Special Polynomials of Ordinary Differential Equations

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[summary by Bruno Salvy]

Abstract

A new algorithm to compute first integrals of quasi-linear ordinary differential equations is presented. Its specialization to the linear case proves useful and in the second order case it leads to an almost rational version of Kovacic's algorithm.

Introduction

Transcendental functions in computer algebra are naturally defined via differential equations. In particular, this makes it possible to test equality between transcendental functions algorithmically, instead of resorting to heuristic rewriting rules. Some functions defined by differential equations occur so often that they have received a special name, and finding an expression in terms of those functions which cancels an equation is called finding a *closed-form* solution. Given an algebraic differential equation of order n or equivalently a polynomial P in $Y, Y', \dots, Y^{(n)}$, a natural generalization of the idea of closed-form solution is provided by *solutions of order* $r < n$. A function y is a solution of order r of P if there exists a polynomial Q in $Y, Y', \dots, Y^{(r)}$ such that $Q(y) = 0$ and $P(y) = 0$, while there is no polynomial of order lower than r cancelling y. Finding these solutions is the topic of [9] and [8], of which we give here an overview.

The quest for these solutions is simplified by a well-known *reduction algorithm* [5, p. 5–7] which provides an analogous of Euclidean division in the differential case. If P and Q are two polynomials as above, with $r \leq n$, this reduction proceeds in two steps. First Q is differentiated $n - r$ times, then P is reduced with respect to these derivatives by Euclidean divisions, except that no division by the leading terms are performed. Thus in a finite number of steps, one gets an identity

$$\operatorname{In}(Q)^m \operatorname{Sep}(Q)^{n-r} P = \sum_{i=0}^{n-r} \alpha_i Q^{(i)} + R, \tag{1}$$

where R is of order less than r; $\operatorname{Sep}(Q) := \partial Q / \partial Y^{(r)}$ is the *separant* of Q; $\operatorname{In}(Q)$ is the *initial* of Q, i.e., the coefficient of its highest power of $Y^{(r)}$; m is the degree of P in $Y^{(n)}$.

If Q is a polynomial of order $r < n$ cancelling a rth order solution y of P, then the polynomial R in (1) must be 0, since otherwise $R(y) = 0$ with R of order less than r.

At this level of generality, not much is known at present. In the sequel, following [9], we reduce ourselves to solutions of order $n - 1$ of a polynomial P *linear* in $Y^{(n)}$ (Section 1); then to the case of a linear operator P (Section 2); and finally to the case when this linear operator is of order 2 (Section 3).

1. Solutions of order $n-1$

In this section, P is a polynomial of the form

$$P = s(Y, Y', \ldots, Y^{(n-1)})Y^{(n)} + t(Y, Y', \ldots, Y^{(n-1)}).$$

Let Q be any polynomial of order $n-1$. Reducing the polynomial Q' of order n by P leads to

$$\mathrm{In}(P)Q' = \mathrm{Sep}(Q)P + R_1, \tag{2}$$

with R_1 a polynomial of order less than n. This polynomial is then reduced by Q yielding

$$\mathrm{In}^m(Q)sQ' = \mathrm{Sep}(Q)\,\mathrm{In}^m(Q)P + \alpha Q + R, \tag{3}$$

where m is the degree of Q in $Y^{(n-1)}$ and α and R are polynomials of order less than $n-1$.

In view of (3), a polynomial Q of order $n-1$ is called *special* for P if $R = 0$ in (3). In particular, if Q cancels a solution of order $n-1$ of P, it is a special polynomial. Reciprocally, solutions of a special polynomial Q are either solutions of P or solutions of $\mathrm{In}(Q)$ or $\mathrm{Sep}(Q)$.

As a consequence, finding special polynomials of Q is almost equivalent to finding its solutions of order $n-1$. Unfortunately, no general procedure is known to find these polynomials. Even when the degree m of Q is given, finding Q by an undeterminate coefficients method requires some skill. However, when besides m, α in (3) is given, then J.-A. Weil shows in [9] how the method can be reduced to a set of linear differential equations whose rational solutions are the coefficients of Q. Finding these solutions is then a simple routine via Abramov's algorithm [1, 2]. To solve the problem completely, one would need to produce candidates for α algorithmically. Only heuristics or special cases exist at this point.

2. Linear differential equations

In this section P is a polynomial of the form

$$Y^{(n)} + a_{n-2}Y^{(n-2)} + \cdots + a_0 Y.$$

Any linear equation can be reduced into this form, by dividing out by the leading coefficient and then changing the unknown function into $y\exp(-\int a_{n-1}/n)$.

The specificity of the linear equation is that in reduction (2), the remainder R_1 preserves the homogeneous parts of Q. If $Q = \sum Q_i$ is the decomposition of Q as a sum of homogeneous parts,

$$Q' - \mathrm{Sep}(Q)P = \sum Q_i - P\frac{\partial Q_i}{\partial Y^{(n-1)}}$$

is also a decomposition in homogeneous parts. Therefore if Q is a special polynomial, the corresponding α does not depend on Y and it is sufficient to look for monic homogeneous special polynomials.

Given the degree m of a homogeneous special polynomial, write

$$Q = [y^{(n-1)}]^m + f_0[y^{(n-1)}]^{m-1}y^{(n-2)} + \cdots.$$

The leading term of the remainder of Q' mod P is $f_0[y^{(n-1)}]^m$. Thus $\alpha = f_0$, and the system obtained by undeterminate coefficients is linear in all the coefficients except f_0. J.-A. Weil shows in [9] how this leads to a linear system whose solutions with a rational logarithmic derivative is sought. Algorithms to solve this problem efficiently were given by M. Rothstein in 1976, by J. H. Davenport in 1986 and by M. Bronstein in 1990 (see [3] and references therein).

3. Second order and Kovacic's algorithm

By a general Lie-Kolchin theorem (see [6]), a linear differential equation admits Liouvillian solutions if and only if its associated Ricatti equation has an algebraic solution. In the second order case, this Ricatti equation takes the form

$$u' - r + u^2 = 0. \tag{4}$$

Then a special polynomial is one of order 0, i.e., a polynomial whose solutions are algebraic functions solutions of (4). By Kovacic's algorithm [4], the degree of minimal special polynomials can only be one of $\{1, 2, 4, 6, 12\}$. Then the above algorithm applies and the equation it yields for f_0 is the mth symmetric power of the original linear equation. A reason why the second order can be solved efficiently is any special polynomial yields a Liouvillian solution.

More work on reducing as much as possible of the second order case to rational (instead of exponential) solutions of linear differential equations is presented in [8]. As described in the summary of F. Ulmer's talk in these proceedings, there are three cases to consider in order, each step assuming that the previous one failed. At each step, the possible degrees of the minimal special polynomials are known, but their computation may require looking for an exponential solution of $L^{\circledS m}$. The idea in [8] is that in almost every case, one can *choose* m such that a special polynomial is associated with a *rational* solution of $L^{\circledS m}$. Thus, characterizing all possible solutions of the Ricatti equation (e.g., by group-theoretic properties) also characterizes the corresponding special polynomials. The characterization goes as follows.

3.1. Reducible case. If the differential operator can be written $(d/dx - b(x))(d/dx - a(x))y(x)$, then obviously it has an exponential solution (i.e., a solution whose logarithmic derivative is rational). Before looking for such a solution, one should look for rational solutions, which is much easier. Then one can look for a basis of rational solutions of $L^{\circledS 2}$. If it contains only one solution, [8] proves that the special polynomial of degree 2 it induces must factor. If its factors are distinct, they yield two independent solutions of the differential equation. Otherwise, reduction of order leads to the second one. Only if this fails, one should look for exponential solutions.

3.2. Imprimitive case. According to [4], there exists a solution whose logarithmic derivative is algebraic of degree 2. Besides, a careful analysis of the tables of characters of the possible groups is used in [7] to show that the Galois group is imprimitive if and only if $L^{\circledS 4}$ has a rational solution. Such a solution leads to a special polynomial of degree 4. Since at this stage it is impossible that the Ricatti equation has a rational solution, the special polynomial is either irreducible or has factors of degree 2. Consideration of the possible groups shows that either the basis of rational solutions is reduced to one element and the induced special polynomial is the square of the minimal special polynomial or the basis contains two elements and a linear combination of the special polynomials they induce is a perfect square which can be found by resultant and gcd computation.

3.3. Primitive case. This is the case when solutions are algebraic. It is not necessarily a good idea to look for a special polynomial in this case (see the summary of Ulmer's talk). However, rational solutions of $L^{\circledS 6}$, $L^{\circledS 8}$ or $L^{\circledS 12}$ lead to irreducible special polynomial of these degrees.

If none of the above yields a special polynomial, then there is no Liouvillian solution. A byproduct of this method is that algebraic extensions are never necessary except in the reducible case, where examples show that they can be unavoidable. Another useful property of this approach is that it is not limited to equations of the form $Y'' + b(x)Y = 0$, but extends to the more general case, provided the coefficient of Y' is the logarithmic derivative of a rational function (possibly in an algebraic extension). Then it is not necessary to perform the change of variable suggested at the beginning of Section 2.

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