Lectures on optimization
Basic camp

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Definition of an optimization problem

◇ An optimization problem has the form

$$\text{Find } \bar{x} \in \mathbb{R}^n \text{ such that } \quad f(\bar{x}) = \min \{ f(x) \mid x \in \mathcal{K} \}, \qquad (P)$$

where $\mathcal{K} \subseteq \mathbb{R}^n$ is a given set. By definition, this mean to find $\bar{x} \in \mathcal{K}$ such that

$$f(\bar{x}) \leq f(x) \ \forall \ x \in \mathcal{K}.$$

◇ In the above, f is called an objective function, $\mathcal{K}$ is called a feasible set (or constraint set) and any $\bar{x}$ solving (P) is called a global solution to problem (P).

◇ Usually one also considers the weaker notion, but easier to characterize, of local solution to problem (P). Namely, $\bar{x} \in \mathcal{K}$ is a local solution to (P) if there exists $\delta > 0$ such that $f(\bar{x}) \leq f(x)$ for all $x \in \mathcal{K} \cap B(\bar{x}, \delta)$, where

$$B(\bar{x}, \delta) := \{ x \in \mathbb{R}^n \mid \|x - \bar{x}\| \leq \delta \}.$$

◇ In optimization theory one usually studies the following features of problem (P):

1.- Does there exist a solution $\bar{x}$ (global or local)?

2.- Optimality conditions, i.e. properties satisfied by the solutions (or local solutions).

3.- Computation algorithms for finding approximate solutions.

◇ In this course we will mainly focus on points 1 and 2 of the previous program.

◇ We will also consider mainly two cases for the feasible set $\mathcal{K}$:

◇ $\mathcal{K} = \mathbb{R}^n$ (unconstrained case).

◇ Equality and inequality constraints:

$$\mathcal{K} = \{x \in \mathbb{R}^n \mid g_i(x) = 0, \; i = 1, \dots, m, \; h_j(x) \leq 0, \; j = 1, \dots, \ell\}. \quad (1)$$

◇ In order to tackle point 2 we will assume that f is a smooth function. If the feasible set (1) is considered, we will also assume that g_i and h_j are smooth functions.

Some mathematical tools

◇ [Infimum] Let $A \subseteq \mathbb{R}$. We say that $m \in \mathbb{R}$ is a lower bound of A if $m \leq a$ for all $a \in A$. If m_* is a lower bound of A such that $m_* \geq m$ for every lower bound m of A, then m_* is called the infimum of A and it is denoted by $m_* = \inf A$. If $m_* \in A$, then we say that m_* is the mimimum of A, which is denoted $m_* = \min A$. If no lower bound for A exists, then we set $\inf A := -\infty$. Another convention is that if $A = \emptyset$ then $\inf A = +\infty$.

Example: Suppose that $A = \{1/n \mid n \geq 1\}$. Then, any $m \in]-\infty, 0]$ is a lower bound of A, $\inf A = 0$ and no minima exist.

Lemma 1. *If $\inf A$ is finite or $\inf A = -\infty$, then there exists a sequence $(a_n)_{n\in\mathbb{N}}$ of elements in A such that $a_n \to \inf A$ as $n \to \infty$.*

Proof. Exercise. □

Let $f : \mathbb{R}^n \to \mathbb{R}$ and $\mathcal{K}$ be given. Then, we define

$$\inf_{x\in\mathcal{K}} f(x) := \inf \underbrace{\{f(x) \mid x \in \mathcal{K}\}}_{A}$$

◇ [Supremum] Let $A \subseteq \mathbb{R}$. We say that $M \in \mathbb{R}$ is an upper bound of A if $M \geq a$ for all $a \in A$. If M^* is an upper bound of A such that $M^* \leq M$ for every upper bound M of A, then M^* is called the supremum of A and it is denoted by $M^* = \sup A$. If $M^* \in A$, then we say that M^* is the maximum of A, which is denoted $M^* = \max A$. If no upper bound for A exists, then we set $\sup A := +\infty$. Another convention is that if $A = \emptyset$ then $\sup A = -\infty$.

Example: Suppose that $A = \{-1/n \mid n \geq 1\}$. Then, any $M \in [0, +\infty[$ is an upper bound of A, $\sup A = 0$ and no maxima exist.

Let $f : \mathbb{R}^n \to \mathbb{R}$ and $\mathcal{K}$ be given. Then, we define

$$\sup_{x\in\mathcal{K}} f(x) := \sup \underbrace{\{f(x) \mid x \in \mathcal{K}\}}_{A}$$

◇ [Graph of a function] Let $f : \mathbb{R}^n \to \mathbb{R}$. The graph $\mathsf{Gr}(f) \subseteq \mathbb{R}^{n+1}$ is defined by

$$\mathsf{Gr}(f) := \{(x, f(x)) \mid x \in \mathbb{R}^n\}.$$

◇ [Level sets] Let $c \in \mathbb{R}$. The level set of value c is defined by

$$\mathsf{Lev}_f(c) := \{x \in \mathbb{R}^n \mid f(x) = c\}.$$

- When $n = 2$, the sets $\mathsf{Lev}_f(c)$ are useful in order to draw the graph of a function.
- These sets will also be useful in order to solve graphically two dimensional linear programming problems, i.e. $n = 2$, and the function f and the set $\mathcal{K}$ are defined by means of affine functions.

Example 1: We consider the function

$$\mathbb{R}^2 \ni (x, y) \mapsto f(x, y) := x + y + 2 \in \mathbb{R},$$

whose level set is given, for all $c \in \mathbb{R}$, by

$$\mathrm{Lev}_f(c) := \left\{(x, y) \in \mathbb{R}^2 \mid x + y + 2 = c\right\}.$$

Note that the optimization problem with this f and $\mathcal{K} = \mathbb{R}^2$ does not have a solution.

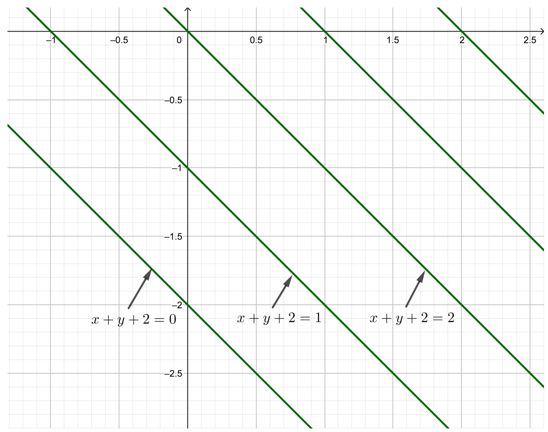
Example 2: Consider the function

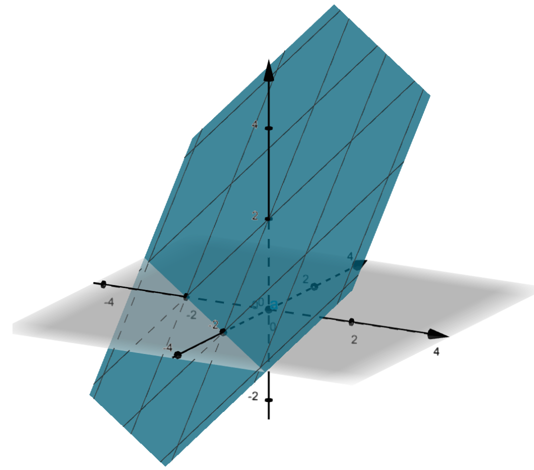
$$\mathbb{R}^2 \ni (x, y) \mapsto f(x, y) := x^2 + y^2 \in \mathbb{R}.$$

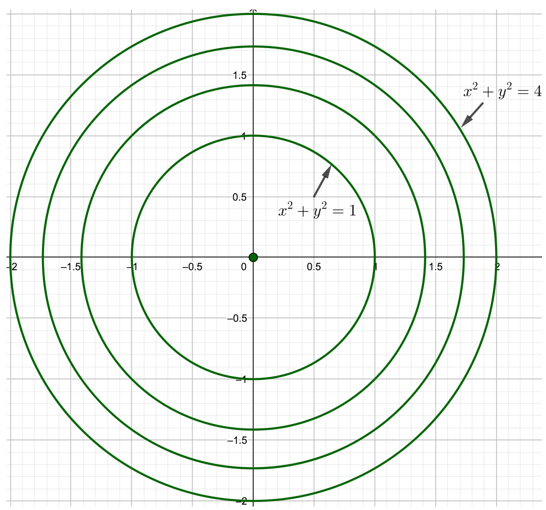
Then $\mathrm{Lev}_f(c) = \emptyset$ if $c < 0$ and, if $c \geq 0$,

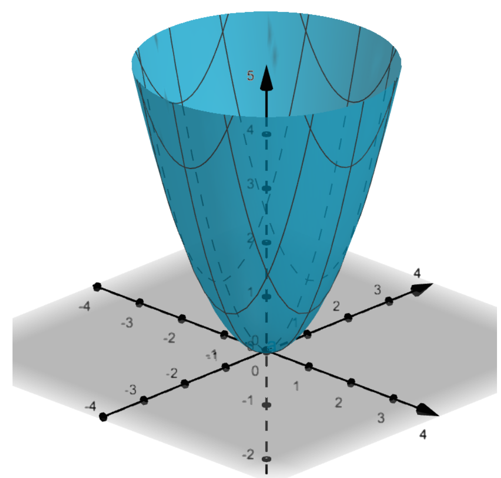
$$\mathrm{Lev}_f(c) = \{(x, y) \mid x^2 + y^2 = c\},$$

i.e. the circle centered at 0 and of radius $\sqrt{c}$.


$x + y + 2 = 0$
$x + y + 2 = 1$
$x + y + 2 = 2$




$x^2 + y^2 = 4$
$x^2 + y^2 = 1$

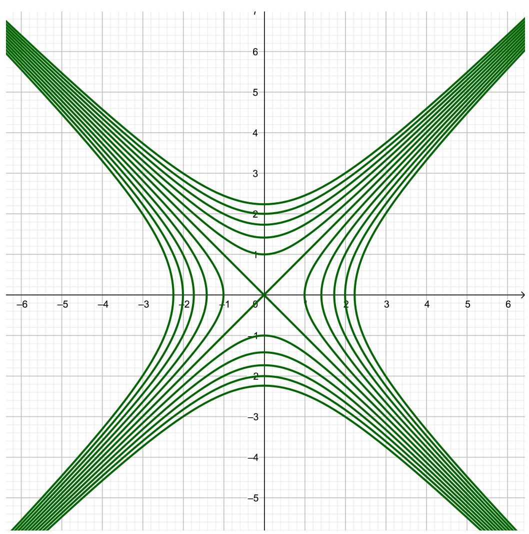


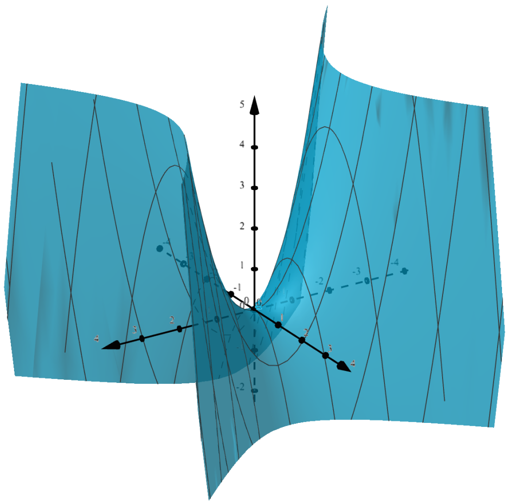
Example 3: Consider the function

$$\mathbb{R}^2 \ni (x, y) \mapsto f(x, y) := x^2 - y^2 \in \mathbb{R}.$$

In this case the level sets are given, for all $c \in \mathbb{R}$, by

$$\mathrm{Lev}_f(c) = \{(x, y) \mid y^2 = x^2 - c\}.$$





◇ [Differentiability] Let $f : \mathbb{R}^n \to \mathbb{R}$. We say that f is differentiable at $\bar{x} \in \mathbb{R}^n$ if for all $i = 1, \ldots, n$ the partial derivatives

$$\frac{\partial f}{\partial x_i}(\bar{x}) := \lim_{\tau \to 0} \frac{f(\bar{x} + \tau \mathbf{e}_i) - f(\bar{x})}{\tau} \quad \text{(where } \mathbf{e}_i := (0, \ldots, \underbrace{1}_{i}, \ldots, 0)\text{)},$$

exist and, defining the gradient of f at $\bar{x}$ by

$$\nabla f(\bar{x}) := \left(\frac{\partial f}{\partial x_1}(\bar{x}), \ldots, \frac{\partial f}{\partial x_n}(\bar{x}) \right) \in \mathbb{R}^n,$$

we have that

$$\lim_{h \to 0} \frac{f(\bar{x} + h) - f(\bar{x}) - \nabla f(\bar{x}) \cdot h}{\|h\|} = 0.$$

If f is differentiable at every x belonging to a set $A \subseteq \mathbb{R}^n$, we say that f is differentiable in A.

Remark 1. Notice that f is differentiable at $\bar{x}$ iff there exists $\varepsilon_{\bar{x}} : \mathbb{R}^n \to \mathbb{R}$, with $\lim_{h\to 0} \varepsilon_{\bar{x}}(h) = 0$ and

$$f(\bar{x} + h) = f(\bar{x}) + \nabla f(\bar{x}) \cdot h + \|h\| \varepsilon_{\bar{x}}(h). \tag{2}$$

In particular, f is continuous at $\bar{x}$.

Lemma 2. *Assume that f is differentiable at $\bar{x}$ and let $h \in \mathbb{R}^n$. Then,*

$$\lim_{\tau\to 0,\, \tau>0} \frac{f(\bar{x} + \tau h) - f(\bar{x})}{\tau} = \nabla f(\bar{x}) \cdot h.$$

Proof. By (2), for every $\tau > 0$, we have

$$f(\bar{x} + \tau h) - f(\bar{x}) = \tau \nabla f(\bar{x}) \cdot h + \tau \|h\| \varepsilon_{\bar{x}}(\tau h).$$

Dividing by τ and letting $\tau \to 0$ gives the result. □

Remark 2. (i) [Simple criterion to check differentiability] Suppose that $A \subseteq \mathbb{R}^n$ is an open set containing $\bar{x}$ and that

$$A \ni x \mapsto \nabla f(x) \in \mathbb{R}^n,$$

is well-defined and continuous at $\bar{x}$. Then, f is differentiable at $\bar{x}$.

As a consequence, if ∇f is continuous in A, then f is differentiable in A. In this case, we say that f is $\mathcal{C}^1$ in A (i.e. differentiability and continuity of ∇f in A). When f is $\mathcal{C}^1$ in $\mathbb{R}^n$ we simply say that f is $\mathcal{C}^1$.

(ii) The notion of differentiability can be extended to a function $f : \mathbb{R}^n \to \mathbb{R}^m$. In this case, f is differentiable at $\bar{x}$ if there exists $L \in \mathcal{M}_{m\times n}(\mathbb{R})$ such that

$$\lim_{\|h\|\to 0} \frac{\|f(\bar{x}+h) - f(\bar{x}) - Lh\|}{\|h\|} \to 0.$$

If f is differentiable at $\bar{x}$, then $L = Df(\bar{x})$, called the Jacobian matrix of f at $\bar{x}$, which is given by

$$
Df(\bar{x}) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(\bar{x}) & \dots & \frac{\partial f_1}{\partial x_n}(\bar{x}) \\ \dots & \dots & \dots \\ \frac{\partial f_i}{\partial x_1}(\bar{x}) & \dots & \frac{\partial f_i}{\partial x_n}(\bar{x}) \\ \dots & \dots & \dots \\ \frac{\partial f_m}{\partial x_1}(\bar{x}) & \dots & \frac{\partial f_m}{\partial x_n}(\bar{x}) \end{pmatrix}
$$

Note that when $m = 1$ we have that $Df(\bar{x}) = \nabla f(\bar{x})^\top$.

The chain rule says that if $f : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at $\bar{x}$ and $g : \mathbb{R}^m \to \mathbb{R}^p$ is differentiable at $f(\bar{x})$, then $g \circ f : \mathbb{R}^n \to \mathbb{R}^p$ is differentiable at $\bar{x}$ and the following identity holds

$$
D(g \circ f)(\bar{x}) = Dg(f(\bar{x}))Df(\bar{x}).
$$

(iii) In the previous definitions the fact that the domain of definition of f is $\mathbb{R}^n$ is not important. The definition can be extended naturally for functions defined on open subsets of $\mathbb{R}^n$.

Basic examples:

(i) Let $c \in \mathbb{R}^n$ and consider the function $f_1 : \mathbb{R}^n \to \mathbb{R}$ defined by $f_1(x) = c \cdot x$. Then, for every $x \in \mathbb{R}^n$, we have $\nabla f_1(x) = c$ and, hence, f is differentiable.

(ii) Let $Q \in M_{n\times n}(\mathbb{R})$ and consider the function $f_2 : \mathbb{R}^n \to \mathbb{R}$ defined by

$$f_2(x) = \tfrac{1}{2}\langle Qx, x\rangle \quad \forall\, x \in \mathbb{R}^n.$$

Then, for all $x \in \mathbb{R}^n$ and $h \in \mathbb{R}^n$, we have

$$\begin{aligned} f_2(x+h) &= \tfrac{1}{2}\langle Q(x+h), x+h\rangle \\ &= \tfrac{1}{2}\langle Qx, x\rangle + \tfrac{1}{2}\left[\langle Qx, h\rangle + \langle Qh, x\rangle\right] + \tfrac{1}{2}\langle Qh, h\rangle \\ &= \tfrac{1}{2}\langle Qx, x\rangle + \langle \tfrac{1}{2}\left(Q + Q^\top\right)x, h\rangle + \tfrac{1}{2}\langle Qh, h\rangle \\ &= f_2(\bar{x}) + \langle \tfrac{1}{2}\left(Q + Q^\top\right)x, h\rangle + \|h\|\varepsilon_x(h), \end{aligned}$$

where $\lim_{h\to 0} \varepsilon_x(h) = 0$. Therefore, f_2 is differentiable and

$$\nabla f_2(x) = \tfrac{1}{2}\left(Q + Q^\top\right) x \quad \forall\, x \in \mathbb{R}^n.$$

In particular, if Q is symmetric, then $\nabla f_2(x) = Qx$.

(iii) Consider the function $f_3 : \mathbb{R}^n \to \mathbb{R}$ defined by $f_3(x) = \|x\|$. Then, since $f_3(x) = \sqrt{\|x\|^2}$, if $x \neq 0$, the chain rule shows that

$$Df(x) = D(\sqrt{\cdot})(\|x\|^2) D(\|\cdot\|^2)(x) = \tfrac{1}{2}\frac{1}{\sqrt{\|x\|^2}}(2x)^\top = \frac{x^\top}{\|x\|},$$

which implies that $\nabla f_3(x) = \frac{x}{\|x\|}$, and, since this function is continuous at every $x \neq 0$, we have that f_3 is $\mathcal{C}^1$ in the set $\mathbb{R}^n \setminus \{0\}$. Let us show that f_3 is not differentiable at $x = 0$. Indeed, if this is not the case, then all the partial derivatives $\frac{\partial f_3}{\partial x_i}(0)$ should exists for all $i = 1, \ldots, n$. Taking, for instance, $i = 1$, we have

$$\lim_{\tau\to 0} \frac{\|0 + \tau \mathbf{e_1}\| - \|0\|}{\tau} = \lim_{\tau\to 0} \frac{|\tau|}{\tau},$$

which does not exist, because

$$\lim_{\tau \to 0^-} \frac{|\tau|}{\tau} = \lim_{\tau \to 0^-} \frac{-\tau}{\tau} = -1 \neq 1 = \lim_{\tau \to 0^+} \frac{\tau}{\tau} = \lim_{\tau \to 0^+} \frac{|\tau|}{\tau}.$$

◇ [Second order derivative and Taylor expansion] Suppose that $f : \mathbb{R}^n \to \mathbb{R}$ is $\mathcal{C}^1$. In particular, the function $\mathbb{R}^n \ni x \mapsto \nabla f(x) \in \mathbb{R}^n$ is well defined. If this function is differentiable at $\bar{x}$, then we say that f is twice differentiable at $\bar{x}$. If f is twice differentiable at every x belonging to a set $A \subseteq \mathbb{R}^n$, then we say that f is twice differentiable in A.

If this is the case, then, by definition,

$$\frac{\partial^2 f}{\partial x_i \partial x_j}(\bar{x}) := \frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_i}\right)(\bar{x}),$$

exists for all i, $j = 1, \ldots, n$. The following result, due to Clairaut and also known as Schwarz's theorem, says that, under appropriate conditions we can change the derivation order.

Theorem 1. *Suppose that the function f is twice differentiable in an open set $A \subseteq \mathbb{R}^n$ containing $\bar{x}$ and that for all i, $j = 1, \ldots, n$ the function $A \ni x \mapsto \frac{\partial^2 f}{\partial x_i \partial x_j}(x) \in \mathbb{R}$ is continuous at $\bar{x}$. Then,*

$$\frac{\partial^2 f}{\partial x_i \partial x_j}(\bar{x}) = \frac{\partial^2 f}{\partial x_j \partial x_i}(\bar{x}).$$

Under the assumptions of the previous theorem, the Jacobian of $\nabla f(\bar{x})$ takes the form

$$D^2 f(\bar{x}) = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2}(\bar{x}) & \ldots & \frac{\partial^2 f}{\partial x_n \partial x_1}(\bar{x}) \\ \vdots & \ldots & \vdots \\ \frac{\partial^2 f}{\partial x_i \partial x_1}(\bar{x}) & \ldots & \frac{\partial^2 f}{\partial x_n \partial x_i}(\bar{x}) \\ \vdots & \ldots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1}(\bar{x}) & \ldots & \frac{\partial^2 f}{\partial x_n^2}(\bar{x}) \end{pmatrix}.$$

This matrix, called the Hessian matrix of f at $\bar{x}$ belongs to $\mathcal{M}_{n \times n}(\mathbb{R})$ and it is a symmetric matrix by the previous result.

If $f : \mathbb{R}^n \to \mathbb{R}$ is twice differentiable in an open set $A \subseteq \mathbb{R}^n$ and for all i, $j = 1, \ldots, n$ the function

$$A \ni x \mapsto \frac{\partial^2 f}{\partial x_i \partial x_j}(x) \in \mathbb{R}$$

is continuous, we say that f is $\mathcal{C}^2$ in A.

◇ [Taylor's theorem] We admit the following important result:

Theorem 2. *Suppose that* $f : \mathbb{R}^n \to \mathbb{R}$ *is* $\mathcal{C}^2$ *in an open set* $A \subseteq \mathbb{R}^n$*. Then, for all* $x \in A$ *and* h *such that* $x + h \in A$*, we have the following expansion*

$$f(x+h) = f(x) + \nabla f(x) \cdot h + \tfrac{1}{2}\langle D^2 f(x)h, h\rangle + \|h\|^2 R_x(h),$$

where $R_x(h) \to 0$ *as* $h \to 0$*.*

Example: Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x,y) = e^x \cos(y) - x - 1$.

Then,

$$\begin{aligned}
\frac{\partial f}{\partial x}(0,0) &= e^x\cos(y)\big|_{(x,y)=(0,0)} - 1 = 0,\\
\frac{\partial f}{\partial y}(0,0) &= -e^x\sin(y)\big|_{(x,y)=(0,0)} = 0,\\
\frac{\partial^2 f}{\partial x^2}(0,0) &= e^x\cos(y)\big|_{(x,y)=(0,0)} = 1,\\
\frac{\partial^2 f}{\partial y^2}(0,0) &= -e^x\cos(y)\big|_{(x,y)=(0,0)} = -1,\\
\frac{\partial^2 f}{\partial x\partial y}(0,0) &= -e^x\sin(y)\big|_{(x,y)=(0,0)} = 0.
\end{aligned}$$

Note that all the first and second order partial derivatives are continuous in $\mathbb{R}^n$. Therefore, we can apply the previous result and obtain that the Taylor's expansion of f at $(0,0)$ is given by

$$\begin{aligned}
f((0,0)+h) &= f(0,0) + \nabla f(0,0)\cdot h + \tfrac{1}{2}\langle D^2 f(0,0)h, h\rangle + \|h\|^2 R_{\bar{x}}(h),\\
&= 0 + 0 + \tfrac{1}{2}h_1^2 - \tfrac{1}{2}h_2^2 + \|h\|^2 R_{(0,0)}(h),\\
&= \tfrac{1}{2}h_1^2 - \tfrac{1}{2}h_2^2 + \|h\|^2 R_{(0,0)}(h).
\end{aligned}$$

This expansion shows that locally around $(0,0)$ the function f above is similar to the function in Example 3.

Some good reading for the previous part

◇ Chapters 2 and 3 in "Vector calculus", sixth edition, by J. E. Marsden and A. Tromba.

◇ Chapter 14 in "Calculus: Early transcendentals", eight edition, by J. Stewart.

Some basic existence results for problem (P)

- [Compactness] Recall that $A \subseteq \mathbb{R}^n$ is called compact if A is closed and bounded (i.e. A is closed and there exists $R > 0$ such that $\|x\| \leq R$ for all $x \in A$).

  Let us recall an important result concerning the compactness of a set A.

  Theorem 3. [Bolzano-Weierstrass theorem] *A set $A \subseteq \mathbb{R}^n$ is compact if and only if every sequence $(x_k)_{k\in\mathbb{N}}$ of elements of A has a convergence subsequence. This means that there exists $\bar{x} \in A$ and a subsequence $(x_{k_\ell})_{\ell\in\mathbb{N}}$ of $(x_k)_{k\in\mathbb{N}}$ such that*

$$\bar{x} = \lim_{\ell\to\infty} x_{k_\ell}.$$

- [The basic existence results] Note that by definition, if $\inf_{x\in\mathcal{K}} f(x) = -\infty$, then f has no lower bounds in $\mathcal{K}$ and, hence, there are no solutions to (P). On the other hand, if $\inf_{x\in\mathcal{L}} f(x)$ is finite, then the existence of a solution can also fail to hold as the following example shows.

Example: Consider the function $\mathbb{R} \ni x \mapsto f(x) := e^{-x}$ and take $\mathcal{K} := [0, +\infty[$. Then, $\inf_{x \in \mathcal{K}} f(x) = 0$ and there is no $x \in \mathcal{K}$ such that $f(x) = 0$.

Definition 1. *We say that $(x_k)_{k \in \mathbb{N}} \subseteq \mathcal{K}$ is a minimizing sequence for (P) if*

$$\inf_{x \in \mathcal{K}} f(x) = \lim_{k \to \infty} f(x_k).$$

By definition, a minimizing sequence always exists if $\mathcal{K}$ is non-empty.

Theorem 4. [Weierstrass theorem, $\mathcal{K}$ compact] *Suppose that $f : \mathbb{R}^n \to \mathbb{R}$ is continuous and that $\mathcal{K}$ is non-empty and compact. Then, problem (P) admits at least one global solution.*

Proof. Let $(x_k)_{k \in \mathbb{N}} \in \mathcal{K}$ be a minimizing sequence. By compactness, there exists $\bar{x} \in \mathcal{K}$ and a subsequence $(x_{k_\ell})_{\ell \in \mathbb{N}}$ of $(x_k)_{k \in \mathbb{N}}$ such that $\bar{x} = \lim_{\ell \to \infty} x_{k_\ell}$. Then, by continuity

$$f(\bar{x}) = \lim_{\ell \to \infty} f(x_{k_\ell}) = \inf_{x \in \mathcal{K}} f(x).$$

□

Example: Suppose that $f : \mathbb{R}^3 \to \mathbb{R}$ is given by $f(x, y, z) = x^2 - y^3 + \sin z$ and $\mathcal{K} = \{(x, y, z) \mid x^4 + y^4 + z^4 \leq 1\}$. Then f is continuous and $\mathcal{K}$ is compact. As a consequence, problem (P) admits at least one solution.

Theorem 5. [$\mathcal{K}$ closed but not bounded] *Suppose that $\mathcal{K}$ is non-empty, closed, and that f is continuous and "coercive" or "infinity at the infinity" in $\mathcal{K}$, i.e.*

$$\lim_{x \in \mathcal{K},\ \|x\| \to \infty} f(x) = +\infty. \tag{3}$$

Then, problem (P) admits at least one global solution.

Proof. Let $(x_k)_{k \in \mathbb{N}} \in \mathcal{K}$ be a minimizing sequence. Since $\inf_{x \in \mathcal{K}} f(x) = -\infty$ or $\inf_{x \in \mathcal{K}} f(x) \in \mathbb{R}$ and $\lim_{k \to \infty} f(x_k) = \inf_{x \in \mathcal{K}} f(x)$, there exists $R > 0$ such that $(x_k)_{k \in \mathbb{N}} \subseteq \mathcal{K}_R := \{x' \in \mathcal{K} \mid f(x') \leq R\} \subseteq \mathcal{K}$. By continuity of f, this set is closed and bounded because f is coercive. Arguing as in the previous proof, the compactness of $\mathcal{K}_R$ implies the existence of $\bar{x} \in \mathcal{K}_R$ such that a subsequence of $(x_k)_{k \in \mathbb{N}}$ converges to $\bar{x}$, which, by continuity of f, implies that $\bar{x}$ solves (P). □

Example: Suppose that $f : \mathbb{R}^n \to \mathbb{R}$ is given by

$$f(x) = \langle Qx, x\rangle + c^\top x \quad \forall\, x \in \mathbb{R}^n,$$

where $Q \in \mathcal{M}_{n,n}(\mathbb{R})$ is symmetric and positive definite, and $c \in \mathbb{R}^n$. Since

$$\langle Qx, x\rangle \geq \lambda_{\min}(Q)\|x\|^2 \quad \forall\, x \in \mathbb{R}^n$$

(where $\lambda_{\min}(Q) > 0$ is the smallest eigenvalue of Q), we have that

$$f(x) \geq \lambda_{\min}(Q)\|x\|^2 - \|c\|\|x\| \quad \forall\, x \in \mathbb{R}^n.$$

This implies that $f(x) \to \infty$ as $\|x\| \to \infty$. Therefore,

$$\lim_{x\in\mathcal{K},\|x\|\to\infty} f(x) = \infty, \tag{4}$$

holds for every closed set $\mathcal{K}$. Since f is also continuous, problem (P) admits at least one global solution for any given non-empty closed set $\mathcal{K}$.

Example: Suppose that $f : \mathbb{R}^2 \to \mathbb{R}$ is given by

$$f(x, y) = x^2 + y^3 \quad \forall\, (x, y) \in \mathbb{R}^2,$$

and

$$\mathcal{K} = \{(x, y) \in \mathbb{R}^2 \mid y \geq -1\}.$$

Then,

$$\lim_{x \in \mathcal{K}, \|x\| \to \infty} f(x) = +\infty \tag{5}$$

holds (exercise) and, hence, (P) admits at least one global solution.

Optimality conditions for unconstrained problems

◇ Notice that, by the second existence theorem, if f is continuous and satisfies that

$$\lim_{\|x\|\to\infty} f(x) = +\infty,$$

then, if $\mathcal{K} = \mathbb{R}^n$, problem (P) admits at least one global solution.

◇ [First order optimality conditions for unconstrained problems]

We have the following result

Theorem 6. [Fermat's rule] *Suppose that $\mathcal{K} = \mathbb{R}^n$ and that $\bar{x}$ is a local solution to problem (P). If f is differentiable at $\bar{x}$, then $\nabla f(\bar{x}) = 0$.*

Proof. For every $h \in \mathbb{R}^n$ and $\tau > 0$, the local optimality of $\bar{x}$ yields

$$f(\bar{x}) \leq f(\bar{x} + \tau h) = f(\bar{x}) + \tau \nabla f(\bar{x}) \cdot h + \tau \|h\| \varepsilon_{\bar{x}}(\tau h),$$

where $\lim_{z\to 0} \varepsilon_{\bar{x}}(z) = 0$. Therefore,

$$0 \leq \tau \nabla f(\bar{x}) \cdot h + \tau \|h\| \varepsilon_{\bar{x}}(\tau h).$$

Dividing by τ and letting $\tau \to 0$, we get

$$\nabla f(\bar{x}) \cdot h \geq 0.$$

Since h is arbitrary, we get that $\nabla f(\bar{x}) = 0$ (take for instance $h = -\nabla f(\bar{x})$ in the previous inequality). □

◇ [Second order optimality conditions for unconstrained problems]

We have the following second order necessary condition for local optimality:

Theorem 7. *Suppose that $\mathcal{K} = \mathbb{R}^n$ and that $\bar{x}$ is a local solution to problem (P). If f is $\mathcal{C}^2$ in an open set A containing $\bar{x}$, then $D^2 f(\bar{x})$ is positive semidefinite. In other words,*

$$\langle D^2 f(\bar{x}) h, h \rangle \geq 0 \quad \forall\, h \in \mathbb{R}^n.$$

Proof. Let us fix $h \in \mathbb{R}^n$. By Taylor's theorem, for all $\tau > 0$ small enough, we have

$$f(\bar{x} + \tau h) \quad = \quad f(\bar{x}) + \nabla f(\bar{x}) \cdot (\tau h) + \tfrac{1}{2}\langle D^2 f(\bar{x})\tau h, \tau h\rangle + \|\tau h\|^2 R_{\bar{x}}(\tau h),$$

where $R_{\bar{x}}(\tau h) \to 0$ as $\tau \to 0$. Using the local optimality of $\bar{x}$, the previous result implies that $\nabla f(\bar{x}) = 0$ and, hence,

$$0 \leq f(\bar{x} + \tau h) - f(\bar{x}) = \frac{\tau^2}{2}\langle D^2 f(\bar{x})h, h\rangle + \tau^2 \|h\|^2 R_{\bar{x}}(\tau h).$$

Dividing by τ^2 and passing to the limit with $\tau \to 0$, we get the result. □

We have the following second order sufficient condition for local optimality.

Theorem 8. *Suppose that* $f : \mathbb{R}^n \to \mathbb{R}$ *is* $\mathcal{C}^2$ *in an open set* A *containing* $\bar{x}$ *and that*

(i) $\nabla f(\bar{x}) = 0$.

(ii) *The matrix* $D^2 f(\bar{x})$ *is positive definite. In other words,*

$$\langle D^2 f(\bar{x})h, h \rangle > 0 \quad \forall\, h \in \mathbb{R}^n,\ h \neq 0.$$

Then, $\bar{x}$ *is a local solution to* (P).

Proof. Let $\lambda > 0$ be the smallest eigenvalue of $D^2 f(\bar{x})$, then

$$\forall h \in \mathbb{R}^n, \quad \langle D^2 f(\bar{x})h, h \rangle \geq \lambda \|h\|^2.$$

Using this inequality, the hypothesis $\nabla f(\bar{x}) = 0$, and the Taylor's expansion, for all

$h \in \mathbb{R}^n$ such that $\bar{x} + h \in A$ we have that

$$\begin{aligned} f(\bar{x}+h) - f(\bar{x}) &= \nabla f(\bar{x}) \cdot h + \frac{1}{2}\langle D^2 f(\bar{x})h, h\rangle + \|h\|^2 R_{\bar{x}}(h) \\ &\geq \frac{\lambda}{2}\|h\|^2 + \|h\|^2 R_{\bar{x}}(h) \\ &= \left(\frac{\lambda}{2} + R_{\bar{x}}(h)\right) \|h\|^2. \end{aligned}$$

Since $R_{\bar{x}}(h) \to 0$ as $h \to 0$, we can choose $\delta > 0$ such that $\|h\| \leq \delta$, $\bar{x} + h \in A$ and $|R_{\bar{x}}(h)| \leq \frac{\lambda}{4}$. As a consequence,

$$f(\bar{x}+h) - f(\bar{x}) \geq \frac{\lambda}{4}\|h\|^2 \quad \forall\, h \in \mathbb{R}^n \text{ with } \|h\| \leq \delta,$$

which proves the local optimality of $\bar{x}$. □

Example: Let us study problem (P) with $\mathcal{K} = \mathbb{R}^2$ and

$$\mathbb{R}^2 \ni (x, y) \mapsto f(x, y) = 2x^3 + 3y^2 + 3x^2y - 24y.$$

First, consider the sequence $(x_k, y_k) = (-k, 0)$ for $k \in \mathbb{N}$. Then,

$$f(x_k, y_k) = -2k^3 \to -\infty \text{ as } k \to \infty.$$

Therefore, $\inf_{(x,y)\in\mathbb{R}^2} f(x, y) = -\infty$ and problem (P) does not admit global solutions. Let us look for local solutions. We know that if (x, y) is a local solution, then it should satisfy $\nabla f(x, y) = (0, 0)$. This equation gives

$$\begin{aligned} 6x^2 + 6xy &= 0, \\ 6y + 3x^2 &= 24. \end{aligned}$$

From the first equation, we get that $x = 0$ or $x = -y$. In the first case, the second equation yields $y = 4$, while in the second case we obtain that $x^2 - 2x - 8 = 0$ which yields the two solutions $(4, -4)$ and $(-2, 2)$. Therefore, we have the three candidates $(0, 4)$, $(4, -4)$ and $(-2, 2)$. Let us check what can be obtained from the Hessian at these three points. We have that

$$D^2 f(x, y) = \begin{pmatrix} 12x + 6y & 6x \\ 6x & 6 \end{pmatrix}.$$

For the first candidate, we have

$$D^2 f(0, 4) = \begin{pmatrix} 24 & 0 \\ 0 & 6 \end{pmatrix}.$$

which is positive definite. This implies that $(0, 4)$ is a local solution of (P). For the second candidate, we have

$$D^2 f(4, -4) = \begin{pmatrix} 24 & 24 \\ 24 & 6 \end{pmatrix} = 6 \begin{pmatrix} 4 & 4 \\ 4 & 1 \end{pmatrix},$$

whose determinant is given by $36(-12) < 0$, which implies that $D^2 f(4, -4)$ is indefinite (the sign of the eigenvalues is not constant). Finally,

$$D^2 f(-2,2) = \begin{pmatrix} -12 & -12 \\ -12 & 6 \end{pmatrix}$$

which is also indefinite because the sign of the diagonal terms are not constant. Therefore, $(0,4)$ is the unique local solution to (P).

◇ [Maximization problems] If instead of problem (P) we consider the problem

$$\text{Find } \bar{x} \in \mathbb{R}^n \text{ such that} \quad f(\bar{x}) = \max\{f(x) \mid x \in \mathcal{K}\}, \qquad (P')$$

then $\bar{x}$ is a local (resp. global) solution to (P') iff $\bar{x}$ is a local (resp. global) solution to (P) with f replaced by $-f$. In particular, if $\bar{x}$ is a local solution to (P') and f is regular enough, then we have the following first order necessary condition

$$\nabla f(\bar{x}) = 0,$$

as well as the following second order necessary condition

$$\langle D^2 f(\bar{x})h, h\rangle \leq 0 \quad \forall\, h \in \mathbb{R}^n.$$

In other words, $D^2 f(\bar{x})$ is negative semidefinite.

Conversely, if $\bar{x} \in \mathbb{R}^n$ is such that $\nabla f(\bar{x}) = 0$ and

$$\langle D^2 f(\bar{x})h, h\rangle < 0 \quad \forall\, h \in \mathbb{R}^n,\ h \neq 0.$$

Then, $\bar{x}$ is a local solution to (P').

Convexity

- ◇ [Convexity of a set] A non-empty set $C \subseteq \mathbb{R}^n$ is called convex if for any $\lambda \in [0,1]$ and x, $y \in C$, we have that

$$\lambda x + (1-\lambda)y \in C.$$

Let us fix a convex set $C \subseteq \mathbb{R}^n$.

- ◇ [Convexity of a function] A function $f : C \to \mathbb{R}$ is said to be convex if for any $\lambda \in [0,1]$ and x, $y \in C$, we have that

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y).$$

- ◇ [Relation between convex functions and convex sets] Given $f : \mathbb{R}^n \to \mathbb{R}$, let us define its epigraph epi(f) by

$$\text{epi}(f) := \left\{(x,y) \in \mathbb{R}^{n+1} \mid y \geq f(x)\right\}.$$

Proposition 1. *The function f is convex iff the set epi(f) is convex.*

Proof. Indeed, suppose that f is convex and let (x_1, z_1), $(x_2, z_2) \in \text{epi}(f)$. Then, given $\lambda \in [0, 1]$ set

$$P_\lambda := \lambda(x_1, z_1) + (1 - \lambda)(x_2, z_2) = (\lambda x_1 + (1 - \lambda)x_2, \lambda z_1 + (1 - \lambda)z_2)$$

Since, by convexity,

$$f(\lambda x_1 + (1 - \lambda)x_2) \leq \lambda f(x_1) + (1 - \lambda)f(x_2) \leq \lambda z_1 + (1 - \lambda)z_2,$$

we have that $P_\lambda \in \text{epi}(f)$. Conversely, assume that epi(f) is convex and let x_1, $x_2 \in \mathbb{R}^n$ and $\lambda \in [0, 1]$. Since $(x_1, f(x_1))$, $(x_2, f(x_2)) \in \text{epi}(f)$, we deduce that

$$(\lambda x_1 + (1 - \lambda)x_2, \lambda f(x_1) + (1 - \lambda)f(x_2)) \in \text{epi}(f),$$

and, hence,

$$f(\lambda x_1 + (1 - \lambda)x_2) \leq \lambda f(x_1) + (1 - \lambda)f(x_2),$$

which proves the convexity of f. □

◇ [Strict convexity of a function] A function $f : C \to \mathbb{R}$ is said to be strictly convex if for any $\lambda \in (0,1)$ and x, $y \in C$, with $x \neq y$, we have that

$$f(\lambda x + (1-\lambda)y) < \lambda f(x) + (1-\lambda)f(y).$$

◇ [Concavity and strict concavity of a function] A function $f : C \to \mathbb{R}$ is said to be concave if $-f$ is convex. Similarly, the function f is strictly concave if $-f$ is strictly convex.

Example: Let us show that the function $\mathbb{R}^n \ni x \mapsto \|x\| \in \mathbb{R}$ is convex but not strictly convex. Indeed, the convexity follows from the triangle inequality

$$\|\lambda x + (1-\lambda)y\| \leq \|\lambda x\| + \|(1-\lambda)y\| = \lambda\|x\| + (1-\lambda)\|y\|.$$

Now, if we have that for some $\lambda \in (0,1)$

$$\|\lambda x + (1-\lambda)y\| = \lambda\|x\| + (1-\lambda)\|y\|$$

the equality case in the triangle inequality ($\|a+b\| = \|a\| + \|b\|$ iff $a = 0$ and $b = 0$ or $a = \alpha b$ with $\alpha > 0$) shows that the previous inequality holds iff that $x = y = 0$

or $x = \gamma y$ for some $\gamma > 0$. By taking $x \neq 0$ and $y = \gamma x$ with $\gamma \in (0, \infty) \setminus \{1\}$ we conclude that $\|\cdot\|$ is not strictly convex.

Example: Let $\beta \in (1, +\infty)$. Let us show that the function $\mathbb{R}^n \ni x \mapsto \|x\|^\beta \in \mathbb{R}$ is strictly convex. Indeed, the real function $[0, +\infty) \ni t \mapsto \alpha(t) := t^\beta \in \mathbb{R}$ is increasing and strictly convex because

$$\alpha'(t) = \beta t^{\beta-1} > 0 \text{ and } \alpha''(t) = \beta(\beta-1)t^{\beta-2} > 0 \ \forall\, t \in (0, +\infty).$$

As a consequence, for any $\lambda \in [0, 1]$, using that

$$\|\lambda x + (1-\lambda)y\| \leq \lambda\|x\| + (1-\lambda)\|y\|,$$

we get that

$$\begin{aligned} \|\lambda x + (1-\lambda)y\|^\beta &\leq (\lambda\|x\| + (1-\lambda)\|y\|)^\beta \\ &\leq \lambda\|x\|^\beta + (1-\lambda)\|y\|^\beta, \end{aligned} \tag{6}$$

which implies the convexity of $\|\cdot\|^\beta$. Now, in order to prove the strict convexity, assume that for some $\lambda \in (0,1)$ we have

$$\|\lambda x + (1-\lambda)y\|^\beta = \lambda\|x\|^\beta + (1-\lambda)\|y\|^\beta,$$

and let us prove that $x = y$. Then, all the inequalities in (6) are equalities and, hence,

$$\begin{aligned} &\|\lambda x + (1-\lambda)y\| = \lambda\|x\| + (1-\lambda)\|y\|, \\ \text{and } &(\lambda\|x\| + (1-\lambda)\|y\|)^\beta = \lambda\|x\|^\beta + (1-\lambda)\|y\|^\beta. \end{aligned}$$

The equality case in the triangle inequality and the first relation above imply that $x = y = 0$ or $x = \gamma y$ for some $\gamma > 0$. The strict convexity of α and the second inequality above imply that $\|x\| = \|y\|$. Therefore, either $x = y = 0$ or both x and y are not zero and $x = \gamma y$ for some $\gamma > 0$ and $\|x\| = \|y\|$. In the latter case, we get that $\alpha = 1$ and, hence, $x = y$ from which the strict convexity follows.

◇ [Convexity and differentiability] We have the following result:

Theorem 9. *Let $f : C \to \mathbb{R}$ be a differentiable function. Then,*

(i) *f is convex in $\mathbb{R}^n$ if and only if for every $x \in C$ we have*

$$f(y) \geq f(x) + \nabla f(x) \cdot (y - x), \quad \forall\, y \in C. \tag{7}$$

(ii) *f is strictly convex in $\mathbb{R}^n$ if and only if for every $x \in C$ we have*

$$f(y) > f(x) + \nabla f(x) \cdot (y - x), \quad \forall\, y \in C,\ y \neq x. \tag{8}$$

Proof. (i) By definition of convex function, for any x, $y \in C$ and $\lambda \in (0, 1)$, we have

$$f(\lambda y + (1 - \lambda)x) - f(x) \leq \lambda\,(f(y) - f(x))$$

Since, $\lambda y + (1 - \lambda)x = x + \lambda(y - x)$, we get

$$\frac{f(x + \lambda(y - x)) - f(x)}{\lambda} \leq f(y) - f(x).$$

Letting $\lambda \to 0$, Lemma 2 yields

$$\nabla f(x) \cdot (y - x) \leq f(y) - f(x).$$

Conversely, take x_1 and x_2 in C, $\lambda \in]0,1[$ and define $x_\lambda := \lambda x_1 + (1-\lambda)x_2$. By assumption,

$$\forall i \in \{1,2\}, \quad f(x_i) \geq f(x_\lambda) + \nabla f(x_\lambda) \cdot (x_i - x_\lambda),$$

and we obtain, by taking the convex combination, that

$$\lambda f(x_1) + (1-\lambda) f(x_2) \geq f(x_\lambda) + \nabla f(x_\lambda) \cdot (\lambda x_1 + (1-\lambda)x_2 - x_\lambda) = f(x_\lambda),$$

which shows that f is convex.

(ii) Since f is convex, by (i) we have that

$$f(y) \geq f(x) + \nabla f(x) \cdot (y - x), \quad \forall\, y \in C. \tag{9}$$

Suppose that there exists $y \in C$ such that $y \neq x$ and

$$f(y) = f(x) + \nabla f(x) \cdot (y - x).$$

Let $z = \frac{1}{2}x + \frac{1}{2}y$. Then, by (9), with $y = z$, and strict convexity, we get

$$f(x)+\nabla f(x)\cdot(\tfrac{1}{2}y-\tfrac{1}{2}x) \leq f(z) < \tfrac{1}{2}f(x)+\tfrac{1}{2}f(y) = f(x)+\nabla f(x)\cdot(\tfrac{1}{2}y-\tfrac{1}{2}x),$$

which is impossible. The proof that (8) implies that f is strictly convex is completely analogous to the proof that (7) implies convexity. The result follows. □

Theorem 10. *Let $f : C \to \mathbb{R}$ be $\mathcal{C}^2$ in C and suppose that C is open (besides being convex). Then*

(i) *f is convex if and only if $D^2 f(x)$ is positive semidefinite for all $x \in C$.*
(ii) *f is strictly convex if $D^2 f(x)$ is positive definite for all $x \in C$*

Proof. (i) Suppose that f is convex. Then, by Taylor's theorem for every $x \in C$,

$h \in \mathbb{R}^n$ and $\tau > 0$ small enough such that $x + \tau h \in C$ we have

$$f(x+\tau h) = f(x) + \tau \nabla f(x) \cdot h + \frac{\tau^2}{2} \langle D^2 f(x) h, h \rangle + \tau^2 \|h\|^2 R_x(\tau h),$$

which implies, by the first order characterization of convexity, that

$$0 \leq \tfrac{1}{2} \langle D^2 f(x) h, h \rangle + \|h\|^2 R_x(\tau h).$$

Using that $\lim_{\tau \to 0} R_x(\tau h) = 0$, and the fact that h is arbitrary, we get that

$$\langle D^2 f(x) h, h \rangle \geq 0 \;\; \forall \; h \in \mathbb{R}^n.$$

Suppose that $D^2 f(x)$ is positive semidefinite for all $x \in C$ and assume, for the time being, that for every x, $y \in C$ there exists $c_{xy} \in \{\lambda x + (1-\lambda) y \mid \lambda \in (0,1)\}$ such that

$$f(y) = f(x) + \nabla f(x) \cdot (y - x) + \tfrac{1}{2} \langle D^2 f(c_{xy})(y-x), y - x \rangle. \tag{10}$$

Then, have that

$$f(y) \geq f(x) + \nabla f(x) \cdot (y - x) \quad \forall\, x,\ y \in C,$$

and, hence, f is convex. It remains to prove (10). Defining $g(\tau) := f(x + \tau(y - x))$ for all $\tau \in [0, 1]$, formula (10) follows from the equality

$$g(1) = g(0) + g'(0) + \tfrac{1}{2} g''(\hat{\tau})$$

for some $\hat{\tau} \in (0, 1)$.
(ii) The assertion follows directly from (10), with $y \neq y$, and Theorem 9(ii). □

Remark 3. Note that the positive definiteness of $D^2 f(x)$, for all $x \in C$, is only a sufficient condition for strict convexity but not necessary. Indeed, the function $\mathbb{R} \ni x \mapsto f(x) = x^4 \in \mathbb{R}$ is strictly convex but $f''(0) = 0$.

Example: Let $Q \in \mathcal{M}_{n,n}(\mathbb{R})$ be symmetric and let $f : \mathbb{R}^n \to \mathbb{R}$ be defined by

$$f(x) = \tfrac{1}{2} x^\top Q x + c^\top x.$$

Then, $D^2 f(x) = Q$ and hence f is convex if Q is semidefinite positive and strictly convex if Q is definite positive.
In this case, the fact that Q is definite positive is also a necessary condition for strict convexity. Indeed, for simplicity suppose that $c = 0$ and write $Q = PDP^\top$, where the set of columns of P is an orthonormal basis of eigenvectors of Q (which exists because Q is symmetric), and D is the diagonal matrix containing the corresponding eigenvalues $\{\lambda_i\}_{i=1}^N$ in the diagonal. Set $y(x) = P^\top x$. Then,

$$f(x) = \sum \lambda_{i=1}^n y_i(x)^2.$$

If Q is not positive definite, then there exists $j \in \{1, \ldots, N\}$ such that $\lambda_j \leq 0$. Then, it is easy to see that f is not strictly convex on the set $\{x \in \mathbb{R}^n \mid y_i(x) = 0, \text{ for all } i \in \{1, \ldots, n\} \setminus \{j\}\}$.

Optimization with constraints

◇ [Optimality conditions for convex problems] Let us begin with a definition.

Definition 2. *Problem (P) is called convex if f is convex and $\mathcal{K}$ is a non-empty closed and convex set.*

We have the following result.

Theorem 11. [Characterization of solutions for convex problems] *Suppose that problem (P) is convex and that $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable in $\mathcal{K}$. Then, the following statements are equivalent:*

(i) *$\bar{x}$ is a local solution to (P).*

(ii) *The following inequality holds:*

$$\langle \nabla f(\bar{x}), x - \bar{x} \rangle \geq 0 \quad \forall\, x \in \mathcal{K}. \tag{11}$$

(iii) *$\bar{x}$ is a global solution to (P).*

Proof. Let us prove that (i) implies (ii). Indeed, by convexity of $\mathcal{K}$ we have that given $y \in \mathcal{K}$ for any $\tau \in [0,1]$ the point $\tau y + (1-\tau)\bar{x} = \bar{x} + \tau(y - \bar{x}) \in \mathcal{K}$. Therefore, by the differentiability of f, if τ is small enough, we have

$$0 \leq f(\bar{x} + \tau(y - \bar{x})) - f(\bar{x}) = \tau \nabla f(\bar{x}) \cdot (y - \bar{x}) + \tau \|y - \bar{x}\| \varepsilon_{\bar{x}}(\tau \|y - \bar{x}\|),$$

where $\lim_{h \to 0} \varepsilon_{\bar{x}}(h) = 0$. Dividing by τ and letting $\tau \to 0$, we get (ii).

The proof that (ii) implies (iii) follows directly from the inequalities

$$f(y) \geq f(\bar{x}) + \nabla f(\bar{x}) \cdot (y - \bar{x}) \geq f(\bar{x}) \quad \forall\, y \in \mathcal{K}.$$

Finally, (iii) implies (i) is trivial. The result follows. □

Remark 4. In particular, if $f : \mathbb{R}^n \to \mathbb{R}$ is convex and differentiable and $\mathcal{K} = \mathbb{R}^n$, the relation

$$\nabla f(\bar{x}) = 0,$$

is a necessary and sufficient condition for $\bar{x}$ to be a global solution to (P).

Proposition 2. *Suppose that $\mathcal{K}$ is convex and that f is strictly convex in $\mathcal{K}$. Then, there exists at most one solution to problem (P).*

Proof. Assume, by contradiction, that $x_1 \neq x_2$ are both solutions to (P). Then, $\frac{1}{2}x_1 + \frac{1}{2}x_2 \in \mathcal{K}$ and

$$f(\tfrac{1}{2}x_1 + \tfrac{1}{2}x_2) < \tfrac{1}{2}f(x_1) + \tfrac{1}{2}f(x_2) = \min_{x \in \mathcal{K}} f(x).$$

◇ [Least squares] Let $A \in \mathcal{M}_{m,n}(\mathbb{R})$, $b \in \mathbb{R}^m$ and consider the system $Ax = b$. Suppose that $m > n$. This type of systems appear, for instance, in data fitting problem and it is often ill-posed, in the sense that there is no x satisfying the equation. In this case, one usually considers the optimization problem

$$\min_{x \in \mathcal{K} := \mathbb{R}^n} f(x) := \|Ax - b\|^2. \tag{12}$$

Note that

$$f(x) = \langle A^\top A x, x \rangle - 2\langle A^\top b, x \rangle + \|b\|^2.$$

and, hence, $D^2 f(x) = 2A^\top A$, which is symmetric positive semidefinite, and, hence, f is convex. Let us assume that the columns of A are linearly independent. Then, for any $h \in \mathbb{R}^n$,

$$\langle A^\top A h, h\rangle = 0 \Leftrightarrow Ah = 0 \Leftrightarrow h = 0,$$

i.e. for all $x \in \mathbb{R}^n$, the matrix $D^2 f(x)$ is symmetric positive definite and, hence, f is strictly convex. Moreover, denoting by $\lambda_{min} > 0$ the smallest eigenvalue of $2A^\top A$, we have

$$f(x) \geq \lambda_{min}\|x\|^2 - 2\langle A^\top b, x\rangle + \|b\|^2.$$

and, hence, f is infinity at the infinity. Therefore, problem (12) admits only one solution $\bar{x}$. By Remark 4, the solution $\bar{x}$ is characterized by

$$A^\top A\bar{x} = A^\top b, \quad \text{i.e.} \quad \bar{x} = (A^\top A)^{-1} A^\top b.$$

◇ [Projection on a closed and convex set] Suppose that $\mathcal{K}$ is a nonempty closed and convex set and let $y \in \mathbb{R}^n$. Consider the problem the projection problem

$$\inf\left\{\|x - y\| \mid x \in \mathcal{K}\right\}. \qquad (Proj_{\mathcal{K}})$$

Note that $\mathcal{K}$ being closed and the cost functional being coercive, we have the existence of at least one solution $\bar{x}$. In order, to characterize $\bar{x}$ notice that the set of solutions to $(Proj_{\mathcal{K}})$ is the same as the set of solutions to the problem

$$\inf \left\{ \frac{1}{2}\|x-y\|^2 \mid x \in \mathcal{K} \right\}.$$

Then, since the cost functional of the problem above is strictly convex, Proposition 2 implies that $\bar{x}$ is its unique solution and, hence, is also the unique solution to $(Proj_{\mathcal{K}})$. Moreover, by Theorem 11(ii), we have that $\bar{x}$ is characterized by the inequality

$$(y-\bar{x})\cdot(x-\bar{x}) \leq 0 \quad \forall\, x \in \mathcal{K}. \tag{13}$$

Example: Let $b \in \mathbb{R}^m$ and $A \in \mathcal{M}_{m\times n}$ be such that

$$b \in \text{Im}(A) := \{Ax \mid x \in \mathbb{R}^m\}.$$

Suppose that

$$\mathcal{K} = \{x \in \mathbb{R}^n \mid Ax = b\}. \tag{14}$$

Then, $\mathcal{K}$ is closed, convex and nonempty. Moreover, for any $h \in \mathrm{Ker}(A)$ we have that $\bar{x} + h \in \mathcal{K}$. As a consequence, (13) implies that

$$(y - \bar{x}) \cdot h \leq 0 \quad \forall\, h \in \mathrm{Ker}(A),$$

and, using that $h \in \mathrm{Ker}(A)$ iff $-h \in \mathrm{Ker}(A)$, we get that

$$(y - \bar{x}) \cdot h = 0 \quad \forall\, h \in \mathrm{Ker}(A). \tag{15}$$

Conversely, since for every $x \in \mathcal{K}$ we have $x - \bar{x} \in \mathrm{Ker}(A)$, relation (15) implies (13), and, hence, (15) characterizes $\bar{x}$. Note that (15) can be written as[1]

$$y - \bar{x} \in \mathrm{Ker}(A)^{\perp} = \left\{ v \in \mathbb{R}^n \mid v^{\top} h = 0 \quad \forall\, h \in \mathrm{Ker}(A) \right\},$$

[1] Recall that given a subspace V of $\mathbb{R}^n$, the orthogonal space $V^{\perp}$ is defined by

$$V^{\perp} := \{ z \in \mathbb{R}^n \mid z^{\top} v = 0 \ \forall\, v \in V \}.$$

Two important properties of the orthogonal space are $V \oplus V^{\perp} = \mathbb{R}^n$, and $(V^{\perp})^{\perp} = V$.

or, equivalently,

$$y = \bar{x} + z \text{ for some } z \in \mathrm{Ker}(A)^{\perp}. \tag{16}$$

□

◇ [Convex problems with equality constraints] Now, we consider the same set $\mathcal{K}$ as in (14) but we consider a general differentiable convex objective function $f : \mathbb{R}^n \to \mathbb{R}$. We will need the following result from Linear Algebra.

Lemma 3. *Let $A \in \mathcal{M}_{m,n}(\mathbb{R})$. Then, $\mathrm{Ker}(A)^{\perp} = \mathrm{Im}(A^{\top})$.*

Proof. By the previous footnote, the desired relation is equivalent to $\mathrm{Im}(A^{\top})^{\perp} = \mathrm{Ker}(A)$. Now, $x \in \mathrm{Im}(A^{\top})^{\perp}$ iff $\langle x, A^{\top}y\rangle = 0$ for all $y \in \mathbb{R}^m$, and this holds iff $\langle Ax, y\rangle = 0$ for all $y \in \mathbb{R}^m$, i.e. $x \in \mathrm{Ker}(A)$. □

Proposition 3. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be differentiable and suppose that the set $\mathcal{K}$ in* (14) *is nonempty. Then $\bar{x}$ is a global solution to (P) iff $\bar{x} \in \mathcal{K}$ and there exists $\lambda \in \mathbb{R}^m$ such that*

$$\nabla f(\bar{x}) + A^{\top}\lambda = 0. \tag{17}$$

Proof. We are going to show that (17) is equivalent to (11) from which the result follows. Indeed, exactly as in the previous example, we have that (11) is equivalent to

$$\nabla f(\bar{x}) \cdot h = 0 \quad \forall\, h \in \mathrm{Ker}(A),$$

i.e.

$$\nabla f(\bar{x}) \in \mathrm{Ker}(A)^{\perp}.$$

Lemma 3 implies the existence of $\mu \in \mathbb{R}^m$ such that $\nabla f(\bar{x}) = A^{\top}\mu$. Setting $\lambda = -\mu$ we get (17). □

Example: Let $Q \in \mathcal{M}_{n,n}(\mathbb{R})$ be symmetric and positive definite, and $c \in \mathbb{R}^n$. In the framework of the previous proposition, suppose that f is given by

$$f(x) = \tfrac{1}{2}\langle Qx, x\rangle + c^{\top}x \quad \forall\, x \in \mathbb{R}^n,$$

and that A has m linearly independent columns. A classical linear algebra result states that this is equivalent to the fact that the m lines of A are linearly independent. In this case, we say that A has full rank.
Under the previous assumptions on Q, we have seen that f is strictly convex. Moreover, the condition on the columns of A implies that $\mathrm{Im}(A) = \mathbb{R}^m$ and, hence,

$\mathcal{K} \neq \emptyset$. Now, by Proposition 3 the point $\bar{x}$ solves (P) iff $\bar{x} \in \mathcal{K}$ and there exists $\lambda \in \mathbb{R}^m$ such that (17) holds. In other words, there exists $\lambda \in \mathbb{R}^m$ such that

$$A\bar{x} = b, \quad \text{and} \quad Q\bar{x} + c + A^\top \lambda = 0.$$

The second equation above yields $\bar{x} = -Q^{-1}(c + A^\top \lambda)$ and, hence, by the first equation, we get

$$AQ^{-1}c + AQ^{-1}A^\top \lambda + b = 0. \tag{18}$$

Let us show that $M := AQ^{-1}A^\top$ is invertible. Indeed, since $M \in \mathcal{M}_{m,m}(\mathbb{R})$ it suffices to show that $My = 0$ implies that $y = 0$. Now, let $y \in \mathbb{R}^m$ such that $My = 0$. Then, $\langle My, y \rangle = 0$ and, hence, $\langle Q^{-1}A^\top y, A^\top y \rangle = 0$, which implies, since Q^{-1} is also positive definite, that $A^\top y = 0$. Now, since the columns of $A^\top$ are also linearly independent, we deduce that $y = 0$, i.e. M is invertible. Using this fact, we can solve for λ in (18), obtaining

$$\lambda = -M^{-1}\left(AQ^{-1}c + b\right).$$

We deduce that

$$\bar{x} = -Q^{-1}\left(c - A^\top M^{-1}\left(AQ^{-1}c + b\right)\right), \tag{19}$$

is the unique solution to this problem.

Example: Let us now consider the projection problem

$$\begin{aligned} \min\ & \tfrac{1}{2}\|x - y\|^2 \\ \text{s.t.}\quad & Ax = b. \end{aligned}$$

Noting that $\frac{1}{2}\|x - y\|^2 = \frac{1}{2}\|x\|^2 - y^\top x + \frac{1}{2}\|y\|^2$, the previous problem has the same solution than

$$\begin{aligned} \min\ & \tfrac{1}{2}\|x\|^2 - y^\top x \\ \text{s.t.}\quad & Ax = b, \end{aligned}$$

which corresponds to $Q = I_{n\times n}$ (the $n \times n$ identity matrix) and $c = -y$. Then, (19) implies that the solution of this problem is given by

$$\bar{x} = (I - A^\top(AA^\top)^{-1}A)y + A^\top(AA^\top)^{-1}b.$$

Note that if $h \in \mathrm{Ker}(A)$

$$\begin{aligned}\langle y - \bar{x}, h \rangle &= \langle A^\top (AA^\top)^{-1} Ay - A^\top (AA^\top)^{-1} b, h \rangle \\ &= \langle (AA^\top)^{-1} Ay - (AA^\top)^{-1} b, Ah \rangle \\ &= 0,\end{aligned}$$

confirming (16).

Optimality conditions for problems with equality and inequality constraints

◇ [An introductory example: linear programming] A firm produces two kind of products. Let x_1, x_2 be, respectively, the quantity of product 1 and 2 (in tons) made in one month. Assume that there are some constraints on the quantity of x_1 and x_2:

- the factory cannot produce more than 3 units of x_1.
- fabrication process implies the following linear constraints on x_1 and x_2

$$-2x_1 + x_2 \leq 2, \qquad -x_1 + x_2 \leq 3.$$

The optimization problem is to chose the quantities x_1 and x_2 in order to maximize the benefits of the firm if the monthly revenue is $x_1 + 2x_2$.

The problem can be written as

$$\begin{aligned} &\sup \; x_1 + 2x_2 \\ &-2x_1 + x_2 \leq 2, \\ &-x_1 + x_2 \leq 3 \\ &0 \leq x_1 \leq 3, \quad 0 \leq x_2. \end{aligned} \tag{LP}$$

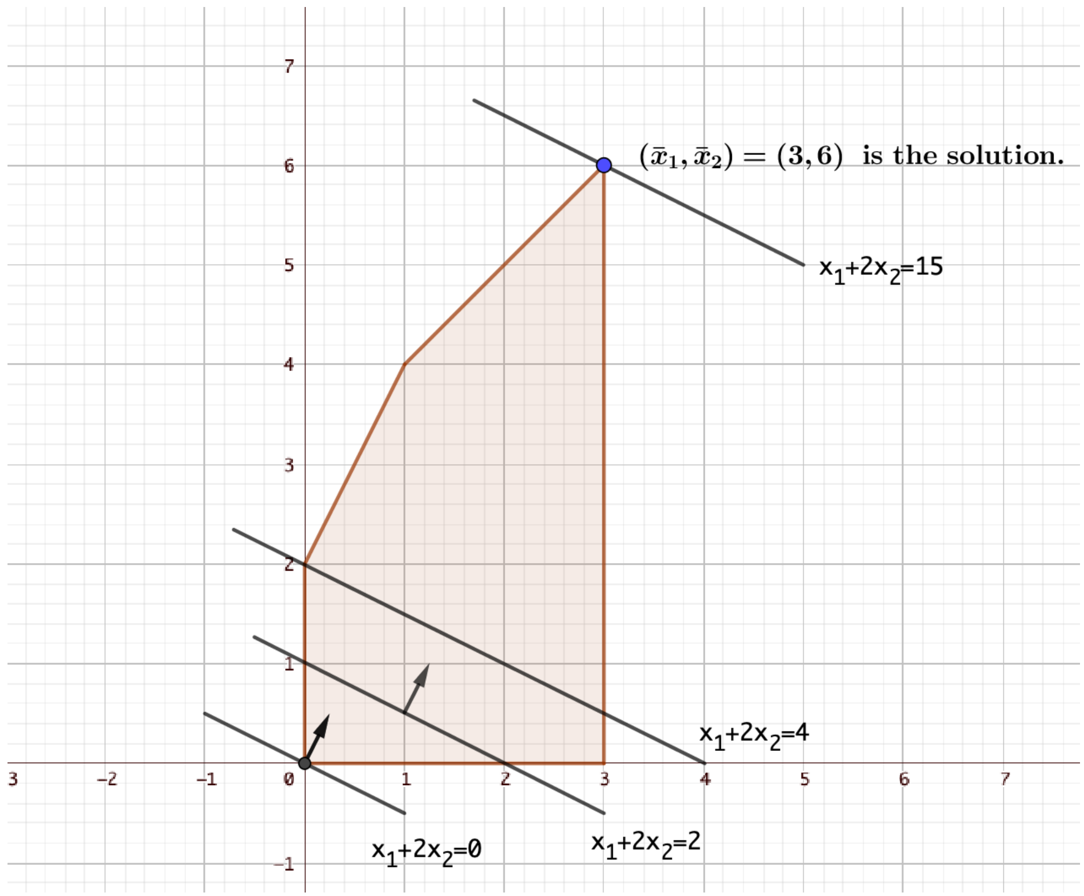
This two dimensional example can be solved graphically. See the figure below.

- For $f(x_1, x_2) = x_1 + 2x_2$, we consider the level sets $\text{Lev}_f(c)$ with $c \in \mathbb{R}$.
- $(\bar{x}_1, \bar{x}_2)$ solves the (LP) iff $\bar{c} := f(\bar{x}_1, \bar{x}_2)$ is the maximum $c \in \mathbb{R}$ such that $\text{Lev}_f(c) \cap P \neq \emptyset$, where P is the polygon defined by

$$P := \left\{ (x_1, x_2) \in \mathbb{R}^2 \mid -2x_1 + x_2 \leq 2, \;\; -x_1 + x_2 \leq 3, \;\; 0 \leq x_1 \leq 3, \;\; 0 \leq x_2 \right\}.$$

- In order to find such $\bar{c}$, we start with any $c \in \mathbb{R}$ such that $\text{Lev}_f(c) \cap P \neq \emptyset$ and then we vary c by moving the line $x_1 + 2x_2 = c$ in the normal direction given by $(1, 2)$ until we find $\bar{c}$.

- In larger dimensions $(n > 2)$, in practice this procedure cannot be applied. The most popular method to solve linear programming problems being the simplex method.



◇ [Nonlinear optimization problems with equality constraints] Consider problem (P) with

$$\mathcal{K} := \{x \in \mathbb{R}^n \mid g_1(x) = 0, \ldots, g_m(x) = 0\},$$

where, for all $i = 1, \ldots, m$, $g_i : \mathbb{R}^n \to \mathbb{R}$ is a given function. In this case, Problem (P) is usually written as

$$\left.\begin{array}{ll} \min & f(x) \\ \text{s.t.} & g_1(x) = 0, \\ & \vdots \\ & g_m(x) = 0. \end{array}\right\} \qquad (P)$$

In what follows we will assume that $n > m$. Indeed, if $n \leq m$, then, unless some of the constraints are redundant, the set $\mathcal{K}$ will eventually be empty or a singleton, and then (P) becomes trivial.

The main result in this section is the following first order necessary condition for optimality.

Theorem 12. [Lagrange] *Let* $\bar{x} \in \mathcal{K}$ *be a local solution to* (P)*. Assume that* f *and* g_i $(i = 1, \ldots, m)$ *are* $\mathcal{C}^1$*, and that*

$$\textit{the set of vectors } \{\nabla g_1(\bar{x}), \ldots, \nabla g_m(x)\} \textit{ are linearly independent.} \qquad (CQ)$$

Then, there exists $(\lambda_1, \ldots, \lambda_m) \in \mathbb{R}^m$ *such that*

$$\nabla f(\bar{x}) + \sum_{i=1}^{m} \lambda_i \nabla g(\bar{x}) = 0. \tag{20}$$

[Sketch of the proof] The technical point is the use of Assumption (CQ). Indeed, let us set $g(x) = (g_1(x), \ldots, g_m(x))$ and let $h \in \mathbb{R}^n$ be such that $h \in \mathrm{Ker}(Dg(\bar{x}))$. Under (CQ), the Implicit Function Theorem allows us to prove the existence of $\delta > 0$ and $\mathcal{C}^1$ function $\phi : (-\delta, \delta) \to \mathbb{R}^m$ such that $\phi(0) = \bar{x}$, $\phi(t) \in \mathcal{K}$ for all $t \in (-\delta, \delta)$ and $\phi'(0) = h$. Then, by the optimality of $\bar{x}$, and diminishing δ, if necessary, we get

$$f(\bar{x}) \leq f(\phi(t)) \ \forall\, t \in (-\delta, \delta),$$

which gives, after a Taylor expansion,

$$\nabla f(\bar{x})^\top h \geq 0.$$

Since $h \in \mathrm{Ker}(Dg(\bar{x}))$ is arbitrary we get that $\nabla f(\bar{x})^\top h = 0$, for all $h \in \mathrm{Ker}(Dg(\bar{x}))$, which implies that

$$\mathrm{Ker}(Dg(\bar{x})) \subseteq \mathrm{Ker}(\nabla f(\bar{x})^\top),$$

and, hence, from Lemma 3 we get

$$\mathsf{Im}(\nabla f(\bar{x})) = \mathrm{Ker}(\nabla f(\bar{x})^\top)^\perp \subseteq \mathrm{Ker}(Dg(\bar{x}))^\perp = \mathsf{Im}(Dg(\bar{x})^\top). \tag{21}$$

Relation (20) follows directly from (21).

Remark 5. (i) If $m = 1$, then (20) means that $\nabla f(\bar{x})$ and $\nabla g_1(\bar{x})$ are collinear.
(ii) The same optimality condition (20) holds if instead of considering minimization problem, we consider the maximization problem

$$\left.\begin{array}{l} \max\ f(x) \\ \text{s.t.}\ \ g_1(x) = 0, \\ \qquad\quad \vdots \\ \qquad\ \ g_m(x) = 0. \end{array}\right\}$$

(iii) Condition (CQ), called constraint qualification qualification condition, plays a important role. Indeed, let us consider the problem

$$\left.\begin{array}{l} \min\ x \\ \text{s.t.}\ \ x^3 - y^2 = 0, \end{array}\right\}$$

whose unique solution is $(\bar{x}, \bar{y}) = (0, 0)$. Relation (20) reads: there exists $\lambda \in \mathbb{R}$ such that

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 3x^2 \\ -2y \end{pmatrix}\Bigg|_{(\bar{x},\bar{y})=(0,0)} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

which clearly does not holds. The reason for this is that (CQ) does not holds. Indeed,

$$\left(\begin{array}{c} 3x^2 \\ -2y \end{array} \right) \Big|_{(\bar{x},\bar{y})=(0,0)} = \left(\begin{array}{c} 0 \\ 0 \end{array} \right),$$

which is not linearly independent.

(iv) Under (CQ) if $(\bar{x}, \lambda)$ and $(\bar{x}, \mu)$ satisfy (20), then $\lambda = \mu$. Indeed, we have

$$\sum_{i=1}^{m} (\lambda_i - \mu_i) \nabla g_i(\bar{x}) = 0,$$

and (CQ) implies that $\lambda_i = \mu_i$ for all $i = 1, \ldots, m$.

(v)[Affine constraints] We have seen that, in this case, (20) holds without having (CQ). However, in this case, the uniqueness of λ may not hold.

Definition 3. (i) *Given* $\lambda = (\lambda_1, \ldots, \lambda_m) \in \mathbb{R}^m$ *satisfying* (12) *and* $i \in \{1, \ldots, m\}$, *we say that* λ_i *is a Lagrange multiplier associated to the constraint* $g_i(x) = 0$.
(ii) *The function* $L : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ *defined by*

$$L(x, \lambda) = f(x) + \langle \lambda, g(x) \rangle,$$

is called the Lagrangian of problem (P).

Theorem 12 says that if $\bar{x}$ is a local solution to (P), then, there exists $\lambda \in \mathbb{R}^m$ such that

$$\nabla_x L(\bar{x}, \lambda) = 0.$$

Note that $\bar{x} \in \mathcal{K}$, which is equivalent to $g(x) = (g_1(\bar{x}), \ldots, g_m(\bar{x})) = 0$ for all $i = 1, \ldots, m$. Thus, $\nabla_\lambda L(\bar{x}, \lambda) = g(\bar{x}) = 0$, and, hence, $(\bar{x}, \lambda)$ satisfies

$$\nabla_x L(\bar{x}, \lambda) = 0, \qquad \nabla_\lambda L(\bar{x}, \lambda) = 0, \tag{22}$$

which is a system of $n + m$ equations for $n + m$ unknowns.

Example: Let us consider the problem

$$\left.\begin{array}{l} \min\ xy \\ \text{s.t.}\ \ x^2+(y+1)^2=1. \end{array}\right\}$$

In this case $f:\mathbb{R}^2\to\mathbb{R}$, is given by $f(x,y)=xy$, and $\mathcal{K}=\{(x,y)\in\mathbb{R}^2\mid g(x,y)=0\}$, with $g:\mathbb{R}^2\to\mathbb{R}$ being given by $g(x,y)=x^2+(y+1)^2-1$.

Note that $\mathcal{K}$ is given by the cercle centered at $(0,-1)$ with radius 1. Hence, $\mathcal{K}$ is a compact subset of $\mathbb{R}^2$. The function f being continuous, the Weierstrass theorem implies that the optimization problem has at least one solution $(\bar{x},\bar{y})\in\mathcal{K}$.

Let us check study (CQ). We have $\nabla g(x,y)=(2x,2(y+1))$ and, hence, $\nabla g(x,y)=0$ iff $x=0, y=-1$. Thus, every $(x,y)\in\mathbb{R}^2\setminus\{(0,-1)\}$ satisfies (CQ). Since $(0,-1)\notin\mathcal{K}$ we deduce that (CQ) holds for every $(x,y)\in\mathcal{K}$.

The Lagrangian $L:\mathbb{R}^2\times\mathbb{R}\to\mathbb{R}$ of this problem is given by

$$L(x,y,\lambda)=xy+\lambda(x^2+(y+1)^2-1).$$

By Theorem 12, we have the existence of $\lambda\in\mathbb{R}$ such that (22) holds at $(\bar{x},\bar{y},\lambda)$.

Now,

$$\nabla_{(x,y)} L(\bar{x}, \bar{y}, \lambda) = 0 \quad \Leftrightarrow \quad \begin{array}{rcl} \bar{y} + 2\lambda\bar{x} & = & 0, \\ \bar{x} + 2\lambda(\bar{y}+1) & = & 0, \end{array}$$
$$\Leftrightarrow \quad \begin{array}{rcl} \bar{y} & = & -2\lambda\bar{x}, \\ (1-4\lambda^2)\bar{x} & = & -2\lambda. \end{array} \tag{23}$$

Now, $1 - 4\lambda^2 = 0$ iff $\lambda = 1/2$ or $\lambda = -1/2$, and both cases contradict the last equality above. Therefore, $1 - 4\lambda^2 \neq 0$ and, hence,

$$\bar{x} = \frac{2\lambda}{4\lambda^2 - 1} \quad \text{and} \quad \bar{y} = \frac{-4\lambda^2}{4\lambda^2 - 1}.$$

Since $\nabla_\lambda L(\bar{x}, \bar{y}, \lambda) = g(\bar{x}, \bar{y}) = 0$, we get

$$\begin{array}{l} \left(\frac{2\lambda}{4\lambda^2-1}\right)^2 + \left(1 - \frac{4\lambda^2}{4\lambda^2-1}\right)^2 = 1, \\ \Leftrightarrow 4\lambda^2 + 1 = (4\lambda^2 - 1)^2 \\ \Leftrightarrow (4\lambda^2 - 1)^2 - (4\lambda^2 - 1) - 2 = 0, \end{array}$$

which yields

$$4\lambda^2 - 1 = \tfrac{1+\sqrt{9}}{2} \quad \text{or} \quad 4\lambda^2 - 1 = \tfrac{1-\sqrt{9}}{2}$$

$$\text{i.e. } \lambda^2 = 3/4 \quad \text{or} \quad \lambda^2 = 0.$$

If $\lambda = 0$, then (23) yields $\bar{x} = \bar{y} = 0$. If $\lambda = \sqrt{3}/2$ we get $\bar{x} = \sqrt{3}/2$ and $\bar{y} = -3/2$. If $\lambda = -\sqrt{3}/2$ we get $\bar{x} = -\sqrt{3}/2$ and $\bar{y} = -3/2$. Thus, the candidates to solve the problem are

$$(\bar{x}_1, \bar{y}_1) = (0, 0), \quad (\bar{x}_2, \bar{y}_2) = (\sqrt{3}/2, -3/2) \text{ and } (\bar{x}_3, \bar{y}_3) = (-\sqrt{3}/2, -3/2)$$

We have $f(\bar{x}_1, \bar{y}_1) = 0$, $f(\bar{x}_2, \bar{y}_2) = -3\sqrt{3}/4$ and $f(\bar{x}_3, \bar{y}_3) = 3\sqrt{3}/4$. Therefore, the global solution is $(\bar{x}_2, \bar{y}_2)$. □