# Linear differential equations and liouvillian solutions

*Felix Ulmer*
Université de Rennes I

May 30 1994

[summary by Jacques-Arthur Weil]

Let $k$ be a differential field (e.g $k = \mathbb{Q}(x)$ or $k = \mathbb{C}(x)$ with derivation $\frac{d}{dx}$. This aim of this talk is to review the methods of differential Galois theory used for solving the equation $L(y) = a_n y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_0 y = 0$ (with $a_i \in k$). For effectivity and simplicity, we will consider $k = \overline{\mathbb{Q}}(x)$ in the sequel.

## 1. What is solving

Here we describe some classes of solutions that one usually searches for.

**1.1.** First, we say that a solution is *rational* if it belongs to $k$. For example, the equation $a_2 y'' + xy' - y = 0$ has the solution $y = x$ which is in $k$. Algorithms for computing such solutions have been known for long. The first one is due to Liouville (1833). Some faster or more general versions have been given by Abramov ([1]), Bronstein ([2]), and Singer ([7], for the case when $k$ contains a wider class of functions).

If there is no rational solution, then one must perform a field extension to find a solution. Let $K$ be a differential field which is an extension of $k$, and $\Delta$ be the derivation on $K$ (resp $\delta$ on $k$). We say $K$ is a differential field extension of $k$ if $\Delta$ and $\delta$ coincide on $k$.

**1.2.** Now, we say that a solution of $L$ is *algebraic* if it belongs to an algebraic extension of $k$. In other words, this means that there is an irreducible polynomial $P$ with coefficients in $k$ such that $P(y) = 0$. For example, if we define $y$ as a zero of the polynomial $y^2 - x$, then $y$ is a solution of $2xy' = y$. Work on caracterising such solutions has been performed for example by Pépin, Klein, Jordan, Fuchs, Baldassari& Dwork, Singer (see e.g [5, 11, 8] for further references).

**1.3.** Then, a solution that is not algebraic is transcendental. Again, there is an interesting class of solutions that corresponds to the notion of "integrability by quadratures". We say that a solution $y$ of $L$ is *Liouvillian* if it belongs to a field $K$ such that:

a) $K = K_n \supseteq \cdots \supseteq K_1 \supseteq K_0 = k$
b) $K_i = K_{i-1}(\eta_i)$ for $i = 1 \cdots n$ and :
  i) $\eta_i$ is algebraic over $K_{i-1}$, or
  ii) $\eta_i' \in K_{i-1}$ (case of an integral), or

iii) $\frac{\eta_i'}{\eta_i} \in K_{i-1}$ (case of exponential of an integral)

For example, if we take $L(y) = y'' - \frac{1}{2(x+1)}y' - (x+1)y = 0$, then we have the following basis of liouvillian solutions: $\{e^{\int\sqrt{1+x}}, e^{-\int\sqrt{1+x}}\}$.

**1.4.** There is a very important subclass of the liouvillian solutions: we say that a solution $y$ is *exponential* if its logarithmic derivative is in $k$, i.e $y'/y \in k$. For example, the equation $y" - (2+4x^2)y = 0$ has the solution $y = e^{x^2}$ $(y'/y = 2x)$. Method for computing such solutions have been given, for example, by Singer or Bronstein (see e.g [7, 2]).

The main known tool to compute liouvillian solutions of linear differential equations is differential galois theory. Roughly, the idea is to look at the group of transformations that send a solution of the equation to another solution of the equation; from the knowledge of this group, one can derive algebraic properties of the solutions. We now outline this formalism.

## 2. Differential Galois theory

**2.1. Picard-Vessiot.** To a given vector space of solutions of $L$, one can associate a field extension the following way. Suppose we want to adjoin an element $y_1$ to $k$. As we work in a differential context, we must also add all its derivatives. So, we will write $k\langle y\rangle := k(y, y', y'', \cdots)$.

Now, we say that $K \supset k$ is a *Picard-Vessiot Extension* (P.V.E) if

i) $K = k\langle y_1, \cdots, y_n\rangle$, where $\{y_1, \cdots, y_n\}$ is a basis of the solution space of $L(y) = 0$, and
ii) $K$ and $k$ have the same field of constants $C$ (elements with zero-derivative).

Then, we proceed as in classical Galois theory: The *differential Galois group* of $L$ is the set of the automorphisms of $K$ that let $k$ point-wise fixed and that commute with the derivation (one can show that this definition does not depend on $K$). We call it $\mathrm{Gal}(L)$. As in classical Galois theory, we will have that an element is in $k$ if and only if it is left fixed by $\mathrm{Gal}(L)$; also, the subfields of $K$ appear as fixed fields of some algebraic subgroup of $\mathrm{Gal}(L)$.

**2.2. Galois group.** Now, call $V$ the vector space of solutions of $L$. As $G$ acts on $V$, we can decompose its action on a basis of $V$. In other words, the image of a solution of $L$ is still a solution of $L$, so the image of an element of $K$ is completely caracterised by the images of the $y_i$ in the basis $\{y_1, \cdots, y_n\}$. This provides a faithful matrix representation of degree $n$ of the Galois group: $\mathrm{Gal}(L)$ can be viewed as a subgroup of $GL(n, C)$ (the group of invertible $n \times n$ matrices with entries in $C$).

In fact, $\mathrm{Gal}(L)$ is a *linear algebraic group* (its entries are solutions of a set of polynomial equations). So, its entries have a structure of an algebraic variety. In particular, there is a component of this variety in which lies the origin; we denote it by $\mathrm{Gal}(L)^o$. We shall forget some details here, but a key fact is that $L$ has a liouvillian solution if and only if $\mathrm{Gal}(L)^o$ is solvable (Picard-Vessiot, Kolchin). In this sense, finding liouvillian solutions is the differential analog of searching for solutions by radicals in the classical case. Then, using a theorem of Lie-Kolchin on triangularization of matrix groups, one can show that this happens if the elements of $\mathrm{Gal}(L)^o$ have a common eigenvector $y$: $\forall\sigma \in \mathrm{Gal}(L)^o, \exists c_\sigma \in C, \sigma(y) = c_\sigma y$.

This has the following consequence:

$$\sigma(\frac{y'}{y}) = \frac{\sigma(y')}{\sigma(y)} = \frac{c_\sigma y'}{c_\sigma y} = \frac{y'}{y}.$$

This means that $y'/y$ is in the fixed field $K^o$ of $\mathrm{Gal}(L)^o$. One can show that this implies that $y'/y$ is algebraic over $k$.

**2.3. Ricatti equation.** As a consequence, there exists a $u = \frac{y'}{y}$ that is a solution of $P(u) = u^N + b_{N-1}u^{N-1} + \cdots + b_0 = 0$ (in other words, $y = e^{\int u}$ is a solution of $L(y) = 0$).

If we let $y' = uy$, then $y^{(i)} = R_i(u, u', \dots)y$, with $R_i = R'_{i-1} + uR_{i-1}$. Replacing in $L$, we get that $\sum a_i R_i(u, u', \dots) = 0$: this is a non-linear differential equation of order $n-1$ satisfied by $u$, called the *Ricatti equation*. For example, if $L = y'' - ry$, then the Ricatti equation is $u' + u^2 - r = 0$.

Now, it follows from the preceding discussion that finding a liouvillian solution is reduced to finding an algebraic solution of the Ricatti equation, which again splits into the following two subproblems:

(1) find a bound for the degree $N$ of $P$.
(2) Given $N$, compute the coefficients of a polynomial $P$ such that its zeroes are logarithmic derivatives of zeroes of $L$

By group-theoretic considerations, one can answer problem 1: it follows from works of Kovacic or Singer that there is a function $f(n)$ such that $N \leq f(n)$ (e.g, $f(2) = 60, f(3) = 360, f(4) \leq 5040, f(5) \leq 25920, f(6) \leq 604800, \dots$). In fact, recent works of Ulmer and Singer & Ulmer show that the sharp bounds are $N \leq 12$ for $n = 2$ and $N \leq 36$ for $n = 3$. We later come back to this point.

So, we may now focus on the actual computation of the coefficients of the polynomial $P$.

## 3. Computing a solution

**3.1. Symmetric powers.** Let us now introduce a technical tool. Suppose, for a moment, that we work in an algebraic closure of $k$. There, $P$ has $N$ zeroes $u_1, \dots, u_N$, and $P(u) = \prod (u - u_i)$. But all zeroes of $P$ are logarithmic derivatives of solutions of $L(y) = 0$; thus, there are $N$ solutions $y_i$ such that:

$$P(u) = \prod (u - \frac{y_i'}{y_i}) = u^N - (\frac{y_1'}{y_1} + \cdots + \frac{y_N'}{y_N})U^{N-1} + \cdots + \prod \frac{y_i'}{y_i}.$$

So, the coefficient $b_{N-1}$ verifies $b_{N-1} = \frac{y_1'}{y_1} + \cdots + \frac{y_N'}{y_N} = \frac{(y_1 \cdots y_N)'}{y_1 \cdots y_N}$

Now, for any integer $m$, one can construct a linear differential equation $L^{\circledS m}$, called the *m-th symmetric power of* $L$, whose solution space is spanned by all monomials of degree $m$ in the $y_1, \dots, y_n$. In particular, we have that $b_{N-1}$ is the logarithmic derivative of a solution of $L^{\circledS N}$: our problem is now reduced to finding exponential solutions of $L^{\circledS N}$. By similar techniques, one can find the other coefficients.

Now, there are three main cases to study in more details.

**3.2. Reducible operators.** Let $D = \frac{d}{dx}$. Then, $L(y)$ can be viewed as the action of the operator $\sum a_i D^i$ on $y$. Such operators form a non-commutative multiplicative ring $\mathcal{D} = k[D]$ in the following way: for $a \in k$, we have $D(ay) = aD(y) + a'y$, so the multiplication on $\mathcal{D}$ follows from the rule $Da = aD + a'$ (precisely, $\mathcal{D}$ is called an Ore ring of type "derivation"). In fact, $\mathcal{D}$ is a left and right euclidean ring; given some element $L$ of $\mathcal{D}$, one may search if $L$ factors in $\mathcal{D}$. For example, we have $D^2 = D.D = (D + 1/x)(D - 1/x)$.

Before searching for solutions, one should first search if $L$ factors. In terms of solution space, $\mathrm{Gal}(L)$ has an invariant subspace of dimension $m$ if and only if $L$ has a factor of order $m$. In that case, we say that $\mathrm{Gal}(L)$ (resp $L$) is *reducible*. Algorithms for performing such factorizations (or detecting reducibility) exist on $\mathbb{C}(x)$. The classical algorithm dates to Beke/Schlesinger (1895); Grigor'ev, Singer, or Van Hoeij have recently proposed alternative methods.

**3.3. Irreducible operators.** Now, we assume that $\mathrm{Gal}(L)$ is irreducible. We say that $\mathrm{Gal}(L)$ is *imprimitive* if $V$ is a direct sum of subspaces that are permuted transitively under the action of $\mathrm{Gal}(L)$. Else, we say that $\mathrm{Gal}(L)$ is *primitive*. This distinction is not at all artificial, as show the following theorems:

**Theorem 1 (Kovacic, 1986).** *Let $L$ be of order 2 and $Gal(L) \subseteq SL(2, \mathbb{C})$, then:*

(1) *$Gal(L)$ is reducible, or*
(2) *$Gal(L)$ is imprimitive and then $\exists y$ with $[k(y'/y) : k] = 2$, or*
(3) *$Gal(L)$ is primitive and $\exists y$ with $[k(y'/y) : k] = 4, 6, 12$, or*
(4) *$Gal(L) = SL(2, \mathbb{C})$ and $L(y) = 0$ has no liouvillian solution.*

**Theorem 2 (Singer-Ulmer, 1993).** *Let $L$ be of order 3 and $Gal(L) \subseteq SL(3, \mathbb{C})$, then:*

(1) *$Gal(L)$ is reducible and $L = L_1(L_2)$ or*
(2) *$Gal(L)$ is imprimitive and then $\exists y$ with $[k(y'/y) : k] = 3$, or*
(3) *$Gal(L)$ is primitive finite and $\exists y$ with $[k(y'/y) : k] = 6, 9, 21, 36$, or*
(4) *Else, $L(y) = 0$ has no liouvillian solutions.*

In general, if one has that, if $\mathrm{Gal}(L)$ is irreducible then: either $\mathrm{Gal}(L)$ is imprimitive and $\exists y$ with $[k(y'/y) : k]$ small, either $\mathrm{Gal}(L)$ is primitive finite and $\exists y$ with $[k(y'/y) : k]$ big, or $\mathrm{Gal}(L)$ is primitive infinite and there is no liouvillian solution.

## 4. Algebraic solutions of $L$

**4.1.** It is, in general difficult to compute $y$ from the knowledge of $y'/y$ (Abel's problem), but one can compute directly compute $y$ in the case of a known finite primitive group because $y$ is then algebraic. It follows that $y$ is algebraic over $k(y'/y)$, and one can show that there is an integer $m$ such that $y^m \in k(y'/y)$. Thus, if $d$ is one of the possible degrees for $[k(y'/y) : k]$, then the minimum polynomial of $y$ is of the form $P(y) = y^{m.d} + a_{d-1}y^{m.(d-1)} + \cdots + a_1 y^m + a_0$. Note that this polynomial has the same number of coefficients as the minimum polynomial of an algebraic solution of the Ricatti equation.

**4.2.** To show that the Ricatti had an algebraic solution, we showed that there was a subgroup of $L$ with a common eigenvector. Such a subgroup is called *1-reducible*. To find the group or a solution, we must therefore find a 1-reducible subgroup $H$ of $\mathrm{Gal}(L)$ of minimal index. Suppose we have found such an $H$ and let $\mathfrak{S} = \{\sigma_1, \ldots, \sigma_d\}$ be a system of representants of $\mathrm{Gal}(L)/H$. Then, if $y_0$ is the eigenvector of $H$, then

$$P(y) = \prod_{\sigma \in \mathfrak{S}} (y^m - \sigma(y_0)^m)$$

Now, as the $a_i$ are rational, they are invariant under $\mathrm{Gal}(L)$. So, one can decompose the $a_i$ in terms of invariants (or semi-invariants) of the group. Recall that a homogeneous polynomial $M(y_1, \ldots, y_n)$ is called an *invariant* of the group if it is left invariant under the action of the group $(\sigma(M)(y_i) = M(\sigma(y_i)) = M(y_i))$.

Now, to detect if the group has invariants of degree $m$ (resp semi-invariants), one just has to search for rational solutions (resp exponential solutions) of $L^{\circledS m}$, and we are almost done: as these solutions are given up to multiplication by constants, we just adjust the constants so as to really obtain the desired polynomials.

Examples and more precise descriptions of this process are given in [8, 9].

**4.3.** The whole philosophy was to reduce the computation of Liouvillian solutions to the computation of exponential (and sometimes rational) solutions to some symmetric powers of $L$. In fact, by group-theoretic considerations, one can show that one can detect the presence of liouvillian solutions to the reducibility of some symmetric powers. Conversely, reduciblity of some symetric powers helps finding the Galois group of a given linear differential equation. This gives elegant criteria, as shows this last result from [8]:

**Theorem 3 (Singer-Ulmer).** *Liouvillian solutions and symmetric powers are linked the following way:*

- *The equation $y'' - ry$ has a liouvillian solution if and only if $L^{\circledS 6}$ is reducible.*
- *The equation $L(y) = y''' - a_1 y' - a_0 y = 0$ has a Liouvillian solution if and only if*
  (1) *$L^{\circledS 4}$ has order less than 5 or is reducible* AND
  (2) (a) *$L^{\circledS 2}$ has order 6 and is irreducible,* OR
  (b) *$L^{\circledS 3}$ has a factor of order 4.*

## References

1. Abramov S.A, Kvansenko K.Yu *Fast algorithms for the rational solutions of linear differential equations with polynomial coefficients* Proceedings ISSAC 91, ACM Press 1991.
2. Bronstein M *On solutions of linear ordinary differential equations in their coefficient field* J. Symb. Comp **13** (1992), 413-439.
3. Kaplansky I *An introduction to differential algebra* Second edition, Hermann, Paris 1976.
4. Kovacic J *An algorithm for solving second order linear homogeneous differential equations* J.Symb.Comp (1986) **2** pp 3-43.
5. Singer M.F *Liouvillian solutions of linear differential equations* Amer.J.Mat. **103** 1981.

6. SINGER M.F *An outline of differential Galois theory* in CADE (Ed E. Tournier), 1989, Academic Press
7. SINGER M.F *Liouvillian solutions of linear differential equations with liouvillian coefficients* J.Symb.Comp (1991) vol **11**, pp 251-273.
8. SINGER M.F & ULMER F *Liouvillian and algebraic solutions of second and third order linear differential equations* J.Symb.Comp (1993) vol **16**, pp 37-73.
9. SINGER M.F & ULMER F *Galois groups of second and third order linear differential equations* J.Symb.Comp (1993) vol **16**, pp 1-36.
10. SINGER M.F & ULMER F *Necessary conditions for liouvillian solutions of (third order) linear differential equations* To appear in AAECC Journal.
11. ULMER F *On Liouvillian solutions of linear differential equations* AAECC Journal **2**, 1992.