Subdifferential representation of convex functions: refinements and applications

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Abstract Every lower semicontinuous convex function can be represented through its subdifferential by means of an “integration” formula introduced in [10] by Rockafellar. We show that in Banach spaces with the Radon-Nikodym property this formula can be significantly refined under a standard coercivity assumption. This yields an interesting application to the convexification of lower semicontinuous functions.

Key words Convex function, subdifferential, epi-pointed function, cusco mapping, strongly exposed point.

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1 Introduction

Let $X$ be a Banach space and $g : X \to \mathbb{R} \cup \{+\infty\}$ a proper lower semicontinuous convex function. Rockafellar [10] has shown that $g$ can be represented through its subdifferential $\partial g$ as follows:

$$g(x) = g(x_0) + \sup \left\{ \sum_{i=0}^{n-1} \langle x_i^*, x_{i+1} - x_i \rangle + \langle x_n^*, x - x_n \rangle \right\},$$

for every $x \in X$, where $x_0$ is an arbitrary point in the domain of $\partial g$ and where the above supremum is taken over all integers $n$, all $x_1, x_2, \ldots, x_n$ in $X$ and all $x_0^* \in \partial g(x_0), x_1^* \in \partial g(x_1), \ldots, x_n^* \in \partial g(x_n)$ (for $n = 0$ we take the convention $\sum_{i=0}^{-1} = 0$). In this paper we show that, in Banach spaces with the Radon-Nikodym property (see Definition 2), and under a standard coercivity assumption on $g$, formula (1) can be considerably simplified. Namely, it suffices to estimate the above supremum among the strongly exposed points of $g$ (see Definition 10), instead of the much larger set of all points of the domain of $\partial g$.

This simple geometrical fact has also the following consequence: the closed convex envelope of a non-convex function $f$ satisfying the same coercivity condition can be recovered by the Fenchel subdifferential $\partial f$ of $f$ through formula (1), and this besides the fact that for non-convex functions this subdifferential is generally poor. This last result generalizes the ones obtained in [1, Proposition 2.7], [2, Theorem 3.5] in finite dimensions.

2 Preliminaries

Throughout the paper we denote by $X$ a Banach space and by $X^*$ its dual space. In the sequel, we denote by $i : X \hookrightarrow X^{**}$ the isometric embedding of $X$ into its second dual space $X^{**}$. Given $x \in X$, $x^* \in X^*$ and $x^{**} \in X^{**}$, we denote by $\langle x^*, x \rangle$ (respectively, $\langle x^*, x^{**} \rangle$) the value of the functional $x^*$ at $x$ (respectively, the value of $x^{**}$ at $x^*$). Note also that with this notation we have $\langle x^*, i(x) \rangle = \langle x^*, x \rangle$. For $x \in X$ and $\rho > 0$ we denote by $B(x, \rho)$ the open ball centered at $x$ with radius $\rho$, and by $\Delta_d$ ($d \geq 1$) the unit simplex of $\mathbb{R}^d$, that is,

$$\Delta_d := \{ (\lambda_1, \lambda_2, \ldots, \lambda_d) \in \mathbb{R}^d : \sum_{i=1}^d \lambda_i = 1 \text{ and } \lambda_i \geq 0 \text{ for all } i = 1, \ldots, d \}.$$

If $f : X \to \mathbb{R} \cup \{+\infty\}$ is an extended real valued function, we denote by

$$\text{epi} f = \{ (x,t) \in X \times \mathbb{R} : f(x) \leq t \}$$


its epigraph, and by
\[ \text{dom } f := \{ x \in X : f(x) \in \mathbb{R} \} \]
its domain. When the domain of \( f \) is nonempty we say that \( f \) is proper. By the term subdifferential, we always mean the Fenchel subdifferential \( \partial f \) defined for every \( x \in \text{dom } f \) as follows
\[ \partial f(x) = \{ x^* \in X^* : f(y) \geq f(x) + \langle x^*, y - x \rangle, \forall y \in X \}. \] (2)
If \( x \in X \setminus \text{dom } f \), we set \( \partial f(x) = \emptyset \). Then the domain of the subdifferential of \( f \) is defined by
\[ \text{dom } \partial f = \{ x^* \in X^* : \partial f(x) \neq \emptyset \}. \]

For a proper lower semicontinuous function \( f \), its closed convex envelope \( \overline{\text{co}} f : X \rightarrow \mathbb{R} \cup \{ +\infty \} \) can be defined through its epigraph via the formula
\[ \overline{\text{co}}(\text{epi} f) = \text{co}(\text{epi} f), \]
where \( \overline{\text{co}}(\text{epi} f) \) is the closed convex hull of \( \text{epi} f \) in the Banach space \( X \times \mathbb{R} \) endowed with the norm \( (x, t) \mapsto (|x|^2 + |t|^2)^{1/2} \) for all \( (x, t) \in X \times \mathbb{R} \). If \( f^{**} : X^{**} \rightarrow \mathbb{R} \cup \{ +\infty \} \) denotes the Legendre-Fenchel biconjugate of \( f \), then it is well-known that \( \overline{\text{co}} f = f^{**} \circ i \), that is, for every \( x \in X \)
\[ (\overline{\text{co}} f)(x) = f^{**}(i(x)) = \sup_{x^* \in X^*} \{ \langle x^*, x \rangle - f^*(x^*) \}, \]
where \( f^* : X^* \rightarrow \mathbb{R} \cup \{ +\infty \} \) is the Legendre-Fenchel conjugate of \( f \), that is, the following lower semicontinuous convex function
\[ f^*(x^*) = \sup_{x \in X} \{ \langle x^*, x \rangle - f(x) \}. \]
Note also that for any \( x \in X \) and \( x^* \in X^* \) we have:
\[ x^* \in \partial f(x) \iff i(x) \in \partial(f^*)(x^*). \] (3)

Let \( C \) be a non-empty closed convex subset of \( X \). We denote by \( \sigma_C : X^* \rightarrow \mathbb{R} \cup \{ +\infty \} \) the Legendre-Fenchel conjugate of the indicator function of \( C \), that is, for all \( p \in X^* \)
\[ \sigma_C(p) = \sup_{u \in C} (p, u). \]
Note that \( \sigma_C \) is a positively homogeneous convex function. Its relationship with the Legendre-Fenchel conjugate of a proper lower semicontinuous convex function \( g \) is as follows:
\[ t g^*(t^{-1} x^*) = \sigma_{\text{epi } g}(x^*, -t), \]
for all \( t > 0 \) and all \( x^* \in X^* \). In particular, using the fact that \( \text{dom } \sigma_{\text{epi } g} \) and \( \text{intdom } \sigma_{\text{epi } g} \) are convex cones, it is easily seen that
\[ x^* \in \text{intdom } g^* \iff (x^*, -1) \in \text{intdom } \sigma_{\text{epi } g}. \] (4)

We denote by \( N_C(u) \) the set of normal directions of \( C \) at a point \( u \in C \), that is
\[ N_C(u) = \{ p \in X^* : \langle p, v - u \rangle \leq 0, \forall v \in C \}. \]
Its relationship with the subdifferential of a proper lower semicontinuous convex function \( g \) is as follows
\[ t^{-1} x^* \in \partial g(x) \text{ if and only if } (x^*, -t) \in N_{\text{epi } g}(x, g(x)), \]
for all \( t > 0, x \in X \) and all \( x^* \in X^* \).
2.1 Strongly exposed points and Radon-Nikodym property

Let us recall from [9, Definition 5.8] the following definition.

**Definition 1** Let \( u \in C \). The point \( u \) is called a strongly exposed point of \( C \) if there exists \( p \in X^* \) such that for each sequence \( \{u_n\} \subset C \) the following implication holds

\[
\lim_{n \to +\infty} \langle p, u_n \rangle = \sigma_C(p) \Rightarrow \lim_{n \to +\infty} u_n = u.
\]

In such a case \( p \in X^* \) is said to be a “strongly exposing” functional for the point \( u \) in \( C \). We denote by \( \text{Exp}(C, u) \) the set of all functionals of \( X^* \) satisfying this property.

Let us further denote by \( \text{exp} C \) the set of strongly exposed points of \( C \). Clearly, \( u \in \text{exp} C \) if, and only if, \( \text{Exp}(C, u) \neq \emptyset \).

We denote by \( \text{Exp} C \) the set of all strongly exposing functionals, that is,

\[
\text{Exp} C = \bigcup_{u \in \text{exp} C} \text{Exp}(C, u).
\]

We also recall (see [9, Theorem 5.21], for example) the following definition.

**Definition 2** A Banach space \( X \) is said to have the Radon-Nikodym property (in short, \( X \) is an RNP space), if every non-empty closed convex bounded subset \( C \) of \( X \) can be represented as the closed convex hull of its strongly exposed points, that is,

\[
C = \overline{\text{co}(\text{exp} C)}.
\]  

(5)

Examples of Radon-Nikodym spaces are reflexive Banach spaces and separable dual spaces.

Let us mention that, in RNP spaces, the set \( \text{Exp} C \) of strongly exposing functionals of a nonempty closed convex bounded set \( C \) is dense in \( X^* \). Moreover, the boundedness of \( C \) implies that \( \text{dom} \sigma_C = X^* \). In case of unbounded sets, one has the following result.

**Proposition 3** Suppose that \( X \) is an RNP space and \( C \) is a nonempty closed convex set. Then \( \text{Exp} C \) is dense in \( \text{int} \text{dom} \sigma_C \).

**Proof** If \( \text{int} \text{dom} \sigma_C = \emptyset \) the assertion holds trivially. Let us assume that \( U := \text{int} \text{dom} \sigma_C \neq \emptyset \) and let us note that the \( \text{w}^* \)-lower semicontinuous convex function \( \sigma_C \) is continuous on the open set \( U \), see [9, Proposition 3.3]. Using Collier’s characterization of the Radon-Nikodym property ([7, Theorem 1]), we conclude that \( \sigma_C \) is Fréchet differentiable in a dense subset \( D \) of \( U \). For every \( p_0 \in D \), Smulian’s duality guarantees that there exists \( u_0 \in \text{exp} C \) such that \( u_0 = \nabla^F \sigma_C(p_0) \) (see [9], for example). In particular, \( p_0 \in \text{Exp}(C, u_0) \), hence \( p_0 \in \text{Exp} C \). The proof is complete. \( \square \)

2.2 Cyclically monotone operators

Given a multivalued operator \( T : X \rightrightarrows X^* \), we denote its domain by \( \text{dom}(T) = \{x \in X : T(x) \neq \emptyset\} \), its image by

\[
\text{Im}(T) = \bigcup_{x \in \text{dom}(T)} T(x)
\]

and its graph by

\[
\text{Gr}(T) := \{(x, x^*) \in X \times X^* : x^* \in T(x)\}.
\]
We also denote by $T^{-1} : X^* \Rightarrow X$ the inverse operator, that is, for every $x \in X$ and $x^* \in X^*$

$$x \in T^{-1}(x^*) \iff x^* \in T(x).$$

The operator $T$ is called cyclically monotone (respectively, monotone) if for every $n \geq 1$ (respectively, for $n = 2$), every $\{x_i\}_{i=1}^n$ in $X$ and every $\{x_i^*\}_{i=1}^n$ in $X^*$ such that $x_i^* \in T(x_i)$, for $i = 1, \ldots, n$

$$\sum_{i=1}^n (x_i^*, x_{i+1} - x_i) \leq 0,$$

where $x_{n+1} := x_1$. It is called maximal cyclically monotone (respectively, maximal monotone), if its graph cannot be strictly contained in the graph of any other cyclically monotone (respectively, monotone) operator.

We recall from [10] (see also [9]) the following fundamental results:

**Proposition 4** The subdifferential $\partial g$ of a proper lower semicontinuous convex function $g$ is both a maximal monotone and a maximal cyclically monotone operator.

**Proposition 5** Let $T$ be cyclically monotone operator and $x_0 \in \text{dom}(T)$. Then the formula below defines a proper lower semicontinuous convex function $\hat{h}$ (depending on $T$ and $x_0$) such that $\text{Gr}(T) \subset \text{Gr}(\partial \hat{h})$:

$$\hat{h}(x) := \sup \left\{ \sum_{i=0}^{n-1} (x_i^*, x_{i+1} - x_i) + (x_n^*, x - x_n) \right\},$$

where the supremum is taken for all $n \geq 1$, all $x_1, x_2, \ldots, x_n$ in $\text{dom}(T)$ and all $x_0^* \in T(x_0), x_1^* \in T(x_1), \ldots, x_n^* \in T(x_n)$.

In the sequel, we shall refer to (7) by the term “Rockafellar’s integration formula”.

### 2.3 w*-cusco and minimal w*-cusco mappings

The operator $T$ is said to be w*-upper-semicontinuous at $x \in X$, if for every w*-open set $W \supset T(x)$ there exists $\rho > 0$ such that $T(u) \subset W$ for every $u \in B(x, \rho)$.

We recall from [4] (see also [5]) the following definition.

**Definition 6** Let $U$ be an open subset of $X$. An operator $T : X \rightrightarrows X^*$ is said to be w*-cusco on $U$, if it is w*-upper semicontinuous with nonempty w*-compact convex values at each point of $U$. It is said to be minimal w*-cusco on $U$ if its graph does not strictly contain the graph of any other w*-cusco mapping on $U$.

In the sequel, we shall need the following result (see [5, Theorem 2.23]).

**Proposition 7** Let $U$ be an open set and $T$ be a maximal monotone operator with $U \subset \text{dom}(T)$. Then $T$ is minimal w*-cusco on $U$.

Further, given a multivalued operator $S : X \rightrightarrows X^*$ we can consider w*-cusco mappings $T$ that are minimal under the property of containing the graph of $S$. We recall from [5, Proposition 2.3] the following “uniqueness” result that will be in use in the sequel.

**Proposition 8** Let $U$ be an open set and let $S : X \rightrightarrows X^*$ be a multivalued operator whose domain $\text{dom}(S)$ is dense in $U$. If the graph of $S$ is contained in the graph of some w*-cusco mapping on $U$, then there exists a unique w*-cusco mapping on $U$ that contains the graph of $S$ and that is minimal under this property.
3 Refined representations of convex functions

Throughout this section $g : X \to \mathbb{R} \cup \{+\infty\}$ will denote a proper lower semicontinuous convex function. Let further $T : X \rightrightarrows X^*$ be a multivalued operator satisfying
\[ \text{Gr}(T) \subset \text{Gr}(\partial g) \text{ and } x_0 \in \text{dom}(T). \] (8)

Applying (7) to the cyclically monotone operator $T$ we obtain a lower semicontinuous convex function $\hat{h}$ with $\text{Gr}(T) \subset \text{Gr}(\partial \hat{h})$. Evoking (1) we easily obtain that for every $x \in X$
\[ \hat{h}(x) \leq g(x) - g(x_0). \] (9)

The following result gives conditions where (9) holds with equality.

**Theorem 9** Let $g : X \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous convex function and $T : X \rightrightarrows X^*$ be a multivalued operator satisfying (8). Consider the following conditions:

(A1) intdom $g \neq \emptyset$ and dom($T$) is dense in intdom$g$.

(A2) intdom $g^* \neq \emptyset$ and im($T$) is dense in intdom$g^*$.

If either (A1) or (A2) holds true, then for every $x \in X$
\[ g(x) = g(x_0) + \sup \left\{ \sum_{i=0}^{n-1} \langle x_i^*, x_{i+1} - x_i \rangle + \langle x_n^*, x - x_n \rangle \mid n \geq 1, \text{ all } x_1, x_2, ..., x_n \in \text{dom}(T) \text{ and all } x_0^* \in T(x_0), x_i^* \in T(x_i), ..., x_n^* \in T(x_n) \right\}. \] (10)

where the supremum is taken for all $n \geq 1$, all $x_1, x_2, ..., x_n$ in dom($T$) and all $x_0^* \in T(x_0), x_i^* \in T(x_i), ..., x_n^* \in T(x_n)$.

**Proof** Let us denote by $\hat{g}$ the lower semicontinuous convex function defined by the right part of (10). By (9) we have $\text{Gr}(T) \subset \text{Gr}(\partial \hat{g})$, which yields
\[ \hat{g} \leq g \text{ and } g(x_0) = \hat{g}(x_0). \] (11)

(A1) Set $U = \text{intdom } g \neq \emptyset$ and note that $U \subset \text{intdom } \partial g$. We deduce from (11) that $U \subset \text{dom } \hat{g}$. Since $U$ is open, it also follows that $U \subset \text{intdom } \partial \hat{g}$. Hence, by Proposition 7, the operators $\partial g$ and $\partial \hat{g}$ are minimal w*-cuscus on $U$. By (8) we have $\text{Gr}(T) \subset \text{Gr}(\partial g)$, while by Proposition 5, $\text{Gr}(T) \subset \text{Gr}(\partial \hat{g})$. Since dom$T$ is dense in $U$, Proposition 8 yields that $\partial g = \partial \hat{g}$ on $U$. Consequently, in view of (11) $\hat{g} = g$ on $U = \text{intdom } g$. A standard argument shows that the equality is extended in the whole space.

(A2) Set $V = \text{intdom } g^* \neq \emptyset$. Set $S = i \circ T^{-1}$ and note that $S$ and $T$ have essentially the same graph. Note also that (3) yields that $\text{Gr} \left( i \circ (\partial g)^{-1} \right) \subset \text{Gr} \left( \partial (g^*) \right)$ and $\text{Gr} \left( i \circ (\partial \hat{g})^{-1} \right) \subset \text{Gr} \left( \partial (\hat{g}^*) \right)$. Consequently, $\text{Gr}(T) \subset \text{Gr}(\partial g)$ (respectively, $\text{Gr}(T) \subset \text{Gr}(\partial \hat{g})$) yields easily that $\text{Gr}(S) \subset \text{Gr}(\partial (g^*))$ (respectively, $\text{Gr}(S) \subset \text{Gr}(\partial (\hat{g}^*))$). Since dom$S = \text{Im } T$ is dense in $V$, and since both $\partial (g^*)$ and $\partial (\hat{g}^*)$ are minimal w*-cuscus on $V$ (as operators from $X^*$ with values on $X^{**}$), it follows by Proposition 8 that $\partial (g^*) = \partial (\hat{g}^*)$ on $V$, hence
\[ g^* = \hat{g}^* + k \text{ on } \text{int}(\text{dom } g^*) \] (12)

where $k$ is an additive constant. Since int(dom $g^*$) is nonempty, the above equality can be extended to $X^*$, provided that
\[ \text{int}(\text{dom } g^*) = \text{int}(\text{dom } \hat{g}^*). \] (13)

Let us now prove this last equality. According to equality (12) we have int(dom $g^*) \subset \text{dom } \hat{g}^*$ and so int(dom $g^*) \subset \text{int}(\text{dom } \hat{g}^*)$. Conversely, taking conjugates in both sides of the inequality in (11) we have $g^* \leq \hat{g}^*$. Hence in particular dom $\hat{g}^* \subset \text{dom } g^*$ and so int(dom $\hat{g}^*) \subset \text{int}(\text{dom } g^*)$. We conclude that equality (13) holds as desired. It follows that
\[ g^* = \hat{g}^* + k. \]

Taking conjugates and considering the restriction on $X$ we obtain $g = \hat{g} - k$. Since $g(x_0) = \hat{g}(x_0)$ we conclude that $k = 0$ and thus $g = \hat{g}$. \[\square\]
3.1 Application: Representation of convex epi-pointed functions

The following definition will be useful in the sequel.

**Definition 10** A point \( x \in \text{dom } g \) is called strongly exposed for the function \( g \) if
\[
(x, g(x)) \in \text{exp}(\text{epi } g).
\]
We denote by \( \text{exp } g \) the set of strongly exposed points of \( g \).

For every \( x \in \text{exp } g \) we also denote by \( \text{Exp}(g, x) \) the set of all \( x^* \in X^* \) satisfying
\[
(x^*, -1) \in \text{Exp}(\text{epi } g, (x, f(x))),
\]
and we set
\[
\text{Exp}(g) = \bigcup_{x \in \text{exp } g} \text{Exp}(g, x).
\]
It may happen that the set of strongly exposed points is empty, for instance when \( g \) is a constant function. We shall avoid this situation since, as we shall show \( \text{exp } g \) is non-empty under the following coercivity assumption that we recall from [3, p. 1669].

**Definition 11** A proper lower semicontinuous function \( f : X \rightarrow \mathbb{R} \cup \{+\infty\} \) is called epi-pointed if
\[
\text{int}(\text{dom } f^*) \neq \emptyset.
\]
As shown in [3, Proposition 4.5], the above definition is equivalent to the following coercivity condition. (This has been established in finite dimensions; only minor modifications are needed for the general case.)

- there exists \( x^* \in X^*, \rho > 0 \) and \( r \in \mathbb{R} \) such that
\[
f(x) \geq \langle x^*, x \rangle + \rho \|x\| - r \quad \text{for all } x \in X.
\]

**Remark 12** A proper lower semicontinuous function \( f \) is epi-pointed if, and only if, \( \overline{\text{co } f} \) is epi-pointed.

Let us now state the following consequence of Proposition 3.

**Proposition 13** Let \( g \) be a lower semicontinuous convex epi-pointed function defined on an RNP space. Then \( \text{Exp}(g) \) is dense in \( \text{int dom } (g^*) \).

**Proof** Let \( x_0^* \in \text{int dom } g^* \) and \( \varepsilon > 0 \) such that \( B(x_0^*, \varepsilon) \subset \text{int dom } g^* \). Set
\[
\delta := \min \left\{ 1/2, \varepsilon(2\|x_0^*\| + 1)^{-1} \right\}.
\]
By (4) we have that \( (x_0^*, -1) \in \text{int dom } g^* \). By Proposition 3, there exists \( z^* \in B(x_0^*, \delta) \) and \( s \in (1 - \delta, 1 + \delta) \), such that \( (z^*, -s) \in \text{Exp}(\text{epi } g) \). Then obviously \( (s^{-1}z^*, -1) \in \text{Exp}(\text{epi } g) \), that is, \( x^* := s^{-1}z^* \in \text{Exp}(g) \). A direct calculation now yields
\[
\|x^* - x_0^*\| \leq \|x^* - z^*\| + \|z^* - x_0^*\| < s^{-1}(1 - s)\|z^*\| + \delta \leq 2\delta(\|x_0^*\| + \delta) + \delta \leq \varepsilon,
\]
that is, \( x^* \in B(x_0^*, \varepsilon) \cap \text{Exp}(g) \). This completes the proof.

We are ready to state the following subdifferential representation result for epi-pointed functions.
Theorem 14 Let $X$ be a Radon-Nikodym space, let $g$ be a lower semicontinuous convex and epi-pointed function and let $x_0 \in \text{dom } \partial g$. Then for every $x \in X$ we have

$$g(x) = g(x_0) + \sup \left\{ \sum_{i=0}^{n-1} \langle x_i^*, x_{i+1} - x_i \rangle + \langle x_n^*, x - x_n \rangle \right\},$$

where the supremum is taken over all integers $n$, all $x_1, x_2, ..., x_n$ in $\text{exp } g$, all $x_i^* \in \partial g(x_i)$, $(i = 0, 1, ..., n)$.

Proof Let us define the multivalued operator $T : X \to X^*$ as follows:

$$T(x) = \begin{cases} \partial g(x), & \text{si } x \in \{x_0\} \cup \text{exp}(g) \\ \emptyset, & \text{si } x \notin \{x_0\} \cup \text{exp}(g). \end{cases}$$

Since $\text{Gr } (T) \subset \text{Gr } (\partial g)$, the operator $T$ is also cyclically monotone.

We claim that the right part of (14) coincides (up to a constant) with the Rockafellar’s integration formula (7) for the operator $T$. Indeed, given a finite sequence $\{x_i\}_{i=1}^n$ in $\text{dom } (T)$, and denoting by $i_0$ the smaller index in $\{1, ..., n\}$ such that $x_i \neq x_0$ for all $i \geq i_0$, one has (using the cyclic monotonicity of $T$)

$$\sum_{i=0}^{i_0} \langle x_i^*, x_{i+1} - x_i \rangle \leq 0.$$ 

Omitting these terms (that do not contribute in the supremum), the sequence $\{x_i\}_{i=1}^n$ in $\text{dom } (T)$ can be replaced by the sequence $\{x_i\}_{i=i_0}^n$ in $\text{exp } g$.

By Proposition 13, $\text{Im}(T)$ is dense in $\text{int}(\text{dom } g^*)$. Thus, the result follows from Theorem 9. \qed

Remark 15 (i) Formula (14) fails for non-epi-pointed functions, even in finite dimensions. Consider for instance the proper lower semicontinuous convex function $g : \mathbb{R}^2 \to \mathbb{R}$ defined as follows:

$$g(x, y) = \frac{1}{2} y^2.$$

In this case $\text{exp } g = \emptyset$ and for $x_0 = (0, 0)$ formula (14) yields $g(x) = 0$, which is not true.

(ii) Formula (14) also fails in Banach spaces without the Radon-Nikodym property. Indeed let $X = c_0(\mathbb{N})$ and let $g$ be the indicator function of the closed unit ball of $X$. Then $g$ is a proper lower semicontinuous convex function which is also epi-pointed, since $g^*$ coincides with the norm of $X^* = \ell^1(\mathbb{N})$. Let further $x_0 = 0$ and note that $\partial g(x_0) = \{0\}$. Since the closed unit ball of $X$ has no extreme points, it follows easily that $\text{exp } g = \emptyset$. Thus formula (14) yields $g(x) = 0$, which is again not true.

3.2 Application: convexification of epi-pointed functions

Throughout this section we denote by $f : X \to \mathbb{R} \cup \{+\infty\}$ a proper lower semicontinuous epi-pointed function and we set

$$g = \overline{\text{co }} f.$$

We easily check that

$$x \in \text{dom } \partial f \implies (g(x) = f(x) \text{ and } \partial g(x) = \partial f(x)).$$

The following lemma gives an interesting particular case where the above situation occurs.

Lemma 16 Let $x \in \text{exp } g$. Then

$$g(x) = f(x), \quad \text{and } \partial g(x) = \partial f(x).$$
Proof We set $C := \text{epi } g$, $A := \text{epi } f$ and $u := (x, g(x))$. Note that $g(x) = f(x)$ if, and only if, $u \in A$. Let us suppose, towards a contradiction, that $g(x) > f(x)$, that is $u \notin A$. Since $A$ is closed, there exists $\varepsilon > 0$ such that
$$A \cap B(u, \varepsilon) = \emptyset.$$  
By assumption $u \in \exp C$, so there exists $p \in X^* \times \mathbb{R}$ and $\delta > 0$ such that $C \cap H \subset B(u, \varepsilon)$, where $H$ is the open half-space $\{v \in X \times \mathbb{R} : \langle p, v \rangle > \langle p, u \rangle - \delta\}$. Then, recalling that $A \subset C$, relation (16) implies $A \cap H = \emptyset$, or equivalently, taking the closed convex hull of the set $A$, that $C \cap H = \emptyset$. We obtain a contradiction since $u \in C \cap H$. Consequently, $f(x) = g(x)$. The equality of subdifferentials is now straightforward. \hfill \Box

As a consequence of the above lemma we obtain a representation formula for the closed convex envelope $g$ of an epi-pointed function $f$ based on the Fenchel subdifferential of $f$.

Corollary 17 Let $x_0 \in \text{dom } \partial f$. Then for every $x \in X$, we have
$$\text{co } f(x) = f(x_0) + \sup \left\{ \sum_{i=0}^{n-1} \langle x_i^n, x_{i+1} - x_i \rangle + \langle x_n^n, x - x_n \rangle \mid n \in \mathbb{Z}, x_1, x_2, \ldots, x_n \text{ in } \text{dom } \partial f \text{ and all } x_0^0 \in \partial f(x_0), x_1^1 \in \partial f(x_1), \ldots, x_n^n \in \partial f(x_n) \right\},$$

where and the supremum is taken over all integers $n$, all $x_1, x_2, \ldots, x_n$ in $\text{dom } \partial f$ and all $x_0^0 \in \partial f(x_0), x_1^1 \in \partial f(x_1), \ldots, x_n^n \in \partial f(x_n)$.

Proof According to formula (1) and using relations (15), the right hand side of (17) defines a proper lower semicontinuous convex function $\hat{f}$ satisfying $\hat{f} \leq g$ (note that $g(x_0) = f(x_0)$). On the other hand, according to Theorem 14 and Lemma 16, we obtain $\hat{f} \geq g$. This finishes the proof. \hfill \Box

References


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