Elliptic PDEs with singular coefficients

Michel Théra
LACO, UMR-CNRS 6090, Université de Limoges

michel.thera@unilim.fr

reporting joint work with Teemu Pennanen et Julian Revalski

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Credits

Talk based on:

- T. Pennanen, J.P. Revalski and MT, Variational composition of a monotone operator with a linear mapping *J. Functional Analysis*, vol 198/1 pp 84 - 105, 2003.
- H. Attouch, J.-B. Baillon and M. Théra, Variational sum of monotone operators, *J. Convex Anal.*, **1**(1994), 1-29.

Contents

- Motivation
- Notations and basic definitions
- The variational composition
- Measurability of composite mappings
- Application to PDE's with singular coefficients

Setting

Let $\Omega \subset \mathbb{R}^N$ be open and let $Q : \Omega \mapsto \mathbb{R}^{N \times N}$ be measurable with $Q(x)$ symmetric and positive semidefinite a.e. on Ω. In physics, for instance, we are often faced to minimize over $u \in H_0^1(\Omega)$, the extended-real-valued functional $g : H_0^1(\Omega) \longrightarrow \mathbb{R} \cup \{+\infty\}$ given by

$$g(u) = \begin{cases} \frac{1}{2} \int_\Omega \nabla u(x)^T Q(x) \nabla u(x) dx & \text{if } \nabla u(\cdot)^T Q(\cdot) \nabla u(\cdot) \in L^1(\Omega), \\ +\infty & \text{otherwise.} \end{cases}$$

Therefore, we are equivalently interested in solving the inclusion :

$$\partial g(u) \ni 0$$

Important to find an explicit expression for ∂g .

Set I_f for the convex and lower semicontinuous function on $L^2(\Omega; H)$ defined by

$$I_f(u) = \begin{cases} \int_\Omega f(\omega, u(\omega))d\omega & \text{if } f(\cdot, u(\cdot)) \in L^1(\Omega) \\ +\infty & \text{otherwise} \end{cases}$$

f for the mapping given by $f(x,v) = \frac{1}{2}v^T Q(x) v$
and $\nabla : H_0^1(\Omega) \to L^2(\Omega; \mathbb{R}^N)$, for the continuous linear map defined by $\nabla u = \{\frac{\partial u}{\partial x_i}\}_{i=1}^N$

We can express g in the composite form $g = I_f \circ \nabla$.

When a qualification condition holds

In cases where the constraint qualification

$$0 \in \mathrm{Int}(\mathrm{rge}\, A - \mathrm{Dom}\, f)$$

holds, e.g. when

$$Q(x) \in L^\infty(\Omega; \mathbb{R}^{N\times N}).$$

Then,

$$\mathrm{Dom}\, I_f = L^2(\Omega; \mathbb{R}^N),$$

the classical chain rule gives:

$$\mathrm{Dom}\, \partial g = \left\{ u \in H_0^1(\Omega) \;\middle|\; Q(\cdot)\nabla u(\cdot) \in L^2(\Omega; \mathbb{R}^N) \right\}$$
$$\partial g(u) = -\mathrm{div}[Q(\cdot)\nabla u(\cdot)].$$

Without any qualification condition, what to do?

When the constraint qualification does not hold, the classical chain rule for subdifferentials of composite convex functions no longer apply. However, it may still be possible to find the right expression for ∂g through the computation of the so-called variational composition.

The Main result

This is demonstrated in the following result that allows singular coefficients. Recall, that the dual of $H_0^1(\Omega)$ coincides with the space $H^{-1}(\Omega)$ of distributions, and that the divergence is defined (in the distribution sense) for any distribution.

Main Theorem ◑ *Let* Ω *be bounded,* $\alpha > 0$*, and let* $Q \in L^1(\Omega; \mathbb{R}^{N\times N})$ *be such that* $\langle v, Q(x)v\rangle \geq \alpha |v|^2$ *for all* $v \in \mathbb{R}^m$ *and for a.e.* $x \in \Omega$*. Then for each* $u^* \in H^{-1}(\Omega)$*, there exists a* $u \in H_0^1(\Omega)$ *such that*

$$-\mathrm{div}[Q(\cdot)u(\cdot)] = u^*.$$

Objective of the talk

- To propose a regularized notion of a composition of a monotone operator with a linear mapping.
- This new concept, called the variational composition, can be shown to be maximal monotone in many cases where the usual composition is not.
- The two notions coincide, however, whenever the latter is maximal monotone.

The utility of the variational composition is demonstrated by

- applications to subdifferential calculus
- theory of measurable multifunctions
- elliptic PDEs on divergence form

Notations and Definitions

U and X will be real reflexive Banach spaces and U^* and X^*

A set-valued mapping $T : U \rightrightarrows U^*$ is called *monotone* if

$$u_1^* \in T(u_1),\ u_2^* \in T(u_2) \quad \Longrightarrow \quad \langle u_1 - u_2, u_1^* - u_2^* \rangle \geq 0,$$

If a monotone mapping cannot be properly extended to another monotone mapping from U to U^*, it is called *maximal monotone.*

An important example is the subdifferential

$$\partial f(u) = \{u^* \in U^* \mid f(v) \geq f(u) + \langle u^*, v - u \rangle \ \forall v \in U\}$$

of a convex function $f : U \to \mathbb{R} \cup \{+\infty\}$.

Duality mapping- Yosida regularization

$J_U : U \rightrightarrows U^*$, given by $J_U = \partial \frac{1}{2}\|u\|^2$, $u \in U$;

$J_U(u) = \{u^* \in U^* : \langle u, u^* \rangle = \|u\|^2 = \|u^*\|^2\},\ u \in U.$

T – mm, $\lambda > 0$, the Yosida regularization of order $\lambda > 0$

$$T_\lambda = (T^{-1} + \lambda J_U^{-1})^{-1}$$

defined on all U, singled-valued, mm, continuous ;

X-Hilbert: $T_\lambda = (1/\lambda)(I - J_\lambda^T)$ where

$$J_\lambda^T = (I + \lambda A)^{-1}$$

is the resolvent of order λ of T; I–identity.

As usual, the following Minty-Rockafellar criterion for maximal monotonicity will be crucial.

The Minty-Rockafellar Theorem

A monotone mapping $T : U \rightrightarrows U^*$ is maximal if and only if for every $\lambda > 0$, $\mathrm{rge}(T + \lambda J_U) = U^*$. In this case the inverse $(T + \lambda J_U)^{-1}$ is a single-valued maximal monotone operator which is norm to weak continuous.

It follows from this and the properties of the chosen norms that, if T is maximal monotone, then for any $\lambda > 0$, the *Yosida regularization*

$$T_\lambda = (T^{-1} + \lambda J_U^{-1})^{-1}$$

of T is single-valued, strongly continuous and maximal monotone with $\operatorname{Dom} T_\lambda = U$ (see for example Attouch).

The following is well-known :

Let T be maximal monotone.

(a) We have $u^* \in \operatorname{rge} T$ if and only if the family $\{u_\lambda \mid \lambda > 0\}$ of solutions to

$$T(u) + \lambda J_U(u) \ni u^*$$

remains bounded as $\lambda \searrow 0$. When this happens, $\|u_\lambda\| \leq \|\bar{u}\| \; \forall \lambda > 0$, and $u_\lambda \to \bar{u}$ strongly as $\lambda \searrow 0$, where $\bar{u}$ is the minimum norm solution of $T(u) \ni u^*$.

(b) We have $u \in \operatorname{Dom} T$ if and only if the family $\{T_\lambda(u) \mid \lambda > 0\}$ remains bounded as $\lambda \searrow 0$. When this happens, $\|T_\lambda(u)\| \leq \|\bar{u}^*\|$ and $T_\lambda(u) \to \bar{u}^*$ strongly as $\lambda \searrow 0$, where $\bar{u}^*$ is the minimum norm element of $T(u)$.

Let $A : X \to U$ be linear and continuous with adjoint $A^* : U^* \to X^*$.

The composite mapping $A^*TA : X \rightrightarrows X^*$, given by $A^*TA(x) := \cup \{ A^*u^* \mid u^* \in T(Ax) \}$, is monotone.

Examples: partial differential equations in divergence form

contains the pointwise sum of two or more operators as a special case.

Without further conditions, A^*TA may fail to be maximal monotone;

see Rockafellar and Wets 98, Robinson 99 and Pennanen 00 for sufficient conditions.

Penot - 2005 -

$$\mathbb{R}_+(\operatorname{co}\operatorname{Dom}T - R(A)) = U \implies A^*TA \text{ is mm}$$

$$\mathbb{R}_+(\operatorname{co}\operatorname{Dom}T - \operatorname{co}\operatorname{Dom}T) = U \implies S + T \text{ is mm}$$

General Idea

It is then a natural idea to try to approximate A^*TA by something which is guaranteed to be maximal monotone.

A good candidate is $A^*T_\lambda A$, where T_λ is the Yosida regularization of T with parameter $\lambda > 0$.

(after renorming of the space, if necessary) T_λ is a monotone continuous mapping

the same is then true of $A^*T_\lambda A$, which guarantees the maximality.

If one now takes the limit of $A^*T_\lambda A$ as $\lambda \searrow 0$, in the sense of graphical convergence, it turns out that one obtains something that is more likely to be maximal monotone than the pointwise composition A^*TA.

This limit mapping, denoted here $(A^*TA)_v$, (to be given a more precisely later on) is what we call the *variational composition* of A and T.

The purpose of this talk is to study the relation between A^*TA and $(A^*TA)_v$, to give sufficient conditions for maximality of $(A^*TA)_v$, and to give applications of this new concept.

Note that if T_1 and T_2 are set-valued mappings from X to X^*, their pointwise sum can be expressed in the composite form A^*TA, by defining $U = X \times X$, $Ax = (x, x)$, and $T(x_1, x_2) = T_1(x_1) \times T_2(x_2)$. Indeed, then

$$A^*(x_1^*, x_2^*) = x_1^* + x_2^*$$

and so

$$A^*TA(x) = T_1(x) + T_2(x).$$

This fact will allow us to draw connections between the variational composition and the variational sum.

Graph-convergence (in the sense of Painlevé-Kuratowski)

$\{C_\lambda : U \rightrightarrows U^*\}_{\lambda>0}$ mm,

- g-$\liminf_{\lambda \searrow 0} C_\lambda := \{(u,u^*) : \forall \lambda_n \searrow 0,$ $\exists (u_n, u_n^*) \longrightarrow (u,u^*)$ with $u_n^* \in C_{\lambda_n}(u_n)\}$.
- g-$\limsup_{\lambda \searrow 0} C_\lambda := \{(u,u^*) : \exists \lambda_n \searrow 0, (u_n, u_n^*) \longrightarrow$ (u,u^*) with $u_n^* \in C_{\lambda_n}(u_n)\}$.
- If g-$\liminf_{\lambda \searrow 0} C_\lambda =$ g-$\limsup_{\lambda \searrow 0} C_\lambda$, we say that $\{C_\lambda\}_{\lambda>0}$ graph-converges ;

  the limit is denoted by g-$\lim_{\lambda \searrow 0} C_\lambda$.

Properties

- g-$\liminf_{\lambda \searrow 0} C_\lambda$ is monotone;
- If C is mm, g-$\lim_{\lambda \searrow 0} C_\lambda = C$ if and only if g-$\liminf_{\lambda \searrow 0} C_\lambda \supset C$;
- If C is mm, g-$\lim_{\lambda \searrow 0} C_\lambda = C$ if and only if

$$\lim_{\lambda \searrow 0} (C_\lambda + J_U)^{-1}(u^*) = (C + J_U)^{-1}(u^*) \forall u^* \in U^*.$$

(Attouch)

Variational composition

$A \in L(X,U)$, $T : U \rightrightarrows U^*$ max-monotone..

The *variational composition* $(A^*TA)_v : X \rightrightarrows X^*$ of A and T is the mapping

$$(A^*TA)_v = \operatorname*{g\text{-}liminf}_{\lambda \searrow 0} A^*T_\lambda A.$$

Proposition.

- $(A^*TA)_v$ is monotone ;
- $(A^*TA)_v = \text{g-lim}\, A^*T_\lambda A$, if $(A^*TA)_v$ is maximal ;
- $\mathrm{Dom}(A^*TA) \subset \mathrm{Dom}(A^*TA)_v$;
- if $(A^*TA)_v$ est mm, then $A^*TA \subset (A^*TA)_v$.

Comparison with the variational sum

T^1,T^2 maximal monotone in X ; the variational sum (Attouch, Baillon et Théra):

$(T^1 \underset{v}{+} T^2) := \text{g-liminf}_{\lambda,\mu\searrow 0,\ \lambda\mu\neq 0}(T^1_\lambda + T^2_\mu)$.

Let $U = X \times X$, $Ax = (x, x)$ and

$T(x_1, x_2) = T_1(x_1) \times T_2(x_2)$. Then,

$$(A^*TA)_v = \underset{\lambda\searrow 0}{\text{g-liminf}}(T^1_\lambda + T^2_\lambda),$$

Then,

$$T^1 \underset{v}{+} T^2 \subset (A^*TA)_v.$$

In particular,

$(A^*TA)_v = T^1 \underset{v}{+} T^2$, if $T^1 \underset{v}{+} T^2$ is mm.

Comparison with the pointwise composition

Theorem. If the operator $\overline{A^*TA}$ (the closure in $X \times X^*$) is maximal monotone, then

$$(A^*TA)_v = \overline{A^*TA}.$$

Corollary. If A^*TA is maximal monotone, we have

$$(A^*TA)_v = A^*TA.$$

and $(A^*TA)_v = A^*TA = \text{g-lim}_{\lambda \searrow 0} A^*T_\lambda A$.

Corollary. If $T^1, \ldots, T^m$ maxial monotone from X to X^*. If the operator $\overline{T^1 + \cdots + T^m}$is maximal monotone, then

$$\underset{\lambda \searrow 0}{\text{g-lim}}(T^1_\lambda + \cdots + T^m_\lambda) = \overline{T^1 + \cdots + T^m}.$$

Considering subdifferentials

$f :\in \Gamma_0(U)$; $A : X \to U$ linear continuous;

For $f \circ A$ we have

$$\partial(f \circ A) \supset A^* \partial f A,$$

equality holds under a qualification assumption

$$0 \in \mathrm{Int}(A(X) - \mathrm{Dom}\, f)$$

Theorem. Let $A : X \to U$ linear and continuous and f convex, lsc and proper in U.

If $A(X) \cap \mathrm{Dom}\, f \neq \emptyset$, then

$$\partial(f \circ A) = (A^* \partial f A)_v.$$

(Analogous results : Hiriart-Urruty-Phelps, Ioffe, Penot, Thibault, R.-Théra...)

Measurability of composite mappings

Ω denotes a measurable space, and all the other spaces are separable Hilbert spaces.

Given a family of set-valued mappings $\{T(\omega) : H \rightrightarrows H\}_{\omega\in\Omega}$, we define the mapping $\mathcal{L}_2[T] : L^2(\Omega; H) \rightrightarrows L^2(\Omega; H)$ (the canonical extension of T) by

$$\mathcal{L}_2[T](v) = \left\{v^* \in L^2(\Omega; H) \mid v^*(\omega) \in T(\omega)(v(\omega)) \text{ a.e. on } \Omega\right\}.$$

A set-valued mapping $S : \Omega \rightrightarrows H$ is *measurable* if for any open $C \subset H$, the set

$$S^{-1}(C) = \{\omega \in \Omega \mid S(\omega) \cap C \neq \emptyset\}$$

is measurable.

Measurability of set-valued mappings has been studied extensively by many authors; see for example Castaing and Valadier, Attouch, Rockafellar, Rockafellar and Wets.

Let $\{T(\omega)\}_{\omega\in\Omega}$ be a measurable family of maximal monotone mappings on H. Then, $\mathcal{L}_2[T]$ is maximal monotone provided

$$\operatorname{Dom}\mathcal{L}_2[T] \neq \emptyset.$$

The above result is closely related to the theory of convex normal integrands .

A function f on $\Omega \times H$ is said to be a *convex normal integrand* *if the mapping $\omega \mapsto \mathrm{epi}\, f(\omega,\cdot)$ is measurable with closed and convex values. If f is a convex normal integrand, then the integral functional*

$$I_f(u) = \begin{cases} \int_\Omega f(\omega, u(\omega))d\omega & \textit{if } f(\cdot, u(\cdot)) \in L^1(\Omega) \\ +\infty & \textit{otherwise} \end{cases}$$

is a convex and lower semicontinuous function on $L^2(\Omega; H)$.

Attouch's Theorem

f is a normal integrand if and only if $\{\partial f(\omega, \cdot)\}_{\omega\in\Omega}$ is a measurable family of maximal monotone mappings on H, and there is a measurable function $u : \Omega \mapsto H$ such that $f(\cdot, u(\cdot))$ is measurable.
The formula

$$(◐) \quad \partial I_f = \mathcal{L}_2[\partial f]$$

is valid for any convex normal integrand provided $\operatorname{Dom} \mathcal{L}_2[\partial f] \neq \emptyset$.

Theorem Let (Ω, μ) be a positive σ-finite complete measure space, let $\{T(\omega)\}_{\omega\in\Omega}$ be a measurable family of max-monotone mappings on U, and let $A : \Omega \times X \to U$ be a Carathéodory mapping with $A(\omega) \in L(X, U)$ for every $\omega \in \Omega$.

- If the mapping $\overline{A(\omega)^* T(\omega) A(\omega)}$ is maximal monotone a.e. on Ω, then $\{\overline{A(\omega)^* T(\omega) A(\omega)}\}_{\omega\in\Omega}$ is a measurable family.
- If $T(\omega) = \partial f(\omega, \cdot)$ for a convex normal integrand f, and $\mathrm{rge}\, A(\omega) \cap \mathrm{Dom}\, f(\omega, \cdot) \neq \emptyset$ a.e. on Ω, then

$$\{\partial(f(\omega, \cdot) \circ A(\omega))\}_{\omega\in\Omega}$$

is a measurable family, and $f(\omega, \cdot) \circ A(\omega)$ is a convex normal integrand.

Case $U = X \times X$

$$T(\omega)(x_1, x_2) = T_1(\omega)(x_1) \times T_2(\omega)(x_2)$$

and

$$A(\omega)(x) = (x, x)$$

We recover the following result due to Attouch:

Attouch Theorem : Let (Ω, μ) be a positive σ-finite complete measure space, and let $T_1(\omega)$ and $T_2(\omega)$ be measurable families of maximal monotone mappings on U. If for every ω the mapping $T_1(\omega) + T_2(\omega)$ is maximal monotone, then the family $\{T_1(\omega) + T_2(\omega)\}_{\omega \in \Omega}$ is measurable.

Application to PDE's in divergence form

$\Omega \subset \mathbb{R}^N$– nonempty open set, $Q : \Omega \mapsto \mathbb{R}^{N\times N}$ measurable,with $Q(x)$ symmetric and positive semi-definite a.e. in Ω. We consider

$g : H_0^1(\Omega) \to \mathbb{R} \cup \{+\infty\}$

$$g(u) = \begin{cases} \frac{1}{2}\int_\Omega \nabla u(x) Q(x) \nabla u(x) dx & \text{if } \nabla u Q \nabla u \in L^1(\Omega), \\ +\infty & \text{otherwise} \end{cases}$$

Minimize g in $H_0^1(\Omega)$

or

solve $\partial g(u) \ni 0$.

determine ∂g!.

$g = I_f \circ \nabla$

where

$\nabla : H_0^1(\Omega) \to L^2(\Omega;^N)$, is given by $\nabla u = \{\frac{\partial u}{\partial x_i}\}_{i=1}^N$

and $I_f = \int_\Omega f(x,v)dx$, where $f(x,v) = \frac{1}{2} v \cdot Q(x)v$.

g **is convex lsc.**

$\partial f(x,\cdot) = Q(x)$ and $\partial I_f = \mathcal{L}_2[Q]$

where

$\mathcal{L}_2[Q](v) = \{v^* \in L^2(\Omega;\mathbb{R}^N) : v^*(x) = Q(x)v(x)$ p.p. in $\Omega\}$

–the canonical extension of ∂f-

The case with a constraint of qualification

$$0 \in \mathrm{nt}\ (A(X) - \mathrm{Dom}\, I_f)$$

For example: If $Q(x) \in L^\infty(\Omega; \mathbb{R}^{N\times N})$ then

$$\mathrm{Dom}\, I_f = L^2(\Omega; \mathbb{R}^N)$$

We have
$\partial g = \nabla^* \mathcal{L}_2[Q]\nabla$, where $\nabla^* = -\mathrm{div}$ (the divergence) :

$$\mathrm{Dom}\, \partial g = \{u \in H_0^1(\Omega) : Q\nabla u \in L^2(\Omega; \mathbb{R}^N)\}$$
$$\partial g(u) = -\mathrm{div}(Q\nabla u).$$

Le case without constraint of qualification

Theorem.
If $Q \in L^1_{loc}(\Omega; \mathbb{R}^{N\times N})$, then $C_c^\infty(\Omega) \subset \operatorname{Dom} g$, $Q\nabla u \in L^1_{loc}(\Omega; \mathbb{R}^N)$ $\forall u \in \operatorname{Dom} g$, and

$$\operatorname{Dom}\partial g = \Big\{ u \in H^1_0(\Omega) \,|\, u \in \operatorname{Dom} g,\ \operatorname{div}(Q\nabla u) \in H^{-1}(\Omega),$$

$$\langle w, -\operatorname{div}(Q\nabla u)\rangle = \int_\Omega \nabla w \cdot Q\nabla u \quad \forall w \in \operatorname{Dom} g \Big\},$$

$$\partial g(u) = -\operatorname{div}(Q\nabla u).$$

In particular, $\partial g = (\nabla^* \mathcal{L}_2[Q]\nabla)_v$

Corollary.
Let Ω be bounded, $\alpha > 0$, and let $Q \in L^1(\Omega; \mathbb{R}^{N\times N})$ be such that $v \cdot Q(x)v \geq \alpha|v|^2$ for all $v \in \mathbb{R}^m$ and for a.e. $x \in \Omega$. Then for each $u^* \in H^{-1}(\Omega)$, there exists a **unique** $u \in H_0^1(\Omega)$ such that

$$-\mathrm{div}(Q\nabla u) = u^*,$$

$$\nabla u \cdot Q\nabla u \in L^1(\Omega), \quad Q\nabla u \in L^1(\Omega; \mathbb{R}^N),$$

$$\text{and} \quad \langle w, -\mathrm{div}(Q\nabla u)\rangle = \int_\Omega \nabla w \cdot Q\nabla u$$

for all $w \in H_0^1(\Omega)$ such that $\nabla w \cdot Q\nabla w \in L^1(\Omega)$.

Thank you for your attention