ABSTRACT. The aim of this paper is to give sufficient conditions for a quasiconvex set-valued mapping to be convex. In particular, we recover several known characterizations of convex real-valued functions, given in terms of quasiconvexity and Jensen-type convexity by K. Nikodem [1], F.A. Behringer [2], and X.M. Yang, K.L. Teo and X.Q. Yang [3].

1. Introduction

Throughout the paper we denote by $X$ a linear space and by $Y$ a topological linear space, partially ordered by a closed convex cone $K$ having a nonempty interior in $Y$. Let $F : D \to 2^Y$ be a set-valued mapping, defined on a nonempty convex subset $D$ of $X$.

Recall (see e.g. [4]) that $F$ is said to be $K$-convex if the inclusion

\begin{equation}
F(\{x\} + (1-t)F(x')) \subseteq F(tx + (1-t)x') + K
\end{equation}

holds for all $x, x' \in D$ and for every $t \in [0,1]$.

By analogy to vector-valued functions, we say that $F$ is $K$-quasiconvex if for each $y \in Y$ the level set $L_F(y) := \{x \in D : y \in F(x) + K\}$ is convex. Since $K$ is a convex cone, it can be easily seen that $F$ is $K$-quasiconvex whenever it is $K$-convex.

1991 Mathematics Subject Classification. Primary 26B25; Secondary 49J53.

Key words and phrases. Generalized convexity, set-valued mappings.

This work was supported by a research grant of CNCSU under Contract Nr. 46174.
In order to get sufficient conditions for a $K$-quasiconvex mapping to be $K$-convex, we shall consider the following concept of generalized convexity:

$F$ will be called \textit{weakly $K$-convex with respect to a nonempty set $T \subset ]0,1[$} if for all $x, x' \in D$ there exists some $t \in T$ for which (1) holds.

Note that this concept extends several notions of generalized convexity, which were intensively studied in the literature in the particular case of real-valued functions. Indeed, if $T \subset ]0,1[$ is a singleton, we recover the Jensen-type convexity (see e.g. [5] and references therein), which is nowadays also known as nearly convexity (see e.g. [6]). On the other hand, for $T = ]0,1[$ we recover the notion of weakly convexity, introduced by A. Aleman in [7]. As an intermediate case, if $T = [\delta, 1 - \delta]$ with $\delta \in ]0, 1/2[$, we recover the notion of uniform convexlikeness, which has been introduced by H. Hartwig in [8].

Our aim here is to study the set-valued mappings, but our main result also focus on vector-valued functions. Actually, if $f : D \to Y$ is a function defined on a nonempty convex subset $D$ of $X$, then $f$ will be called $K$-convex (respectively $K$-quasiconvex, or weakly $K$-convex with respect to a nonempty set $T \subset ]0,1[)$ if and only if the set-valued mapping $F : D \to 2^Y$, defined by $F(x) = \{f(x)\}$ for all $x \in D$, is $K$-convex (respectively $K$-quasiconvex, or weakly $K$-convex with respect to $T$).

2. Main result

\textbf{Theorem 2.1.} Let $F : D \to 2^Y$ be a set-valued mapping defined on a nonempty convex subset $D$ of $X$. If $F$ has $K$-closed values (i.e. $F(x) + K$ is a closed set for every $x \in D$), then the following assertions are equivalent:

(i) $F$ is $K$-convex;
(ii) $F$ is both $K$-quasiconvex and weakly $K$-convex with respect to a nonempty compact set $T \subset ]0,1[.$

**Proof.** Obviously (i) implies (ii), the conclusion being true for any $T \subset ]0,1[.$

Conversely suppose that (ii) holds and let $T$ be a nonempty compact subset of $]0,1[$ for which $F$ is weakly $K$-convex. Let us denote, for all $x, x' \in D$,

$$T_{x,x'} := \{ t \in [0,1] : (1) \text{ holds } \}.$$  

In order to prove (i), we just have to show that $T_{x,x'} = [0,1]$ for all $x, x' \in D$. To this end, consider two arbitrary points $x_0, x_1 \in D$ and let us firstly prove that $T_{x_0,x_1}$ is dense in $[0,1]$. Suppose on the contrary that this is not the case. Then there exist some $a, b \in [0,1], a < b$, such that

$$[a, b] \cap T_{x_0,x_1} = \emptyset.$$ 

Since $[0,1] \subset T_{x_0,x_1}$, we can define the real numbers

$$\alpha := \sup [0, a] \cap T_{x_0,x_1} \text{ and } \beta := \inf [b, 1] \cap T_{x_0,x_1}.$$ 

Obviously $\alpha \leq a < b \leq \beta$ and, by (2) and (3), we have

$$]\alpha, \beta[ \cap T_{x_0,x_1} = \emptyset.$$ 

Let us denote, for all $t \in [0,1]$,

$$x_t := tx_0 + (1-t)x_1 \text{ and } Y_t := tF(x_0) + (1-t)F(x_1).$$ 

Recalling (3) and taking into account that $T$ is compact, we can find some numbers $u \in [0,\alpha] \cap T_{x_0,x_1}$ and $v \in [\beta,1] \cap T_{x_0,x_1}$ such that $tu + (1-t)v \in ]\alpha, \beta[ \text{ for all } t \in T$. On
the other hand, since $F$ is weakly $K$-convex with respect to $T$, we can choose a number $\tau \in T_{x_u,x_v} \cap T$. Hence

$$\gamma := \tau u + (1 - \tau)v \in ]\alpha, \beta[.$$  

(5)

Since $u, v \in T_{x_0,x_1}$, we have $Y_u \subset F(x_u) + K$ and $Y_v \subset F(x_v) + K$. Hence

$$\tau Y_u + (1 - \tau)Y_v \subset \tau F(x_u) + (1 - \tau)F(x_v) + K.$$  

Recalling that $\tau \in T_{x_u,x_v}$, i.e. $\tau F(x_u) + (1 - \tau)F(x_v) \subset F(\tau x_u + (1 - \tau)x_v) + K$, we obtain:

$$Y_{\gamma} = \tau Y_u + (1 - \tau)Y_v \subset F(\tau x_u + (1 - \tau)x_v) + K = F(x_{\gamma}) + K,$$

which means that $\gamma \in T_{x_0,x_1}$. By (5) it follows that $]\alpha, \beta[ \cap T_{x_0,x_1} \neq \emptyset$, contradicting (4).

So, we have proved that $T_{x_0,x_1}$ is dense in $[0,1]$. Now, let us show that $T_{x_0,x_1} = [0,1]$. Obviously $\{0,1\} \subset T_{x_0,x_1} \subset [0,1]$. Consider an arbitrary $t \in ]0,1[$. We just need to prove that $t \in T_{x_0,x_1}$, i.e. $Y_t \subset F(x_t) + K$. If $Y_t = \emptyset$ the conclusion is obvious. Otherwise let $y \in Y_t$. By definition of $Y_t$ we have $y = tz_0 + (1 - t)z_1$ for some $z_0 \in F(x_0)$ and $z_1 \in F(x_1)$.

Consider a point $e \in \text{int}K$. By density of $T_{x_0,x_1}$ in $[0,1]$, we infer the existence of two sequences: $(t_n^-)_{n \in \mathbb{N}}$ in $T_{x_0,x_1} \cap [0,t]$ and $(t_n^+)_{n \in \mathbb{N}}$ in $T_{x_0,x_1} \cap [t,1]$, such that

$$\{y_n^-, y_n^+\} \subset y + \frac{1}{n}e - \text{int}K, \text{ for all } n \geq 1,$$

where $y_n^- = t_n^- z_0 + (1 - t_n^-)z_1$ and $y_n^+ = t_n^+ z_0 + (1 - t_n^+)z_1$. Then, we have

$$y + \frac{1}{n}e \in t_n^- F(x_0) + (1 - t_n^-)F(x_1) + \text{int}K \subset F(x_{t_n^-}) + K + \text{int}K \subset F(x_{t_n^-}) + K;$$

$$y + \frac{1}{n}e \in t_n^+ F(x_0) + (1 - t_n^+)F(x_1) + \text{int}K \subset F(x_{t_n^+}) + K + \text{int}K \subset F(x_{t_n^+}) + K;$$
implying that \( \{x_{t_n}^{-}, x_{t_n}^{+}\} \subset L_f(y + \frac{1}{n}e) \), for all \( n \geq 1 \). Recalling that \( F \) is \( K \)-quasiconvex and taking into account that \( x_t \in [x_{t_n}^{-}, x_{t_n}^{+}] \) for each \( n \in \mathbb{N} \), we can deduce that

\[
x_t \in L_f(y + \frac{1}{n}e), \text{ i.e. } y + \frac{1}{n}e \in F(x_t) + K, \text{ for all } n \geq 1.
\]

Finally, by letting \( n \to \infty \), we infer that \( y \in \overline{F(x_t) + K} = F(x_t) + K \).

\[\square\]

**Corollary 2.2.** Let \( f : D \to Y \) be a function defined on a nonempty convex subset \( D \) of \( X \). Then \( f \) is \( K \)-convex if and only if it is both \( K \)-quasiconvex and weakly \( K \)-convex with respect to a nonempty compact set \( T \subset ]0, 1[ \).

**Proof.** It follows by Theorem 2.1, where \( F : D \to 2^Y \) is defined by \( F(x) = \{f(x)\} \) for all \( x \in D \). In this case \( F(x) + K \) is closed for every \( x \in D \), since the cone \( K \) is closed.

\[\square\]

**Remark 2.3.** The assumption on the compactness of \( T \) is essential. Indeed, consider \( X = Y = \mathbb{R} \) and \( C = \mathbb{R}_+ \), and let \( f : D = [0, 1] \to \mathbb{R} \) be defined by: \( f(x) = 1 \) if \( x \in [0, 1/2] \), and \( f(x) = 0 \) if \( x \in ]1/2, 1] \). Then \( f \) is both quasiconvex and weakly convex with respect to \( T = ]0, 1[ \), but \( f \) is not convex.

**Remark 2.4.** Corollary 2.2 generalizes some known characterization theorems given for real-valued convex functions, such as:

(a) Proposition 3 in [1], where \( X = \mathbb{R}^n \), \( Y = \mathbb{R} \), \( K = \mathbb{R}_+ \), \( D \subset \mathbb{R}^n \) is a nonempty convex open set, and \( T = \{1/2\} \).

(b) Theorem 2 in [2], where \( X \) is a linear space, \( Y = \mathbb{R} \), \( C = \mathbb{R}_+ \), \( D \subset X \) is a nonempty convex set, and \( T = \{1/2\} \).
(c) Theorem 3 in [3], where $X = \mathbb{R}^n$, $Y = \mathbb{R}$, $K = \mathbb{R}_+$, $D \subset \mathbb{R}^n$ is a nonempty convex set, and $T = \{\alpha\}$ with $\alpha \in ]0, 1[$.

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