{|\mathbf{x} - \mathbf{y}|} : \mathbf{x} \in C,\ |\mathbf{x} - \mathbf{y}| \geq R \right\}.$$


C
y
R
$M^C_{y,R}$

Analytical characterization of W.P

A nonempty closed convex set C of a reflexive Banach space X is well-positioned $\iff$ the following two assumptions are satisfied:

- C contains no lines;
- $\nexists \{x_n\}_{n\in\mathbb{N}} \subset C,\ \|x_n\| \to +\infty$, such that $\frac{x_n}{\|x_n\|} \rightharpoonup 0$

When X is finite dimensional, a nonempty closed convex set is well-positioned $\iff$ C_∞ is pointed, i.e., $C^\infty \cap -C^\infty = \{0\}$.

In particular, every compact and convex set is well-positioned in $\mathbb{R}^n$.

A characterization

Let C be a nonempty subset of a reflexive Banach space X. The following two conditions are equivalent:

- The barrier cone of C has a nonempty interior;
- C is well-positioned.

Moreover, if $\mathrm{Int}\ \mathcal{B}(C) \neq \emptyset$, then,

$$\mathrm{Int}\ \mathcal{B}(C) = \mathrm{Int}\ (C^{\infty})^{\circ}.$$

Well-positioned functionals

We say that a proper convex lower semicontinuous functional $\Psi : X \to \mathbb{R} \cup \{+\infty\}$ is well-positioned if the epigraph of Ψ,

$$\operatorname{epi} \Psi = \{(x, \lambda) \in X \times \mathbb{R} : \lambda \geq \Psi(x)\},$$

is a well-positioned subset of $X \times \mathbb{R}$.

Let Ψ be a proper lower semicontinuous convex function on a reflexive Banach space. Then,

$$\mathrm{Int}\ \ \mathrm{Dom}\Psi^* \neq \emptyset \quad \Longleftrightarrow \quad \Psi \text{ is well} - \text{positioned}.$$

Well-positioned functionals

$\Psi \in \Gamma_0(X)$ is well-positioned $\iff$ if the two following assumptions hold:

- $\operatorname{Ker}(\Psi_\infty)$ contains no lines;
- $\nexists (x_n)_{n\in\mathbb{N}} \subset \mathrm{Dom}\,\Psi,\ \|x_n\| \to +\infty$, such that $\frac{x_n}{\|x_n\|} \rightharpoonup 0$ and $\frac{\Psi(x_n)}{\|x_n\|} \to 0$.

We also racapture the following result :

$g \in \mathrm{Int}\ \mathrm{Dom}\Psi^*$ if and only if the functional $\Psi - g$ is coercive, i.e.,

$$\liminf_{\|x\|\to+\infty} \frac{\Psi(x) - \langle g, x\rangle}{\|x\|} > 0.$$

Stability of the existence of the solution for semi-coercive variational inequalities

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Framework :

$VI(A, f, \Phi, K)$: find $u \in K \cap \text{Dom}\Phi$ such that

$$\langle Au - f, v - u\rangle \ + \ \Phi(v) - \Phi(u) \geq 0, \ \ \forall v \in K, \qquad (1)$$

Assumptions :

- K is a closed convex set in a reflexive Banach space X
- $f \in X^*$
- $\Phi \in \Gamma_0(X)$ is assumed to be bounded below
- $K \cap \text{Dom}\Phi \neq \emptyset$
- A is a semi-coercive operator from X to X^*
- A is pseudomonotone in the sense of Brezis.

Stability of the existence of the solution for semi-coercive variational inequalities

Recall that A is semicoercive if

$$\langle Av - Au, v - u\rangle \geq \kappa \left(\mathrm{dist}_U(v-u)\right)^2 \quad \forall u, v \in X \tag{2}$$

$$A(x+u) = A(x) \quad \forall x \in X \text{ and } u \in U, \text{ and } A(X) \subseteq U^{\perp},$$

for some positive constant κ and some closed subspace U of X.

Stability of the existence of the solution for semi-coercive variational inequalities

In other words, we characterize all data (A, f, Φ, K) for which there is some $\varepsilon > 0$ such that the variational inequality $VI(A_\varepsilon, f_\varepsilon, \Phi_\varepsilon, K_\varepsilon)$ has solutions for every instance involving a bounded and semi-coercive operator A_ε, a linear continuous functional f_ε, a proper lower semicontinuous and convex functional Φ_ε that is bounded below, and a closed convex set K_ε such that $K_\varepsilon \cap \mathrm{Dom}\Phi_\varepsilon \neq \emptyset$, and

$$\|A(x) - A_\varepsilon(x)\|_* < \varepsilon, \quad \forall x \in X$$

$$\|f - f_\varepsilon\|_* < \varepsilon,$$

$$K \subset K_\varepsilon + \varepsilon\mathbb{B}_X \text{ and } K_\varepsilon \subset K + \varepsilon\mathbb{B}_X,$$

$$\Phi(x) - \varepsilon \leq \Phi_\varepsilon(x) \leq \Phi(x) + \varepsilon, \quad \forall x \in X.$$

Stability of the existence of the solution for semi-coercive variational inequalities

In this framework, it was proved [AET CCM, Proposition 3.1] that a sufficient and necessary condition ensuring the uniform stability of the solution set of the given variational inequality is that $f \in \mathrm{Int}\ \mathrm{Dom}\Psi^*$, where

- $\mathrm{Int}\ R(A, \Phi, K) = \mathrm{Int}\ \mathrm{Dom}\Psi^*$
- $R(A, \Phi, K) = \{f \in X^* \mid V.I.(A, f, \Phi, K) \text{ has at least a solution}\}$
- $\Psi(x) := \ \kappa(\mathrm{dist}_U(x))^2 + I_K(x) + \Phi(x) \quad \forall x \in X.$

Analytical characterization of the stability of the solution set.

$V.I.(A_\varepsilon, f_\varepsilon, \Phi_\varepsilon, K_\varepsilon)$ has solutions for every sufficiently small uniform perturbations $A_\varepsilon, f_\varepsilon, \Phi_\varepsilon, K_\varepsilon$, if and only if the following three conditions hold:

(i) The set $U \cap K_\infty \cap \ker(\Phi_\infty)$ contains no lines;

(ii) There is no sequence $(x_n)_{n\in\mathbb{N}} \in K$ such that

$$\frac{x_n}{\|x_n\|} \rightharpoonup 0 \text{ and } \frac{\kappa(\mathrm{dist}_U(x_n))^2 + \Phi(x_n)}{\|x_n\|} \to 0;$$

and

(iii) $\langle f, u\rangle < \Phi_\infty(u), \quad \forall u \in (K_\infty \cap U), u \neq 0.$

Consequence

We have stability of the existence of a solution $\iff$ $\Psi - f$ is coercive.

This result relates the stability of the solution of a variational inequality and the coerciveness of an associated energy-type functional.

Closedness of the algebraic difference of two convex closed sets

Another objective of this presentation is to determine, in a reflexive Banach space, the broadest condition which should be added to the recession condition $C_1^\infty \cap C_2^\infty = \{0\}$ in order to ensure the closedness of the algebraic difference of two convex closed sets.

Namely, we will characterize the class $\mathcal{C}$ of those closed convex sets C_1 containing no lines for which the algebraic difference $C_1 - C_2$ is closed for every closed convex set C_2 fulfilling $C_1^\infty \cap C_2^\infty = \{0\}$.

Closedness of the algebraic difference of two convex closed sets: a geometrical condition [AET JMPA 2003]

$$\boxed{C_1 \in \mathcal{C} \quad \Longleftrightarrow \quad C \text{ well-positioned}}$$

Thus, the statement of the broadest setting for the Dieudonné Theorem in a reflexive setting is the following:

The algebraic difference of two closed convex sets containing no lines remains closed whenever the both following conditions are fulfilled:

- $C_1^\infty \cap C_2^\infty = \{0\}$
- at least one of the sets C_1, C_2 is well-positioned.

Closedness of the algebraic difference of two convex closed sets

Observation : the algebraic difference of two closed and convex sets may be closed even if none of the terms of the difference is well-positioned.

The closedness of the algebraic difference is, for instance, ensured if

$$\lim_{r\to\infty} \mathrm{d}(C_1 \setminus r\mathbb{B}_X, C_2 \setminus r\mathbb{B}_X) = \infty. \tag{3}$$

This condition is sufficient but is obviously not necessary.

Take

- $C_1 = \{(x, y) \in \mathbb{R}^2 : x \geq 0, y \geq 0\}$
- $C_2 = \{(x, y) \in \mathbb{R}^2 : x \geq 0, y \leq 0\}$

$C_1 - C_2 = \{(x, y) \in \mathbb{R}^2 : y \geq 0\}$ although relation (3) is violated.

Closedness of the algebraic difference of two convex closed sets

For closed convex sets containing no lines, there is an important difference between the cases when $C_1 - C_2$ is closed while the sets C_1 and C_2 fulfill condition (3) and those when $C_1 - C_2$ is closed while condition (3) is not verified.

Let us first remark that if condition (3) is fulfilled by two closed convex sets C_1 and C_2, then it is also verified by any two uniform perturbations $C_{1,\varepsilon}$ and $C_{2,\varepsilon}$ of C_1 and C_2, that is closed convex sets fulfilling

$$C_{1,\varepsilon} \subseteq C_1+\varepsilon\mathbb{B}_X\,, C_1 \subseteq C_{1,\varepsilon}+\varepsilon\mathbb{B}_X\,, \quad C_{2,\varepsilon} \subseteq C_2+\varepsilon\mathbb{B}_X\,, C_2 \subseteq C_{2,\varepsilon}+\varepsilon\mathbb{B}_X\,. \tag{4}$$

Closedness of the algebraic difference of two convex closed sets

Consequently, the algebraic difference $C_{1,\varepsilon} - C_{2,\varepsilon}$ is closed and therefore relation (3) ensures not only the closure of the algebraic difference of the initial sets, but also the closure of the algebraic difference of any small uniform perturbations of the initial sets.

Closedness of the algebraic difference of two convex closed sets

The closedness of the algebraic difference is unstable with respect to small uniform perturbations of the initial sets when condition (3) is violated.

More precisely, [AET, JMPA, 2003] if the algebraic difference of two closed convex sets, C_1 and C_2 containing no lines is a closed proper subset of a reflexive Banach space X and if condition (3) is violated, then, for every $\varepsilon > 0$ there are $C_{1,\varepsilon}$ and $C_{2,\varepsilon}$, two uniform perturbations of C_1 and C_2 such that $C_{1,\varepsilon} - C_{2,\varepsilon}$ is not a closed set.

Closedness of the algebraic difference of two convex closed sets

Condition (3), although only a sufficient condition when the mere closedness of the algebraic difference is requested, turns out to be both sufficient and necessary, if we are seeking on stable closedness.

Slice-continuous sets

The last part of the lecture will be devoted to the class of slice-continuous sets.

This class extends the concept of continuous sets introduced by Gale and Klee (see also Auslender and Coutat), i.e. closed convex sets such that their support functional

$$\sigma_C : X^* \to \mathbb{R}$$

is continuous on $X^* \setminus \{0\}$.

A nonempty closed convex sets C is slice-continuous if $C \cap L$ is continuous with respect to every closed linear manifold L which meets C.

This class coincides with the class of well-positioned closed convex sets with no boundary rays and no asymptotes.

Asymptote

Let C be a nonempty, closed convex subset of the normed vector space X. We say that the half-line $A := y_0 + \mathbb{R}_+ w$ (with $y_0, w \in X$ and $\|w\| = 1$) is an asymptote of C, and that w is an asymptotic direction of C, if $A \cap \operatorname{Int} C = \emptyset$ and $\operatorname{gap}(A \setminus r\mathbb{B}_X, C) = 0$ for every $r \geq 0$.

Lemma A

Let X be a Banach space and $w \in X$ with $\|w\| = 1$. The following two statements are equivalent:

(a) w is an asymptotic direction of C;

(b) $w \in C^\infty$ and the half-line $B := z_0 + \mathbb{R}_+ w$ is disjoint from C for some $z_0 \in X$.

A characterization

Proposition (@)

Let C be a nonempty proper closed convex subset of X. The following statements are equivalent:

- C is slice-continuous;
- C is continuous and has no asymptotes;
- C has no asymptotic linear manifolds;
- C is well-positioned and admits no asymptotes;
- for every closed linear manifold L which meets C, the barrier cone of $C \cap L$ is the union between $V(L)^{\perp}$ and a nonempty norm-open set.

Observation

In a Banach space X, the following are equivalent:

$$f \in \Gamma_0(X) \text{ is norm} - \text{continuous at } x \in X$$

$$\Longleftrightarrow$$

$$x \in (X \setminus \operatorname{Dom} h) \cup \operatorname{Int}(\operatorname{Dom} h).$$

Apply this remark to $\sigma_C : X^* \to \mathbb{R} \cup \{+\infty\}$ to derive

$$C \text{ is continuous}$$

$$\Longleftrightarrow$$

$$\mathcal{B}(C) = \{0\} \cup \operatorname{Int}(\mathcal{B}(C)).$$

Known

We recall that

- in $\mathbb{R}^n$, a nonempty closed convex set can be strictly separated from any other disjoint nonempty closed convex set if and only if it is continuous.
- on the basis of the results from [AC] and [GK], it can be established that a non-constant, real-valued, convex function attains its infimum on every nonempty closed convex subset of $\mathbb{R}^n$ if and only if the function attains its infimum on $\mathbb{R}^n$ and all its level sets are continuous.

Theorem ($\star$) ETZ, JFA, 2005

We will conclude the presentation by the following result:

Let $\Phi : X \to \mathbb{R}$ be a non-constant, convex and continuous function which attains its infimum on a reflexive Banach space X. Then the following statements are equivalent:

- Φ attains its infimum on each nonempty closed convex subset of X;
- every nonempty level set of Φ is a slice-continuous set.

Finally, we want to point out that the same condition characterizes the class of nonempty closed and convex sets which may be strictly separated by a closed hyperplane from any disjoint nonempty closed convex set.

Strict separation of convex sets

We use the last Theorem to characterize all the nonempty closed and convex subsets C of a reflexive Banach space X which can be strictly separated from every disjoint nonempty closed and convex set D, *i.e.*,

$$\exists f \in X^* \text{ s.t. } \sup_{x \in C} \langle f, x \rangle < \inf_{y \in D} \langle f, y \rangle .$$

Indeed, it is well known that two nonempty closed and convex subsets C and D of X can be strictly separated $\iff \operatorname{gap}(C, D) > 0$.

For every nonempty closed and convex subset C of the reflexive Banach space X, set $\Phi_C : X \longrightarrow \mathbb{R}$ for the real-valued function defined by

$$\Phi_C(x) = \inf_{y \in C} \|y - x\|, \quad \forall\, x \in X.$$

It is straightforward to prove that Φ_C is convex and continuous, and that its level sets satisfy

$$C_m = \begin{cases} \emptyset & \text{if} \quad m < 0, \\ C + m\mathbb{B}_X & \text{if} \quad m \geq 0. \end{cases}$$

Let us also remark that $\mathrm{gap}(C, D) = \inf_{x \in D} \Phi_C(x)$.

Lemma ($\star\star$)

The following result allows us to use our Main Theorem in deciding whether the nonempty closed and convex set C may be strictly separated from every disjoint nonempty closed and convex set D.

Let C be a nonempty closed and convex subset of a reflexive Banach space X. The two following assertions are equivalent:

- For every nonempty closed and convex subset D of X such that $C \cap D = \emptyset$ one has that $\operatorname{gap}(C, D) > 0$;
- The function Φ_C attains its infimum on every nonempty closed and convex subset of X.

Theorem ETZ, JFA, 2005

Proposition (@), Theorem ($\star$) and Lemma ($\star\star$) allow us to claim:

Let C be a nonempty closed and convex proper subset of a reflexive Banach space X. The two following assertions are equivalent:

- C can be strictly separated from every disjoint nonempty closed and convex subset of X;
- C is a slice-continuous set.

Sketch of the proof

Lemma ($\star\star$) implies that (a) holds $\iff$ Φ_C attains its infimum on every nonempty closed and convex subset of X.

From Theorem ($\star$), (a) holds $\iff$ $C + r\mathbb{B}_X$ is a slice-continuous set for every $r \geq 0$.

The proof will be achieved if we show that for each slice-continuous set C and each $r \geq 0$, the sets $C + r\mathbb{B}_X$ are slice-continuous.

Sketch of the proof

Assume that C is a slice-continuous set.

It is obvious that $\mathcal{B}(C + r\mathbb{B}_X) = \mathcal{B}(C)$ for every $r \geq 0$.

We know that C is well-positioned $\iff$ $\operatorname{Int}\mathcal{B}(C) \neq \emptyset$.

So $C + r\mathbb{B}_X$ is well-positioned for every $r \geq 0$.

Assume that $C + r\mathbb{B}_X$ has asymptotes for some $r \geq 0$.

Then, by Lemma A, there exist $w \in (C + r\mathbb{B}_X)^\infty = C^\infty$ with $\|w\| = 1$ and $z_0 \in X$ such that $B := z_0 + \mathbb{R}_+ w$ is disjoint from $C + r\mathbb{B}_X$.

It follows that B and C are disjoint, and so, using again Lemma A, C has asymptotes, a contradiction

Thank You!

ESAIM

COCV

Control, Optimisation
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