A non-integrability criterion for hamiltonian systems illustrated on the planar three-body problem
Preliminary Version

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1 Introduction

The *hamiltonian systems* are differential systems which describe the equations of motion of mechanical systems whose mechanical energy (the Hamiltonian) is conserved. One says that the mechanical energy is a *first integral* for the hamiltonian system. The *complete integrability* of a hamiltonian system will be ensured by a sufficient supply of first integrals.
The problem we are interested in here is the problem of three bodies moving in a Newtonian reference system with the only forces acting on them being their mutual gravitational attraction (say the sun, Earth and moon). One knows classical integrals for this problem but they don't ensure the complete integrability of the system ([M-H]). The question that we address is: can one find other (meromorphic) integrals for this system making it (meromorphically) completely integrable ?
In 1890, Poincaré proved that the three-body problem does not have any additional analytic first integral besides the known integrals ([Poin]). To obtain this result, he studied a *variational equation* which is a *linear* differential equation computed along a particular solution of the hamiltonian system.
During the last twenty years many significant improvements regarding complete (meromorphic) integrability of hamiltonian systems have been obtained by Ziglin ([Zig1], [Zig2], 1982); Churchill, Rod and Singer ([C-R-S], 1996) and Morales and Ramis ([M-R], 1998). They all found necessary conditions of complete (meromorphic) integrability based on the monodromy group ([Zig1], [Zig2]) or the differential Galois group ([C-R-S], [M-R]) of this *variational equation*.
Our study will rely on the criterion of Morales and Ramis (from [M-R], see section 2 for details) : If the system (S) is completely integrable, then the connected component of the identity in the group G, denoted G^0, is an abelian group. Before applying this theorem to the example of the three-body problem, we deduce from it a new criterion based on a local and global formal study (detection of logarithmics and factorization) of this *variational equation*.

Theorem 1 *Let (S) be a hamiltonian system and $L(y) = 0$ be the normal variational equation computed along a particular solution of (S).*
If the linear differential operator L has a right factor M such that

- *the equation $M(y) = 0$ is completely reducible,*
- *the equation $M(y) = 0$ has formal solutions with logarithmic terms*

then the system (S) is not completely integrable.

The main interest of this criterion is that one can easily apply it when the coefficients of the variational equation lie in $k(x)$ where k is an algebraic extension of $\mathbb{Q}$. Indeed there exists algorithms to compute formal solutions ([Sch], [Bar1], [Bar2], [D-C-T], [Hila], [Hoe1], [Ince],[Le], [Mal], [Som], [Tour], ...); to factorize linear differential operators ([Sch], [Beke], [Hoe], [Hoe2], [Pflu], [Sin1],[B-P], ...) or to detect whether an equation is completely reducible or not ([Sin1]).
However the *variational equation* has often the particularity to depend on a finite number of *parameters* (the masses of the bodies for the three-body problem). So we need to

adapt the existing algorithms from the non parameterized (well mastered) situation to a parameterized situation. In [Bou2], the first author studied this adaptation and met two kinds of situations:
First, she shows that some computations (for example the computation of the generalized exponents at a point) are reduced to computations on constructible sets (which can be achieved by a computer ([Bou4]); In such cases, computing with parameters is similar (but just more difficult for computers) to computing without parameters.
Secondly, there may appear *arithmetic* conditions on the parameters, namely one may have to look for polynomial solutions whose degree depends on parameters and is no longer bounded or, equivalently, to try to solve a linear system whose size is a parameter. This induces indecidability results ([Bou3]).

Here the symplectic structure of the variational equation enables to reduce the number of these complicated situations where there appears arithmetic conditions on the parameters (this point will be more developped in a further work, [BW01]). Furthermore, the physical constraints on the parameters enable also to bound degrees which yet depend on parameters.
So, using our criterion (theorem 1), some of the tools developped in [Bou2] and taking the constraints on the parameters into consideration, we establish the non complete (meromorphic) integrability of the planar three-body problem along Lagrange' solution.

Theorem 2 *The planar three-body problem is not meromorphically completely integrable.*

This result was also established by A. Tsygvintsev (see [Tsy1], [Tsy2]). A (restricted) version was published in [Bou1]. Other results on three-body problems involving the Morales-Ramis theorem appear in [JTo] (see also [Au2])

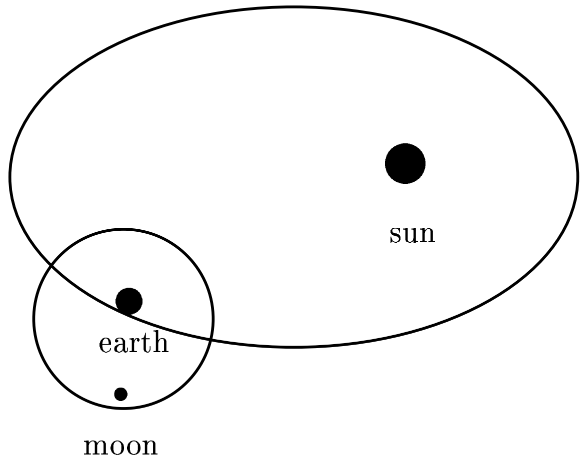
In section 2, we define the *hamiltonian systems* and the notion of *complete integrability*. We illustrate this part with the planar three-body problem. Then we recall Ramis and Morales' theorem (theorem 3) and we present our criterion (theorem 1).
In section 3 we consider the planar three-body problem along Lagrange' solution. We first give some directions for use of our criterion and prove our theorem 2 separating the two cases $m_1 = 1$ and $m_1 \neq 1$. The way our criterion is applied in this section should also apply to other situations from hamiltonian mechanics.

2 Complete integrability of hamiltonian systems

In section 2.1, we define *hamiltonian systems*, *first integrals* and *complete integrability of hamiltonian systems* (see [Au1], [Chur], [M-R]). In section 2.2, we present the non-integrability criterion of J.J. Morales and J.P. Ramis ([M-R]). Lastly, in section 2.3, we deduce from this criterion a new criterion which is simpler to test.

2.1 Some definitions

We illustrate this part with the three-body problem (see section 3).
Let us consider three bodies in a newtonian reference system and let us assume that the only forces acting on them are their mutual gravitational attraction ([M-H]).



Each body is represented by its mass m_i, its position q_i and its moment p_i ($p_i = m_i \frac{dq_i}{dt}$). According to the Newton law and the law of gravitation the equations of the motion of these bodies can be written in the following form:

$$\begin{cases} \frac{dq_i}{dt} = \frac{\partial H}{\partial p_i} \\ \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i} \\ i = 1, 2, 3 \end{cases}$$

where $H = \sum_{j=1}^{3} \frac{||p_j||^2}{2m_j} - \sum_{1 \le j,k \le 3} \frac{m_j m_k}{||q_j - q_k||}$.
These equations form a differential system, called *hamiltonian system*. It depends on the three parameters m_1, m_2 and m_3 (we assume $m_3 = 1$).

Definition 1 *Let* $n \in \mathbb{N}^*$, $\mathrm{x} = (x_1, \dots, x_{2n}) = (q_1, \dots, q_n, p_1, \dots, p_n) \in \mathbb{R}^{2n}$.
A hamiltonian system *on a non empty domain* U *of* $\mathbb{R}^{2n}$ *is a system of differential equations of the form:*

$$\begin{cases} \frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}(q, p) \\ \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}(q, p) \\ j = 1, \dots, n \end{cases}$$

where $H : U \to \mathbb{R}$ *is the* hamiltonian *function.*
The variables p_i *and* q_i *are* conjugate variables. *The positive integer* n *is called the* number of degrees of freedom.

For the three-body problem the number of degrees of freedom is 9 in the space and 6 in the plan.

Remark 1 *A hamiltonian system can also be written in the following form:*

$$\mathrm{x}'(t) = J\,\nabla H(\mathrm{x}(t))$$

where $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$ *and* $\nabla H(\mathrm{x})$ *is the gradient of* H *at* x.

The hamiltonian H represents the mechanical energy and is conserved. Indeed, for each solution x of the system, $dH(\mathrm{x}(t)) = 0$. One says that H is a *first integral* for the hamiltonian system.

Definition 2 *A function* $G : U \to \mathbb{R}$ *is a* first integral *of the hamiltonian system if, for all solution* $\mathrm{x}(t)$ *of the system,*

$$\frac{d}{dt} G(\mathrm{x}(t)) = 0.$$

Remark 2 *The first integral* G *of the hamiltonian system is also characterized by the equality*

$$\{G(\mathrm{x}), H(\mathrm{x})\} = 0$$

where

$$\{G_1, G_2\} = \sum_{i=1}^{n} \left(\frac{\partial G_1}{\partial p_i} \frac{\partial G_2}{\partial q_i} - \frac{\partial G_1}{\partial q_i} \frac{\partial G_2}{\partial p_i} \right) = \nabla G_1(\mathrm{x}), J\,\nabla G_2(\mathrm{x})$$

is the Poisson bracket of G_1 *and* G_2.

The first integrals give the geometry of the solution curve. Indeed the solutions of the system lie on the hypersurfaces $G = cte$.
There are ten classical first integrals for the three-body problem ([M-H]): the hamiltonian H; the three components of the total linear moment $(L = p_1 + p_2 + p_3)$; the three components of the vector C_0 defined by $C = Lt + C_0$ where C is the center of masses of the system $(C = m_1\, q_1 + m_2\, q_2 + m_3\, q_3)$ and the three components of the total angular moment $(A = q_1\, p_1 + q_2\, p_2 + q_3\, p_3)$.

Definition 3 *A hamiltonian system with* n *degrees of freedom is said to be* completely integrable *if it has* n *first integrals* $G_1, \ldots, G_n$ *such that:*

- $G_1, \ldots, G_n$ *are functionally independant*
  $(\nabla G_1, \ldots, \nabla G_n$ *are linearly independant)*

- $G_1, \ldots, G_n$ *are in involution: for all solution* x *of the hamiltonian system,*
  $\{G_i(\mathrm{x}),\, G_j(\mathrm{x})\} = \{G_j(\mathrm{x}),\, G_i(\mathrm{x})\} = \nabla G_i(\mathrm{x}), J\,\nabla G_j(\mathrm{x}) = 0$
  $(G_1, \ldots, G_n$ *commute for the Poisson bracket).*

It is necessary to give a more precise sense to this notion of complete integrability asking which class of functions one wants the first integrals to belong to (analytic, algebraic functions, ...).
In 1890, Poincaré proved that the three-body problem is not *analytically* completely integrable ([Poin]):
'...I establish for example that the three-body problem admits, beside the known integrals, no uniform analytic integral. Many other circumstances make us think that the complete solution, if ever one can discover it, will require analytic instruments absolutely different from those we own and infinitely more complicated ...'
To answer the question of the analytic complete integrability, Poincaré studied the solutions which are infinitesimaly close to a particular solution of the hamiltonian system. The behavior of these solutions is given by a homogeneous linear differential system, the *variational system*, which is obtained by linearization of the hamiltonian system along the particular solution.

Definition 4 *The* variational system *along a solution* $\mathrm{x}_0(t)$ *of a hamiltonian system is the linear differential system:*

$$h'(t) = J\ \mathcal{H}(H, \mathrm{x}_0(t))\ h(t)$$

where $\mathcal{H}(H, \mathrm{x}_0(t))$ *is the hessian of* H *at* $\mathrm{x}_0(t)$ *and* $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$.

Remark 3 $\mathrm{x}_0'(t)$ *is a particular solution of the variational system along the solution* $\mathrm{x}_0(t)$.

During the last twenty years many results on the *meromorphic* non integrability of the hamiltonian systems were deduced from the study of this variational system. In 1982, Ziglin ([Zig1], [Zig2]) established necessary conditions of meromorphic complete integrability on the monodromy group of the variational system. In 1995, Baider, R.C. Churchill, Rod and M. Singer ([C-R-S]) and in 1998, J.J. Morales and J.P. Ramis ([M-R]) established necessary conditions of meromorphic complete integrability on the differential Galois group of the variational system.
Before presenting Morales and Ramis' theorem in section 2.2, let us come back to the planar three-body problem. The question one askes is: can we find 6 meromorphic first integrals satisfying furthermore involution and independance properties ?
In [Tsy1], A. Tsygvintsev reduces the number of degrees of freedom of the planar three-body problem from 6 to 3 and gets a new hamiltonian (that we denote H again):

$$H = \frac{1}{2}\left(\frac{1}{m_1}+1\right)\left({p_1}^2 + \frac{(p_3\,q_2 - p_2\,q_3 - c)^2}{{q_1}^2}\right) + \frac{1}{2}\left(\frac{1}{m_2}+1\right)\left({p_2}^2+{p_3}^2\right) + p_1\,p_2$$
$$-\frac{p_3\,(p_3\,q_2 - p_2\,q_3 - c)}{q_1} - \frac{m_2}{\sqrt{{q_2}^2+{q_3}^2}} - \frac{m_1}{q_1} - \frac{m_1\,m_2}{\sqrt{(q_1-q_2)^2+{q_3}^2}}.$$

The parameter c represents the integral of the angular moment of the system. The non-integrability of this hamiltonian system will imply the non-integrability of the initial hamiltonian system: if one cannot find 3 independant meromorphic first integrals in involution for this hamiltonian system, then the planar three-body problem will not be

completely integrable.
He chooses the Lagrange solution as particular solution of the system ([Tsy1]). The three bodies then form a configuration which is homographically equivalent to an equilateral triangle :each body describes a parabola centered on a vertex of the equilateral triangle.

2.2 J.J. Morales and J.P. Ramis theorem

Theorem 3 *Let (S) be a hamiltonian system, $\mathrm{x}_0(t)$ a particular solution of (S), $L(y(t)) = 0$ the normal variational equation of (S) computed along the solution $\mathrm{x}_0(t)$ and G the differential Galois group of $L(y(t)) = 0$.*
If the system (S) is completely integrable, then the connected component of the identity in the group G, denoted G^0, is an abelian group.

Remark 4 *One also says that G is virtually abelian (see [Au2]).*

This theorem is a simple criterion of non-integrability of the hamiltonian systems, however in practice it is not easy to apply it. Indeed it is theoretically possible to compute the Galois group of a completely reducible equation ([C-S]) but it is difficult to achieve this computation practically, especially when this equation depends on a finite number of parameters. In the next section we propose a new criterion of non-integrability which is based on a local study (the detection of logarithms in the formal solutions) and a global study (the factorization of linear differential operators). That way, the above theorem can be applied also by people not familiar with differential Galois theory.

2.3 A new criterion of non-integrability

Proposition 1 *Let $G \subset GL(n, C)$ be a completely reducible linear algebraic group acting on $V = C^n$. The following assertions are equivalent:*
(a) G^0 is solvable,
(b) G^0 is diagonalizable,
(c) G^0 is abelian.

Proof
The implications $(b) \Rightarrow (c)$ and $(c) \Rightarrow (a)$ are immediate ; Now assume that (a) holds. First we assume that V is irreducible under G. Because G° is solvable, it is triangularizable. In particular, all its elements have a common eigenvector v_1. Let $g \in G$ and $h \in G^\circ$. Because G° is normal in G, we have $h(g(v_1)) = g(\tilde{h}(v_1))$ with $\tilde{h} \in G^\circ$. But, as v_1 is a common eigenvector for G°, we have $\tilde{h}(v_1) = \chi_{\tilde{h}}.v_1$ with $\chi_{\tilde{h}} \in C$. But now $h(g(v_1)) = g(\chi_{\tilde{h}}.v_1) = \chi_{\tilde{h}}.g(v_1)$ so all $g(v_1$ are eigenvectors of G°. The linear space spanned by the $g(v_1(forg \in G)$ is a subspace of V invariant under G. By irreducibility, it is equal to V. Because it it is generated by the $g(v_1)$, G° acts diagonally on it.
Now, if V is reducible, it is a direct sum of subspaces which are irreducible under G, and we apply inductively the above reasonning to these irreducible summands. $\square$

Remark 5 *While we were writing this paper, a result analogous to the above proposition appeared in [Bro00], lemma 3.6.*

Now, if G is the differential Galois group of a linear differential equation $L(y) = 0$, then it is easier to find sufficient conditions of non diagonality of G° using the local information that we can read in the formal solutions.

Proposition 2 *Let $L(y) = 0$ be a homogeneous linear differential equation with Galois group G.*
Assume that the equation $L(y) = 0$ is completely reducible.

1. *If it has formal solutions around some point which contain logarithmic terms, then the connected component of the identity in the group G is not an abelian group.*

2. *If it has a non-trivial Stokes multiplier around some point (an irregular singularity), then the connected component of the identity in the group G is not an abelian group.*

Proof
Let L be a differential operator of degree n with coefficients in a field k with differential Galois group G.
Let us assume that L is irreducible. If the group G^0 is abelian, then according to the proposition 1, it is diagonalizable. Furthermore if a logarithmic term appears locally, then the corresponding local group has a non trivial unipotent subgroup. So the group G^0 contains non trivial unipotent elements, which contradicts the diagonality of G^0.
If L is completely reducible one concludes in the same way as in the proof of proposition 1.
Part (b) is proved similarly because the presence of a Stokes multiplier induces a unipotent element in the differential Galois group ([PS01]). $\square$ From the two previous propositions one can establish the following criterion of non abelianity for the group G^0.

Theorem 4 *Let $L(y) = 0$ be a homogeneous linear differential equation et let M be a right factor of L.*
If the equation $M(y) = 0$ is completely reducible and has formal solutions with logarithmic terms, then the component of the identity in the group G is not an abelian group.

Proof
Let G_M be the differential Galois group of the equation $M(y) = 0$. There exists a surjection from G^0 to G_M^0 (see lemma 5.10 page 38 of [Ka]) so the non abelianity of G_M^0 implies the non abelianity of the group G^0. $\square$

Remark 6 *Similarly, as in the previous proposition, the presence of logarithms can be replaced by the presence of a non-trivial Stokes multplier in this theorem. The key is again that a unipotent element yields an obstruction to diagonalizability, and hence to integrability.*

From this theorem and the theorem 3 we deduce a criterion for the non complete integrability of the hamiltonian systems:

Theorem 1 *Let (S) be a hamiltonian system and $L(y) = 0$ be the normal variational equation computed along a particular solution of (S).*
If the linear differential operator L has a right factor M such that

- *the equation $M(y)=0$ is completely reducible,*
- *the equation $M(y)=0$ has formal solutions with logarithmic terms*

then the system (S) is not completely integrable.
The advantage of this criterion is that there exists algorithms to compute the formal solutions at a point ([Sch], [Bar1], [Bar2], [D-C-T], [Hila], [Hoe1], [Ince],[Le], [Mal], [Som], [Tour], ...) and to factorize the equation ([Sch], [Beke], [Hoe], [Hoe2], [Pflu], [Sin1], [B-P], ...).
The only difficulty consists in adapting these algorithms from the non parameterized case to the parameterized case. One meets two kinds of situations in the parameterized case: one situation where the algorithms can be adapted (generalized exponents) and one situation where the presence of the parameters imply problems of indecidability (polynomial solutions).
In the example of the planar three-body problem we are going to overcome this last difficulty thanks to strong constraints on the parameters: they are positive real.

3 The non-integrability of the planar three-body problem

3.1 Some directions for use of our criterion

We explain here the main steps to apply our criterion.

What we have at our disposal before applying the criterion.

1. The **Hamiltonian** with 3 degrees of freedom ([Tsy1]):

$$H=\frac{1}{2}\left(\frac{1}{m_1}+1\right)\left({p_1}^2+\frac{(p_3\,q_2-p_2\,q_3-c)^2}{{q_1}^2}\right)+\frac{1}{2}\left(\frac{1}{m_2}+1\right)\left({p_2}^2+{p_3}^2\right)+p_1\,p_2$$

$$-\frac{p_3\,(p_3\,q_2-p_2\,q_3-c)}{q_1}-\frac{m_2}{\sqrt{{q_2}^2+{q_3}^2}}-\frac{m_1}{q_1}-\frac{m_1\,m_2}{\sqrt{(q_1-q_2)^2+{q_3}^2}}.$$

2. A **particular solution of the hamiltonian system**, Lagrange' solution ([Tsy1]):

$$\mathrm{x}_0(t)={}^t\left(\ \frac{2}{3}\left(w^2-\frac{\sqrt{3}}{3}\,c\,w+\frac{c^2}{3}\right),\quad \frac{q_1}{2},\quad \frac{\sqrt{3}}{2}\,q_1,\quad \frac{w}{q_1},\quad -\frac{\sqrt{3}}{3\,q_1},\quad \frac{2\sqrt{3}}{3}\,p_1-\frac{c}{3\,q_1}\ \right)$$

where

$$w'(t)=\frac{27}{4}(3\,w(t)^2-\sqrt{3}\,c\,w(t)+c^2).$$

3. The **variational system along Lagrange' solution** (6×6 linear differential system)

$$Y'(t)=J\,\mathcal{H}(H,\mathrm{x}_0(t))\,Y(t)$$

transformed in another differential system after a change of variable:

$$Y'(x) = M(x)\, Y(x).$$

4. A **symplectic transformation** (see annex 1 page 19) which enables to reduce the variational system to a **normal variational system** (4×4 differential linear system, annex 2 page 21)

$$\nu'(x) = \begin{pmatrix} A_1(x) & A_2(x) \\ A_3(x) & -{}^t A_1(x) \end{pmatrix} \nu(x).$$

Remark 7 *A. Tsygvintsev gets a normal variational system using Whittaker' reductions ([Tsy1]).*

5. A cyclic vector which enables to transform the normal variational system to a **(scalar) normal variational equation** $L(y(x)) = 0$ (see [Sin2] for references to the cyclic vector method for transforming linear differential systems to linear differential equations).

How we apply our criterion to our normal variational equation.

The normal variational equation $L(y(x)) = 0$ has the particularity to depend on parameters (the masses of the bodies).

If we fix these parameters (for example $m_1 = m_2 = m_3 = 1$) then the equation $L(y(x)) = 0$ is an equation with coefficients in $\mathbb{Q}(x)$.
In this case (non parameterized case), there exists algorithms to compute the formal solutions at a point ([Sch], [Bar1], [Bar2], [D-C-T], [Hila], [Hoe1], [Ince],[Le], [Mal], [Som], [Tour], ...) and to factor the equation ([Sch], [Beke], [Hoe], [Hoe2], [Pflu], [Sin1],[B-P], ...).
So when the masses are equal, one can quickly prove that the equation is completely reducible and that there are logarithmic terms in some formal solutions. Thanks to our criterion we can conclude that the planar three-body problem (with equal masses) is not completely integrable (see [Bou1]).

Now let us assume that we don't give any particular numerical value to the parameters (except m_3 that we can assume to be equal to one).
The two questions one then deals with to apply our criterion are the following ones:

How to detect whereas there are logarithmic terms locally in the formal solutions of a *parameterized* linear differential equation ?

How to factor a *parameterized* linear differential operator ?

The main difficulty regarding these two last questions comes from the possible presence of arithmetic conditions on the parameters. In particular there appears indecidability problems during the computation of the polynomial solutions which we need to factorize operators. In [Bou2], the first author proposes detailed tools helping to answer these two last questions. Using these tools and taking the physical constraints into consideration (i.e. the parameters are posistive real), we are going to prove now the following proposition in the two situations: $m_1 = 1$ and $m_1 \neq 1$:

Proposition 3 *The differential operator L has at least one irreducible right factor M such that the equation $M(y) = 0$ has formal solutions with logarithmic terms.*

3.2 The case $m_1 = 1$

A scalar normal variational equation (obtained with the cyclic vector $(1, 0, 0, 0)$) is

$$L(y(x)) = a_4\, y^{(4)}(x) + a_3\, y^{(3)}(x) + a_2\, y^{(2)}(x) + a_1\, y'(x) + a_0\, y(x) = 0$$

with

$$a_0 = (36\, m_2 - 9)\, x^2\, (x-1)^2 - 9\, (m_2+2)\, (2\, m_2+5)\, x\, (x-1) - 6\, (m_2+2)^2$$

$$a_1 = 3\, (m_2+2)\, x\, (2\, x-1)\, (x-1)\, (3\, x\, (x-1) + 4 + 2\, m_2)$$

$$a_2 = -3\,(m_2+2)^2\,(2\,x\,(x-1)+1)\;x^2\,(x-1)^2$$
$$a_3 = (m_2+2)^2\,(2\,x-1)\,x^3\,(x-1)^3$$
$$a_4 = (m_2+2)^2\,x^4\,(x-1)^4\,.$$

Let us first prove that there are formal solutions at zero with logarithmic terms.
Let $\sum c_k x^{k+\rho}$ be a series solution of the equation $L(y)=0$.
The coefficients c_k satisfy the recurrence relation

$$f_0(k+\rho)\,c_k + f_1(k+\rho-1)\,c_{k-1} + f_2(k+\rho-2)\,c_{k-2} + f_3(k+\rho-3)\,c_{k-3} + f_4(k+\rho-4)\,c_{k-4} = 0$$

where

$$f_0(k) = (m_2+2)^2\,(k-1)\,(k-2)\,(k-3)\,(k+1).$$

There are four exponents at zero, 3, 2, 1 and -1. They do not depend on the paremeters and they differ each other from an integer. The formal solution associated to the exponent 3 has no logarithmic term. The formal solution associated to the exponent 2 has no logarithmic term if and only if $f_1(2)=0$ (see [Ince] page 405 for example). But

$$f_1(k) = -(m_2+2)(k-1)\left(8\,k^3 - 30\,k^2 + 4\,k + 45 + 4\,m_2\,k^3 - 15\,m_2\,k^2 + 2\,km_2 + 18\,m_2\right)$$

so

$$f_1(2) = 3\,(m_2+2)\,(2m_2+1).$$

As the parameter m_2 is positive, $f_1(2)$ never cancels and there always exists a formal solution at zero with a logarithmic term. The only presence of logarithmic terms does not suffice to contradict the abelianity of the group G^0. One needs to study the irreducibility of G so we are going to look for the factors of L.
The exponents at the point 0 are 3, 2, 1 and -1. Those at the point 1 are also 3, 2, 1 and -1. The exponents at ∞ are the solutions of

$$((m_2+2)\,n\,(n+1)\;-9)\,((m_2+2)\,n\,(n+3)+1-4m_2) = 0.$$

We notice that the exponents at ∞ depend on the parameter m_2 whereas two sums of two exponents at infinity are integers (independant of m_2). This last property was foreseeable: it comes from the symplectic structure of the equation (see [BW01]). So it is easier to look for factors of degree two than to look for factors of degree one. We adapt here the algorithm of [B-P]. The second exterior equation associated to L (see chapter 2 and 4 of [PS01]) has got two exponential solutions,

$$z_1 = 1 + \frac{m_2+2}{12}\,\frac{1}{(x-1)}$$

and

$$z_2 = x(x-1)$$

(these solutions are in fact rational).
One can construct a differential operator of degree two associated to the rational solution $\lambda\,z_1 + \mu\,z_2$, which divides L if and only if λ and μ satisfy the Pl"ucker relation

$$\mu\,(3(m_2+2)\mu + (2m_2^2 - 10m_2 - 1)\,\lambda) = 0.$$

We denote M_1 the factor associated to $(\lambda,\mu)=(1,0)$ and M_2 the factor associated to $(\lambda,\mu)=(\frac{-3(m_2+2)}{2\,m_2^2-10\,m_2-1},1)$ when $2\,m_2^2-10\,m_2-1$ is non zero.
The operator L is not irreducible, so one cannot conclude immediately.
One has now to see whether one of the equations $M_1(y)=0$ or $M_2(y)=0$ is completely reducible and if there are logarithmic terms in its formal solutions.
Although it is difficult to answer this question for these operators separately, we will show the following lemma - this approach should be usable in other situations as well:

Lemma 1 *The operators M_1 and M_2 cannot be simultaneously reducible.*

Proof
Let us first study the formal solutions at 0 of the equations $M_1(y)=0$ and $M_2(y)=0$.
The exponents at 0 and 1 for $M_1(y)=0$ and $M_2(y)=0$ are equal to -1 and 1.
For each of this factor there is a formal solution at 0 which contains a logarithmic term. It can be easily proved using the criterion of [Ince] page 405.
Here, as the exponents do not depend on the parameters, one can use the program formalsol of maple and compute a basis of formal solutions at 0 for $M_1(y)=0$ and for $M_2(y)=0$.
We get the following basis of solutions at 0 for $M_1(y)=0$:

$$(\sigma_1(x),\ \frac{9(2m_2+1)}{4(m_2+2)^2}\,\ln(x)\,\sigma_1(x)+\sigma_2(x))$$

where

$$\begin{cases}\sigma_1(x)=x+\frac{(m_2-7)x^2}{m_2+2}+\cdots\\[2ex] \sigma_2(x)=-\frac{1}{2}\frac{1}{x}+\frac{1}{2}\frac{m_2-1}{m_2+2}+\frac{1}{16}\frac{\left(-142\,m_2+413+8\,m_2{}^2\right)x}{(m_2+2)^2}+\frac{1}{16}\frac{\left(8\,m_2{}^3-294\,m_2{}^2+2787\,m_2-2339\right)x^2}{(m_2+2)^3}+\cdots\end{cases}$$

A basis of formal solutions at 0 for $M_2(y)=0$ is:

$$(\sigma_1(x),\ \frac{9(2m_2+1)}{4(m_2+2)^2}\,\ln(x)\,\sigma_1(x)+\sigma_2(x))$$

where

$$\begin{cases}\sigma_1(x)=x+\frac{(m_2-7)x^2}{m_2+2}+\cdots\\[2ex] \sigma_2(x)=-\frac{1}{2}\frac{1}{x}+\frac{1}{2}\frac{m_2-1}{m_2+2}+\frac{1}{16}\frac{\left(40\,m_2{}^2-302\,m_2+397\right)x}{(m_2+2)^2}+\frac{1}{16}\frac{\left(40\,m_2{}^3-294\,m_2{}^2+1971\,m_2-2419\right)x^2}{(m_2+2)^3}+\cdots\end{cases}$$

As m_2 is positive, the coefficient $\frac{9(2m_2+1)}{4(m_2+2)^2}$ never cancels, which ensures the presence of a logarithmic term.
Let us now see whether the operators M_1 and M_2 are irreducible or not.
The exponents at infinity of $M_1(y)=0$ are the solutions of

$$P_1(n)=(m_2+2)\,(n^2+n)-9=0.$$

Those of M_2 are the solutions of

$$P_2(n) = (m_2 + 2)\,(n^2 + 3\,n) + 1 - 4\,m_2 = 0.$$

To test whether the equation $M_1(y) = 0$ (resp. $M_2(y) = 0$) is reducible, it suffices to look for its exponential solutions. These solutions can be written in the following form:

$$\frac{p(x)}{x\,(x-1)}$$

where $p(x)$ is a polynomial whose degree satisfies $P_1(2-d) = 0$ (resp. $P_2(2-d) = 0$). A necessary condition for reducibility of M_1 (resp. M_2) is that the equation $P_1(n) = 0$ (resp. $P_2(n) = 0$) has an integer solution less than or equal to 2.
We prove that M_1 is irreducible when $2\,m_2^2 - 10\,m_2 - 1$ is equal to 0 and that M_1 and M_2 cannot both be reducible when $2m_2^2 - 10m_2 - 1$ is nonzero.
If $2\,m_2^2 - 10\,m_2 - 1$ is equal to 0, that is to say if m_2 is equal to $\frac{5+3\sqrt{3}}{2}$, then the solutions of $P_1(n) = 0$ are $-\sqrt{3}$ and $\sqrt{3} - 1$.
If $2\,m_2^2 - 10\,m_2 - 1$ is non zero then
$P_1(n_1) = P_2(n_2) = 0,\ n_1 \in \mathbb{Z},\ n_2 \in \mathbb{Z}$
$\Rightarrow n_1^2 + n_1 - \frac{9}{m_2+2} = n_2^2 + 3n_2 + \frac{1-4m_2}{m_2+2} = 0,\ n_1 \in \mathbb{Z},\ n_2 \in \mathbb{Z}$
$\Rightarrow n_1^2 + n_2^2 + n_1 + 3\,n_2 - 4 = 0,\ n_1 \in \mathbb{Z},\ n_2 \in \mathbb{Z}$
$\Rightarrow (2\,n_1 + 1)^2 + (2\,n_2 + 3)^2 = 26,\ n_1 \in \mathbb{Z},\ n_2 \in \mathbb{Z}$
$\Rightarrow (n_1, n_2) \in \{(0,1), (0,-4), (-1,1), (-1,-4), (-2,-1), (-2,-2), (-3,-1), (-3,-2)\}$.
But $P_1(0)$ and $P_1(-1)$ cannot cancel.
$P_2(-1) = 0 \Rightarrow m_2 = \frac{-1}{2}$
$P_2(-2) = 0 \Rightarrow m_2 = \frac{-1}{2}$
so, M_1 and M_2 cannot be simultaneously reducible. $\square$ To conclude, the operator L has always at least one factor M ($M = M_1$ or $M = M_2$) satisfying the two properties:
$M(y) = 0$ has formal solutions at 0 with logarithmic terms;
M is irreducible.
So the proposition 3 is fulfilled.

3.3 The case $m_1 \neq 1$

A scalar normal variational equation (obtained with the cyclic vector $(-r\sqrt{3}, 1, 0, 0)$) is

$$L(y(x)) = a_4\,y^{(4)}(x) + a_3\,y^{(3)}(x) + a_2\,y^{(2)}(x) + a_1\,y'(x) + a_0\,y(x) = 0$$

with

$$\begin{gathered} a_0 = 2\,(-m_1 + 3\,m_1\,r + 3\,r + 1)\,(m_1 - 1 - 6\,r^2 + 6\,m_1\,r^2 + 3\,m_1\,r + 3\,r)\,x^2\,(x-1)^2 - \\ 6\,(6\,m_1\,r + 3\,m_1\,r^2 - m_1 - 3\,r^2 + 6\,r + 1)\,(9\,m_1\,r^2 + 21\,m_1\,r - 4\,m_1 - 9\,r^2 + 21\,r + 4) \\ x\,(x-1) - 6\,(6\,m_1\,r + 3\,m_1\,r^2 - m_1 - 3\,r^2 + 6\,r + 1)^2 \end{gathered}$$

$$\begin{gathered} a_1 = 6\,(6\,m_1\,r + 3\,m_1\,r^2 - m_1 - 3\,r^2 + 6\,r + 1) \\ (2\,x - 1)\,x\,(x-1) \\ ((-m_1 + 3\,m_1\,r + 3\,r + 1)\,x\,(x-1) + 6\,m_1\,r + 3\,m_1\,r^2 - m_1 - 3\,r^2 + 6\,r + 1) \\ -3\,(6\,m_1\,r + 3\,m_1\,r^2 - m_1 - 3\,r^2 + 6\,r + 1)^2\,(1 + 2\,x^2 - 2\,x)\,x^2\,(x-1)^2 \end{gathered}$$

$$a_2 = -3\left(6\,m_1\,r + 3\,m_1\,r^2 - m_1 - 3\,r^2 + 6\,r + 1\right)^2 \left(1 + 2\,x^2 - 2\,x\right)\,x^2\,(x-1)^2$$

$$a_3 = \left(6\,m_1\,r + 3\,m_1\,r^2 - m_1 - 3\,r^2 + 6\,r + 1\right)^2 (2\,x - 1)\,x^3\,(x-1)^3$$

$$a_4 = \left(6\,m_1\,r + 3\,m_1\,r^2 - m_1 - 3\,r^2 + 6\,r + 1\right)^2 x^4\,(x-1)^4$$

The parameter r is defined by the following relation:

$$3(m_1 - 1)r^2 + 2(m_1 + 1 - 2m_2)r + 1 - m_1 = 0 \tag{1}$$

The exponents at the points 0 and 1 are again 3, 2, 1 and -1 and one can easily prove that the formal solution associated to the exponent 2 contains a logarithmic term.
The exponents at ∞ are the solutions of

$$(2(1+m_1+m_2)(n^2+n)-9(1+m_1)+3\frac{m_1-1}{r})(2(1+m_1+m_2)(n^2+3n)+m_1+1-8m_2-3\frac{m_1-1}{r}) = 0.$$

They depend on the parmeters but two of their sums two by two are integers.
The second exterior equation has two exponential solutions

$$z_1 = 1 + \frac{1}{8}\,\frac{6\,m_1\,r + 3\,m_1\,r^2 - m_1 - 3\,r^2 + 6\,r + 1}{x(x-1)(3\,m_1\,r + 3\,r + 1 - m_1)}$$

and

$$z_2 = x(x-1)$$

which are in fact rational.
The rational solution $\lambda\,z_1 + \mu\,z_2$ satisfies Pl"ucker relation if, and only if, $\mu\,(2\,f(m_1,m_2)\,(4\,m_2\,r - 2\,r - 2\,m_1\,r + m_1 - 1)\lambda + 3\,(1+m_2+m_1)(4\,r m_2\,m_1 + 4\,m_2\,r + {m_1}^2 - 3\,{m_1}^2 r - 2\,m_1\,r - 1 - 3\,r)\mu = 0$
where

$$f(m_1,m_2) = 2\,{m_2}^2 + 2\,{m_1}^2 - 5\,m_2 - 5\,m_2\,m_1 - 5\,m_1 + 2.$$

We denote M_1 the factor associated to $(\lambda,\mu) = (1,0)$.
If $f(m_1,m_2)$ is non zero, we denote M_2 the factor associated to
$(\lambda,\mu) = (1, -\frac{2\,f(m_1,m_2)\,(4\,m_2\,r - 2\,r - 2\,m_1\,r + m_1 - 1)}{3\,(1+m_2+m_1)(4\,r m_2\,m_1 + 4\,m_2\,r + {m_1}^2 - 3\,{m_1}^2 r - 2\,m_1\,r - 1 - 3\,r)}) \neq (1,0)$.
These two factors are given in the annex 3 page 22.
As the exponents at 0 and at 1 are equal to -1 and 1, the exponential solutions of $M_1(y) = 0$ (resp. $M_2(y) = 0$) are of the type $\frac{p(x)}{x(x-1)}$ and the degree d of the polynomial $p(x)$ satisfies $P_1(2-d) = 0$ (resp. $P_2(2-d) = 0$) where

$$P_1(n) = 2(1 + m_1 + m_2)(n^2 + n) - 9(1 + m_1) + 3\frac{m_1 - 1}{r},$$

$$P_2(n) = 2(1 + m_1 + m_2)(n^2 + 3n) + m_1 + 1 - 8m_2 - 3\frac{m_1 - 1}{r}.$$

We prove that if $f(m_1,m_2)$ is equal to 0, then M_1 is irreducible and if $f(m_1,m_2)$ is non zero then M_1 and M_2 cannot be simultaneously reducible.
If $f(m_1,m_2)$ is equal to zero,

$$P_1(n) = 0 \Rightarrow (m_1 - 1)^2(m_1^2 + m_1 + 1)^2(n^4 + 2n^3 - 5n^2 - 6n + 6) = 0.$$

This result is obtained thanks to the package *Groebner* of maple.
So $P_1(n)$ cannot have any integer solution and M_1 is irreducible.
Let us assume that $f(m_1, m_2)$ is non zero and that M_1 and M_2 are both reducible. Then there exists two integers n_1 and n_2 such that $P_1(n_1) = P_2(n_2) = 0$.
But
$P_1(n_1) = P_2(n_2) = 0,\ n_1 \in \mathbb{Z},\ n_2 \in \mathbb{Z}$
$\Rightarrow n_1^2 + n_2^2 + n_1 + 3\,n_2 - 4 = 0,\ n_1 \in \mathbb{Z},\ n_2 \in \mathbb{Z}$
$\Rightarrow (n_1, n_2) \in \{(0,1),\ (0,-4),\ (-1,1),\ (-1,-4),\ (-2,-1),\ (-2,-2),\ (-3,-1),\ (-3,-2)\}$.
Furthermore
$P_1(-1) = P_1(0)$ and $P_2(-1) = P_2(-2)$
$P_1(-1) = 0 \Rightarrow r = \frac{m_1-1}{3(m_1+1)} \Rightarrow m_2 = \frac{-m_1}{m_1+1}$
$P_2(-1) = 0 \Rightarrow r = -\frac{m_1+1}{m_1-1} \Rightarrow m_2 = \frac{-m_1}{m_1+1}$.
As m_1 and m_2 are both positive, the operators M_1 and M_2 cannot be simultaneously reducible.

To conclude, like in the section 3.2, the proposition 3 is again satisfied.

The non-integrability criterion 1 and the proposition 3 enable us to conclude that the planar three-body problem along Lagrange' solution is not completely integrable.

Theorem 2 *The planar three-body problem is not meromorphically completely integrable.*

4 Conclusion

In this paper we have established the non complete (meromorphic) integrability of the planar three-body problem along Lagrange' solution. This result has also been found by A. Tsygvintsev ([Tsy1], [Tsy2]).
Our proof relies on many points:

- Morales and Ramis' theorem ([M-R]) which gives a theoretical galoisian criterion of non complete integrability for the hamiltonian systems;
- a new (practical) criterion deduced from this theorem and from ingredients coming from differential Galois theory;
- algorithms to compute formal solutions ([Sch], [Bar1], [Bar2], [D-C-T], [Hila], [Hoe1], [Ince],[Le], [Mal], [Som], [Tour], ...) and to factorize linear differential operators ([Sch], [Beke], [Hoe], [Hoe2], [Pflu], [Sin1],[B-P], ...);
- the (partial) study of the adaptation of these algorithms to a parameterized situation, with the effort to separate the automatisable parts from the non automatisable parts ([Bou2]);
- strong constraints on the parameters coming from the physical structure of the problem.

One can hope that all these tools can be used to treat many other problems on hamiltonian systems.

ANNEX 1

Symplectic transformation of the system

The variational system along Lagrange' solution is

$$Y'(t) = J\,\mathcal{H}(H, \mathrm{x}_0(t))\,Y(t).$$

The vector $\mathrm{x}_0'(t)$ is a particular solution of this system.
After the changes of variables

$$u = w(t) \text{ and } u = \frac{-c\,m_1\,(m_2+2)}{2\,(m_1+m_2+m_1\,m_2)}\,x + \frac{c\sqrt{3}\,m_1\,m_2}{2\,(m_1+m_2+m_1\,m_2)},$$

we get a 6×6 variational system

$$Y'(x) = M(x)\,Y(x)$$

which has the following particular solution:

$$W_0 = \begin{pmatrix} 2\,\frac{x(m_1\,m_2+m_2+m_1)}{(1+x^2)c} \\ \frac{x(m_1\,m_2+m_2+m_1)}{(1+x^2)c} \\ \frac{\sqrt{3}x(m_1\,m_2+m_2+m_1)}{(1+x^2)c} \\ -2\,\frac{\left(-m_2-2+2\,x\sqrt{3}m_2+x^2m_2+2\,x^2\right)m_1\,(m_1\,m_2+m_2+m_1)^4}{(m_1+m_2+1)^2(1+x^2)^3c^4} \\ 2\,\frac{\left(-m_1+1+2\,m_1\,\sqrt{3}x+2\,\sqrt{3}x+m_1\,x^2-x^2\right)m_2\,(m_1\,m_2+m_2+m_1)^4}{(m_1+m_2+1)^2(1+x^2)^3c^4} \\ -\frac{2}{3}\,\frac{\sqrt{3}\left(-3\,m_1-3+2\,\sqrt{3}x-2\,m_1\,\sqrt{3}x+3\,m_1\,x^2+3\,x^2\right)m_2\,(m_1\,m_2+m_2+m_1)^4}{(m_1+m_2+1)^2(1+x^2)^3c^4} \end{pmatrix}$$

. The matrix $M(x)$ is symplectic:

$$M(x) = \begin{pmatrix} M_1(x) & M_2(x) \\ M_3(x) & -{}^tM_1(x) \end{pmatrix}$$

where $M_2 =^t M_2$ and $M_3 =^t M_3$.
Let us consider the following symplectic matrix P (it satisfies ${}^tP\,J\,P = J$):

$$P(x) = \begin{pmatrix} -W_0[1] & 0 & 0 & 0 & 0 & 0 \\ -W_0[2] & -1 & 0 & 0 & 0 & 0 \\ -W_0[3] & 0 & -1 & 0 & 0 & 0 \\ -W_0[4] & -\frac{W_0[5]}{W_0[1]} & -\frac{W_0[6]}{W_0[1]} & -\frac{1}{W_0[1]} & \frac{W_0[2]}{W_0[1]} & \frac{W_0[3]}{W_0[1]} \\ -W_0[5] & 0 & 0 & 0 & -1 & 0 \\ -W_0[6] & 0 & 0 & 0 & 0 & -1 \end{pmatrix}.$$

The system satisfied by Z defined by $Y = P\,Z$ is

$$Z'(x) = \tilde{M}(x)\,Z(x)$$

where $\tilde{M}(x) = P(x)^{-1}\,(M(x)\,P(x) - P'(x))$.
One can easily prove that the matrix $\tilde{M}$ is also symplectic ([M-R]). Furthermore as $-W_0$ is a solution of the equation $Y'(x) = M(x)\,Y(x)$, the first row of the matrix $\tilde{M}(x)$ is equal to zero. So $\tilde{M}(x)$ can be written in the following form:

$$\tilde{M}(x) = \begin{pmatrix} 0 & \times\ \ \times & \times & \times\ \ \times \\ 0 & \multirow{2}{*}{$A_1(x)$} & \times & \multirow{2}{*}{$A_2(x)$} \\ 0 & & \times & \\ 0 & 0\ \ 0 & 0 & 0\ \ 0 \\ 0 & \multirow{2}{*}{$A_3(x)$} & \times & \multirow{2}{*}{$-{}^tA_1(x)$} \\ 0 & & \times & \end{pmatrix}$$

and the system $Z'(x) = \tilde{M}(x)\,Z(x)$ is equivalent to the system

$$Z'(x) = \begin{pmatrix} \multicolumn{2}{c}{\multirow{2}{*}{$A_1(x)$}} & \multicolumn{2}{c}{\multirow{2}{*}{$A_2(x)$}} & p_1 & 0 \\ & & & & p_2 & 0 \\ \multicolumn{2}{c}{\multirow{2}{*}{$A_3(x)$}} & \multicolumn{2}{c}{\multirow{2}{*}{$-{}^tA_1(x)$}} & p_3 & 0 \\ & & & & p_4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ p_5 & p_6 & p_7 & p_8 & p_9 & p_{10} \end{pmatrix} Z(x).$$

One can extract from this variational equation the normal variational equation

$$\nu'(x) = \begin{pmatrix} A_1(x) & A_2(x) \\ A_3(x) & -{}^tA_1(x) \end{pmatrix} \nu(x).$$

ANNEX 2

Normal variational system along Lagrange' solution.

$$\nu'(x) = \begin{pmatrix} \boxed{\begin{matrix} s_1 & s_2 \\ s_3 & s_4 \end{matrix}} & \boxed{\begin{matrix} s_5 & 0 \\ 0 & s_5 \end{matrix}} \\ & \\ \boxed{\begin{matrix} s_6 & s_7 \\ s_7 & s_8 \end{matrix}} & \boxed{\begin{matrix} -s_1 & -s_3 \\ -s_2 & -s_4 \end{matrix}} \end{pmatrix} \nu(x)$$

with

- $s_1 = \frac{m_2\left(\left(-5\,{m_1}^2-5-14\,m_1\right)x^2+4\,\sqrt{3}(m_1-1)(m_1+1)x-(m_1-1)^2\right)}{4\,(m_1+m_2+1)\,m_1\,x\,(x^2+1)}$

- $s_2 = \frac{-3\,\sqrt{3}(m_1-1)(m_1+1)m_2\,x^2-4\,(m_1+1)(m_1\,m_2-2\,m_1+m_2)x+\sqrt{3}(m_1-1)(m_1+1)m_2}{4\,(m_1+m_2+1)\,m_1\,x\,(x^2+1)}$

- $s_3 = \frac{-3\,\sqrt{3}(m_1-1)(m_1+1)m_2\,x^2+\left(-4\,{m_1}^2m_2-8\,m_1-4\,m_2-24\,m_1\,m_2-8\,{m_1}^2\right)x+\sqrt{3}(m_1-1)(m_1+1)m_2}{4\,(m_1+m_2+1)\,m_1\,x\,(x^2+1)}$

- $s_4 = \frac{m_2\left(\left({m_1}^2+1+10\,m_1\right)x^2-4\,\sqrt{3}(m_1-1)(m_1+1)\,x-3\,(m_1+1)^2\right)}{4\,(m_1+m_2+1)\,m_1\,x\,(x^2+1)}$

- $s_5 = \frac{c^3\,(m_1+m_2+1)\,(x^2+1)}{2\,m_1\,m_2\,(m_1+m_2+m_1\,m_2)^2}$

- $s_6 = \frac{(m_1+1)m_2\,(m_1\,m_2+m_2+m_1)^3\left(\left(-13\,{m_1}^2m_2-2\,{m_1}^2-24\,m_1\,m_2-2\,m_1-13\,m_2\right)x^2+4\,\sqrt{3}(m_1-1)(m_1+1)m_2\,x-m_2\,(m_1-1)^2\right)}{2\,m_1\,(1+x^2)^2x^2c^3(m_1+m_2+1)^3}$

- $s_7 =$
$\frac{m_2\,(m_1\,m_2+m_2+m_1)^3\,\left(-3\,\sqrt{3}\left({m_1}^2m_2+2\,{m_1}^2+4\,m_1\,m_2+2\,m_1+m_2\right)(m_1-1)x^2-4\,m_2\,(m_1+1)\left({m_1}^2+4\,m_1+1\right)x+\sqrt{3}(m_1-1)m_2\,(m_1+1)^2\right)}{2\,m_1\,(1+x^2)^2x^2c^3(m_1+m_2+1)^3}$

- $s_8 =$
$\frac{m_2\,(m_1+1)(m_1\,m_2+m_2+m_1)^3\left(\left(-7\,{m_1}^2m_2+10\,{m_1}^2+12\,m_1\,m_2+10\,m_1-7\,m_2\right)x^2-4\,\sqrt{3}(m_1-1)(m_1+1)m_2\,x-3\,m_2\,(m_1+1)^2\right)}{2\,m_1\,(1+x^2)^2x^2c^3(m_1+m_2+1)^3}$

ANNEX 3

Factors of the operator L $(\partial = \frac{d}{dx})$

- First case: $m_1 = 1$.

$$M_1 = \tilde{b_2}\,\partial^2 + \tilde{b}_1\,\partial + \tilde{b}_0$$

$$\begin{cases} \tilde{b}_0 = -108\,x^2(x-1)^2 + (-27\,m_2 - 54)x(x-1) - (2+m_2)^2, \\ \tilde{b}_1 = x(2+m_2)^2(2\,x-1)(x-1), \\ \tilde{b}_2 = x^2(2+m_2)(x-1)^2(12\,x^2 - 12\,x + 2 + m_2) \end{cases}$$

$$M_2 = \tilde{c}_2\,\partial^2 + \tilde{c}_1\,\partial + \tilde{c}_0$$

$$\begin{cases} \tilde{c}_0 = -4\,\left(4\,m_2 - 1\right)\left(2\,{m_2}^2 - 10\,m_2 - 1\right)x^3\left(x-1\right)^3 - 24\,\left(2+m_2\right)\left({m_2}^2 - 5\,m_2 - 5\right)x^2\left(x-1\right)^2 \\ +27\,\left(2+m_2\right)^2 x\left(x-1\right) + \left(2+m_2\right)^3, \\ \\ \tilde{c}_1 = x\left(x-1\right)\left(2\,x-1\right)\left(-4\,\left(2+m_2\right)\left(2\,{m_2}^2 - 10\,m_2 - 1\right)x^2\left(x-1\right)^2 - \left(2+m_2\right)^3\right), \\ \\ \tilde{c}_2 = x^2\left(x-1\right)^2 \\ \left(4\,\left(2+m_2\right)\left(2\,{m_2}^2 - 10\,m_2 - 1\right)x^2\left(x-1\right)^2 - 12\,\left(2+m_2\right)^2 x\left(x-1\right) - \left(2+m_2\right)^3\right) \end{cases}$$

- Second case : $m_1 \neq 1$.

$$M_1 = b_2\,\partial^2 + b_1\,\partial + b_0$$

$$\left\{\begin{array}{l}
b_0 = -48\ (3\,m_1\,r - m_1 + 1 + 3\,r)^2\,x^2\,(x-1)^2 \\
-18\ (6\,m_1\,r + 3\,m_1\,r^2 - m_1 - 3\,r^2 + 6\,r + 1)\,(3\,m_1\,r - m_1 + 1 + 3\,r)\,x\,(x-1) - \\
(6\,m_1\,r + 3\,m_1\,r^2 - m_1 - 3\,r^2 + 6\,r + 1)^2 \\
\\
b_1 = x\,(6\,m_1\,r + 3\,m_1\,r^2 - m_1 - 3\,r^2 + 6\,r + 1)^2\,(2\,x-1)\,(x-1) \\
\\
b_2 = x^2\,(x-1)^2\,(6\,m_1\,r + 3\,m_1\,r^2 - m_1 - 3\,r^2 + 6\,r + 1) \\
(8\ (3\,m_1\,r - m_1 + 1 + 3\,r)\,x\,(x-1) + (6\,m_1\,r + 3\,m_1\,r^2 - m_1 - 3\,r^2 + 6\,r + 1))
\end{array}\right.$$

$$M_2 = c_2\,\partial^2 + c_1\,\partial + c_0$$

$$\left\{\begin{array}{l}
c_0 = -16\,(6\,m_1\,r^2 + 3\,m_1\,r + m_1 - 1 + 3\,r - 6\,r^2) \\
(9\,r^4{m_1}^2 - 18\,{m_1}^2r^3 - 6\,{m_1}^2r^2 + 6\,{m_1}^2r + {m_1}^2 - 18\,m_1\,r^4 - 60\,m_1\,r^2 - 2\,m_1 + \\
9\,r^4 + 18\,r^3 - 6\,r^2 - 6\,r + 1)x^3\,(x-1)^3 - \\
24\ (6\,m_1\,r + 3\,m_1\,r^2 - m_1 - 3\,r^2 + 6\,r + 1) \\
(-18\,{m_1}^2r^3 - 18\,r + 9\,r^4 - {m_1}^2 + 2\,m_1 + 18\,r^3 - 24\,{m_1}^2r^2 - 18\,m_1\,r^4 - 1 + \\
18\,{m_1}^2r - 96\,m_1\,r^2 + 9\,r^4{m_1}^2 - 24\,r^2)x^2\,(x-1)^2 + \\
18\ (6\,m_1\,r + 3\,m_1\,r^2 - m_1 - 3\,r^2 + 6\,r + 1)^2 \\
(3\,m_1\,r - m_1 + 3\,r + 1)\,x\,(x-1) + \\
(6\,m_1\,r + 3\,m_1\,r^2 - m_1 - 3\,r^2 + 6\,r + 1)^3, \\
\\
c_1 = x\,(x-1)\,(2\,x-1)\,(-8\ (6\,m_1\,r + 3\,m_1\,r^2 - m_1 - 3\,r^2 + 6\,r + 1) \\
(9\,r^4{m_1}^2 - 18\,{m_1}^2r^3 - 6\,{m_1}^2r^2 + 6\,{m_1}^2r + {m_1}^2 - 18\,m_1\,r^4 - 60\,m_1\,r^2 - 2\,m_1 + \\
9\,r^4 + 18\,r^3 - 6\,r^2 - 6\,r + 1)x^2\,(x-1)^2 - \\
(6\,m_1\,r + 3\,m_1\,r^2 - m_1 - 3\,r^2 + 6\,r + 1)^3), \\
\\
c_2 = x^2\,(x-1)^2\,(8\ (6\,m_1\,r + 3\,m_1\,r^2 - m_1 - 3\,r^2 + 6\,r + 1) \\
(9\,r^4{m_1}^2 - 18\,{m_1}^2r^3 - 6\,{m_1}^2r^2 + 6\,{m_1}^2r + {m_1}^2 - 18\,m_1\,r^4 - 60\,m_1\,r^2 - 2\,m_1 + \\
9\,r^4 + 18\,r^3 - 6\,r^2 - 6\,r + 1)x^2\,(x-1)^2 - \\
8\ (6\,m_1\,r + 3\,m_1\,r^2 - m_1 - 3\,r^2 + 6\,r + 1)^2\,(3\,m_1\,r - m_1 + 3\,r + 1)\,x\,(x-1) - \\
(6\,m_1\,r + 3\,m_1\,r^2 - m_1 - 3\,r^2 + 6\,r + 1)^3)
\end{array}\right.$$

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