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ASYMPTOTICS FOR SOME VIBRO-IMPACT PROBLEMS WITH A LINEAR DISSIPATION TERM

ALEXANDRE CABOT AND LAETITIA PAOLI

Abstract. Given $\gamma \geq 0$, let us consider the following differential inclusion

$$(S) \qquad \ddot{x}(t) + \gamma\, \dot{x}(t) + \partial\Phi(x(t)) \ni 0, \qquad t \in \mathbb{R}_+,$$

where $\Phi : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ is a lower semicontinuous convex function such that $\operatorname{int}(\operatorname{dom}\Phi) \neq \emptyset$. The operator $\partial\Phi$ denotes the subdifferential of Φ. When $\Phi = f + \delta_K$ with $f : \mathbb{R}^d \to \mathbb{R}$ a smooth convex function and $K \subset \mathbb{R}^d$ a closed convex set, inclusion (S) describes the motion of a discrete mechanical system subjected to the perfect unilateral constraint $x(t) \in K$ and submitted to the conservative force $-\nabla f(x)$ and the viscous friction force $-\gamma\, \dot{x}$. We define the notion of *dissipative* solution to (S) and we prove the existence of such solutions with conservation (resp. loss) of energy at impacts. If $\gamma > 0$ and $\Phi_{|\operatorname{dom}\Phi}$ is locally Lipschitz continuous, any dissipative solution to (S) converges, as $t \to +\infty$, to a minimum point of Φ. When Φ is strongly convex, the speed of convergence is exponential. Assuming as above that $\Phi = f + \delta_K$, suppose that the boundary of K is smooth enough and that the normal component of the velocity is reversed and multiplied by a restitution coefficient $r \in [0, 1]$ while the tangential component is conserved whenever $x(t) \in \operatorname{bd}(K)$. We prove that any dissipative solution to (S) satisfying the previous impact law with $r < 1$ is contained in the boundary of K after a finite time. The case $r = 1$ is also addressed and leads to a qualitatively different behavior.

1. Introduction

Throughout the paper, the space $\mathbb{R}^d$ is endowed with the Euclidean inner product $(\cdot,\cdot)$ and the corresponding norm $|\cdot|$. Given $\gamma \geq 0$, let us consider the second-order in time differential inclusion

$$(S) \qquad \ddot{x}(t) + \gamma\, \dot{x}(t) + \partial\Phi(x(t)) \ni 0, \qquad t \in \mathbb{R}_+,$$

where $\Phi : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ is a lower semicontinuous convex function such that $\operatorname{int}(\operatorname{dom}\Phi) \neq \emptyset$. The function Φ is called the potential function and the operator $\partial\Phi$ stands for the subdifferential of Φ in the sense of convex analysis: for every $x \in \operatorname{dom}\Phi$,

$$\xi \in \partial\Phi(x) \quad \Longleftrightarrow \quad \forall y \in \mathbb{R}^d, \quad \Phi(y) \geq \Phi(x) + (\xi, y - x).$$

The nonnegative parameter γ is called the friction parameter. When the function Φ is smooth, the subdifferential $\partial\Phi$ coincides with the gradient $\nabla\Phi$ and inclusion (S) becomes

$$(HBF) \qquad \ddot{x}(t) + \gamma\, \dot{x}(t) + \nabla\Phi(x(t)) = 0, \qquad t \in \mathbb{R}_+,$$

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called the “Heavy Ball with Friction” dynamical system, see [1, 3]. The (HBF) system is dissipative as soon as $\gamma > 0$, and can be studied in the classical framework of the theory of dissipative dynamical systems (see for example Hale [9] and Haraux [10]). The main interest of the (HBF) system in optimization is that it is not a descent method: it allows to go up and down along the graph of Φ.
In view of application to constrained optimization problems, one has to be able to deal with a nonsmooth potential Φ. Indeed, the problem of minimizing a given function $f : \mathbb{R}^d \to \mathbb{R}$ over a set $K \subset \mathbb{R}^d$ amounts to minimizing the extended real-valued function $\Phi := f + \delta_K$ over $\mathbb{R}^d$ (recall that the indicator function δ_K takes the value 0 over K and $+\infty$ elsewhere). This justifies the introduction of a nonsmooth term in the heavy ball equation. The general system (S) associated with a nonsmooth convex function Φ has been studied in [2] under the terminology of “Generalized Heavy Ball with Friction” system.

Let us now comment on the mechanical origin of the system (S). Let $f : \mathbb{R}^d \to \mathbb{R}$ be a smooth convex potential and let $K \subset \mathbb{R}^d$ be a closed convex set. Consider a discrete mechanical system with d degrees of freedom and a trivial mass matrix, whose position and velocity are respectively denoted by x and $\dot{x}$. We assume that the system is subjected to the action of the conservative force $-\nabla f(x)$ and the viscous friction force $-\gamma\,\dot{x}$. An immediate application of the Fundamental Principle of Dynamics shows that the unconstrained motion is described by the following ODE

$$\ddot{x} = -\gamma\,\dot{x} - \nabla f(x).$$

We assume that the trajectory must remain in the set K, *i.e.* $x(t) \in K$ for all $t \geq 0$. This unilateral constraint may lead to some discontinuities for the velocity. Indeed, let us assume for instance that $x(t) \in \operatorname{int}(K)$ for all $t \in (t_0, t_1) \cup (t_1, t_2) \subset \mathbb{R}_+$ and $x(t_1) \in \operatorname{bd}(K)$. Then the constraint implies that $\dot{x}^-(t_1) \in -T_K(x(t_1))$ and $\dot{x}^+(t_1) \in T_K(x(t_1))$, where $T_K(\xi)$ denotes the tangent cone to K at ξ given by

$$T_K(\xi) = \overline{\cup_{\lambda>0}\lambda(K-\xi)}.$$

Hence, if $\dot{x}^-(t_1) \notin T_K(x(t_1))$, it is clear that $\dot{x}$ is discontinuous at $t = t_1$. It follows that the equation of the motion has to be modified by adding a measure μ to the right-hand side, *i.e.*

$$\ddot{x} = -\gamma\,\dot{x} - \nabla f(x) + \mu.$$

This measure μ describes the reaction force due to the unilateral constraint and

$$\operatorname{Supp}(\mu) \subset \{t \geq 0, \quad x(t) \in \operatorname{bd}(K)\}.$$

Assuming moreover that the constraint is perfect, we infer (see [13]) that $-\mu \in N_K(x)$. Recall that the normal cone $N_K(\xi)$ to K at $\xi \in \mathbb{R}^d$ is defined by $N_K(\xi) = T_K(\xi)^{\perp}$. It ensues that the motion is described by the following measure differential inclusion

$$\ddot{x} + \gamma\,\dot{x} + \nabla f(x) + N_K(x) \ni 0,$$

and we recover the dynamical system (S) associated with $\Phi := f + \delta_K$. The discontinuities of the velocity satisfy

$$\dot{x}^+(t) - \dot{x}^-(t) = \mu(\{t\}) \in -N_K(x(t)),$$

but this is not sufficient to define uniquely $\dot{x}^+(t)$ as a function of $\dot{x}^-(t)$. Thus, we should add an assumption on the behavior of the energy at impacts. Since we have in mind the asymptotic study of (S) as $t \to +\infty$, we will assume that the energy is dissipated at impacts, *i.e.* $|\dot{x}^+(t)| \le |\dot{x}^-(t)|$ for every $t \ge 0$, which yields the mechanical consistency of the model when $\Phi = f + \delta_K$. This leads us to the notion of *dissipative* solution (see definition 2.1 below). The limit case corresponding to the conservation of energy, gives the so-called *elastic* shocks. The existence of dissipative solutions conserving the energy at shocks can be easily derived from [20, 28] or also [2]. In the first two references, the energy conserving solution is obtained as a limit of Moreau-Yosida approximate solutions, while in the third one it is obtained by using a general epiconvergent approximation of Φ. Section 5 is devoted to these questions of existence of dissipative solutions with conservation of energy at shocks. A quite different situation corresponds to the so-called *inelastic* shocks, for which the normal component of the velocity vanishes after each impact, *i.e.*

$$\dot{x}^+(t) = \mathrm{Proj}\big(T_K\big(x(t)\big), \dot{x}^-(t)\big), \qquad \forall t \ge 0. \tag{1.1}$$

The notion of standard inelastic shocks was introduced by Moreau [14, 15, 17] and the reader is referred to [24, 25] for a mathematical justification of this impact law by a penalty method. The existence of a solution satisfying the inelastic bounce law (1.1) is an open problem in the case of a general convex set K. When the boundary of K is smooth enough, the set K can be described at least locally by a single inequality, *i.e.* $K = \big\{\xi \in \mathbb{R}^d;\, \varphi(\xi) \ge 0\big\}$. In this single-constraint case, several existence results have been obtained. The corresponding proofs rely on the study of a sequence of approximate solutions which are built either by means of a time-discretization of the differential inclusion (see [12, 19, 11, 26]) or by means of a penalization (see [19, 23, 29]). When K is described by several inequalities (multi-constraint case), *i.e.*

$$K = \big\{\xi \in \mathbb{R}^d;\, \varphi_\alpha(\xi) \ge 0, \quad 1 \le \alpha \le \nu\big\}, \quad \nu \ge 1,$$

Paoli [21] has recently brought to light a geometric condition on the active constraints, which ensures the existence of a solution satisfying the inelastic bounce law (1.1). The proof is based on the use of a suitable time-discretization scheme, already proposed in [19, 26]. We cannot apply immediately the results of [21], but the same techniques allow to prove the existence of a dissipative solution to (S) satisfying the inelastic bounce law (1.1) (see section 6).

Once the existence of dissipative solutions to (S) is acquired, our main purpose is the asymptotic study of such solutions as $t \to +\infty$. The assumption $\gamma > 0$ plays now a crucial role since it corresponds to the dissipative character of the dynamical system (S). Assuming that the function Φ is bounded from below, we prove that the value of Φ tends toward $\inf \Phi$ on each trajectory associated to (S) as $t \to +\infty$. If moreover $\mathrm{argmin}\,\Phi \ne \emptyset$, any dissipative solution x to (S) tends toward a minimum of Φ. These results are gathered in Theorem 3.4, which is a generalization of a former result due to Attouch, Cabot and Redont [2]. In this paper, the authors deal with a nonsmooth potential Φ, but the study is restricted to the elastic shock solutions which are obtained as limits of approximate trajectories via penalization techniques. If moreover the function Φ is minorized by some definite positive quadratic term (which is the case if Φ is strongly convex), we prove that any trajectory x converges toward the unique minimum of Φ and the speed of convergence is exponential (see Theorem 3.5).

Coming back to the mechanical interpretation of (S), let us assume that the function Φ can be decomposed as $\Phi = f+\delta_K$, with $f : \mathbb{R}^d \to \mathbb{R}$ a smooth convex function and $K \subset \mathbb{R}^d$ a closed convex set with smooth boundary. As explained above, the assumption of perfect unilateral constraint shows that $\dot{x}^+(t) - \dot{x}^-(t) \in -N_K\big(x(t)\big)$ which implies that the tangential component of the velocity is conserved. To complete the model, we choose a Newton's impact law, *i.e.* we assume that the normal velocity is reversed and multiplied by a restitution coefficient $r \in [0, 1]$ whenever $x(t) \in \mathrm{bd}(K)$. When $r < 1$, we prove that any trajectory of (S) is contained in the boundary of K after a finite time. Of course, in the particular case of a punctual particle submitted to gravity and bouncing on the floor (supposed horizontal and plane), we recover that the dynamics stops after a finite time. On the other hand, when $r = 1$, either the trajectory is contained in the boundary of K after a finite time, or there is a countable infinity of impacts. Denoting by $(t_n)_{n\in\mathbb{N}}$ the instants of impact, we obtain in addition the estimate $t_n \sim \frac{3}{\gamma} \ln n$ as $n \to +\infty$. These results, based on the linearization of the dynamics near the equilibrium, are stated in Theorem 4.3.

The article is organized as follows: in section 2, we precisely state the notion of dissipative solution to (S). We also define the mechanical energy E of the dynamical system (S). The decay properties of the function E are crucial for the study of the asymptotic behavior of (S), which is analyzed in section 3. The main results of this section are the convergence of the trajectory as soon as $\mathrm{argmin}\, \Phi \neq \emptyset$ (*cf.* Theorem 3.4) and the exponential decay of the energy under some further assumption (*cf.* Theorem 3.5). In section 4, we assume that the function Φ can be decomposed as $\Phi = f + \delta_K$, where $f : \mathbb{R}^d \to \mathbb{R}$ is a smooth convex function and $K \subset \mathbb{R}^d$ is a closed convex set with smooth boundary. We show that any dissipative solution to (S) satisfying the Newton's law with restitution coefficient $r \in [0, 1)$ remains in the boundary of K after a finite time (*cf.* Theorem 4.3). For a matter of readability, the questions of existence of dissipative solutions to (S) are postponed to sections 5 and 6.

2. Notion of dissipative solution

Let us now recall the basic definitions of the functional spaces that will serve throughout the paper. The set $\mathcal{C}^0_c(\mathbb{R}_+, \mathbb{R}^d)$ is the space of continuous functions from $\mathbb{R}_+$ into $\mathbb{R}^d$ with compact support in $\mathbb{R}_+$. The set $\mathcal{M}(\mathbb{R}_+, \mathbb{R}^d)$ is the space of Radon measures on $\mathbb{R}_+$ with values in $\mathbb{R}^d$, that is the dual space to $\mathcal{C}^0_c(\mathbb{R}_+, \mathbb{R}^d)$ equipped with its usual inductive limit topology; it may be identified with the space of regular Borel measures on $\mathbb{R}_+$ with values in $\mathbb{R}^d$ that are of finite variation (Dinculeanu [8, §19], Moreau [16, section 7]). The set $BV_{loc}(\mathbb{R}_+, \mathbb{R}^d)$ is the space of functions from $\mathbb{R}_+$ into $\mathbb{R}^d$ that are of locally bounded variation, see [8, 16]. Every $u \in BV_{loc}(\mathbb{R}_+, \mathbb{R}^d)$ has respectively left and right limits, $u^-(t)$ and $u^+(t)$ at any point $t \geq 0$ (with the convention $u^-(0) = u(0)$). Recall also that the set of discontinuities of every $u \in BV_{loc}(\mathbb{R}_+, \mathbb{R}^d)$ is at most countable.

Given $\gamma \geq 0$, let us consider the second-order in time differential inclusion

$$(S) \qquad \ddot{x}(t) + \gamma\, \dot{x}(t) + \partial\Phi(x(t)) \ni 0, \qquad t \in \mathbb{R}_+,$$

where $\Phi : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ is a lower semicontinuous convex function such that $\mathrm{int}\,(\mathrm{dom}\, \Phi) \neq \emptyset$. We supplement system (S) with initial data satisfying

$(x_0, \dot{x}_0) \in \operatorname{dom}\Phi \times T_{\overline{\operatorname{dom}\Phi}}(x_0)$, *i.e.* admissible initial data. As we have observed, for a nonsmooth potential Φ, we cannot expect the previous dynamical system to have a regular solution. We have to accept, as possible solutions, functions whose second derivatives are measures defined on $\mathbb{R}_+$ with values in $\mathbb{R}^d$. Let us now make precise the notion of dissipative solution to (S) associated with the initial data $(x_0, \dot{x}_0)$.

Definition 2.1. Let $\gamma \geq 0$ and $\Phi : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ be a convex, lower semicontinuous function such that $\operatorname{int}(\operatorname{dom}\Phi) \neq \emptyset$. A map $x : \mathbb{R}_+ \to \mathbb{R}^d$ is called a dissipative solution to (S) for the initial data $(x_0, \dot{x}_0) \in \operatorname{dom}\Phi \times T_{\overline{\operatorname{dom}\Phi}}(x_0)$ if and only if it satisfies:
(i) x is locally Lipschitz continuous on $\mathbb{R}_+$ and $x(t) \in \operatorname{dom}\Phi$ for every $t \in \mathbb{R}_+$.
(ii) $\dot{x} \in BV_{loc}(\mathbb{R}_+; \mathbb{R}^d)$; $\ddot{x} = d\dot{x} \in \mathcal{M}(\mathbb{R}_+; \mathbb{R}^d)$.
(iii) For every $(t_1, t_2) \in \mathbb{R}_+^2$ such that $t_1 < t_2$ and $\dot{x}^-(t_1) = \dot{x}(t_1)$ and $\dot{x}(t_2) = \dot{x}^+(t_2)$ and for every function $y \in \mathcal{C}^0([t_1, t_2]; \mathbb{R}^d)$, we have

$$\int_{t_1}^{t_2} (\Phi(y(t)) - \Phi(x(t))\, dt \geq -\langle \ddot{x} + \gamma\, \dot{x}\, dt, y - x\rangle_{\mathcal{M}([t_1,t_2];\mathbb{R}^d), \mathcal{C}^0([t_1,t_2];\mathbb{R}^d)},$$

with the convention $\dot{x}^-(0) = \dot{x}(0) = \dot{x}_0$.
(iv) The initial conditions are fulfilled in the following sense:

$$x(0) = x_0, \quad \dot{x}^+(0) = \dot{x}_0.$$

(v) For every $t \in \mathbb{R}_+^*$, we have $|\dot{x}^+(t)| \leq |\dot{x}^-(t)|$.

Items (i) to (iv) are standard in the formalism of vibro-impact problems. Close definitions of generalized solutions to (S) may already be found in [2, 19, 20, 21, 23, 28]. Point (v) expresses that the solution x dissipates energy at shocks. Let us state the first properties of dissipative solutions.

Proposition 2.2. *Let $\gamma \geq 0$ and $\Phi : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ be a convex, lower semicontinuous function such that* $\operatorname{int}(\operatorname{dom}\Phi) \neq \emptyset$*. Let x be a dissipative solution to (S) and let $\ddot{x} = \ddot{x}_a\, dt + \ddot{x}_s$ be the decomposition of $\ddot{x}$ with respect to the Lebesgue's measure. Then, we have:*
(i) $\ddot{x}_a(t) + \gamma\dot{x}(t) + \partial\Phi(x(t)) \ni 0$ for almost every $t \in \mathbb{R}_+$,
(ii) $\dot{x}^+(t) - \dot{x}^-(t) \in -N_{\overline{\operatorname{dom}\Phi}}(x(t))$ for all $t \in \mathbb{R}_+$.

Proof. Let $T > 0$ be such that $\dot{x}^+(T) = \dot{x}(T)$ and let us denote by μ the Stieltjes measure associated to $-\dot{x} - \gamma\, x$ on $[0, T]$. Condition (iii) of Definition 2.1 implies that

$$\int_0^T \Phi(y(t)) - \Phi(x(t))\, dt \geq \langle \mu, y - x\rangle_{\mathcal{M}([0,T];\mathbb{R}^d), \mathcal{C}^0([0,T];\mathbb{R}^d)}$$

for every function $y \in \mathcal{C}^0([0, T]; \mathbb{R}^d)$. Therefore, the measure μ belongs to the subdifferential set $\partial J_\Phi(x)$, where the functional J_Φ is defined by

$$J_\Phi : \begin{cases} \mathcal{C}^0([0,T];\mathbb{R}^d) & \to \quad \mathbb{R} \cup \{+\infty\} \\ \quad y & \mapsto \quad \int_0^T \Phi(y(t))\, dt. \end{cases}$$

Let $\mu = g dt + \mu_s$ be the decomposition of μ with respect to the Lebesgue's measure on $[0, T]$, where $g \in L^1([0, T]; \mathbb{R}^d, dt)$ and μ_s is a singular measure with respect to the Lebesgue's measure. From the Radon-Nikodym theorem, we also have $\mu_s = h\, d|\mu_s|$ with $h \in L^1([0, T]; \mathbb{R}^d, d|\mu_s|)$. Corollary 5.A of Rockafellar [27] shows that:

$$g(t) \in \partial\Phi(x(t)) \qquad dt\text{ -almost everywhere on } [0, T], \tag{2.1}$$

$$h(t) \in N_{\overline{\mathrm{dom}\,\Phi}}(x(t)) \qquad d|\mu_s| \text{ -almost everywhere on } [0,T]. \tag{2.2}$$

The definition of $\ddot{x}_a$ and g shows immediately that $g = -\ddot{x}_a - \gamma\,\dot{x}$ almost everywhere on $[0,T]$. We deduce from (2.1) that the inclusion of item (i) is satisfied for almost every $t \in [0,T]$. Since T is arbitrary (up to the denumerable set of the discontinuity points of $\dot{x}$), this inclusion actually holds for almost every $t \in \mathbb{R}_+$.
Let $t \in [0,T]$. If $\dot{x}^+(t) = \dot{x}^-(t)$, the inclusion of item (ii) is trivially satisfied since the cone $N_{\overline{\mathrm{dom}\,\Phi}}(x(t))$ contains the origin. Now assume that $\dot{x}^+(t) \neq \dot{x}^-(t)$. Notice that

$$\begin{aligned} \dot{x}^+(t) - \dot{x}^-(t) &= -\mu(\{t\}) = -\mu_s(\{t\}) \\ &= -h(t)\,|\mu_s|(\{t\}). \end{aligned} \tag{2.3}$$

Since $\dot{x}^+(t) \neq \dot{x}^-(t)$, we obtain $|\mu_s|(\{t\}) > 0$. Hence point t is not negligible for the measure $|\mu_s|$ and inclusion (2.2) must hold. Recalling that the set $N_{\overline{\mathrm{dom}\,\Phi}}(x(t))$ is a cone, we derive from (2.3) that $\dot{x}^+(t) - \dot{x}^-(t) \in -N_{\overline{\mathrm{dom}\,\Phi}}(x(t))$. Since T is arbitrary, this concludes the proof of item (ii). □

Point (i) of Proposition 2.2 establishes a consistant link between the notions of classical and dissipative solutions. Point (ii) shows that a discontinuity of the velocity $\dot{x}$ can only occur on the boundary of $\mathrm{dom}\,\Phi$; indeed if $x \in \mathrm{int}\,(\mathrm{dom}\,\Phi)$ then $N_{\overline{\mathrm{dom}\,\Phi}}(x) = \{0\}$. Let us define the mechanical energy E of the dynamical system by:

$$E(t) = \frac{1}{2}|\dot{x}^+(t)|^2 + \Phi(x(t)) \qquad \text{for every } t \in \mathbb{R}_+.$$

The energy E is a nonincreasing function and the following proposition describes its decay rate.

Proposition 2.3. *Let $\gamma \geq 0$ and $\Phi : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ be a convex, lower semicontinuous function such that $\mathrm{int}\,(\mathrm{dom}\,\Phi) \neq \emptyset$. Additionally assume that the function $\Phi_{|\mathrm{dom}\,\Phi}$ is locally Lipschitz continuous. Let x be a dissipative solution to (S) and let E be the associate energy function. For every $(t_1, t_2) \in \mathbb{R}_+^2$ such that $t_1 \leq t_2$, we have:*

$$E(t_2) - E(t_1) = -\gamma \int_{t_1}^{t_2} |\dot{x}(s)|^2\,ds + \frac{1}{2}\sum_{t \in D}(|\dot{x}^+(t)|^2 - |\dot{x}^-(t)|^2),$$

where $D := \{t \in (t_1, t_2], \quad \dot{x} \text{ is discontinuous at } t\}$. Hence, in view of Definition 2.1(v), the following inequality holds:

$$E(t_2) - E(t_1) \leq -\gamma \int_{t_1}^{t_2} |\dot{x}(s)|^2\,ds. \tag{2.4}$$

Proof. Let $T > t_2$ be such that $\dot{x}^+(T) = \dot{x}(T)$ and let us denote by μ the Stieltjes measure associated to $-\dot{x} - \gamma\,x$ on $[0,T]$. Let us decompose the measure μ as in the proof of Proposition 2.2: $\mu = g\,dt + h\,d|\mu_s|$, where μ_s is a singular measure with respect to the Lebesgue's measure and the functions g, h satisfy respectively $g \in L^1([0,T]; \mathbb{R}^d, dt)$ and $h \in L^1([0,T]; \mathbb{R}^d, d|\mu_s|)$. The arguments of the proof are the same as those proposed by Ballard [4, Proposition 7] and they rely on the derivation formula of a bilinear expression (see Moreau [16, p. 38-43]). We have:

$$\begin{aligned}\frac{1}{2}|\dot{x}^+(t_2)|^2 - \frac{1}{2}|\dot{x}^+(t_1)|^2 &= \int_{(t_1,t_2]} \left\langle \frac{\dot{x}^+(t)+\dot{x}^-(t)}{2}, \ddot{x} \right\rangle \\ &= \int_{(t_1,t_2]\setminus D} \langle \dot{x}(t), \ddot{x}\rangle + \int_D \left\langle \frac{\dot{x}^+(t)+\dot{x}^-(t)}{2}, \ddot{x} \right\rangle, \end{aligned} \tag{2.5}$$

where $\langle .,. \rangle$ denotes the duality bracket between the spaces $\mathcal{C}^0([0,T];\mathbb{R}^d)$ and $\mathcal{M}([0,T];\mathbb{R}^d)$. It is immediate that

$$\int_D \left\langle \frac{\dot{x}^+(t)+\dot{x}^-(t)}{2}, \ddot{x} \right\rangle = \frac{1}{2}\sum_{t\in D}(|\dot{x}^+(t)|^2 - |\dot{x}^-(t)|^2). \tag{2.6}$$

On the other hand, recalling that $\ddot{x} = -\gamma\dot{x}\,dt - g\,dt - h\,d|\mu_s|$ on $[0,T]$ and that the set D is negligible for the Lebesgue's measure, we obtain

(2.7)
$$\int_{(t_1,t_2]\setminus D} \langle \dot{x}(t), \ddot{x}\rangle = -\gamma \int_{t_1}^{t_2} |\dot{x}(s)|^2\,ds - \int_{t_1}^{t_2} (\dot{x}(t), g(t))\,dt - \int_{(t_1,t_2]\setminus D} (\dot{x}(t), h(t))\,d|\mu_s|.$$

Let us first compute the term $\int_{t_1}^{t_2}(\dot{x}(t), g(t))\,dt$. Since the maps x and $\Phi_{|\mathrm{dom}\,\Phi}$ are locally Lipschitz continuous and since $x(\mathbb{R}_+) \subset \mathrm{dom}\,\Phi$, the composition $\Phi \circ x$ is also locally Lipschitz continuous, hence locally absolutely continuous. On the other hand, we know from (2.1) that $g(t) \in \partial\Phi(x(t))$ for almost every $t \in [0,T]$ and we deduce from a classical result (see for example Brezis [5, p. 60]) that

$$\frac{d}{dt}\Phi(x(t)) = (\dot{x}(t), g(t)) \qquad \text{for almost every } t \in [0,T]. \tag{2.8}$$

For the sake of completeness, we recall next the proof of this result (see Claim 2.4). Since $\Phi \circ x$ is locally absolutely continuous, we can integrate (2.8) on $[t_1, t_2]$ to obtain

$$\Phi(x(t_2)) - \Phi(x(t_1)) = \int_{t_1}^{t_2} (\dot{x}(t), g(t))\,dt. \tag{2.9}$$

Let us finally evaluate the term $\int_{(t_1,t_2]\setminus D}(\dot{x}(t), h(t))\,d|\mu_s|$. From (2.2), we have $h(t) \in N_{\overline{\mathrm{dom}\,\Phi}}(x(t))$ for $d|\mu_s|$ -almost every $t \in [0,T]$. On the other hand, for every $t \in (t_1, t_2] \setminus D$, the following holds:

$$\dot{x}(t) = \dot{x}^+(t) = \dot{x}^-(t) \in T_{\overline{\mathrm{dom}\,\Phi}}(x(t)) \cap (-T_{\overline{\mathrm{dom}\,\Phi}}(x(t))).$$

It ensues that $(\dot{x}(t), h(t)) = 0$ for $d|\mu_s|$-almost every $t \in (t_1, t_2] \setminus D$ and finally

$$\int_{(t_1,t_2]\setminus D} (\dot{x}(t), h(t))\,d|\mu_s| = 0. \tag{2.10}$$

The conclusion results from the combination of (2.5), (2.6), (2.7), (2.9) and (2.10). □

Let us now prove the above-mentioned result.

Claim 2.4. *Under the assumptions of Proposition 2.3, we have for almost every* $t \in \mathbb{R}_+$

$$\forall \xi \in \partial\Phi(x(t)), \quad \frac{d}{dt}\Phi(x(t)) = (\dot{x}(t), \xi).$$

Proof. Let $\mathcal{E}$ be the subset of $\mathbb{R}_+$ on which the maps $t \mapsto x(t)$ and $t \mapsto \Phi(x(t))$ are derivable. Since x and $\Phi \circ x$ are locally Lipschitz continuous on $\mathbb{R}_+$, it is clear that $\mathbb{R}_+ \setminus \mathcal{E}$ is negligible for the Lebesgue's measure. Fix $t \in \mathcal{E}$. We have, for every $\varepsilon > 0$

$$\Phi(x(t+\varepsilon)) - \Phi(x(t)) \geq (x(t+\varepsilon) - x(t), \xi)$$

where $\xi \in \partial\Phi(x(t))$. Dividing by ε and letting $\varepsilon \to 0$, we obtain $\frac{d}{dt}\Phi(x(t)) \geq (\dot{x}(t), \xi)$. Replacing ε by $-\varepsilon$ yields the converse inequality. □

3. Asymptotic behaviour of dissipative solutions.

Throughout this section, we are interested in the asymptotic study of the dissipative solutions to (S). Since the dissipative character of the dynamical system (S) is enforced by the positivity of γ, we now assume $\gamma > 0$.

3.1. **Asymptotic estimates and convergence of the trajectory.** Let us start by classical L^2 and L^∞ estimates of the velocity $\dot{x}$.

Proposition 3.1. *Let $\gamma > 0$ and let $\Phi : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ be a convex, lower semicontinuous function which is bounded from below and such that* $\operatorname{int}(\operatorname{dom}\Phi) \neq \emptyset$. *Additionally assume that the function $\Phi_{|\operatorname{dom}\Phi}$ is locally Lipschitz continuous. Let x be a dissipative solution to (S). Then $\dot{x} \in L^\infty(\mathbb{R}_+;\mathbb{R}^d) \cap L^2(\mathbb{R}_+;\mathbb{R}^d)$.*

Proof. Setting $t_1 = 0$ and $t_2 = t \in \mathbb{R}_+$ in the energy inequality (2.4), we obtain

$$\forall t \in \mathbb{R}_+, \qquad \frac{1}{2}|\dot{x}^+(t)|^2 + \Phi(x(t)) \leq E(0) - \gamma \int_0^t |\dot{x}(s)|^2\,ds. \tag{3.1}$$

Since Φ is bounded from below, we immediately deduce that for every $t \geq 0$, $|\dot{x}^+(t)|^2 \leq 2\,E(0) - 2\inf\Phi$. Since $\dot{x}(t) = \dot{x}^+(t) = \dot{x}^-(t)$ except for a countable set of values $t \in \mathbb{R}_+$, it ensues that $\dot{x} \in L^\infty(\mathbb{R}_+;\mathbb{R}^d)$. In view of (3.1), we have $\int_0^t |\dot{x}(s)|^2\,ds \leq (E(0) - \inf\Phi)/\gamma$ for every $t \in \mathbb{R}_+$. Taking the limit when $t \to +\infty$, we immediately obtain that $\dot{x} \in L^2(\mathbb{R}_+;\mathbb{R}^d)$. □

For any $z \in \mathbb{R}^d$, let us introduce the function k defined by

$$\forall t \in \mathbb{R}_+, \qquad k(t) = (\dot{x}^+(t), x(t) - z) + \frac{\gamma}{2}\,|x(t) - z|^2. \tag{3.2}$$

This function can be decomposed as the sum of the function $t \mapsto \frac{1}{2}\,|x(t) - z|^2$ (up to the constant γ) and its right derivative. The use of the auxilliary function k is classical in the asymptotic study of second-order dynamical systems with linear damping (see for example [2, 7]). Before stating the main theorem, we need the following lemma.

Lemma 3.2. *Under the assumptions of Proposition 3.1, let x be a dissipative solution to (S), let E be the associate energy function and k be the function defined by (3.2). Let $z \in \operatorname{dom}\Phi$. For any $(t_1, t_2) \in \mathbb{R}_+^2$ such that $t_1 < t_2$ and $\dot{x}^-(t_i) = \dot{x}(t_i) = \dot{x}^+(t_i)$ $(i{=}1{,}2)$, we have*

$$k(t_2) - k(t_1) + \int_{t_1}^{t_2} (E(s) - \Phi(z))\,ds \leq \frac{3}{2}\int_{t_1}^{t_2} |\dot{x}(s)|^2\,ds, \tag{3.3}$$

with the convention $\dot{x}^-(0) = \dot{x}(0) = \dot{x}_0$.

Proof. Let us decompose the function k as $k = k_1 + k_2$ where k_1 and k_2 are respectively defined by

$$k_1(t) = \frac{\gamma}{2}|x(t) - z|^2 \qquad \text{and} \qquad k_2(t) = (\dot{x}^+(t), x(t) - z),$$

for every $t \in \mathbb{R}_+$. The function k_1 is locally Lipschitz continuous on $\mathbb{R}_+$, hence derivable a.e. on $\mathbb{R}_+$ and

$$\dot{k}_1(t) = \gamma\,(\dot{x}(t), x(t) - z) \qquad \text{a.e. on } \mathbb{R}_+.$$

Since k_1 is locally absolutely continuous on $\mathbb{R}_+$, we have

$$k_1(t_2) - k_1(t_1) = \int_{t_1}^{t_2} \dot{k}_1(t)\,dt = \int_{t_1}^{t_2} \gamma\,(\dot{x}(t), x(t) - z)\,dt. \tag{3.4}$$

On the other hand, the function k_2 belongs to $BV_{loc}(\mathbb{R}_+; \mathbb{R}^d)$ and by applying the derivation rules of bilinear expressions (see Moreau [16, p. 38-43]), we find

$$\begin{aligned} dk_2 &= \langle d\dot{x}, x - z\rangle + \langle \dot{x}^+, d(x - z)\rangle \\ &= \langle \ddot{x}, x - z\rangle + \langle \dot{x}^+, \dot{x}\,dt\rangle. \end{aligned}$$

By integrating the previous expression on $[t_1, t_2]$, we derive

$$k_2^+(t_2) - k_2^-(t_1) = \int_{[t_1,t_2]} \langle \ddot{x}, x - z\rangle + \int_{[t_1,t_2]} (\dot{x}^+, \dot{x})\,dt, \tag{3.5}$$

under the convention $k_2^-(0) = k_2(0)$. Since by assumption $\dot{x}^+(t_i) = \dot{x}^-(t_i)$ $(i=1,2)$, we have $k_2^-(t_1) = k_2(t_1)$ and $k_2^+(t_2) = k_2(t_2)$. Recalling that $\dot{x}^+(t) = \dot{x}^-(t) = \dot{x}(t)$ almost everywhere on $[t_1, t_2]$, we can rewrite (3.5) as

$$k_2(t_2) - k_2(t_1) = \int_{[t_1,t_2]} \langle \ddot{x}, x - z\rangle + \int_{t_1}^{t_2} |\dot{x}|^2\,dt.$$

By adding this last equality to (3.4), we finally obtain

$$k(t_2) - k(t_1) = \int_{[t_1,t_2]} \langle \ddot{x} + \gamma\,\dot{x}\,dt, x - z\rangle + \int_{t_1}^{t_2} |\dot{x}|^2\,dt. \tag{3.6}$$

Defining the constant function $y(t) = z$ for every $t \in [t_1, t_2]$, property (iii) of the definition of dissipative solutions gives

$$\int_{[t_1,t_2]} -\langle \ddot{x} + \gamma\,\dot{x}\,dt, z - x\rangle \le \int_{t_1}^{t_2} (\Phi(z) - \Phi(x(t)))\,dt. \tag{3.7}$$

The conclusion follows from the combination of (3.6), (3.7) and the expression of the energy function E. □

We are now able to go further in the asymptotic analysis of the differential inclusion (S). As usual, the first step consists in proving that the energy converges toward its minimum. This is possible owing to precise estimates based on the functions E and k. The next step deals with the convergence of the trajectory toward some minimum of Φ. The main ingredient of the proof is the Opial lemma, that we recall below for the sake of completeness. This type of proof is classical and has been used in a series of recent papers [1, 2, 3, 7] to prove the convergence of the "heavy ball" trajectory. In reference [2] the authors deal with a nonsmooth potential Φ, but the study is restricted to the elastic shock solutions which are obtained via penalization techniques using an epiconvergent approximation of Φ. The new point here is that we treat the general case of dissipative solutions corresponding to Definition (2.1). For the convenience of the reader, let us recall the Opial lemma [18].

Lemma 3.3 (Opial). *Let H be a Hilbert space and $x : [0, +\infty) \to H$ be a function such that there exists a non void set $\mathcal{S} \subset H$ which satisfies :*
(i) $\forall t_n \to +\infty$ *with* $x(t_n) \to x_\infty$ *weakly in* H*, we have* $x_\infty \in \mathcal{S}$.
(ii) $\forall z \in \mathcal{S}$, $\lim_{t\to+\infty} |x(t) - z|$ *exists.*
Then, $x(t)$ *weakly converges as* $t \to +\infty$ *to some element* x_∞ *of* $\mathcal{S}$.

Let us now summarize the main asymptotic properties of the dynamical system (S).

Theorem 3.4. *Let $\gamma > 0$ and let $\Phi : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ be a convex, lower semi-continuous function which is bounded from below and such that* $\mathrm{int}\,(\mathrm{dom}\,\Phi) \neq \emptyset$*. Additionally assume that the function $\Phi_{|\mathrm{dom}\,\Phi}$ is locally Lipschitz continuous. Let x be a dissipative solution to (S) and let E be the associate energy function. Then, we have*
(a) $\lim_{t\to+\infty} E(t) = \inf \Phi$ *and as a consequence* $\lim_{t\to+\infty} \Phi(x(t)) = \inf \Phi$ *and* $\lim_{t\to+\infty} \dot{x}^-(t) = \lim_{t\to+\infty} \dot{x}^+(t) = 0$.
(b) If moreover $\mathrm{argmin}\,\Phi \neq \emptyset$*,*
  (i) $E - \min \Phi \in L^1(\mathbb{R}_+; \mathbb{R})$ *and* $\Phi \circ x - \min \Phi \in L^1(\mathbb{R}_+; \mathbb{R})$.
  (ii) $\lim_{t\to+\infty} t\,(E(t) - \min \Phi) = 0$*, and as a consequence* $\lim_{t\to+\infty} t\,(\Phi(x(t)) - \min \Phi) = 0$ *and* $\lim_{t\to+\infty} t|\dot{x}^+(t)|^2 = 0$.
(c) If $\mathrm{argmin}\,\Phi \neq \emptyset$*, there exists* $x_\infty \in \mathrm{argmin}\,\Phi$ *such that* $\lim_{t\to+\infty} x(t) = x_\infty$.

Proof. (a) From (2.4) the energy function E is nonincreasing; on the other hand, it is clearly bounded from below by $\inf \Phi$, hence $\lim_{t\to+\infty} E(t)$ exists. Let us now fix $z \in \mathrm{dom}\,\Phi$. From Lemma 3.2 applied with $t_1 = 0$, we have, for almost every $t \geq 0$,

$$k(t) + \int_0^t (E(\tau) - \Phi(z))\, d\tau \leq k(0) + \frac{3}{2} \int_0^t |\dot{x}(\tau)|^2\, d\tau. \tag{3.8}$$

On the other hand, from the Cauchy-Schwarz inequality and the fact that $\dot{x} \in L^\infty(\mathbb{R}_+; \mathbb{R}^d)$ we infer that $k(t) \geq -\|\dot{x}\|_{L^\infty(\mathbb{R}_+;\mathbb{R}^d)} |x(t) - z| + \frac{\gamma}{2}|x(t) - z|^2$ for every $t \in \mathbb{R}_+$. From a classical inequality, it ensues that for every $t \in \mathbb{R}_+$

$$k(t) \geq -\frac{\|\dot{x}\|^2_{L^\infty(\mathbb{R}_+;\mathbb{R}^d)}}{\gamma} + \frac{\gamma}{4}|x(t) - z|^2 \tag{3.9}$$

$$\geq -\frac{\|\dot{x}\|^2_{L^\infty(\mathbb{R}_+;\mathbb{R}^d)}}{\gamma}. \tag{3.10}$$

By combining (3.8), (3.10) and the fact that $\dot{x} \in L^2(\mathbb{R}_+; \mathbb{R}^d)$ (Proposition 3.1), we obtain for almost every $t \geq 0$

$$\int_0^t (E(\tau) - \Phi(z))\, d\tau \leq k(0) + \frac{\|\dot{x}\|^2_{L^\infty(\mathbb{R}_+;\mathbb{R}^d)}}{\gamma} + \frac{3}{2} \int_0^{+\infty} |\dot{x}(\tau)|^2\, d\tau. \tag{3.11}$$

Since the map $t \mapsto \int_0^t (E(\tau) - \Phi(z))\, d\tau$ is continuous on $\mathbb{R}_+$, the previous inequality holds for every $t \in \mathbb{R}_+$. We deduce from (3.11) that $\lim_{t\to+\infty} E(t) \leq \Phi(z)$. Indeed, let us argue by contradiction and assume that $\lim_{t\to+\infty} E(t) > \Phi(z)$. Since the function E is nonincreasing (inequality (2.4)), we have $\int_0^t (E(\tau) - \Phi(z))\, d\tau \geq t(\lim_{\tau\to+\infty} E(\tau) - \Phi(z))$. Taking the limit when $t \to +\infty$ we infer that $\int_0^{+\infty} (E(\tau) - \Phi(z))\, d\tau = +\infty$, a contradiction with (3.11). Hence we have $\lim_{t\to+\infty} E(t) \leq \Phi(z)$

and since this is true for any $z \in \operatorname{dom}\Phi$, we deduce that $\lim_{t\to+\infty} E(t) \le \inf \Phi$. Recalling the inequality $E(t) \ge \inf \Phi$ for all $t \in \mathbb{R}_+$, we conclude that $\lim_{t\to+\infty} E(t) = \inf \Phi$. Using now $E(t) - \inf \Phi \ge \Phi(x(t)) - \inf \Phi \ge 0$ and $E(t) - \inf \Phi \ge \frac{1}{2}|\dot{x}^+(t)|^2 \ge 0$, we obtain $\lim_{t\to+\infty} \Phi(x(t)) = \inf \Phi$ and $\lim_{t\to+\infty} \dot{x}^+(t) = 0$. Finally recalling that $\dot{x}^- = (\dot{x}^+)^-$ on $\mathbb{R}_+^*$, we also have $\lim_{t\to+\infty} \dot{x}^-(t) = 0$.
(b)- (i) From now on, let us take $z \in \operatorname{argmin} \Phi$, so that $\Phi(z) = \min \Phi$. Taking the limit when $t \to +\infty$ in inequality (3.11), we observe that $E - \min \Phi \in L^1(\mathbb{R}_+; \mathbb{R})$. Since $E(t) - \min \Phi \ge \Phi(x(t)) - \min \Phi \ge 0$, we conclude that $\Phi \circ x - \min \Phi \in L^1(\mathbb{R}_+; \mathbb{R})$.
(b)- (ii) Since E is nonincreasing, we have for every $t \ge 0$

$$\int_{\frac{t}{2}}^{t} (E(\tau) - \min \Phi)\, d\tau \;\ge\; \frac{t}{2}(E(t) - \min \Phi) \;\ge\; 0.$$

Passing now to the limit when $t \to +\infty$, we obtain $\lim_{t\to+\infty} t\,(E(t) - \min \Phi) = 0$. The other two limits of item (b)- (ii) follow immediately.
(c) We are now going to apply the Opial lemma with $\mathcal{S} = \operatorname{argmin}\Phi \ne \emptyset$. We have to check points (i) and (ii) of Lemma 3.3.
• Let $(t_n)_{n\ge 0}$ be a sequence tending toward $+\infty$ such that $\lim_{n\to+\infty} x(t_n) = x_\infty$. Then we have $\liminf_{n\to+\infty} \Phi(x(t_n)) \ge \Phi(x_\infty)$ because Φ is lower semicontinuous. But, in view of (a), $\lim_{n\to+\infty} \Phi(x(t_n)) = \min \Phi$. Hence $\Phi(x_\infty) \le \min \Phi$ and therefore $x_\infty \in \operatorname{argmin}\Phi$.
• Given $z \in \operatorname{argmin}\Phi$, let us prove that $\lim_{t\to+\infty} |x(t) - z|$ exists. Since $E(t) \ge \min \Phi = \Phi(z)$ for all $t \in \mathbb{R}_+$, we deduce from inequality (3.3) that the map $\theta : t \mapsto k(t) - \frac{3}{2}\int_0^t |\dot{x}(s)|^2\, ds$ is essentially nonincreasing on $\mathbb{R}_+$. It is clear, in view of (3.10) and the fact that $\int_0^{+\infty} |\dot{x}(s)|^2\, ds < +\infty$, that θ is bounded from below. We deduce that $ess - \lim_{t\to+\infty} \theta(t)$ exists and is finite. Since $\lim_{t\to+\infty} \int_0^t |\dot{x}(s)|^2\, ds = \int_0^{+\infty} |\dot{x}(s)|^2\, ds < +\infty$, it ensues that

$$ess - \lim_{t\to+\infty} k(t) = ess - \lim_{t\to+\infty} \left[(\dot{x}^+(t), x(t) - z) + \frac{\gamma}{2}|x(t) - z|^2\right] \quad \text{exists.} \tag{3.12}$$

It is then clear, with (3.9) that the map $t \mapsto |x(t) - z|$ is bounded. On the other hand, we know by (a) that $\lim_{t\to+\infty} \dot{x}^+(t) = 0$. Consequently $\lim_{t\to+\infty} (\dot{x}^+(t), x(t) - z) = 0$ and hence, with (3.12), $ess - \lim_{t\to+\infty} |x(t) - z|^2$ exists. But, $t \mapsto |x(t) - z|$ is a continuous function and therefore $\lim_{t\to+\infty} |x(t) - z|$ exists. □

3.2. **Exponential decay of the energy.** We are now going to show an exponential decay result when the function Φ admits a strong minimum (see definition (3.13)). This extends classical results of exponential convergence relative to the steepest descent dynamical system. To quote only one of them, the steepest descent trajectory is known to exponentially converge toward its limit as soon as the potential Φ is strongly convex[1] (see for example [5, Theorem 3.9]). Exponential decay results have also been proved for the "heavy ball with friction" system when the potential Φ is smooth ([7]).

[1]Recall that the strong convexity of Φ amounts to the strong monotonicity of $\partial\Phi$, *i.e.* there exists $\alpha > 0$ such that

$$\forall (x_1, x_2) \in \mathbb{R}^d \times \mathbb{R}^d, \quad \forall (y_1, y_2) \in \partial\Phi(x_1) \times \partial\Phi(x_2), \qquad (x_2 - x_1, y_2 - y_1) \ge \alpha\, |x_2 - x_1|^2.$$

Theorem 3.5. *Let $\gamma > 0$ and let $\Phi : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ be a convex, lower semi-continuous function satisfying* $\operatorname{int}(\operatorname{dom}\Phi) \neq \emptyset$. *Assume that the function $\Phi_{|\operatorname{dom}\Phi}$ is locally Lipschitz continuous and that there exist $M > 0$ and $a \in \mathbb{R}^d$ such that*

$$\forall x \in \operatorname{dom}\Phi, \quad \Phi(x) - \Phi(a) \geq \frac{M}{2}\,|x-a|^2. \tag{3.13}$$

Let x be a dissipative solution to the system (S) and let E be the associate energy function. Then, there exists some $C \geq 0$ such that for every $t \in \mathbb{R}_+$

$$\int_t^{+\infty} (E(s) - \min\Phi)\, ds \leq C\, e^{-\delta t},$$

with $\delta = \frac{2\gamma(\sqrt{\gamma^2+4M}-\gamma)}{\gamma+3\sqrt{\gamma^2+4M}}$. Moreover, the following estimates hold for every $t \in \mathbb{R}_+$:

$$\int_t^{+\infty} |\dot{x}(s)|^2\, ds \leq 2\,C\, e^{-\delta t} \qquad \text{and} \qquad \int_t^{+\infty} |x(s)-a|^2\, ds \leq \frac{2\,C}{M}\, e^{-\delta t}.$$

Proof. Without any loss of generality, we may assume that $\min\Phi = \Phi(a) = 0$. From Theorem 3.4 (a), the energy function E satisfies: $\lim_{t\to+\infty} E(t) = \min\Phi = 0$. Taking the limit when $t_2 \to +\infty$ in the energy inequality (2.4), we obtain that, for every $t \in \mathbb{R}_+$:

$$-E(t) \leq -\gamma \int_t^{+\infty} |\dot{x}(s)|^2\, ds. \tag{3.14}$$

Since $\operatorname{argmin}\Phi = \{a\}$, it is immediate with Theorem 3.4 (c) that $\lim_{t\to+\infty} x(t) = a$. We define as above the function k by $k(t) = (\dot{x}^+(t), x(t)-a) + \frac{\gamma}{2}|x(t)-a|^2$ for all $t \geq 0$. Taking the limit when $t_2 \to +\infty$ in inequality (3.3) of Lemma 3.2, we find for almost every $t \in \mathbb{R}_+$

$$-k(t) + \int_t^{+\infty} E(s)\, ds \leq \frac{3}{2}\int_t^{+\infty} |\dot{x}(s)|^2\, ds. \tag{3.15}$$

The previous inequality makes sense since the function E belongs to $L^1(\mathbb{R}_+;\mathbb{R})$ in view of Theorem 3.4 (b). Let us multiply this last inequality by $\frac{2\gamma}{3}$ and add to (3.14); we obtain

$$-E(t) - \frac{2\gamma}{3}\, k(t) + \frac{2\gamma}{3}\int_t^{+\infty} E(s)\, ds \leq 0 \qquad \text{a.e. on } \mathbb{R}_+. \tag{3.16}$$

Our purpose now is to deduce from (3.16) a differential equation involving a single function. This is made possible owing to the following relations between the functions k and E

$$\forall t \geq 0, \qquad k(t) \geq -E(t)/\gamma \quad \text{and} \quad E(t) \geq \alpha\, k(t), \tag{3.17}$$

where α is a positive real number that we are going to determine. We classically have, for all $\theta > 0$,

$$|(\dot{x}^+(t), x(t)-a)| \leq \frac{|\dot{x}^+(t)|^2}{2\theta} + \frac{\theta}{2}|x(t)-a|^2,$$

and hence for every $t \geq 0$,

$$-\frac{|\dot{x}^+(t)|^2}{2\theta} + \frac{-\theta+\gamma}{2}|x(t)-a|^2 \leq k(t) \leq \frac{|\dot{x}^+(t)|^2}{2\theta} + \frac{\theta+\gamma}{2}|x(t)-a|^2. \tag{3.18}$$

Taking $\theta = \gamma$ in the first inequality of (3.18), we obtain $k(t) \geq -|\dot{x}^+(t)|^2/(2\,\gamma) \geq -E(t)/\gamma$, which is the first inequality of (3.17). On the other hand, in view of condition (3.13), we have

$$E(t) \geq \frac{1}{2}\,|\dot{x}^+(t)|^2 + \frac{M}{2}\,|x(t)-a|^2. \tag{3.19}$$

Setting $\tau(\theta) := \min\{\theta, M/(\theta+\gamma)\}$, we deduce from the second inequality of (3.18) and (3.19) that

$$E(t) \geq \tau(\theta)\,k(t). \tag{3.20}$$

We let the reader check that the function $\tau : \mathbb{R}_+ \to \mathbb{R}$ achieves its maximum at $\alpha := (\sqrt{\gamma^2+4\,M}-\gamma)/2$ and that $\tau(\alpha) = \alpha$. Taking $\theta = \alpha$ in inequality (3.20), we obtain the second inequality of (3.17). Inequalities (3.17) imply that $|k(t)| \leq E(t)\,\max(1/\alpha, 1/\gamma)$, which combined with $E \in L^1(\mathbb{R}_+;\mathbb{R}_+)$ implies that $k \in L^1(\mathbb{R}_+;\mathbb{R})$. We deduce from (3.16) and the second inequality of (3.17) that

$$-E(t) - \frac{2\gamma}{3}\,k(t) + \frac{2\gamma}{3}\,\alpha\int_t^{+\infty} k(s)\,ds \leq 0 \qquad \text{a.e. on } \mathbb{R}_+. \tag{3.21}$$

Let us multiply (3.16) by α and (3.21) by $\frac{2\gamma}{3}$; adding the two inequalities and setting $G(t) := E(t) + \frac{2\gamma}{3}\,k(t)$, this yields:

$$-(\frac{2\gamma}{3}+\alpha)\,G(t) + \frac{2\gamma}{3}\,\alpha\int_t^{+\infty} G(s)\,ds \leq 0 \qquad \text{a.e. on } \mathbb{R}_+.$$

An elementary integration on $[0,+\infty)$ gives:

$$\forall t \geq 0, \qquad \int_t^{+\infty} G(s)\,ds \leq \left(\int_0^{+\infty} G(s)\,ds\right)\,e^{-\delta\,t}, \tag{3.22}$$

where the parameter δ is given by:

$$\delta := \frac{\frac{2\gamma}{3}\,\alpha}{\frac{2\gamma}{3}+\alpha} = \frac{2\,\gamma\,(\sqrt{\gamma^2+4\,M}-\gamma)}{\gamma+3\,\sqrt{\gamma^2+4\,M}}.$$

From the first inequality of (3.17), we have $G(t) \geq E(t) - \frac{2}{3}\,E(t) = E(t)/3$. Setting $C := 3\,\int_0^{+\infty} G(s)\,ds$, we deduce in view of (3.22) that

$$\forall t \geq 0, \qquad \int_t^{+\infty} E(s)\,ds \leq C\,e^{-\delta t}.$$

The other estimates are straightforward consequences of the previous inequality. □

4. Accumulation of impacts and boundary motion

Consider the dynamics of a punctual particle submitted to gravity and bouncing on the floor, supposed horizontal and plane. The impacts are assumed to obey the Newton's law, with a restitution coefficient $r \in [0,1]$, whenever the particle hits the floor. If $r = 0$ the dynamics immediately stops after the first impact. If $r \in (0,1)$ the motion stabilizes in a finite time with accumulation of impacts (see for example Ballard [4, §6.1]). In this section, we extend this kind of result to the case of a discrete mechanical system moving in a convex set K with smooth boundary, under the action of a smooth potential f. In presence of viscous friction, this motion can be simply modelized by the differential inclusion (S), associated to the function Φ defined by $\Phi = f + \delta_K$. When $r \in [0,1)$, we prove that the trajectory is contained in the boundary of K after a finite time. The case $r = 1$ is also addressed and leads to a qualitatively different behavior. Before stating precisely these results, we need preliminary lemmas. The first one asserts that, in absence of constraints, the free motion associated to the dynamical system (S) obeys an ordinary differential equation.

Lemma 4.1. *Let $f : \mathbb{R}^d \to \mathbb{R}$ be a convex function of class $\mathcal{C}^1$ such that the map ∇f is locally Lipschitz continuous. Let $K \subset \mathbb{R}^d$ be a closed convex set such that $\mathrm{int}\,(K) \neq \emptyset$. Consider the differential inclusion (S) with $\Phi = f + \delta_K$ and $\gamma \geq 0$. Let x be a dissipative solution to (S) and assume that there exist $t_1 \geq 0$ and $t_2 > t_1$ such that $x(t) \in \mathrm{int}\,(K)$ for every $t \in (t_1, t_2)$. Then the map x coincides on $[t_1, t_2]$ with the unique solution to the following Cauchy problem*

$$\begin{cases} \ddot{y}(t) + \gamma\,\dot{y}(t) + \nabla f(y(t)) = 0, \\ y(t_1) = x(t_1), \quad \dot{y}(t_1) = \dot{x}^+(t_1). \end{cases}$$

Proof. Let us first remark that $\mathrm{dom}\,\Phi = K$ and that $\partial\Phi(x) = \nabla f(x) + N_K(x)$ if $x \in K$. Since $x(t) \in \mathrm{int}\,(K)$ for every $t \in (t_1, t_2)$, we infer from Proposition 2.2 (i) that, for almost every $t \in (t_1, t_2)$,

$$\ddot{x}_a(t) + \gamma\dot{x}(t) + \nabla f(x(t)) = 0, \tag{4.1}$$

where $\ddot{x}_a$ is the density of $\ddot{x}$ with respect to the Lebesgue's measure. Let $T > t_2$ be such that $\dot{x}^+(T) = \dot{x}(T)$ and let us denote by μ the Stieltjes measure associated to $-\dot{x} - \gamma\,x$ on $[0, T]$. Let us decompose the measure μ as in the proof of Proposition 2.2: $\mu = g\,dt + h\,d|\mu_s|$, where μ_s is a singular measure with respect to the Lebesgue's measure and the functions g, h satisfy respectively $g \in L^1([0, T]; \mathbb{R}^d, dt)$ and $h \in L^1([0, T]; \mathbb{R}^d, d|\mu_s|)$. Since $x(t) \in \mathrm{int}\,(K)$ for every $t \in (t_1, t_2)$, we derive from (2.2) that $h(t) = 0$, for $d|\mu_s|$-almost every $t \in (t_1, t_2)$. Therefore, we have $\mu = g\,dt$ on (t_1, t_2) and it ensues that the restriction of the measure $\ddot{x}$ to (t_1, t_2) is absolutely continuous with respect to the Lebesgue's measure. By identifying this measure with its density, we deduce from equality (4.1) that $\ddot{x}(t) + \gamma\dot{x}(t) + \nabla f(x(t)) = 0$ for almost every $t \in (t_1, t_2)$. The conclusion follows by use of classical results on ordinary differential equations. □

From now on, we assume that the set K is given by $K = \{\xi \in \mathbb{R}^d,\, g(\xi) \leq 0\}$, where the map $g : \mathbb{R}^d \to \mathbb{R}$ is smooth and convex. Given a dissipative solution x to (S), let us define the function $h : \mathbb{R}_+ \to \mathbb{R}$ by $h := g \circ x$. Since the map x takes its values in the set $K = \{\xi \in \mathbb{R}^d,\, g(\xi) \leq 0\}$, it is clear that $h(t) \leq 0$ for every $t \geq 0$. It is immediate to check that

$$\forall t > 0, \qquad \dot{h}^-(t) = \left(\nabla g(x(t)), \dot{x}^-(t)\right) \quad \text{and} \quad \dot{h}^+(t) = \left(\nabla g(x(t)), \dot{x}^+(t)\right).$$

Notice that, if $x(\bar{t}) \in \mathrm{bd}\,(K)$ for some $\bar{t} > 0$, then

$$\dot{h}^-(\bar{t}) = \lim_{t \to \bar{t}^-} \frac{g(x(t)) - g(x(\bar{t}))}{t - \bar{t}} \geq 0.$$

In the same way, we have $\dot{h}^+(\bar{t}) \leq 0$ for every $\bar{t} \geq 0$ such that $x(\bar{t}) \in \mathrm{bd}\,(K)$.
The next lemma gives the second-order Taylor expansion of the map $h = g \circ x$ in the right (resp. left) neighbourhood of any instant $\bar{t} > 0$ such that $x(\bar{t}) \in \mathrm{bd}\,(K)$.

Lemma 4.2. *Let $f : \mathbb{R}^d \to \mathbb{R}$ be a convex function of class $\mathcal{C}^1$ such that the map ∇f is locally Lipschitz continuous. Let $g : \mathbb{R}^d \to \mathbb{R}$ be a convex function of class $\mathcal{C}^2$*

such that [2] $\inf g < 0$. *Define the set* $K \subset \mathbb{R}^d$ *by* $K = \{\xi \in \mathbb{R}^d,\, g(\xi) \le 0\}$. *Assuming that* $\Phi = f + \delta_K$ *and that* $\gamma \ge 0$, *let* x *be a dissipative solution to (S). Suppose that there exist* $\bar{t} \in \mathbb{R}_+$ *and* $\varepsilon > 0$ *such that* $x(\bar{t}) \in \mathrm{bd}\,(K)$ *and* $x(t) \in \mathrm{int}\,(K)$ *for every* $t \in (\bar{t}, \bar{t} + \varepsilon)$. *Then the following holds when* $t \to \bar{t}^+$

$$\begin{aligned} g(x(t)) &= (t-\bar{t})\,\big(\nabla g(x(\bar{t})), \dot{x}^+(\bar{t})\big) + \frac{1}{2}(t-\bar{t})^2\,\big[-\gamma\,\big(\nabla g(x(\bar{t})), \dot{x}^+(\bar{t})\big) - \\ &\quad - \big(\nabla g(x(\bar{t})), \nabla f(x(\bar{t}))\big) + \big(\nabla^2 g(x(\bar{t})).\,\dot{x}^+(\bar{t}), \dot{x}^+(\bar{t})\big)\big] + o(t-\bar{t})^2. \end{aligned} \tag{4.2}$$

If one assumes that $\bar{t} > 0$ *and* $x(t) \in \mathrm{int}\,(K)$ *for every* $t \in (\bar{t} - \varepsilon, \bar{t})$, *the same formula holds when* $t \to \bar{t}^-$ *by replacing each term* $\dot{x}^+(\bar{t})$ *by* $\dot{x}^-(\bar{t})$.

Proof. From Lemma 4.1, the map x coincides on $[\bar{t}, \bar{t}+\varepsilon]$ with the unique solution $\tilde{x} : [\bar{t}, \bar{t}+\varepsilon] \to \mathbb{R}^d$ to the following Cauchy problem

$$\begin{cases} \ddot{y}(t) + \gamma\,\dot{y}(t) + \nabla f(y(t)) = 0, \\ y(\bar{t}) = x(\bar{t}), \quad \dot{y}(\bar{t}) = \dot{x}^+(\bar{t}). \end{cases}$$

Since ∇f is continuous, $\tilde{x}$ is of class $\mathcal{C}^2$ and since the function g is of class $\mathcal{C}^2$, the map $g \circ \tilde{x}$ is also of class $\mathcal{C}^2$ on $[\bar{t}, \bar{t}+\varepsilon]$. The following second-order Taylor expansion holds when t tends to $\bar{t}^+$

$$g(\tilde{x}(t)) = g(\tilde{x}(\bar{t})) + (t-\bar{t})\,\frac{d}{dt}(g \circ \tilde{x})(\bar{t}) + \frac{1}{2}(t-\bar{t})^2\,\frac{d^2}{dt^2}(g \circ \tilde{x})(\bar{t}) + o(t-\bar{t})^2 \tag{4.3}$$

and $g(\tilde{x}(\bar{t})) = g(x(\bar{t})) = 0$ since $x(\bar{t}) \in \mathrm{bd}\,(K)$. An immediate computation shows that

$$\begin{aligned} \frac{d}{dt}(g \circ \tilde{x})(\bar{t}) &= \big(\nabla g(x(\bar{t})), \dot{x}^+(\bar{t})\big), \\ \frac{d^2}{dt^2}(g \circ \tilde{x})(\bar{t}) &= -\gamma\,\big(\nabla g(x(\bar{t})), \dot{x}^+(\bar{t})\big) - \big(\nabla g(x(\bar{t})), \nabla f(x(\bar{t}))\big) + \big(\nabla^2 g(x(\bar{t}))\,.\dot{x}^+(\bar{t}), \dot{x}^+(\bar{t})\big). \end{aligned}$$

Since $x(t) = \tilde{x}(t)$ for all $t \in [\bar{t}, \bar{t}+\varepsilon]$, we obtain the expected formula owing to equality (4.3). □

Let us recall that an impact obeys the Newton's law when the normal component of the velocity is reversed and multiplied by a restitution coefficient $r \in [0,1]$ and the tangential component is transmitted. Using the function h defined above, the Newton's law shows that $\dot{h}^+(\bar{t}) = -r\,\dot{h}^-(\bar{t})$ for every $\bar{t} > 0$ such that $x(\bar{t}) \in \mathrm{bd}\,(K)$.

Theorem 4.3. *Let* $f : \mathbb{R}^d \to \mathbb{R}$ *be a convex function of class* $\mathcal{C}^1$ *such that the map* ∇f *is locally Lipschitz continuous. Let* $g : \mathbb{R}^d \to \mathbb{R}$ *be a convex function of class* $\mathcal{C}^2$ *such that* $\inf g < 0$. *Define the set* $K \subset \mathbb{R}^d$ *by* $K = \{\xi \in \mathbb{R}^d,\, g(\xi) \le 0\}$. *Suppose that* $\mathrm{argmin}_K f \neq \emptyset$ *and that* $\inf f < \inf_K f$. *Consider the differential inclusion (S) with* $\Phi = f + \delta_K$ *and* $\gamma > 0$. *Let* x *be a dissipative solution to (S) satisfying the Newton's law at impacts, with a restitution coefficient* $r \in [0,1]$. *There exist* $T \in \mathbb{R}_+$ *and* $\alpha > 0$ *such that* $(\nabla g(x(t)), \nabla f(x(t))) \le -\alpha$ *for all* $t \ge T$. *Assume that there exists* $T' \ge T$ *such that* $x(T') \in \mathrm{int}\,(K)$ *and define the set* $\mathcal{E} := \{t > T',\, x(t) \in \mathrm{bd}\,(K)\}$.
(i) If $r = 0$, *there exists* $t_0 > T'$ *such that* $\mathcal{E} = [t_0, +\infty)$.
(ii) If $r \in (0,1)$, *there exists an increasing sequence* $(t_n)_{n \ge 0}$ *tending toward* $t_* \in \mathbb{R}_+^*$ *such that* $\mathcal{E} = \cup_{n \ge 0}\{t_n\} \cup [t_*, +\infty)$. *Moreover, we have* $\lim_{n \to +\infty}(t_{n+2} - t_{n+1})/(t_{n+1} - t_n) = r$.

[2] Here and in the sequel, we adopt the convention $\inf g = -\infty$ if g is not bounded from below.

(iii) If $r = 1$, additionally assume that the map f is of class $\mathcal{C}^2$ and that the map g is of class $\mathcal{C}^3$. Then the set $\mathcal{E}$ equals $\cup_{n\geq 0}\{t_n\}$ for some increasing sequence $(t_n)_{n\geq 0}$ such that $\lim_{n\to+\infty} t_n = +\infty$. Moreover the following equivalences hold as $n \to +\infty$

$$(\nabla g(x(t_n)), \dot{x}^+(t_n)) \sim \frac{3\,(\nabla g(x_\infty), \nabla f(x_\infty))}{2\,\gamma}\,\frac{1}{n} \qquad \text{and} \qquad t_n \sim \frac{3}{\gamma}\ln n,$$

where we have set $x_\infty := \lim_{t\to+\infty} x(t)$.

Proof. From Theorem 3.4 (c), there exists $x_\infty \in \operatorname{argmin}_K f$ such that $\lim_{t\to+\infty} x(t) = x_\infty$. Since $\inf f < \inf_K f = f(x_\infty)$, we infer that $x_\infty \notin \operatorname{argmin} f$ and hence $\nabla f(x_\infty) \neq 0$. The vector x_∞ satisfies the optimality condition $-\nabla f(x_\infty) \in N_K(x_\infty)$. We derive that $N_K(x_\infty) \neq \{0\}$ and hence $x_\infty \in \operatorname{bd}(K)$, *i.e.* $g(x_\infty) = 0$. Since $\inf g < 0$, this implies in turn that $x_\infty \notin \operatorname{argmin} g$ and hence $\nabla g(x_\infty) \neq 0$. From a classical result, we have $N_K(x_\infty) = \mathbb{R}_+.\nabla g(x_\infty)$ and therefore $-\nabla f(x_\infty) \in \mathbb{R}_+.\nabla g(x_\infty)$. From the continuity of the mappings ∇f and ∇g, we have

$$\lim_{t\to+\infty} (\nabla g(x(t)), \nabla f(x(t))) = (\nabla g(x_\infty), \nabla f(x_\infty)) < 0.$$

Hence we deduce the existence of $T \geq 0$ and $\alpha > 0$ such that

$$\forall t \geq T, \quad (\nabla g(x(t)), \nabla f(x(t))) \leq -\alpha. \tag{4.4}$$

Assume that there exists $T' \geq T$ such that $x(T') \in \operatorname{int}(K)$. Let us first prove that there exists $t > T'$ such that $x(t) \in \operatorname{bd}(K)$. Let us argue by contradiction and assume that for every $t \geq T'$, we have $x(t) \in \operatorname{int}(K)$. From Lemma 4.1, it ensues that the equation of the motion reduces to $\ddot{x}(t) + \gamma\dot{x}(t) + \nabla f(x(t)) = 0$. Since $\lim_{t\to+\infty} x(t) = x_\infty$ and $\lim_{t\to+\infty} \dot{x}(t) = 0$, we deduce that $\lim_{t\to+\infty} \ddot{x}(t) = -\nabla f(x_\infty)$. Taking the scalar product with $\nabla f(x_\infty)$, we get $\lim_{t\to+\infty}(\ddot{x}(t), \nabla f(x_\infty)) = -|\nabla f(x_\infty)|^2 < 0$. But this implies in turn that $\lim_{t\to+\infty}(\dot{x}(t), \nabla f(x_\infty)) = -\infty$, a contradiction. Let us set

$$t_0 := \inf\{t > T', \quad x(t) \in \operatorname{bd}(K)\} < +\infty.$$

From the continuity of the map x, we have $x(t_0) \in \operatorname{bd}(K)$ and since $x(T') \in \operatorname{int}(K)$ we derive that $t_0 > T'$. We now distinguish the cases $r = 0$ and $r \in (0, 1]$. As above, we introduce the function $h : \mathbb{R}_+ \to \mathbb{R}$ defined by $h := g \circ x$.

1. Case $r = 0$. We are going to prove that $x(t) \in \operatorname{bd}(K)$ for every $t \geq t_0$. Let us argue by contradiction and assume that there exists $t_1 > t_0$ such that $x(t_1) \in \operatorname{int}(K)$. Let us set

$$\bar{t} := \sup\{t \in [t_0, t_1], \quad x(t) \in \operatorname{bd}(K)\}.$$

From the continuity of the map x, we have $x(\bar{t}) \in \operatorname{bd}(K)$ and since $x(t_1) \in \operatorname{int}(K)$ we deduce that $\bar{t} < t_1$. The definition of $\bar{t}$ then shows that $x(t) \in \operatorname{int}(K)$ for every $t \in (\bar{t}, t_1]$. On the other hand, the restitution law applied at $t = \bar{t}$ gives $\dot{h}^+(\bar{t}) = 0$. By applying Lemma 4.2 we find the following equality as $t \to \bar{t}^+$

$$h(t) = \frac{1}{2}(t-\bar{t})^2\,\left[-\big(\nabla g(x(\bar{t})), \nabla f(x(\bar{t}))\big) + \big(\nabla^2 g(x(\bar{t})).\,\dot{x}^+(\bar{t}), \dot{x}^+(\bar{t})\big) + o(1)\right].$$

Recalling that $\bar{t} \geq T$, we obtain in view of (4.4) that $(\nabla g(x(\bar{t})), \nabla f(x(\bar{t}))) \leq -\alpha$. From the convexity of the function g, we have $\big(\nabla^2 g(x(\bar{t})).\,\dot{x}^+(\bar{t}), \dot{x}^+(\bar{t})\big) \geq 0$. Hence we infer from the above equality that $h(t) > 0$ in some right neighbourhood of $\bar{t}$. But this contradicts the fact that $x(t) \in K$. As a conclusion, we have shown that

$x(t) \in \mathrm{bd}\,(K)$ for every $t \geq t_0$, and hence $\mathcal{E} = [t_0, +\infty)$.

2. Case $r \in (0,1]$**.** By construction of t_0, we have $x(t_0) \in \mathrm{bd}\,(K)$ and $x(t) \in \mathrm{int}\,(K)$ for every $t \in [T', t_0)$. It ensues that $\dot{h}^-(t_0) \geq 0$. Let us prove that $\dot{h}^-(t_0) > 0$. Let us argue by contradiction and assume that $\dot{h}^-(t_0) = 0$. By applying Lemma 4.2 we obtain the following equality when $t \to t_0^-$

$$h(t) = \frac{1}{2}\,(t - t_0)^2\, \left[-\left(\nabla g(x(t_0)), \nabla f(x(t_0))\right) + \left(\nabla^2 g(x(t_0)).\, \dot{x}^-(t_0), \dot{x}^-(t_0)\right) + o(1)\right].$$

Recalling that $t_0 \geq T$, we derive from (4.4) that $(\nabla f(x(t_0)), \nabla g(x(t_0))) \leq -\alpha$. From the convexity of the function g, we have $\left(\nabla^2 g(x(t_0)).\, \dot{x}^-(t_0), \dot{x}^-(t_0)\right) \geq 0$. It ensues that $h(t) > 0$ in some left neighbourhood of t_0, but this contradicts the fact that $x(t) \in K$. Finally we conclude that $\dot{h}^-(t_0) > 0$. Since $r \neq 0$, we deduce from the restitution law applied at $t = t_0$ that $\dot{h}^+(t_0) < 0$. By arguing as above, it is immediate to check that the set $\{t > t_0,\ x(t) \in \mathrm{bd}\,(K)\}$ is non empty; let us define

$$t_1 := \inf\{t > t_0,\ x(t) \in \mathrm{bd}\,(K)\} < +\infty.$$

Since $h(t_0) = 0$ and $\dot{h}^+(t_0) < 0$, the continuous map h is negative on some right neighbourhood of t_0 and hence $t_1 > t_0$. By using the same arguments as above, we obtain that $\dot{h}^+(t_1) < 0$. By iterating this process, we build a sequence $(t_n)_{n \geq 0}$ defined by

$$\forall n \in \mathbb{N}, \quad t_{n+1} := \inf\{t > t_n,\ x(t) \in \mathrm{bd}\,(K)\}$$

and satisfying

$$\forall n \in \mathbb{N}, \quad t_n < t_{n+1} \quad \text{and} \quad \dot{h}^+(t_{n+1}) < 0.$$

Since the sequence $(t_n)_{n \geq 0}$ is increasing, it converges toward some $t_* \in \mathbb{R}_+ \cup \{+\infty\}$. Let us remark that, if $t_* \in \mathbb{R}_+$ then $\dot{h}^-(t_*) = 0$. Indeed, since the increasing sequence $(t_n)_{n \geq 0}$ converges to t_* we have

$$\lim_{n \to +\infty} \dot{h}^-(t_n) = \lim_{n \to +\infty} \dot{h}^+(t_n) = \dot{h}^-(t_*).$$

Recalling that $\dot{h}^-(t_n) > 0$ (resp. $\dot{h}^+(t_n) < 0$) for every $n \in \mathbb{N}$, we deduce that $\dot{h}^-(t_*) \geq 0$ (resp. $\dot{h}^-(t_*) \leq 0$). Therefore we conclude that $\dot{h}^-(t_*) = 0$.
From Lemma 4.1, the map x satisfies the differential equation $\ddot{x} + \gamma\,\dot{x} + \nabla f(x) = 0$ on each interval (t_n, t_{n+1}). Since the map f is of class $\mathcal{C}^1$, we deduce that the map x is of class $\mathcal{C}^2$ on $\mathcal{D} := \cup_{n \in \mathbb{N}}(t_n, t_{n+1})$. Since the map g is of class $\mathcal{C}^2$, the composition $h = g \circ x$ is also of class $\mathcal{C}^2$ on $\mathcal{D}$. Hence we have for every $t \in \mathcal{D}$

$$\begin{aligned} \ddot{h}(t) &= \frac{d}{dt}(\nabla g(x(t)), \dot{x}(t)) \\ &= (\nabla g(x(t)), \ddot{x}(t)) + \left(\nabla^2 g(x(t)).\, \dot{x}(t), \dot{x}(t)\right) \\ &= -\gamma\,(\nabla g(x(t)), \dot{x}(t)) - (\nabla g(x(t)), \nabla f(x(t))) + \left(\nabla^2 g(x(t)).\, \dot{x}(t), \dot{x}(t)\right). \end{aligned}$$

We deduce that $\lim_{\substack{t \to t_*^- \\ t \in \mathcal{D}}} \ddot{h}(t)$ exists (in $\mathbb{R}$) and we denote by λ this limit. We have

$$\lambda = -(\nabla g(x_\infty), \nabla f(x_\infty)) \qquad \text{if} \quad t_* = +\infty, \tag{4.5}$$

since $\lim_{t \to +\infty} \dot{x}^-(t) = \lim_{t \to +\infty} \dot{x}^+(t) = 0$ (*cf.* Theorem 3.4). When t_* is finite, we obtain

$$\lambda = -(\nabla f(x(t_*)), \nabla g(x(t_*))) + \left(\nabla^2 g(x(t_*)).\, \dot{x}^-(t_*), \dot{x}^-(t_*)\right) \qquad \text{if} \quad t_* < +\infty,$$

since $(\nabla g(x(t_*)), \dot{x}^-(t_*)) = \dot{h}^-(t_*) = 0$. Recalling that the function g is convex, we derive that $\lambda \geq -(\nabla f(x(t_*)), \nabla g(x(t_*)))$. Finally, inequality (4.4) shows that the limit λ satisfies $\lambda \geq \alpha > 0$, whether $t_* < +\infty$ or $t_* = +\infty$. Let us now write the following second-order Taylor formula :

$$h(t_{n+1}) = h(t_n) + (t_{n+1} - t_n)\, \dot{h}^+(t_n) + \int_{t_n}^{t_{n+1}} (t_{n+1} - s)\, \ddot{h}(s)\, ds.$$

Since $h(t_n) = h(t_{n+1}) = 0$ and since $\lim_{\substack{t \to t_*^- \\ t \in \mathcal{D}}} \ddot{h}(t) = \lambda$, we infer that

$$\dot{h}^+(t_n) = \left(-\frac{\lambda}{2} + o(1)\right) (t_{n+1} - t_n), \tag{4.6}$$

as $n \to +\infty$. By reversing the roles of t_n and t_{n+1}, we obtain in the same way

$$\dot{h}^-(t_{n+1}) = \left(\frac{\lambda}{2} + o(1)\right) (t_{n+1} - t_n).$$

The restitution law applied at $t = t_{n+1}$ yields $\dot{h}^+(t_{n+1}) = -r\, \dot{h}^-(t_{n+1})$, and we deduce that

$$\dot{h}^+(t_{n+1}) = \left(-r\,\frac{\lambda}{2} + o(1)\right) (t_{n+1} - t_n), \tag{4.7}$$

as $n \to +\infty$. In view of (4.6) and (4.7), we have

$$\lim_{n \to +\infty} \frac{t_{n+2} - t_{n+1}}{t_{n+1} - t_n} = r. \tag{4.8}$$

Let us now distinguish the cases $r \in (0, 1)$ and $r = 1$.

2.a. Case $r \in (0, 1)$. From (4.8) we deduce that $\sum_{n=0}^{+\infty} (t_{n+1} - t_n) < +\infty$ and hence we conclude that $t_* := \lim_{n \to +\infty} t_n < +\infty$. The next step consists in proving that $x(t) \in \mathrm{bd}\,(K)$ for every $t \geq t_*$. Let us argue by contradiction and assume that there exists $\bar{u} \geq t_*$ such that $x(\bar{u}) \in \mathrm{int}\,(K)$. Since $x(t_*) \in \mathrm{bd}\,(K)$ we have $\bar{u} > t_*$ and we set

$$u_0 := \sup\{t \in [t_*, \bar{u}], \quad x(t) \in \mathrm{bd}\,(K)\}.$$

From the continuity of the map x, we have $x(u_0) \in \mathrm{bd}\,(K)$ and since $x(\bar{u}) \in \mathrm{int}\,(K)$ we deduce that $u_0 < \bar{u}$. The definition of u_0 then shows that $x(t) \in \mathrm{int}\,(K)$ for every $t \in (u_0, \bar{u}]$. It ensues that $\dot{h}^+(u_0) \leq 0$. Let us prove that $\dot{h}^+(u_0) < 0$. Let us argue by contradiction and assume that $\dot{h}^+(u_0) = 0$. By applying Lemma 4.2 we obtain the following equality when $t \to u_0^+$

$$h(t) = \frac{1}{2}\,(t - u_0)^2\, \left[-\,(\nabla g(x(u_0)), \nabla f(x(u_0))) + \left(\nabla^2 g(x(u_0)).\, \dot{x}^+(u_0), \dot{x}^+(u_0)\right) + o(1)\right].$$

Recalling that $u_0 \geq T$, we obtain in view of (4.4) that $(\nabla g(x(u_0)), \nabla f(x(u_0))) \leq -\alpha$. From the convexity of the function g, we have $\left(\nabla^2 g(x(u_0)).\, \dot{x}^+(u_0), \dot{x}^+(u_0)\right) \geq 0$. Hence we infer from the above equality that $h(t) > 0$ in some right neighbourhood of u_0. But this contradicts the fact that $x(t) \in K$. Finally we conclude that $\dot{h}^+(u_0) < 0$. The restitution law applied at $t = u_0$ shows that $\dot{h}^-(u_0) > 0$. Recalling that $\dot{h}^-(t_*) = 0$, we deduce that $u_0 > t_*$. Let us now define

$$u_1 := \sup\{t \in [t_*, u_0),\, x(t) \in \mathrm{bd}\,(K)\}.$$

Since $h(u_0) = 0$ and $\dot{h}^-(u_0) > 0$, the continuous map h is negative on some left neighbourhood of u_0 and hence $u_1 < u_0$. By using the same arguments as above, we obtain that $\dot{h}^+(u_1) < 0$ and that $u_1 > t_*$. By iterating this process, we build a sequence $(u_n)_{n\geq 0}$ defined by

$$\forall n \in \mathbb{N}, \quad u_{n+1} := \sup\{t \in [t_*, u_n),\, x(t) \in \mathrm{bd}\,(K)\}.$$

We let the reader check that the sequence $(u_n)_{n\geq 0}$ is decreasing and satisfies

$$\forall n \in \mathbb{N}, \quad u_n > t_* \quad \text{ and } \quad \dot{h}^+(u_n) < 0.$$

Since the sequence $(u_n)_{n\geq 0}$ is decreasing and bounded from below, it converges toward some $u_* \geq t_*$. The arguments are now similar to the ones that were developed in the analysis of the sequence $(t_n)_{n\geq 0}$. Let us set

$$\lambda' = -(\nabla g(x(u_*)), \nabla f(x(u_*))) + \left(\nabla^2 g(x(u_*)).\, \dot{x}^+(u_*), \dot{x}^+(u_*)\right).$$

It is immediate that $\lambda' \geq \alpha > 0$ and we let the reader check the following estimates as $n \to +\infty$

$$\begin{aligned} \dot{h}^+(u_n) &= \left(-r\,\frac{\lambda'}{2} + o(1)\right)(u_n - u_{n+1}), \\ \dot{h}^+(u_{n+1}) &= \left(-\frac{\lambda'}{2} + o(1)\right)(u_n - u_{n+1}). \end{aligned}$$

This implies immediately that

$$\lim_{n\to+\infty} \frac{u_{n+1} - u_{n+2}}{u_n - u_{n+1}} = \frac{1}{r}.$$

Since $1/r > 1$, we deduce that $\lim_{n\to+\infty}(u_n - u_{n+1}) = +\infty$, which contradicts the fact that $(u_n)_{n\geq 0}$ is bounded. As a conclusion, we have shown that $x(t) \in \mathrm{bd}\,(K)$ for every $t \geq t_*$, and hence $\mathcal{E} = \cup_{n\geq 0}\{t_n\} \cup [t_*, +\infty)$.

2.b. Case $r = 1$. Relation (4.8) does not allow us to conclude that $\lim_{n\to+\infty} t_n = +\infty$. From now on, we assume that the map f is of class $\mathcal{C}^2$ and that the map g is of class $\mathcal{C}^3$. We are going to refine estimates (4.6) and (4.7) under these additional hypotheses. From Lemma 4.1, the map x satisfies the differential equation $\ddot{x} + \gamma\,\dot{x} + \nabla f(x) = 0$ on each interval (t_n, t_{n+1}). Since the map f is of class $\mathcal{C}^2$, we deduce that the map x is of class $\mathcal{C}^3$ on $\mathcal{D} := \cup_{n\in\mathbb{N}}(t_n, t_{n+1})$. Since the map g is of class $\mathcal{C}^3$, the composition $h = g \circ x$ is also of class $\mathcal{C}^3$ on $\mathcal{D}$. An elementary computation shows that, for every $t \in \mathcal{D}$

$$\begin{aligned} \dddot{h}(t) &= \gamma\,(\nabla g(x(t)), \nabla f(x(t))) + \gamma^2\,(\nabla g(x(t)), \dot{x}(t)) \\ &\quad - \left(\nabla g(x(t)), \nabla^2 f(x(t)).\, \dot{x}(t)\right) - 3\,\left(\nabla^2 g(x(t)).\, \dot{x}(t), \nabla f(x(t))\right) \\ &\quad -3\,\gamma\,\left(\nabla^2 g(x(t)).\, \dot{x}(t), \dot{x}(t)\right) + \left((\nabla^3 g(x(t)).\, \dot{x}(t)).\, \dot{x}(t), \dot{x}(t)\right). \end{aligned}$$

As above, we deduce that $\lim_{\substack{t \to t_*^- \\ t \in \mathcal{D}}} \dddot{h}(t)$ exists (in $\mathbb{R}$) and we denote by μ this limit. Remark that

$$\mu = \gamma\,(\nabla g(x_\infty), \nabla f(x_\infty)) \qquad \text{if} \quad t_* = +\infty, \tag{4.9}$$

since $\lim_{t\to+\infty} \dot{x}^-(t) = \lim_{t\to+\infty} \dot{x}^+(t) = 0$. Let us now write the following third-order Taylor formula

$$h(t_{n+1}) = h(t_n) + (t_{n+1} - t_n)\,\dot{h}^+(t_n) + \frac{1}{2}\,(t_{n+1} - t_n)^2\,\ddot{h}^+(t_n) + \frac{1}{2}\int_{t_n}^{t_{n+1}} (t_{n+1} - s)^2\,\dddot{h}(s)\,ds.$$

Since $h(t_n) = h(t_{n+1}) = 0$ and since $\lim_{\substack{t \to t_*^- \\ t \in \mathcal{D}}} \dddot{h}(t) = \mu$, we deduce that

$$\dot{h}^+(t_n) = -\frac{1}{2}\,(t_{n+1} - t_n)\,\ddot{h}^+(t_n) - \Big(\frac{\mu}{6} + o(1)\Big)\,(t_{n+1} - t_n)^2, \tag{4.10}$$

as $n \to +\infty$. By reversing the roles of t_n and t_{n+1}, we obtain in the same way

$$\dot{h}^-(t_{n+1}) = \frac{1}{2}\,(t_{n+1} - t_n)\,\ddot{h}^-(t_{n+1}) - \Big(\frac{\mu}{6} + o(1)\Big)\,(t_{n+1} - t_n)^2. \tag{4.11}$$

Since $\lim_{\substack{t \to t_*^- \\ t \in \mathcal{D}}} \dddot{h}(t) = \mu$, we have

$$\begin{aligned} \ddot{h}^-(t_{n+1}) &= \ddot{h}^+(t_n) + \int_{t_n}^{t_{n+1}} \dddot{h}(s)\,ds \\ &= \ddot{h}^+(t_n) + (\mu + o(1))\,(t_{n+1} - t_n), \end{aligned}$$

as $n \to +\infty$. Coming back to equality (4.11), we obtain

$$\dot{h}^-(t_{n+1}) = \frac{1}{2}\,(t_{n+1} - t_n)\,\ddot{h}^+(t_n) + \Big(\frac{\mu}{3} + o(1)\Big)\,(t_{n+1} - t_n)^2.$$

By applying the restitution law at $t = t_{n+1}$, we deduce that

$$\begin{aligned} \dot{h}^+(t_{n+1}) &= -\dot{h}^-(t_{n+1}) \\ &= -\frac{1}{2}\,(t_{n+1} - t_n)\,\ddot{h}^+(t_n) - \Big(\frac{\mu}{3} + o(1)\Big)\,(t_{n+1} - t_n)^2. \end{aligned} \tag{4.12}$$

Let us evaluate the quantity $1/\dot{h}^+(t_{n+1}) - 1/\dot{h}^+(t_n)$ by using the estimates (4.10)-(4.12) and (4.6)-(4.7):

$$\begin{aligned} \frac{1}{\dot{h}^+(t_{n+1})} - \frac{1}{\dot{h}^+(t_n)} &= \frac{\dot{h}^+(t_n) - \dot{h}^+(t_{n+1})}{\dot{h}^+(t_n)\,\dot{h}^+(t_{n+1})} \\ &= \frac{\big(\frac{\mu}{6} + o(1)\big)\,(t_{n+1} - t_n)^2}{\big(-\frac{\lambda}{2} + o(1)\big)^2\,(t_{n+1} - t_n)^2} = \frac{2\,\mu}{3\,\lambda^2} + o(1). \end{aligned}$$

By summing these estimates, we deduce that

$$\frac{1}{\dot{h}^+(t_n)} = \frac{2\,\mu}{3\,\lambda^2}\,n + o(n) \quad \text{as } n \to +\infty. \tag{4.13}$$

In particular, we obtain $1/\dot{h}^+(t_n) = O(n)$ when $n \to +\infty$. Hence there exists $C_1 > 0$ such that $|\dot{h}^+(t_n)| \geq C_1/n$ for n large enough. Since $\dot{h}^+(t_n) \sim -\frac{\lambda}{2}\,(t_{n+1} - t_n)$ as $n \to +\infty$, this implies in turn that $t_{n+1} - t_n \geq C_2/n$, for some $C_2 > 0$. By summation, we derive the existence of $C_3 \in \mathbb{R}$ such that $t_n \geq C_2 \ln n + C_3$, for n large enough. This shows that $t_* = \lim_{n \to +\infty} t_n = +\infty$ and the set $\mathcal{E}$ equals $\cup_{n \geq 0}\{t_n\}$. On the other hand, we deduce respectively from (4.5) and (4.9) that $\lambda = -(\nabla g(x_\infty), \nabla f(x_\infty))$ and $\mu = \gamma\,(\nabla g(x_\infty), \nabla f(x_\infty))$. In view of (4.13), we conclude that

$$\dot{h}^+(t_n) \sim \frac{3\,(\nabla g(x_\infty), \nabla f(x_\infty))}{2\,\gamma}\,\frac{1}{n} \quad \text{and} \quad t_{n+1} - t_n \sim \frac{3}{\gamma}\,\frac{1}{n} \quad \text{as } n \to +\infty.$$

An immediate summation then shows that $t_n \sim \frac{3}{\gamma}\,\ln n$ as $n \to +\infty$. □

When $r \in [0,1)$, the previous theorem shows that any trajectory x of (S) reaches the boundary of K in finite time and then converges toward x_∞ by staying in the set $\mathrm{bd}\,(K)$. The convergence rate of $|x(t)-x_\infty|$ toward 0 depends on the behaviour of $\Phi = f + \delta_K$. For example, if condition (3.13) holds, Theorem 3.5 shows that the speed of convergence is exponential.

Remark 4.4. In his book [6], Brogliato discusses the question of the finite time stabilization of a mechanical system on a surface. The phenomenon of infinite rebounds within a finite time is also evoked. For further details, we refer to [6, p. 425-428] and the references therein, where heuristic examples and applications to Mechanics and Robotics are discussed.

The conclusions of Theorem 4.3 may be false if the condition $\inf f < \inf_K f$ is not satisfied, even in dimension one. To illustrate it, let us take $K = \mathbb{R}_+$ and given some $k > 0$, let us define the function $f : \mathbb{R} \to \mathbb{R}$ by $f(\xi) = \frac{k}{2}\,\xi^2$ for every $\xi \in \mathbb{R}$. In this example, we have $\inf f = \inf_{\mathbb{R}_+} f = 0$ and $\mathrm{argmin}\, f = \mathrm{argmin}_{\mathbb{R}_+} f = \{0\}$. The next result shows that the structure of the set $\mathcal{E} := \{t > 0,\, x(t) = 0\}$ is different from the one described at Theorem 4.3.

Proposition 4.5. *Given some $k > 0$ and $\gamma > 0$, consider the differential inclusion (S) with $\Phi : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ defined by $\Phi(\xi) = \frac{k}{2}\,\xi^2 + \delta_{\mathbb{R}_+}(\xi)$ for every $\xi \in \mathbb{R}$. Let x be a dissipative solution to (S) satisfying the Newton's law at impacts, with a restitution coefficient $r \in [0,1]$. Assume that $\gamma^2 - 4\,k < 0$ and let us set $\omega := \sqrt{4\,k - \gamma^2}/2$. Assume that the initial conditions satisfy $(x_0, \dot{x}_0) \in (\mathbb{R}_+^* \times \mathbb{R}) \cup (\{0\} \times \mathbb{R}_+^*)$. Let $\mathcal{E} := \{t > 0,\, x(t) = 0\}$ be the instants for which the constraint is saturated.*
(i) If $r = 0$, there exists $t_0 \in (0, \frac{\pi}{\omega}]$ such that $\mathcal{E} = [t_0, +\infty)$.
(ii) If $r \in (0,1]$, the set $\mathcal{E}$ equals $\cup_{n \geq 0}\{t_n\}$, where $t_0 \in (0, \frac{\pi}{\omega}]$ and the sequence $(t_n)_{n\geq 1}$ is defined by $t_n = t_0 + \frac{n\pi}{\omega}$, for every $n \geq 1$. Moreover, we have $\dot{x}^+(t_0) > 0$ and

$$\forall n \geq 1, \quad \dot{x}^+(t_n) = \left(r\, e^{-\frac{\pi\,\gamma}{2\,\omega}}\right)^n \dot{x}^+(t_0).$$

Proof. First notice that the set $\mathcal{E}$ is non empty. Let us argue by contradiction and assume that $x(t) > 0$ for every $t > 0$. From Lemma 4.1, it ensues that the equation of the motion reduces to $\ddot{x} + \gamma\dot{x} + k\,x = 0$. Since $\gamma^2 - 4\,k < 0$, the characteristic equation $s^2 + \gamma\, s + k = 0$ has two conjugate roots $-\frac{\gamma}{2} \pm i\,\omega$. The expression of x is classically given by

$$x(t) = A\, e^{-\frac{\gamma}{2}\,t}\, \sin(\omega\, t + \phi), \tag{4.14}$$

where $A \in \mathbb{R}$ and $\phi \in [0, \pi)$ are constants satisfying

$$x_0 = A\, \sin\phi \quad \text{and} \quad \dot{x}_0 = A\,(-\frac{\gamma}{2}\, \sin\phi + \omega\, \cos\phi). \tag{4.15}$$

Since $x(\frac{\pi}{\omega}) = -x_0\, e^{-\frac{\pi\,\gamma}{2\,\omega}} \leq 0$ we obtain a contradiction. Therefore the set $\mathcal{E}$ is non empty and we set $t_0 = \inf \mathcal{E} < +\infty$. An immediate verification shows that $t_0 \in (0, \frac{\pi}{\omega}]$.
(i) **Case** $r = 0$**.** The Newton's law gives $\dot{x}^+(t_0) = 0$. Recalling that the mechanical energy E is non increasing, we find

$$\forall t \geq t_0, \quad \frac{1}{2}\dot{x}^+(t)^2 + \frac{k}{2}\, x(t)^2 \leq \frac{1}{2}\dot{x}^+(t_0)^2 + \frac{k}{2}\, x(t_0)^2 = 0,$$

and hence $x(t) = 0$ for every $t \geq t_0$, or equivalently $\mathcal{E} = [t_0, +\infty)$.
(ii) **Case** $r \in (0,1]$**.** Let us prove that $\dot{x}^+(t_0) > 0$. For every $t \in (0, t_0)$, we have

$x(t) > 0$ and hence the expression of $x(t)$ is given by (4.14). By differentiating this equality and taking the limit when $t \to t_0^-$, we easily find: $\dot{x}^-(t_0) = -A\,\omega\, e^{-\frac{\gamma}{2}\, t_0}$. In view of (4.15), we have $A > 0$ and hence $\dot{x}^-(t_0) < 0$. The restitution law applied at $t = t_0$ then gives $\dot{x}^+(t_0) = r\,A\,\omega\, e^{-\frac{\gamma}{2}\, t_0} > 0$. By the same arguments as above, it is immediate to check that the set $\{t > t_0,\, x(t) = 0\}$ is not empty. Let us define $t_1 := \inf\{t > t_0,\, x(t) = 0\}$. Since $\dot{x}^+(t_0) > 0$, the continuous map x is positive on some right neighbourhood of t_0 and hence $t_1 > t_0$. We let the reader check that the expression of $x(t)$ on $[t_0, t_1)$ is given by

$$x(t) = \frac{\dot{x}^+(t_0)}{\omega}\, e^{-\frac{\gamma}{2}(t-t_0)}\, \sin(\omega\,(t - t_0)).$$

From the previous formula, it ensues that $\omega\,(t_1 - t_0) = \pi$ and hence $t_1 = t_0 + \pi/\omega$. By differentiating the above formula at $t = t_1^-$, we find

$$\dot{x}^-(t_1) = -\dot{x}^+(t_0)\, e^{-\frac{\pi\,\gamma}{2\,\omega}}.$$

On the other hand, the Newton's law applied at $t = t_1$ yields

$$\dot{x}^+(t_1) = -r\,\dot{x}^-(t_1) = r\, e^{-\frac{\pi\,\gamma}{2\,\omega}}\, \dot{x}^+(t_0) > 0.$$

By iterating this process, we define the sequence $(t_n)_{n\geq 0}$ by

$$\forall n \in \mathbb{N}, \quad t_{n+1} := \inf\{t > t_n,\, x(t) = 0\}.$$

By using the same arguments as above, it is immediate to check that

$$\forall n \in \mathbb{N}, \quad t_{n+1} = t_n + \pi/\omega \quad \text{ and } \quad \dot{x}^+(t_{n+1}) = r\, e^{-\frac{\pi\,\gamma}{2\,\omega}}\, \dot{x}^+(t_n).$$

The assertions of (ii) are then immediate consequences. □

Remark 4.6. Now assume that $\gamma^2 - 4\,k \geq 0$ in the statement of Proposition 4.5. In this case, it is easy to check that there is at most one impact at 0 and after the trajectory stays in the interior of $\mathbb{R}_+$.

5. Existence of dissipative solutions. Case I: conservation of energy at shocks

In the paper [2], Attouch-Cabot-Redont have defined a notion of shock solution which is close to items (i)-(iv) of the definition of dissipative solution. The reader is referred to [2, Definition 2.1] for further precisions. According to the terminology of [2], the solution x of (S) is said to be an elastic shock solution if furthermore the following energy equation is satisfied for almost every $t \in \mathbb{R}_+$

$$\frac{1}{2} \mid \dot{x}(t) \mid^2 + \Phi(x(t)) = \frac{1}{2} \mid \dot{x}_0 \mid^2 + \Phi(x_0) - \gamma \int_0^t \mid \dot{x}(s) \mid^2 ds. \tag{5.1}$$

Attouch-Cabot-Redont have shown the existence of elastic shock solutions to (S), obtained as limits of the solutions of a sequence of smooth problems, by using an epiconvergent approximation of Φ (see [2, Theorem 2.1]). We are going to prove that every elastic shock solution of (S) in the sense of [2] is a dissipative solution such that $|\dot{x}^+(t)| = |\dot{x}^-(t)|$ for every $t > 0$.

Proposition 5.1. *Let $\gamma \geq 0$ and $\Phi : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ be a convex, lower semi-continuous function which is bounded from below and such that $\operatorname{int}(\operatorname{dom}\Phi) \neq \emptyset$. Additionally assume that the function $\Phi_{|\operatorname{dom}\Phi}$ is continuous. Then, for any initial data $(x_0, \dot{x}_0) \in \operatorname{dom}\Phi \times T_{\overline{\operatorname{dom}\Phi}}(x_0)$, every elastic shock solution to (S) in the sense* *of* [2, Definition 2.1] *satisfies items (i) to (v) of Definition 2.1. Moreover, we have* $|\dot{x}^+(t)| = |\dot{x}^-(t)|$ *for every* $t > 0$.

Proof. Items (i)-(ii) are immediate consequences of the corresponding items (1.a)-(1.b)-(1.c)-(2.a) of [2, Definition 2.1]. Let us prove that item (iii) is a consequence of (2.b). Our arguments are close to those of [2, § 2.3.6]. Let $(t_1, t_2) \in \mathbb{R}_+^2$ such that $t_1 < t_2$ and $\dot{x}^-(t_1) = \dot{x}(t_1)$ and $\dot{x}(t_2) = \dot{x}^+(t_2)$ with the convention $\dot{x}^-(0) = \dot{x}(0) = \dot{x}_0$ if $t_1 = 0$. Let $y \in \mathcal{C}^0([t_1, t_2]; \mathbb{R}^d)$ and let (k_n) be a sequence of functions in $\mathcal{C}_c^0(\mathbb{R}_+; \mathbb{R}_+)$ such that $k_n(t) = 0$ for every $t \in \mathbb{R}_+ \setminus [t_1, t_2]$ and every $n \in \mathbb{N}$. Hence, by continuity of k_n, we have $k_n(t_2) = 0$ and $k_n(t_1) = 0$ if $t_1 > 0$. From item 2.b of [2, Definition 2.1] and the fact that $\operatorname{supp}(k_n) \subset [t_1, t_2]$, we obtain (5.2)

$$\int_{t_1}^{t_2} k_n(t) \Big(\Phi(y(t)) - \Phi(x(t))\Big)\, dt \geq -\langle \ddot{x} + \gamma\, \dot{x}, k_n\, (y - x)\rangle_{\mathcal{M}([t_1,t_2];\mathbb{R}^d), \mathcal{C}^0([t_1,t_2];\mathbb{R}^d)}.$$

Now assume that the sequence (k_n) converges pointwise toward 1 on (t_1, t_2) if $t_1 > 0$ or on $[t_1, t_2)$ if $t_1 = 0$. Denoting by μ the measure $\mu := -\ddot{x} - \gamma\, \dot{x}\, dt$, the sequence (k_n) converges to 1 on $[t_1, t_2]$ almost everywhere with respect to μ. Indeed, we have $\mu(\{t_2\}) = -(\dot{x}^+(t_2) - \dot{x}^-(t_2)) = 0$ and in the same way $\mu(\{t_1\}) = 0$ if $t_1 > 0$. From Lebesgue's dominated convergence theorem, we infer that

$$\lim_{n \to +\infty} \langle \mu, k_n(y - x)\rangle_{\mathcal{M}([t_1,t_2];\mathbb{R}^d), \mathcal{C}^0([t_1,t_2];\mathbb{R}^d)} = \langle \mu, y - x\rangle_{\mathcal{M}([t_1,t_2];\mathbb{R}^d), \mathcal{C}^0([t_1,t_2];\mathbb{R}^d)} \tag{5.3}$$

Invoking again Lebesgue's dominated convergence theorem, we have

$$\lim_{n \to +\infty} \int_{t_1}^{t_2} k_n(t) \Big(\Phi(y(t)) - \Phi(x(t))\Big)\, dt = \int_{t_1}^{t_2} \Big(\Phi(y(t)) - \Phi(x(t))\Big)\, dt. \tag{5.4}$$

Finally, in view of (5.2), (5.3) and (5.4) we conclude that

$$\int_{t_1}^{t_2} \Big(\Phi(y(t)) - \Phi(x(t))\Big)\, dt \geq \langle \mu, y - x\rangle_{\mathcal{M}([t_1,t_2];\mathbb{R}^d), \mathcal{C}^0([t_1,t_2];\mathbb{R}^d)},$$

which is exactly item (iii) of Definition 2.1.
Let us now prove (iv) by using items (3.a)-(3.b) of [2, Definition 2.1]. Since $x \in \mathcal{C}^0(\mathbb{R}_+; \mathbb{R}^d)$, item (3.a) implies that $\lim_{t \to 0^+} x(t) = x(0) = x_0$. From (3.b), we have $\dot{x}^+(0) - \dot{x}_0 \in -N_{\overline{\operatorname{dom} \Phi}}(x_0)$, *i.e.* $(\dot{x}^+(0) - \dot{x}_0, w) \geq 0$ for every $w \in T_{\overline{\operatorname{dom} \Phi}}(x_0)$. Let us choose successively $w = \dot{x}_0$ and $w = \dot{x}^+(0)$ to obtain:

$$|\dot{x}_0|^2 \leq (\dot{x}^+(0), \dot{x}_0) \leq |\dot{x}^+(0)|^2. \tag{5.5}$$

On the other hand, we have

$$\frac{1}{2}|\dot{x}^+(0)|^2 + \Phi(x_0) = \frac{1}{2}|\dot{x}_0|^2 + \Phi(x_0). \tag{5.6}$$

Indeed, choose a sequence $t_n \to 0$ such that the energy equality (5.1) holds and take the limit when $n \to +\infty$. From (5.5) and (5.6), we deduce that $|\dot{x}^+(0)|^2 = |\dot{x}_0|^2 = (\dot{x}^+(0), \dot{x}_0)$ and hence $\dot{x}^+(0) = \dot{x}_0$.
Let us finally prove (v) by using the energy equality (5.1). Let $t > 0$ and let (t_{1n}) and (t_{2n}) be two sequences such that $t_{1n} \downarrow t$, $t_{2n} \uparrow t$ and

$$\forall n \geq 0, \quad \forall i \in \{1, 2\}, \quad \frac{1}{2} \mid \dot{x}(t_{in}) \mid^2 + \Phi \circ x\, (t_{in}) = \frac{1}{2} \mid \dot{x}_0 \mid^2 + \Phi(x_0) - \gamma \int_0^{t_{in}} \mid \dot{x}(s) \mid^2 ds.$$

Taking the limit when $n \to +\infty$, we infer that

$$\frac{1}{2} \mid \dot{x}^+(t) \mid^2 + \Phi \circ x^+(t) \quad = \quad \frac{1}{2} \mid \dot{x}^-(t) \mid^2 + \Phi \circ x^-(t). \tag{5.7}$$

Since the functions $x : \mathbb{R}_+ \to \operatorname{dom} \Phi$ and $\Phi_{|\operatorname{dom} \Phi}$ are continuous, we deduce that the function $\Phi \circ x$ is also continuous. In view of (5.7) we conclude that $|\dot{x}^+(t)|^2 = |\dot{x}^-(t)|^2$. □

6. Existence of dissipative solutions. Case II: loss of energy at shocks

We consider a convex, closed subset $K \subset \mathbb{R}^d$ with a non empty interior, such that

$$K = \big\{\xi \in \mathbb{R}^d;\ \varphi_\alpha(\xi) \geq 0, \quad 1 \leq \alpha \leq \nu\big\}, \quad \nu \geq 1$$

where $\varphi_\alpha \in \mathcal{C}^1(\mathbb{R}^d)$ such that, for all $\alpha \in \{1, \dots, \nu\}$, $\nabla \varphi_\alpha$ does not vanish in a neighbourhood of $\big\{\xi \in \mathbb{R}^d;\ \varphi_\alpha(\xi) = 0\big\}$. For all $\xi \in \mathbb{R}^d$ we define the set of active constraints at ξ by

$$J(\xi) = \big\{\alpha \in \{1, \dots, \nu\};\ \varphi_\alpha(\xi) \leq 0\big\}.$$

As usually we assume that, for all $\xi \in K$, $\big(\nabla \varphi_\alpha(\xi)\big)_{\alpha \in J(\xi)}$ is linearly independent. We extend the definition of the tangent and normal cones to K by

$$T_K(\xi) = \big\{w \in \mathbb{R}^d; \big(\nabla \varphi_\alpha(\xi), w\big) \geq 0, \quad \forall \alpha \in J(\xi)\big\}$$
$$N_K(\xi) = \big\{w \in \mathbb{R}^d; w = \sum_{\alpha \in J(\xi)} \lambda_\alpha \nabla \varphi_\alpha(\xi), \quad \lambda_\alpha \leq 0\big\}$$

for all $\xi \in \mathbb{R}^d$. Let f be a convex smooth function (at least $\mathcal{C}^1$) from $\mathbb{R}^d$ to $\mathbb{R}$, bounded from below on K. Assuming that $\Phi := f + \delta_K$, the differential inclusion (S) can be rewritten as

$$\ddot{x} + \gamma \dot{x} + \nabla f(x) \in -N_K(x) \tag{6.1}$$

with $\gamma \geq 0$. Let $(x_0, \dot{x}_0) \in K \times T_K(x_0)$ be admissible initial data. We are interested in existence of dissipative solutions for the Cauchy problem associated with (6.1) and the initial data $(x_0, \dot{x}_0)$.

We propose the following time-stepping scheme: let $h > 0$, we define the first approximate positions U^0 and U^1 by

$$U^0 = x_0, \quad U^1 = \operatorname{Proj}\big(K, x_0 + h\dot{x}_0 + hz(h)\big), \quad \text{with } \lim_{h \to 0} z(h) = 0,$$

and, for all $n \geq 1$

$$U^{n+1} = \operatorname{Proj}\big(K, 2U^n - U^{n-1} + h^2 F^n\big)$$

with

$$F^n = -\gamma \frac{U^{n+1} - U^{n-1}}{2h} - \nabla f\big(U^{n+1}\big).$$

Then we define approximate solutions of the Cauchy problem associated to (6.1) and the initial data $(x_0, \dot{x}_0)$ as follows

$$u_h(t) = U^n + (t - nh) \frac{U^{n+1} - U^n}{h} \quad \text{if } t \in \big[nh, (n+1)h\big)$$

for all $n \geq 0$, for all $h > 0$. We will prove the following result:

Theorem 6.1. *The sequence $(u_h)_{h>0}$ admits a subsequence $(u_{h_i})_{h_i>0}$ which converges in the following sense:*

(a) $u_{h_i} \to x$ strongly in $\mathcal{C}^0\big([0,T];\mathbb{R}^d\big)$, for all $T>0$,
(b) $\dot u_{h_i} \to \dot x$ weakly* in $L^\infty(0,T;\mathbb{R}^d)$ and a.e in $[0,T]$, for all $T>0$,
(c) $\ddot u_{h_i} \to \ddot x$ weakly* in $\mathcal{M}\big([0,+\infty);\mathbb{R}^d\big)$,

and the limit x is a dissipative solution of (6.1).

Remark 6.2. The scheme proposed here is directly inspired by the one proposed in [21]. Let us recall that in [21] we consider the differential inclusion

$$\ddot x + N_K(x) \ni \tilde f(t,x,\dot x)$$

where $\tilde f$ is continuous from $[0,T]\times\mathbb{R}^d\times\mathbb{R}^d$ to $\mathbb{R}^d$ and is Lipschitz continuous with respect to its last two arguments. If we define $\tilde f(t,x,v) = -\gamma v - \nabla f(x)$ for all $(x,v)\in\mathbb{R}^d\times\mathbb{R}^d$, we can rewrite (6.1) in the previous form but we should observe that $\tilde f$ is not necessarily Lipschitz continuous with respect to x and v and that we have to deal now with a non bounded time interval.

The proof is organized as follows. First we begin with a priori estimates of the discrete velocities in $L^\infty(\mathbb{R}_+;\mathbb{R}^d)$ and $BV_{loc}(\mathbb{R}_+;\mathbb{R}^d)$. Then, combining Helly's theorem with a diagonal extraction argument, we infer that the announced convergences hold. Moreover we show that the limit x takes its values in K, which completes the proof of properties (i) and (ii) of Definition 2.1. In the third part, we establish that x satisfies the last three properties of Definition 2.1. Finally we study the transmission of the velocities at impacts: we prove that the limit x satisfies the Newton's law with a restitution coefficient $r=0$ *i.e.*

$$\dot x^+(t) = \mathrm{Proj}\big(T_K\big(x(t)\big), \dot x^-(t)\big)$$

whenever the following condition holds:

$$\big(\nabla\varphi_\alpha\big(x(t)\big), \nabla\varphi_\beta\big(x(t)\big)\big) \le 0 \quad \forall(\alpha,\beta)\in J\big(x(t)\big)^2, \quad \alpha\neq\beta.$$

6.1. **A priori estimates.** Let us define the discrete velocities by

$$V^n = \frac{U^{n+1}-U^n}{h} \quad \forall n\ge 0, \quad \forall h>0.$$

First we begin with an a priori estimate of the discrete velocities in $L^\infty(\mathbb{R}_+;\mathbb{R}^d)$.

Proposition 6.3. *There exist $C>0$ and $h_*>0$ such that*

$$|V^n| \le C \quad \forall n\ge 0, \quad \forall h\in(0,h_*].$$

Proof. Let $h>0$ and $n\ge 1$. By definition of the scheme, we have

$$U^{n+1} = \mathrm{Proj}\big(K, 2U^n - U^{n-1} + h^2F^n\big)$$

with

$$F^n = -\gamma\frac{V^n+V^{n-1}}{2} - \nabla f(U^{n+1}).$$

The characterization of the projection on the convex set K implies that $U^{n+1}\in K$ and

$$\big(2U^n - U^{n-1} + h^2F^n - U^{n+1}, z - U^{n+1}\big) \le 0 \quad \forall z\in K.$$

Since U^0 and U^1 belong to K by construction, we infer that $U^k \in K$ for all $k \geq 0$ and

$$\big(V^{n-1} - V^n + hF^n, z - U^{n+1}\big) \leq 0 \quad \forall z \in K.$$

With $z = U^{n-1}$ we get

$$\big(V^n - V^{n-1} - hF^n, V^n + V^{n-1}\big) \leq 0,$$

and, using the convexity of f

$$\begin{aligned}&\big|V^n\big|^2 - \big|V^{n-1}\big|^2 + \frac{\gamma h}{2}\big|V^n + V^{n-1}\big|^2 \leq -h\big(\nabla f(U^{n+1}), V^n + V^{n-1}\big)\\ &\leq f(U^{n-1}) - f(U^{n+1}).\end{aligned}$$

It follows that

$$\big|V^n\big|^2 \leq \big|V^0\big|^2 + f(U^0) + f(U^1) - f(U^n) - f(U^{n+1}).$$

Furthermore, using the definition of U^0 and U^1 we get

$$\big|V^0\big| = \left|\frac{U^1 - U^0}{h}\right| = \frac{1}{h}\big|\mathrm{Proj}\big(K, x_0 + h\dot{x}_0 + hz(h)\big) - x_0\big| \leq \big|\dot{x}_0\big| + \big|z(h)\big|$$

and, since $\lim_{h\to 0} z(h) = 0$, we infer that there exists $h_* > 0$ such that

$$\big|V^0\big| \leq \big|\dot{x}_0\big| + 1 \quad \forall h \in (0, h_*].$$

Moreover, since f is continuous on $\mathbb{R}^d$ and bounded from below on K, we obtain

$$\big|V^n\big|^2 \leq C^2 = \big(\big|\dot{x}_0\big| + 1\big)^2 + 2\Big(\sup_{x\in B} f(x) - \inf_{x\in K} f(x)\Big) \quad \forall n \geq 1,\ \forall h \in (0, h_*],$$

with $B = \overline{B}\big(x_0, h_*\big(|\dot{x}_0| + 1\big)\big)$. □

Let us estimate now the discrete accelerations.

Proposition 6.4. *Let $T > 0$. There exist $C' > 0$ and $h'_* \in (0, h_*]$ such that*

$$TV\big(\dot{u}_h, [0,T]\big) = \sum_{n=1}^{N}\big|V^n - V^{n-1}\big| \leq C' \quad \forall h \in (0, h'_*]$$

where $N = \lfloor T/h \rfloor$ (the integer part of T/h) if $T/h \notin \mathbb{N}$, $N = \lfloor T/h \rfloor - 1$ otherwise.

Proof. We just have to reproduce the proof of Proposition 2.4 in [21]. It should be noted that we can choose h'_* independent of T if K is a bounded subset of $\mathbb{R}^d$, otherwise C' and h'_* depend both on T. □

6.2. **Passage to the limit.** Let us consider now the sequence of time step $(h_k)_{k\in\mathbb{N}^*}$ defined by $h_k = \dfrac{1}{2^k}$ and let $k_1 \in \mathbb{N}^*$ be such that $h_{k_1} \leq h_*$. For the sake of simplicity we will denote the approximate solutions u_{h_k} (with $k \geq k_1$) simply by u_k. From Proposition 6.3 we know that the sequence $(u_k)_{k\geq k_1}$ is C-Lipschitz continuous on $\mathbb{R}_+$. Moreover

Proposition 6.5. *There exists a subsequence $(u_{\phi(k)})_{k\geq k_1}$ such that the convergences (a)-(b)-(c) of Theorem 6.1 hold.*

Proof. Let $T \in \mathbb{R}_+^* \setminus \mathbb{Q}$. First we observe that the sequence $(\dot{u}_k)_{k \geq k_1}$ is bounded in $BV\big([0,T];\mathbb{R}^d\big)$. Indeed, let $h'_* \in (0, h_*]$ and $C' > 0$ be given as in Proposition 6.4. We define $k'_1 \geq k_1$ such that $\frac{1}{2^{k'_1}} \leq h'_*$ and

$$C'_T = \max_{k_1 \leq k \leq k'_1} \big(C', TV\big(\dot{u}_k, [0,T]\big)\big).$$

Then Proposition 6.4 implies that

$$\sup_{k \geq k_1} TV\big(\dot{u}_k, [0,T]\big) \leq C'_T.$$

Using Helly's theorem, we infer that there exist $v_T \in BV\big([0,T];\mathbb{R}^d\big)$ and a subsequence $(u_{\phi_T(k)})_{k \geq k_1}$ (depending on T) such that

$$\begin{array}{l} \dot{u}^+_{\phi_T(k)}(t) \to v_T(t) \quad \forall t \in [0,T], \\ \langle \psi, \ddot{u}_{\phi_T(k)} \rangle \to \langle \psi, dv_T \rangle_{\mathcal{C}^0([0,\tau];\mathbb{R}^d), \mathcal{M}([0,T];\mathbb{R}^d)} \quad \forall \psi \in \mathcal{C}^0\big([0,T];\mathbb{R}^d\big). \end{array}$$

Then, considering $T = j\pi$, $j \in \mathbb{N}^*$, and using a diagonal extraction argument, we obtain that there exist $v \in BV_{loc}\big(\mathbb{R}_+;\mathbb{R}^d\big)$ and a subsequence $(u_{\phi(k)})_{k \geq k_1}$ (independent on T) such that

$$\begin{array}{l} \dot{u}^+_{\phi(k)}(t) \to v(t) \quad \forall t \in \mathbb{R}_+, \\ \langle \psi, \ddot{u}_{\phi(k)} \rangle \to \langle \psi, dv \rangle_{\mathcal{C}^0_c(\mathbb{R}_+;\mathbb{R}^d), \mathcal{M}(\mathbb{R}_+;\mathbb{R}^d)}, \quad \forall \psi \in \mathcal{C}^0_c\big(\mathbb{R}_+;\mathbb{R}^d\big). \end{array}$$

Furthermore, recalling that

$$\big|\dot{u}^+_k(t)\big| = \big|\dot{u}_k(t)\big| \leq C \quad \forall k \geq k_1, \quad \forall t \in \mathbb{R}_+ \setminus \mathbb{Q},$$

we get also

$$\big|v(t)\big| \leq C \quad \forall t \in \mathbb{R}_+ \setminus \mathbb{Q},$$

and Lebesgue's theorem implies that, for all $T > 0$,

$$\dot{u}_{\phi(k)} \to v \qquad \begin{array}{l} \text{weakly* in } L^\infty\big([0,T];\mathbb{R}^d\big) \text{ and} \\ \text{strongly in } L^p\big([0,T];\mathbb{R}^d\big), \text{ for all } p \geq 1. \end{array}$$

Finally, we define

$$x(t) = x_0 + \int_0^t v(s)\,ds \quad \forall t \in \mathbb{R}_+. \tag{6.2}$$

It is obvious that x is continuous on $\mathbb{R}_+$, $x(0) = x_0$ and $\dot{x}^\pm(t) = v^\pm(t)$ for every $t \in \mathbb{R}_+^*$. It follows that x is C-Lipschitz continuous on $\mathbb{R}_+$, $\dot{x} \in BV_{loc}\big(\mathbb{R}_+;\mathbb{R}^d\big)$ and $d\dot{x} = dv$ in $\mathcal{M}(\mathbb{R}_+;\mathbb{R}^d)$ if we choose $\dot{x}(0) = \dot{x}^-(0) = v(0)$. Moreover, observing that for all $t \in \mathbb{R}_+$ and for all $k \geq k_1$

$$u_{\phi(k)}(t) = x_0 + \sum_{p=1}^n hV^p + (t - nh)V^n = x_0 + \int_0^t \dot{u}_{\phi(k)}(s)\,ds$$

with $h = \dfrac{1}{2^{\phi(k)}}$ and $n = \lfloor t/h \rfloor$, we get

$$x(t) - u_{\phi(k)}(t) = \int_0^t \big(v(s) - \dot{u}_{\phi(k)}(s)\big)\,ds \quad \forall t \in \mathbb{R}_+, \quad \forall k \geq k_1$$

which yields the strong convergence of $\big(u_{\phi(k)}\big)_{k \geq k_1}$ to x in $C^0\big([0,T];\mathbb{R}^d\big)$ for all $T > 0$. $\square$

Now we can prove that the limit function x takes its values in K.

Lemma 6.6. *For all $t \in \mathbb{R}_+$, we have $x(t) \in K$.*

Proof. Let $t \geq 0$ and let us use the same notations as in the previous proof. We have

$$u_{\phi(k)}(t) = U^n + (t - nh)\frac{U^{n+1} - U^n}{h}, \quad h = \frac{1}{2^{\phi(k)}}, \quad n = \left\lfloor \frac{t}{h} \right\rfloor,$$

and $U^n, U^{n+1} \in K$. From the convexity of K, we deduce that $u_{\phi(k)}(t) \in K$. Thus, if $T > t$ we get

$$\operatorname{dist}\big(x(t), K\big) \leq \big|x(t) - u_{\phi(k)}(t)\big| \leq \|x - u_{\phi(k)}\|_{\mathcal{C}^0([0,T];\mathbb{R}^d)},$$

which enables us to conclude. $\square$

6.3. **Dissipativity properties.** Let us prove now that the limit x satisfies property (2b) of Definition 2.1 in [2].

Proposition 6.7. *Let $y \in \mathcal{C}^0\big(\mathbb{R}_+; \mathbb{R}^d\big)$ such that $y(t) \in K$ for all $t \in \mathbb{R}_+$. Let $\psi \in \mathcal{C}^0_c\big(\mathbb{R}_+; \mathbb{R}_+\big)$. Then we have*

$$\int_{\mathbb{R}_+} \psi(t)\Big(f\big(y(t)\big) - f\big(x(t)\big)\Big)\, dt \geq -\langle dv + \gamma \dot{x}, \psi(y - x)\rangle_{\mathcal{M}(\mathbb{R}_+;\mathbb{R}^d), \mathcal{C}^0_c(\mathbb{R}_+;\mathbb{R}^d)}.$$

Proof. Let $y \in \mathcal{C}^0\big(\mathbb{R}_+; \mathbb{R}^d\big)$ such that $y(t) \in K$ for all $t \in \mathbb{R}_+$. Let $\psi \in \mathcal{C}^0_c\big(\mathbb{R}_+; \mathbb{R}_+\big)$ and let $T \in \mathbb{R}^*_+ \setminus \mathbb{Q}$ such that $\operatorname{Supp}(\psi) \subset [0, T)$. Then

$$\begin{aligned} \langle dv + \gamma \dot{x}, \psi(y - x)\rangle_{\mathcal{M}(\mathbb{R}_+;\mathbb{R}^d), \mathcal{C}^0_c(\mathbb{R}_+;\mathbb{R}^d)} \quad &= \quad \langle dv, \psi(y - x)\rangle_{\mathcal{M}(\mathbb{R}_+;\mathbb{R}^d), \mathcal{C}^0_c(\mathbb{R}_+;\mathbb{R}^d)} \\ &\quad + \gamma \int_0^T \Big(v(s), \psi(s)\big(y(s) - x(s)\big)\Big)\, ds. \end{aligned}$$

Using the results of Proposition 6.5, we have

$$\langle dv, \psi(y - x)\rangle_{\mathcal{M}(\mathbb{R}_+;\mathbb{R}^d), \mathcal{C}^0_c(\mathbb{R}_+;\mathbb{R}^d)} = \lim_{k \to +\infty} \sum_{n=1}^{N} \Big(V^n - V^{n-1}, \psi(nh)\big(y(nh) - x(nh)\big)\Big)$$

where $h = \dfrac{1}{2^{\phi(k)}}$ and $N = \lfloor T/h \rfloor$. Observing that

$$U^n = u_{\phi(k)}(nh) \quad \text{and} \quad |U^n - U^{n+1}| = h|V^n| \leq Ch,$$

we have also

$$\begin{aligned} &\left|\sum_{n=1}^{N} \Big(V^n - V^{n-1}, \psi(nh)\big(x(nh) - U^{n+1}\big)\Big)\right| \\ &\leq \left(\sum_{n=1}^{N} |V^n - V^{n-1}|\right) \|\psi\|_{\mathcal{C}^0([0,T];\mathbb{R}_+)} \big(\|x - u_{\phi(k)}\|_{\mathcal{C}^0([0,T];\mathbb{R}^d)} + Ch\big). \end{aligned}$$

Thus

$$\langle dv, \psi(y - x)\rangle_{\mathcal{M}(\mathbb{R}_+;\mathbb{R}^d), \mathcal{C}^0_c(\mathbb{R}_+;\mathbb{R}^d)} = \lim_{k \to +\infty} \sum_{n=1}^{N} \Big(V^n - V^{n-1}, \psi(nh)\big(y(nh) - U^{n+1}\big)\Big).$$

Now let us prove the following lemma:

Lemma 6.8.

$$\int_0^T \Big(v(s), \psi(s)\big(y(s)-x(s)\big)\Big)\, ds = \lim_{k\to+\infty} \sum_{n=1}^N h\left(\frac{V^n+V^{n-1}}{2}, \psi(nh)\big(y(nh)-U^{n+1}\big)\right).$$

Proof. Since $(\dot u_{\phi(k)})_{k\geq k_1}$ converges strongly to v in $L^1\big([0,T];\mathbb{R}^d\big)$, we have

$$\int_0^T \Big(v(s), \psi(s)\big(y(s)-x(s)\big)\Big)\, ds = \lim_{k\to+\infty} \int_0^T \Big(\dot u_{\phi(k)}(s), \psi(s)\big(y(s)-x(s)\big)\Big)\, ds. \tag{6.3}$$

First we rewrite the integral term of the right member as follows:

$$\begin{aligned}
&\int_0^T \Big(\dot u_{\phi(k)}(s), \psi(s)\big(y(s)-x(s)\big)\Big)\, ds \\
&= \sum_{n=1}^N \int_{(n-1)h}^{nh} \Big(V^{n-1}, \psi(s)\big(y(s)-x(s)\big)\Big)\, ds + \int_{Nh}^T \Big(V^N, \psi(s)\big(y(s)-x(s)\big)\Big)\, ds \\
&= \sum_{n=1}^N \int_{(n-1)h}^{nh} \Big(V^{n-1}, \psi(s)\big(y(s)-x(s)\big) - \psi(nh)\big(y(nh)-U^n\big)\Big)\, ds \qquad (6.4)\\
&+ \sum_{n=1}^N \int_{(n-1)h}^{nh} \Big(V^{n-1}, \psi(nh)\big(y(nh)-U^n\big)\Big)\, ds + \int_{Nh}^T \Big(V^N, \psi(s)\big(y(s)-x(s)\big)\Big)\, ds.
\end{aligned}$$

The last term of (6.4) can be estimated by

$$\begin{aligned}
\left|\int_{Nh}^T \Big(V^N, \psi(s)\big(y(s)-x(s)\big)\Big)\, ds\right| &\leq (T-Nh)\big|V^N\big| \max_{s\in[Nh,T]} \big|\psi(s)\big(y(s)-x(s)\big)\big| \\
&\leq Ch\|\psi\|_{\mathcal{C}^0([0,T];\mathbb{R}_+)} \|y-x\|_{\mathcal{C}^0([0,T];\mathbb{R}^d)}.
\end{aligned}$$

Moreover, for all $n\in\{1,\dots,N\}$ and for all $s\in\big[(n-1)h, nh\big]$, we have

$$\begin{aligned}
&\big|\psi(s)\big(y(s)-x(s)\big) - \psi(nh)\big(y(nh)-U^n\big)\big| \\
&\leq \big|\psi(s)\big(y(s)-x(s)\big) - \psi(nh)\big(y(nh)-x(nh)\big)\big| + \big|\psi(nh)\big(x(nh)-u_{\phi(k)}(nh)\big)\big| \\
&\leq \omega_{\psi(y-x)}(h) + \|\psi\|_{\mathcal{C}^0([0,T];\mathbb{R}_+)} \|x-u_{\phi(k)}\|_{\mathcal{C}^0([0,T];\mathbb{R}^d)}
\end{aligned}$$

where $\omega_{\psi(y-x)}$ denotes the continuity modulus of the function $\psi(y-x)$ on $[0,T]$. Then we can estimate the first term of the right hand side of (6.4) by

$$\begin{aligned}
&\left|\sum_{n=1}^N \int_{(n-1)h}^{nh} \Big(V^{n-1}, \psi(s)\big(y(s)-x(s)\big) - \psi(nh)\big(y(nh)-U^n\big)\Big)\, ds\right| \\
&\qquad\qquad \leq CT\Big(\omega_{\psi(y-x)}(h) + \|\psi\|_{\mathcal{C}^0([0,T];\mathbb{R}_+)} \|x-u_{\phi(k)}\|_{\mathcal{C}^0([0,T];\mathbb{R}^d)}\Big).
\end{aligned}$$

Recalling that $(u_{\phi(k)})_{k\geq k_1}$ converges uniformly to x on every compact subset of $\mathbb{R}_+$, we get in view of (6.3)

$$\int_0^T \Big(v(s), \psi(s)\big(y(s)-x(s)\big)\Big)\, ds = \lim_{k\to+\infty} \sum_{n=1}^N h\Big(V^{n-1}, \psi(nh)\big(y(nh)-U^n\big)\Big).$$

Hence, there remains only to prove that

$$\lim_{k\to+\infty} \left(\sum_{n=1}^N h\left(\frac{V^n+V^{n-1}}{2}, \psi(nh)\big(y(nh)-U^{n+1}\big)\right) - \sum_{n=1}^N h\Big(V^{n-1}, \psi(nh)\big(y(nh)-U^n\big)\Big)\right) = 0.$$

But

$$\sum_{n=1}^{N} h\left(\frac{V^n+V^{n-1}}{2}, \psi(nh)\big(y(nh)-U^{n+1}\big)\right) - \sum_{n=1}^{N} h\Big(V^{n-1}, \psi(nh)\big(y(nh)-U^n\big)\Big)$$
$$= \sum_{n=1}^{N} h\left(\frac{V^n-V^{n-1}}{2}, \psi(nh)\big(y(nh)-U^{n+1}\big)\right) + \sum_{n=1}^{N} h\Big(V^{n-1}, \psi(nh)\big(U^{n+1}-U^n\big)\Big).$$

The second term of the right hand side can be estimated by

$$\left|\sum_{n=1}^{N} h\Big(V^{n-1}, \psi(nh)\big(U^{n+1}-U^n\big)\Big)\right| \leq \sum_{n=1}^{N} h^2\big|V^{n-1}\big|\big|V^n\big|\big|\psi(nh)\big| \leq C^2 Th\|\psi\|_{\mathcal{C}^0([0,T];\mathbb{R}_+)}.$$

In order to estimate the first term, we may observe that

$$\big|U^{n+1}-x_0\big| = \big|U^{n+1}-U^0\big| \leq \sum_{p=0}^{n} h|V^p| \leq C(T+h)$$

and $h = \dfrac{1}{2^{\phi(k)}} \in (0,1]$. Hence

$$\left|\sum_{n=1}^{N} h\left(\frac{V^n-V^{n-1}}{2}, \psi(nh)\big(y(nh)-U^{n+1}\big)\right)\right|$$
$$\leq \frac{h}{2}\left(\sum_{n=1}^{N}\big|V^n-V^{n-1}\big|\right)\|\psi\|_{\mathcal{C}^0([0,T];\mathbb{R}_+)}\big(\|y\|_{\mathcal{C}^0([0,T);\mathbb{R}^d)}+M\big)$$

with $M = |x_0| + C(T+1)$. Finally, using the results of Proposition 6.5, we obtain

$$\left|\sum_{n=1}^{N} h\left(\frac{V^n-V^{n-1}}{2}, \psi(nh)\big(y(nh)-U^{n+1}\big)\right)\right| \leq \frac{h}{2}C'_T\|\psi\|_{\mathcal{C}^0([0,T];\mathbb{R}_+)}\big(\|y\|_{\mathcal{C}^0([0,T);\mathbb{R}^d)}+M\big)$$

which enables us to conclude. □

In order to conclude the proof of Proposition 6.7, there remains now to establish that

$$\lim_{k\to+\infty}\sum_{n=1}^{N}\left(V^{n-1}-V^n-\gamma h\frac{V^n+V^{n-1}}{2}, \psi(nh)\big(y(nh)-U^{n+1}\big)\right)$$
$$\leq \int_0^T \psi(t)\big(f\big(y(t)\big)-f\big(x(t)\big)\big)\,dt.$$

From the definition of the scheme, we know that

$$\big(V^{n-1}-V^n+hF^n, z-U^{n+1}\big) \leq 0 \quad \forall z\in K, \quad \forall n\geq 1$$

with

$$F^n = -\gamma\frac{V^n+V^{n-1}}{2} - \nabla f(U^{n+1}).$$

Thus, recalling that $y(t)\in K$ for all $t\in\mathbb{R}_+$, we infer that

$$\left(V^{n-1}-V^n-\gamma h\frac{V^n+V^{n-1}}{2}, y(nh)-U^{n+1}\right) \leq h\big(\nabla f(U^{n+1}), y(nh)-U^{n+1}\big).$$

Moreover, $\psi(nh) \geq 0$ for all $n \geq 1$ and f is convex, thus

$$\sum_{n=1}^{N} \Big(V^{n-1} - V^n - \gamma h \frac{V^n + V^{n-1}}{2}, \psi(nh)\big(y(nh) - U^{n+1}\big) \Big) \leq \sum_{n=1}^{N} h\psi(nh)\big(f\big(y(nh)\big) - f(U^{n+1})\big).$$

But

$$\lim_{k\to+\infty} \sum_{n=1}^{N} h\psi(nh)\Big(f\big(y(nh)\big) - f(U^{n+1})\Big) = \int_0^T \psi(t)\Big(f\big(y(t)\big) - f\big(x(t)\big)\Big)\,dt.$$

Indeed,

$$\begin{aligned}
&\int_0^T \psi(t)\big(f\big(y(t)\big) - f\big(x(t)\big)\big)\,dt - \sum_{n=1}^{N} h\psi(nh)\big(f\big(y(nh)\big) - f(U^{n+1})\big) \\
&= \sum_{n=1}^{N} \int_{(n-1)h}^{nh} \Big(\psi(t) f\big(y(t)\big) - \psi(nh) f\big(y(nh)\big)\Big)\,dt \\
&+ \sum_{n=1}^{N} \int_{(n-1)h}^{nh} \big(\psi(t) - \psi(nh)\big) f\big(x(t)\big)\,dt + \sum_{n=1}^{N} \int_{(n-1)h}^{nh} \psi(nh)\Big(f\big(x(t)\big) - f(U^{n+1})\Big)\,dt \\
&+ \int_{Nh}^{T} \psi(t)\Big(f\big(y(t)\big) - f\big(x(t)\big)\Big)\,dt.
\end{aligned}$$

Arguing as in the previous lemma, we can estimate the first term of the right hand side by $T\omega_{\psi\cdot f\circ y}(h)$ where $\omega_{\psi\cdot f\circ y}$ is the continuity modulus of $\psi \cdot f \circ y$ on $[0,T]$. The second term of the right hand side can be estimated by $T\|f\circ x\|_{\mathcal{C}^0([0,T];\mathbb{R})}\omega_\psi(h)$, where ω_ψ is the continuity modulus of ψ on $[0,T]$ and the last term can be estimated by $h\|\psi\|_{\mathcal{C}^0([0,T];\mathbb{R}_+)}\|f\circ y - f\circ x\|_{\mathcal{C}^0([0,T];\mathbb{R})}$.

For the third term, we recall that U^{n+1} belongs to $\overline{B}\big(x_0, C(T+1)\big)$ for all $n \in \{1,\dots,N\}$, and since x is C-Lipschitz continuous, we have also $x(t) \in \overline{B}\big(x_0, C(T+1)\big)$ for all $t \in [0,T]$. Moreover $f \in \mathcal{C}^1(\mathbb{R}^d)$, thus f is Lipschitz continuous on $\overline{B}\big(x_0, C(T+1)\big)$. Let us denote by L_f the Lipschitz constant of f on $\overline{B}\big(x_0, C(T+1)\big)$, we get

$$\left| \sum_{n=1}^{N} \int_{(n-1)h}^{nh} \psi(nh)\Big(f\big(x(t)\big) - f(U^{n+1})\Big)\,dt \right| \leq \sum_{n=1}^{N} \int_{(n-1)h}^{nh} L_f \big|\psi(nh)\big|\big|x(t) - U^{n+1}\big|\,dt$$

and

$$\begin{aligned}
\big|x(t) - U^{n+1}\big| &\leq \big|x(t) - u_{\phi(k)}(t)\big| + \big|U^{n-1} + \big(t - (n-1)h\big)V^{n-1} - U^{n+1}\big| \\
&\leq \|x - u_{\phi(k)}\|_{\mathcal{C}^0([0,T];\mathbb{R}^d)} + 3Ch.
\end{aligned}$$

Thus

$$\left| \sum_{n=1}^{N} \int_{(n-1)h}^{nh} \psi(nh)\Big(f\big(x(t)\big) - f(U^{n+1})\Big)\,dt \right| \leq L_f T \|\psi\|_{\mathcal{C}^0([0,T];\mathbb{R}_+)} \Big(\|x - u_{\phi(k)}\|_{\mathcal{C}^0([0,T];\mathbb{R}^d)} + 3Ch \Big).$$

Finally we get

$$\lim_{k\to+\infty}\sum_{n=1}^{N} h\psi(nh)\Big(f\big(y(nh)\big)-f(U^{n+1})\Big)=\int_0^T \psi(t)\Big(f\big(y(t)\big)-f\big(x(t)\big)\Big)\,dt$$

which enables us to conclude. □

Then we observe that

Lemma 6.9. *We have* $v(0)=\dot{x}_0$.

Proof. By definition of U^1 we have

$$\big(U^0+h\dot{x}_0+hz(h)-U^1, z-U^1\big)\leq 0 \quad \forall z\in K, \tag{6.5}$$

i.e.

$$\big(\dot{x}_0+z(h)-V^0, z-x_0\big)\leq h\big(\dot{x}_0+z(h)-V^0, V^0\big) \quad \forall z\in K.$$

Since $v(0)=\lim_{k\to+\infty}\dot{u}_{\phi(k)}(0)=\lim_{k\to+\infty}V^0$, we can pass to the limit as k tends to $+\infty$ and we obtain

$$\big(\dot{x}_0-v(0), z-x_0\big)\leq 0 \quad \forall z\in K.$$

It follows that

$$\big(\dot{x}_0-v(0), w\big)\leq 0 \quad \forall w\in T_K(x_0). \tag{6.6}$$

Moreover $V^0=\dfrac{U^1-U^0}{h}=\dfrac{U^1-x_0}{h}$ and $U^1\in K$, so $V^0\in T_K(x_0)$ and $v(0)=\lim_{k\to+\infty}V^0\in\overline{T_K(x_0)}=T_K(x_0)$. By choosing successively $w=\dot{x}_0$ and $w=v(0)$ in (6.6) we get

$$|\dot{x}_0|^2\leq\big(\dot{x}_0, v(0)\big)\leq\big|v(0)\big|^2.$$

But, with $z=x_0$ in (6.5), we have also

$$\big(V^0-\dot{x}_0-z(h), V^0\big)\leq 0$$

and passing to the limit as k tends to $+\infty$, we infer that

$$\big|v(0)\big|^2\leq\big(\dot{x}_0, v(0)\big)\leq|\dot{x}_0|\big|v(0)\big|.$$

Finally we have

$$|\dot{x}_0|^2=\big(\dot{x}_0, v(0)\big)=\big|v(0)\big|^2$$

which yields $v(0)=\dot{x}_0$. □

Now starting from the results of Proposition 6.7 and using the same arguments as in Proposition 5.1, we can conclude that the limit x satisfies property (iii) of Definition 2.1.

Let us establish that the limit x satisfies the initial data in the sense of property (iv) of Definition 2.1. More precisely, let us prove that

Lemma 6.10. *We have* $x(0)=x_0$ *and* $\dot{x}^+(0)=\dot{x}_0$.

Proof. The first part of the result is a direct consequence of the definition of x (see (6.2)). Let us establish now that

$$\big|\dot{x}^+(0)\big| \leq |\dot{x}_0|.$$

Let $T > 0$ and $h = \dfrac{1}{2^{\phi(k)}}$ with $k \geq k_1$. From the definition of the scheme, we know that

$$(V^{n-1} - V^n + hF^n, z - U^{n+1}) \leq 0 \quad \forall z \in K, \quad \forall n \geq 1,$$

thus, with $z = U^n$, we get

$$\big|V^n\big| \leq \big|V^{n-1}\big| + h\big|F^n\big| \quad \forall n \geq 1.$$

It follows that

$$\big|V^n\big| \leq \big|V^0\big| + h\sum_{p=1}^{n}\big|F^p\big| \leq |\dot{x}_0| + \big|z(h)\big| + h\sum_{p=1}^{n}\big|F^p\big| \quad \forall n \geq 1.$$

Moreover, we know that

$$U^{p+1} \in \overline{B}\big(x_0, C(T+1)\big) \quad \forall p \in \big\{0, \dots, \lfloor T/h \rfloor\big\}$$

thus

$$|F^p| = \left|\gamma\frac{V^p + V^{p-1}}{2} + \nabla f(U^{p+1})\right| \leq M_1$$

with

$$M_1 = \gamma C + \sup_{x \in \overline{B}\left(x_0, C(T+1)\right)} \big|\nabla f(x)\big|.$$

Finally, recalling that, for all $t \in [0,T] \setminus \mathbb{Q}$, we have

$$\dot{u}_{\phi(k)}(t) = V^n \quad \text{with} \quad n = \lfloor t/h \rfloor,$$

we obtain

$$\big|\dot{u}_{\phi(k)}(t)\big| = |V^n| \leq |\dot{x}_0| + \big|z(h)\big| + tM_1 \quad \forall t \in [0,T] \setminus \mathbb{Q}, \quad \forall k \geq k_1.$$

We can pass to the limit successively when k tends to $+\infty$ then when t tends to 0 and we get

$$\big|v^+(0)\big| = \big|\dot{x}^+(0)\big| \leq |\dot{x}_0|.$$

Let us consider now $T \in \mathbb{R}_+^*$ such that $\dot{x}^+(T) = \dot{x}(T)$. We recall that the restriction of the measure $\mu = -dv - \gamma\dot{x}$ to $[0,T]$ belongs to the subdifferential of the functional J_Φ

$$J_\Phi : \left\{ \begin{array}{ccc} \mathcal{C}^0\big([0,T];\mathbb{R}^d\big) & \rightarrow & \mathbb{R} \cup \{+\infty\} \\ y & \mapsto & \int_0^T \Phi\big(y(t)\big)\,dt \end{array} \right.$$

where Φ is the convex function $\delta_K + f$. With the results of [27] (corollary 5.A) we infer that

$$\mu\big(\{0\}\big) \in N_{\overline{\operatorname{dom}\Phi}}\big(x(0)\big) = N_K(x_0).$$

But

$$\mu\big(\{0\}\big) = -dv\big(\{0\}\big) = v(0) - v^+(0) = \dot{x}_0 - \dot{x}^+(0).$$

Hence we get $\dot{x}_0 - \dot{x}^+(0) \in N_K(x_0)$ which implies

$$\big(\dot{x}_0 - \dot{x}^+(0), w\big) \leq 0 \quad \forall w \in T_K(x_0).$$

By choosing successively $w = \dot{x}^+(0)$ and $w = \dot{x}_0$, we obtain

$$|\dot{x}_0|^2 \leq \big(\dot{x}^+(0), \dot{x}_0\big) \leq |\dot{x}^+(0)|^2$$

which allows us to conclude. □

There remains now to prove that the limit x satisfies property (v) of Definition 2.1. More precisely, let us prove that

Lemma 6.11. *For every $t \in \mathbb{R}_+^*$, we have $\big|\dot{x}^+(t)\big| \leq \big|\dot{x}^-(t)\big|$.*

Proof. Let $(t_1, t_2) \in \mathbb{R}_+^* \setminus \mathbb{Q}$ such that $t_1 < t_2$ and let $T > t_2$. First we will prove that

$$\big|v(t_2)\big| \leq \big|v(t_1)\big| + (t_2 - t_1)M_1$$

where M_1 is the constant defined at the previous lemma. Let $h = \dfrac{1}{2^{\phi(k)}}$ $(k \geq k_1)$, be such that $h \in (0, t_2 - t_1)$. We have

$$\dot{u}_{\phi(k)}(t_i) = V^{n_i}, \quad \text{with} \;\; n_i = \left\lfloor \frac{t_i}{h} \right\rfloor, \; i = 1, 2$$

and

$$\big|V^{n_2}\big| \leq \big|V^{n_1}\big| + h \sum_{p=n_1+1}^{n_2} \big|F^p\big| \leq \big|V^{n_1}\big| + (t_2 - t_1 + h)M_1.$$

Thus, we get

$$\big|\dot{u}_{\phi(k)}(t_2)\big| \leq \big|\dot{u}_{\phi(k)}(t_1)\big| + (t_2 - t_1 + h)M_1.$$

We can pass to the limit when k tends to $+\infty$ and we obtain

$$\big|v(t_2)\big| \leq \big|v(t_1)\big| + (t_2 - t_1)M_1.$$

Let us consider now $t \in \mathbb{R}_+^*$ and let $T > t$. There exist two sequences $(t_{1n})_{n\geq 0}$ and $(t_{2n})_{n\geq 0}$ which converge to t and such that $t_{1n} \notin \mathbb{Q}$, $t_{2n} \notin \mathbb{Q}$ and $0 < t_{1n} < t < t_{2n} < T$ for all $n \geq 0$. With the previous inequality we get

$$\big|v(t_{2n})\big| \leq \big|v(t_{1n})\big| + (t_{2n} - t_{1n})M_1 \quad \forall n \geq 0.$$

Finally we pass to the limit when n tends to $+\infty$ and we obtain

$$\big|\dot{x}^+(t)\big| = \big|v^+(t)\big| \leq \big|\dot{x}^-(t)\big| = \big|v^-(t)\big|.$$

□

6.4. **Impact law.** In this section we prove that the solution x obtained as the limit of the approximate solutions $u_{\phi(k)}$ satisfies the following Newton's impact law

$$\dot{x}^+(t) = \mathrm{Proj}\big(T_K\big(x(t)\big), \dot{x}^-(t)\big) \quad \forall t > 0, \tag{6.7}$$

whenever

$$\big(\nabla\varphi_\alpha\big(x(t)\big), \nabla\varphi_\beta\big(x(t)\big)\big) \leq 0 \quad \forall (\alpha, \beta) \in J\big(x(t)\big)^2, \; \alpha \neq \beta. \tag{6.8}$$

It should be observed that the impact law (6.7) corresponds to inelastic shocks, *cf.* [15]. Indeed, if we decompose the velocity in normal and tangential components, relation (6.7) means that the normal component of the velocity vanishes at impact, while the tangential component is conserved. It follows that the solution is "really" dissipative since the kinetic energy decreases strictly whenever an impact occurs with $\mathrm{Proj}\big(N_K\big(x(t)\big), \dot{x}^-(t)\big) = \dot{x}_N^-(t) \neq 0$.

The condition (6.8) is directly related to the geometry of the active constraints at impact: it means that the active constraints create acute or right angles. This condition is quite natural here since it is a necessary and sufficient condition to ensure continuity on data for the Cauchy problem (see [22]).

Let us prove now that (6.7) holds whenever the condition (6.8) is satisfied. Let $T > 0$ and define $\tilde{f} : [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ by

$$\tilde{f}(t, x, v) = -\gamma v - \nabla f(x) \quad \forall (t, x, v) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^d.$$

Then we define $\tilde{F} : [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \times [0, 1] \to \mathbb{R}^d$ by

$$\tilde{F}(t, x, x', v, h) = -\gamma v - \nabla f(x' + 2hv) \quad \forall (t, x, x', v, h) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \times [0, 1].$$

Then, for all $h \in (0, 1]$ and for all $n \in \big\{1, \ldots, \lfloor T/h \rfloor\big\}$, we have

$$U^{n+1} = \mathrm{Proj}\big(K, 2U^n - U^{n-1} + h^2 F^n\big)$$

with

$$F^n = \tilde{F}\left(nh, U^n, U^{n-1}, \frac{U^{n+1} - U^{n-1}}{2h}, h\right)$$

Observing that $\tilde{F}$ is consistent with respect to $\tilde{f}$ *i.e.*

$$\tilde{F}(t, x, x, v, 0) = \tilde{f}(t, x, v) \quad \forall (t, x, v) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^d$$

the scheme proposed here corresponds to the time-discretization proposed in [21] for the measure differential inclusion

$$\ddot{x} - \tilde{f}(t, x, \dot{x}) \in -N_K(x).$$

Hence we can obtain (6.7) by the same proof as in [21] (Proposition 2.9).

Remark 6.12. Let us outline that we cannot prove the convergence of the approximate solutions $(u_h)_{h>0}$ by applying directly the results of [21] since we deal here with the unbounded time interval $[0, +\infty)$ and that the functions $\tilde{f}$ and $\tilde{F}$ are not necessarily Lipschitz continuous with respect to the positions.

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