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STABILITY IN FRICTIONAL UNILATERAL ELASTICITY REVISITED: AN APPLICATION OF THE THEORY OF SEMI-COERCIVE VARIATIONAL INEQUALITIES

SAMIR ADLY, EMIL ERNST, AND MICHEL THÉRA

ABSTRACT. In this paper we show how recent results concerning the stability of semi-coercive variational inequalities on reflexive Banach spaces, obtained by the authors in [3] can be applied to establish the existence of an elastic equilibrium to any small uniform perturbation of statical loads in frictional unilateral linear elasticity. The Fenchel duality is one of the key techniques that we use.

1. INTRODUCTION AND NOTATION

The purpose of this article is to revisit recent stability results for semi-coercive variational inequalities on real reflexive Banach spaces obtained by the authors in [3] by the way of Fenchel's duality to deduce the existence of an elastic equilibrium to any small uniform perturbation of statical loads for a classical problem in frictional unilateral linear elasticity.

The theory of variational inequalities initiated in the early sixties by G. Stampacchia and his collaborators for the calculus of variations associated with the minimization of infinite dimensional functionals covers a large spectrum of problems and is a very attractive area of study in applied mathematics (calculus of variations, control theory, free boundary problems defined by non-linear partial differential equations) with a wide range of applications. It modelizes, in particular many classes of problems arising from unilateral problems in mechanics or in plasticity theory, as well as in finance (pricing american options), economics (Walrasian equilibrium problems), industry and engineering.

We begin this section by fixing the notations, definitions and preliminaries that will be used later in the paper. $(X, \|\cdot\|)$ will be a real reflexive Banach space with topological dual $X^\star$ and $\langle\cdot,\cdot\rangle$ will be the duality pairing between X and $X^\star$. As usual $\mathbb{B}_X$ is the closed unit ball of X.

We now recall two very useful notions. Firstly, following Brézis, we say that an operator $A : X \to X^\star$ is *pseudomonotone* if for every sequence

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(u_n) such that:

$$u_n \rightharpoonup u \text{ and } \limsup_{n\to+\infty}\langle Au_n, u_n - u\rangle \le 0,$$

then,

$$\langle Au, u - v\rangle \le \liminf_{n\to+\infty}\langle Au_n, u_n - v\rangle, \quad \forall v \in X.$$

Note that the class of pseudomonotone operators is large enough since its contains linear and monotone operators, monotone and hemicontinuous operators, demi-continuous operators satisfying the (S_+) property and strongly continuous operators (for more details, see Zeidler [12, Proposition 27.6]).

Secondly, we call *semi-coercive*, every operator $A : X \to X^\star$ such that there exist a constant $\kappa > 0$ and a closed linear subspace U of X such that:

$$(\star) \quad \begin{cases} \langle Av - Au, v - u\rangle \ge \kappa \operatorname{dist}_U(v-u)^2, \ \forall u, v \in X \\ A(v+u) = A(v), \ \forall v \in X, \ \forall u \in U \text{ and } A(X) \subset U^\perp, \end{cases}$$

where $\operatorname{dist}_U(\cdot)$ denotes the distance to U, i.e., $\operatorname{dist}_U(x) = \inf\{\|x - u\| : u \in U\}$.

The class of semi-coercive operators includes for example

- The projection operator over a closed linear subspace in the Hilbert space setting;
- If $X = H^1(\Omega)$, where Ω is an open bounded subset of $\mathbb{R}^n$ with a smooth boundary, then the operator $A : X \to X^\star$ defined by

$$\langle Au, v\rangle = \int_\Omega \nabla u \cdot \nabla v dx,$$

  is semi-coercive with $U = \ker A = \mathbb{R}$ (the space of constant functions).

This article concerns the stability of the solution set of the following variational inequality:

$$\mathrm{VI}(A, f, \Phi, K) \quad \begin{cases} \text{Find } u \in K \cap \operatorname{Dom}\Phi \text{ such that:} \\ \langle Au - f, v - u\rangle + \Phi(v) - \Phi(u) \ge 0, \ \forall v \in K, \end{cases}$$

where:

(1) $A : X \to X^\star$ is pseudomonotone and semi-coercive operator;
(2) $K \subset X$ is a non-empty closed and convex subset;
(3) $f \in X^\star$;
(4) $\Phi : X \to \mathbb{R} \cup \{+\infty\}$ is an extended-real-valued convex, lower semi-continuous function with non-empty effective domain

$$\operatorname{Dom}\Phi = \{v \in X \ : \ \Phi(v) < +\infty\} \neq \emptyset$$

(this class of functions will hereafter be called $\Gamma_0(X)$).

Some existence results are well known for Problem $\mathrm{VI}(A, f, \Phi, K)$ when the operator A is linear and coercive, i.e., when there exists a real $\alpha > 0$ such that :

$$\langle Au, u\rangle \geq \alpha \|u\|^2, \ \forall u \in X,$$

or when A is non-linear and coercive in the following sense:

$$\lim_{\|u\| \to +\infty} \frac{\langle Au, u\rangle}{\|u\|} = +\infty.$$

The reader is refereed for instance to the classical contributions of J.L. Lions and G. Stampacchia [11], H. Brézis [4], [5] , G. Fichera [8] and the references cited therein, as well as in finite dimension to the book by F. Facchinei and J. S. Pang [7].

The organization of the paper is as follows. We briefly recall in Section 2 stability results for $\mathrm{VI}(A, f, \Phi, K)$ obtained in our previous article [3]. The main technical result - [3, Proposition 3.1] - is completed and its proof is simplified by making use of duality techniques in Lemma 2.1. Theorem 2.2 in subsection 2.1 specificates this stability result when the underlying space X is a Hilbert space.

Section 3 is concerned with the study of the stability of the existence of the solution to a classical unilateral problem from the Von Kármán theory of linear thin plates. The case when the frictional contact takes place on the border of the plate leads us to a semi-coercive variational inequality for which U is a finite dimensional subspace of X.

Let us also recall some background results from convex analysis (for details about these notions, the reader is invited to consult for instance the book by J.B. Hiriart-Urruty and C. Lemaréchal [10]).

Let K be a non-empty closed convex subset of X. The recession cone K^∞ of K is the set defined by

$$K^\infty := \{d \in X \ : \ x_0 + \lambda d \in X, \ \ \forall \lambda > 0\},$$

where x_0 is an arbitrary element of K. If $\Phi \, : \, X \ \to \ \mathbb{R} \cup \{+\infty\}$ is an extended-real-valued function , we define the epigraph of Φ as

$$\text{epi } \Phi = \{(x, \lambda) \in X \times \mathbb{R} \ : \ \Phi(x) \leq \lambda\}.$$

When Φ belongs to $\Gamma_0(X)$, the recession function Φ^∞ of Φ is defined by the relation:

$$(\text{epi } \Phi)^\infty = \text{epi } \Phi^\infty.$$

Equivalently, we have

$$\Phi^\infty(x) := \lim_{\lambda \to +\infty} \frac{\Phi(x_0 + \lambda x) - \Phi(x_0)}{\lambda}, \tag{1.1}$$

where x_0 is an arbitrary element of $\mathrm{Dom}\ \Phi$. We set $\ker\ \Phi^\infty = \{x \in X : \Phi^\infty(x) = 0\}$, which is a closed convex cone in X.

The Fenchel conjugate $\Phi^\star : X^\star \to \mathbb{R} \cup \{+\infty\}$ of Φ is defined by:

$$\Phi^\star(x^\star) = \sup_{x \in X} \Big\{ \langle x^\star, x\rangle - \Phi(x) \Big\}.$$

The indicator function to a convex set K is given by:

$$I_K(x) = \begin{cases} 0 & \text{if} \quad x \in K \\ +\infty & \text{if} \quad x \notin K. \end{cases}$$

The support function to K is defined by:

$$\sigma_K(x^\star) = (I_K)^\star(x^\star) = \sup_{x \in K} \langle x^\star, x\rangle.$$

We recall that the barrier cone of K is defined by:

$$\mathcal{B}(K) = \{x^\star \in X^\star \ : \ \sup_{x \in K} \langle x^\star, x\rangle < +\infty\},$$

i.e., $\mathcal{B}(K) = \mathrm{Dom}\ \sigma_K$.
If K is a closed cone, its polar is defined by:

$$K^\circ = \{x^\star \in X^\star \ : \ \langle x^\star, x\rangle \leq 0, \ \ \forall x \in K\}.$$

It is well known that if K is a non-empty closed and convex subset, then

$$\mathcal{B}(K)^\circ = K^\infty. \tag{1.2}$$

Therefore,

$$\overline{\mathcal{B}(K)} = (K^\infty)^\circ. \tag{1.3}$$

A closed and convex subset K is said to be linearly bounded if and only if $K^\infty = \{0\}$. Hence, for linearly bounded subsets, the barrier cone is dense in $X^\star$, i.e., $\overline{\mathcal{B}(K)} = X^\star$.

We recall that for $\Phi_1, \ \Phi_2 \in \Gamma_0(X)$, the infimal convolution (or the epigraphical sum) is defined by:

$$(\Phi_1 \Box \Phi_2)(x) = \inf_{y+z=x} \{\Phi_1(y) + \Phi_2(z)\}. \tag{1.4}$$

We say that the infimal convolution is exact provided the infimum appearing in (1.4) is achieved. It is worth noting that if Φ_1 (or Φ_2) is continuous on X, then

$$(\Phi_1 + \Phi_2)^\star = \Phi_1^\star \Box \Phi_2^\star \tag{1.5}$$

and the infimal convolution $\Phi_1^\star \Box \Phi_2^\star$ is exact. In particular, if $\partial\Phi(x)$ denotes the subdifferential of $\Phi \in \Gamma_0(X)$ at $x \in \mathrm{Dom}\ \Phi$, i.e., the non-empty weak*-closed convex set

$$\partial\Phi(x) = \{v \in X^\star \ : \ \langle v, y - x\rangle \leq \Phi(y) - \Phi(x), \ \ \forall y \in \mathrm{Dom}\ \Phi\},$$

it is worth noting that, for convex functions Φ_1 and $\Phi_2 \in \Gamma_0(X)$ with $\mathrm{Dom}\ \Phi_1 \cap \mathrm{Dom}\ \Phi_2 \neq \emptyset$, if $(\Phi_1 + \Phi_2)^\star = \Phi_1^\star \Box \Phi_2^\star$ and the infimal convolution is exact, then $\partial(\Phi_1 + \Phi_2)(x) = \partial\Phi_1(x) + \partial\Phi_2(x)$, for each $x \in \mathrm{Dom}\ \Phi_1 \cap \mathrm{Dom}\ \Phi_2$.

2. THE ABSTRACT STABILITY RESULT

In the paper [3], we discussed the stability of the solution set of the variational inequality $\mathrm{VI}(A, f, \Phi, K)$. More precisely, we characterized all data (A, f, Φ, K) for which there is some real number $\varepsilon > 0$ such the variational inequality has a solution for all data $(A_\varepsilon, f_\varepsilon, \Phi_\varepsilon, K_\varepsilon)$ satisfying the following conditions (hereafter denoted by (☾)):

- $A_\varepsilon : X \to X^\star$ is pseudomonotone and U-semi-coercive such that:

$$\|A(x) - A_\varepsilon(x)\| < \varepsilon,\ \forall x \in X;$$

- $f_\varepsilon \in X^\star$ such that $\|f - f_\varepsilon\| < \varepsilon$;
- $K \subset K_\varepsilon + \varepsilon\mathbb{B}_X$ and $K_\varepsilon \subset K + \varepsilon\mathbb{B}_X$ with K_ε a non-empty closed and convex subset of X;
- $\Phi_\varepsilon \in \Gamma_0(X)$ is bounded below and such that:

$$\Phi(x) - \varepsilon \leq \Phi_\varepsilon(x) \leq \Phi(x) + \varepsilon, \quad \forall x \in X.$$

To this respect, an important role is played by the resolvent set associated to $\mathrm{VI}(A, f, \Phi, K)$,

$$\mathcal{R}(A, \Phi, K) = \{f \in X^\star \ :\ \mathrm{Sol}(A, f, \Phi, K) \neq \emptyset\},$$

or equivalently,

$$\mathcal{R}(A, \Phi, K) = \bigcup_{u \in X} Au + \partial(\Phi + I_K)(u).$$

Indeed, as obviously observed, if the variational inequality has solutions for all data satisfying (☾), then f must belong to the interior of the resolvent set. In order to describe this set denoted by $\mathrm{Int}\,\mathcal{R}(A, \Phi, K)$, let us associate to problem $\mathrm{VI}(A, f, \Phi, K)$ the following function:

$$\Psi(x) := \ \kappa(\mathrm{dist}_U(x))^2 + \Phi(x) + I_K(x) \quad \forall x \in X. \tag{2.1}$$

A crucial step - [3, Proposition 3.1] - in achieving the desired characterization of stable data (A, f, Φ, K) says that $\mathrm{Int}\,\mathcal{R}(A, \Phi, K) = \mathrm{Int}\,\mathrm{Dom}\ \Psi^\star$. The following new Lemma give a complete description of the domain of $\Psi^\star$, the conjugate function of Ψ, in terms of data U, Φ and K, allowing thus a new and simpler proof for the first part (step 1) of [3, Proposition 3.1], namely of the inclusion $\mathcal{R}(A, \Phi, K) \subseteq \mathrm{Dom}\ \Psi^\star$.

Lemma 2.1. *We have*

$$\mathrm{Dom}\ \Psi^\star = U^\perp\ +\ \mathrm{Dom}\ [\Phi + I_K]^\star\,,$$

and consequently,

$$\mathcal{R}(A, \Phi, K) \subseteq \mathrm{Dom}\ \Psi^\star.$$

Proof. Let us compute the domain $\mathrm{Dom}\ \Psi^\star$ of the conjugate $\Psi^\star$. We have $\Psi = \kappa(\mathrm{dist}_U(\cdot))^2 + (\Phi + I_K)$. We set $\Phi_1 = \kappa(\mathrm{dist}_U(\cdot))^2 \in \Gamma_0(X)$ and $\Phi_2 = (\Phi + I_K) \in \Gamma_0(X)$. Since Φ_1 is continuous on X, then using (1.5), we get

$$\Psi^\star = \left[\kappa \mathrm{dist}_U(\cdot)^2\right]^\star\ \square\ [\Phi + I_K]^\star\,.$$

Hence,

$$\mathrm{Dom}\ \Psi^\star = \mathrm{Dom}\ \left[\kappa \mathrm{dist}_U(\cdot)^2\right]^\star\ +\ \mathrm{Dom}\ [\Phi + I_K]^\star\,.$$

On the other hand, we have:

$$\frac{1}{2}\mathrm{dist}_U(\cdot)^2 = \frac{1}{2}\|\cdot\|^2\ \square\ I_U.$$

Since $\|\cdot\|^2$ is continuous on X, using (1.5) again, we obtain

$$\left[\frac{1}{2}\mathrm{dist}_U(\cdot)^2\right]^\star = \frac{1}{2}\|\cdot\|_*^2\ +\ I_{U^\perp}.$$

Therefore,

$$\left[\kappa \mathrm{dist}_U(\cdot)^2\right]^\star = 2\kappa\Big(\frac{1}{2}\|\frac{\cdot}{2\kappa}\|_*^2\ +\ I_{U^\perp}(\frac{\cdot}{2\kappa})\Big).$$

Hence,

$$\mathrm{Dom}\ \left[\kappa \mathrm{dist}_U(\cdot)^2\right]^\star = U^\perp.$$

Consequently,

$$\mathrm{Dom}\ \Psi^\star = U^\perp\ +\ \mathrm{Dom}\ [\Phi + I_K]^\star\,. \tag{2.2}$$

Using assumptions (☼) and the fact that the range of $\partial\Phi$ is included in $\mathrm{Dom}\ \Phi^\star$, we get:

$$\mathcal{R}(A, \Phi, K) \subset U^\perp + \mathrm{Dom}\ [\Phi + I_K]^\star. \tag{2.3}$$

Relations (2.2) and (2.3) complete the proof of the lemma. □

A standard convex analysis result says that the interior of the domain of a convex function (such as $\Phi^\star$) defined on a real Banach space (such as $X^\star$) is non-empty if and only if the function is bounded above on some closed ball, $\overline{x} + r\mathbb{B}_X$, $\overline{x} \in X$, $r > 0$. Accordingly, the interior of the resolvent is non-empty if and only if

$$\Psi(x) \geq \alpha\|x\| + \langle g, x\rangle + \beta\ \forall x \in X,$$

for some $\alpha > 0$, $\beta \in \mathbb{R}$ and $g \in X^\star$. On the other hand, if $\mathrm{Int\,Dom}\,\Psi^\star$ is non-empty, then we have:

$$\mathrm{Int\,Dom}\,\Psi^\star = \Big\{ g \in X^\star \ : \ \langle g, w\rangle < \Psi^\infty(w),\ \forall w \in X \setminus \{0\} \Big\}. \tag{2.4}$$

A simple computation of the recession function Ψ^∞, of Ψ given in (2.1), gives us:

$$\Psi^\infty(w) = I_U(w) + \Phi^\infty(w) + I_{K^\infty}(w). \tag{2.5}$$

Hence,

$$\begin{aligned} f \in \mathrm{Int\,Dom}\,\Psi^\star &\iff \langle f, w\rangle < \Phi^\infty(w),\ \ \forall w \in U \cap K^\infty \setminus \{0\} \\ &\iff \exists \alpha > 0,\ \beta \in \mathbb{R}\ :\ \Psi(x) - \langle f, x\rangle \geq \alpha\|x\| + \beta, \end{aligned} \tag{2.6}$$

which means that the functional $x \mapsto \mathcal{F}(x) := \Psi(x) - \langle f, x\rangle$ is coercive.

Let us recall the the main result in [3].

Theorem 2.1. [3] *There is a real $\varepsilon > 0$ such that the set of solution of the variational inequality $\mathrm{IV}(A_\varepsilon, f_\varepsilon, \Phi_\varepsilon, K_\varepsilon)$ is non-empty for every perturbed data $(A_\varepsilon, f_\varepsilon, \Phi_\varepsilon, K_\varepsilon)$ satisfying the conditions $(☾)$ if and only if the following hold:*

(i) $U \cap K^\infty \cap \ker\, \Phi^\infty$ *contains no lines;*
(ii) $\nexists (x_n) \subset K$ *such that:*
$\|x_n\| \to +\infty,\ \frac{x_n}{\|x_n\|} \rightharpoonup 0$ *and* $\lim_{n\to+\infty} \frac{\kappa(\mathrm{dist}_U(x_n))^2 + \Phi(x_n)}{\|x_n\|} = 0;$
(iii) $\langle f, w\rangle < \Phi^\infty(w),\ \forall w \in U \cap K^\infty \setminus \{0\}.$

Remark 2.1. We note that the compatibility condition (iii) in Theorem 2.1, is equivalent to the coercivity of the following energy functional :

$$\mathcal{F}(x) := \kappa \mathrm{dist}_U(x)^2 + \Phi(x) + I_K(x) - \langle f, x\rangle \tag{2.7}$$

in the sense of (2.6).

2.1. **Application to Hilbert space linear semi-coercive variational inequalities.** When the closed linear subspace U has a finite dimension and the underlying space X is a Hilbert space, the above cited stability result takes a simpler form.

Namely, let us suppose that $(X, \langle\cdot,\cdot\rangle$ is a Hilbert space with associated norm $\|\cdot\|)$ and that $A : X \to X$ is a bounded symmetric linear operator such that:

$$\dim_{\mathbb{R}} \ker A < +\infty. \tag{2.8}$$

Let us suppose that in addition, A is semi-coercive according to definition $(☼)$, or, equivalently:

$$\exists \kappa > 0\ :\ \langle Au, u\rangle \geq \kappa\|Qu\|^2,\ \forall u \in X, \tag{2.9}$$

where $Q = I - P$, while $P : X \to \ker A$ is the orthogonal projection on $\ker A$. Note that, in this case, the linear subspace U is $\ker A$, and $\|Q(x)\|$ coincides with the distance from a point x in X to U.

Example 2.1.

(i) More generally, it is easy to see ([9]) that a linear monotone (i.e., satisfying $\langle Au, u\rangle \geq 0,\ \forall u \in X$) operator A is semi-coercive provided that the image of A, Im$A = \{y \in X^\star :\ \text{for some}\ x \in X,\ y = A(x)\}$ is closed;

(ii) Moreover, if A is a linear and monotone operator, the following statements are equivalent :

  (a) A is semi-coercive and $\dim_{\mathbb{R}} \ker A < +\infty$;

  (b) there is a strongly continuous operator $C : X \to X$ such that $A + C$ is coercive.

(iii) Let $(H, |\cdot|)$ be a Hilbert space and $X \hookrightarrow H$ be a compact mapping. If $A : X \to X$ is a bounded linear and monotone operator fulfilling the following Gärding inequality :

$$\exists\lambda > 0,\ \exists c > 0 \text{ such that : } \langle Au, u\rangle + \lambda|u|^2 \geq c\|u\|^2,\ \forall u \in X,$$

then A is semi-coercive and $\dim_{\mathbb{R}} \ker A < +\infty$.

Proofs of these statements as well as further details may be found in [9].

The announced specification of Theorem 2.1 reads as follows.

Theorem 2.2. *Suppose that A is a bounded symmetric linear operator defined on a Hilbert space X with a finite dimensional kernel. The linear variational inequality* $\mathrm{VI}(A, f, \Phi, K)$ *is stable with respect to uniform perturbations according to relation* ($\Subset$) *if and only if the two following conditions are satisfied:*

(i) $\ker A \cap K^\infty \cap \ker \Phi^\infty$ *contains no line;*

(ii) $\langle f, w\rangle < \Phi^\infty(w),\ \forall w \in \ker A \cap K^\infty,\ w \neq 0.$

Proof. This result is a simple consequence of Theorem 2.1. Indeed, as conditions (i) and (iii) of Theorem 2.1 are verified, let us prove that condition (ii) is also satisfied. To this end, we suppose that there is some sequence $(u_n)_{n\in\mathbb{N}^*} \subset K$ such that: $t_n := \|u_n\| \to +\infty$, $w_n := \frac{u_n}{\|u_n\|} \rightharpoonup 0$, and $\frac{\kappa\|Qu_n\|^2+\Phi(u_n)}{\|u_n\|} \to 0$ when $n \to +\infty$.

We have

$$\frac{\kappa\|Qu_n\|^2 + \Phi(u_n)}{\|u_n\|} = \kappa\, t_n\|Qw_n\|^2 + \frac{\Phi(t_n w_n)}{t_n}. \tag{2.10}$$

Since $\lim_{n\to+\infty}\frac{\kappa\|Qu_n\|^2+\Phi(u_n)}{\|u_n\|}=0$, then

$$\limsup_{n\to+\infty}\kappa\, t_n\|Qw_n\|^2+\liminf_{n\to+\infty}\frac{\Phi(t_nw_n)}{t_n}\leq 0.$$

On the other hand, as $w_n\rightharpoonup 0$ we have $\liminf_{n\to+\infty}\frac{\Phi(t_nw_n)}{t_n}\geq\Phi^\infty(0)=0$, and consequently

$$\limsup_{n\to+\infty}\; t_n\|Qw_n\|^2\leq 0.$$

Thus,

$$\lim_{n\to+\infty}\; t_n\|Qw_n\|^2=0.$$

Hence,

$$\lim_{n\to+\infty}\|Qw_n\|^2=0.$$

It follows that $Qw_n\to 0$ strongly in X.
On the other hand, as $\dim_{\mathbb{R}}\ker A<+\infty$, then $Pw_n\to 0$ strongly in X.
Since $w_n=Pw_n+Qw_n$, this infer the norm-convergence to 0 of the sequence (w_n). This relation contradicts the fact that $\|w_n\|=1$, and completes the proof of the Theorem. ■

3. STABILITY OF THE ELASTIC EQUILIBRIUM IN UNILATERAL FRICTIONAL PROBLEMS IN LINEAR PLATE THEORY.

In this section we apply (Theorem 2.2) (the specification of our general method to the Hilbert space setting) to the equilibrium problem for a linear elastic thin plate subjected to unilateral limit conditions.

We follow the monograph [6, Chapter 4] by G. Duvaut & J.L. Lions as a reference textbook. Let us consider a thin plate occupying a bounded plane domain $\Omega\subset\mathbb{R}^2$ with a smooth (for instance C^1) boundary denoted by Γ. The unknown parameter of our problem is the vertical displacement of the plate (also called vertical deflection in [6, page 201]) and is denoted by $u:\Omega\to\mathbb{R}$.

A vertical load $f\in L^2(\Omega)$ acts on every point of the plate. It is proved in [6, page 210] that the equilibrium vertical displacement fulfills the partial differential equation $\triangle^2u=f$ on Ω, where $\triangle^2$ means $\triangle\circ\triangle$ and represents the biharmonic operator.

The frictional unilateral boundary conditions imposed on Γ may be modelized through the variational inequality $\overline{\mathrm{VI}}(A,f,\Phi,K)$, where the data are defined as follows: $K=X=H^2(\Omega)$, $A:X\to X$ is a bounded linear symmetric and monotone operator defined by:

$$\langle Au, v\rangle = \int_\Omega \Big(\frac{\partial^2 u}{\partial x_1^2}\frac{\partial^2 v}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2}\frac{\partial^2 vu}{\partial x_2^2} + \nu(\frac{\partial^2 u}{\partial x_1^2}\frac{\partial^2 v}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2}\frac{\partial^2 vu}{\partial x_2^2}) + 2(1-\nu)\frac{\partial^2 u}{\partial x_1 \partial x_2}\frac{\partial^2 v}{\partial x_1 \partial x_2}\Big)\, dx,$$

where ν is the Poisson coefficient supposed such that $0 < \nu < \frac{1}{2}$, and finally $\Phi : X \to [-\infty, +\infty]$ is of form

$$\Phi(v) = \int_\Gamma j(v(x))d\sigma,$$

where $j : \mathbb{R} \to \mathbb{R}_+ \cup \{+\infty\}$ is a convex lower semi-continuous function such that $j(0) = 0$ and Γ is the boundary of Ω.

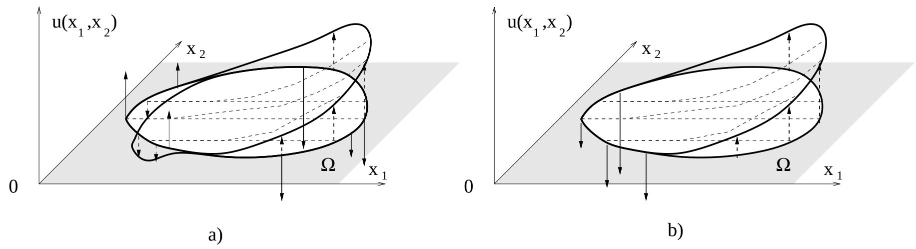

FIGURE 1. Unilateral problems for thin plates

Indeed, for every choice of the convex function j, the $H^2(\Omega)$-solution to the variational inequality $\overline{\mathrm{VI}}(A, f, \Phi, K)$ verifies $\triangle^2 u = f$. Moreover, the case $j(s) = \alpha|s|$, with $\alpha > 0$, corresponds to a unilateral frictional (Coulomb) condition on the boundary of the plate (Fig 1-a) while the function j defined by $j(s) = 0$ when $s \geq 0$ and $j(s) = +\infty$ when $s < 0$, modelizes the perfect (frictionless) unilateral contact (Fig 1-b).

In order to apply the stability result stated by Theorem 2.2 to the variational inequality $\overline{\mathrm{VI}}(A, f, \Phi, K)$, let us first determine the recession function Φ^∞ of Φ. We have

$$\Phi^\infty(v) = \int_\Gamma j^\infty(v(x))d\sigma = \int_\Gamma (j_1 v^+ - j_2 v^-)d\sigma,$$

where

$$j_1 = \lim_{s\to+\infty}\frac{j(s)}{s},\; j_2 = \lim_{s\to-\infty}\frac{j(s)}{s}, v^+ = \max(v,0) \text{ and } v^- = (v,0).$$

On the other hand, it is well known that the kernel of the operator A (that is the subspace U)

$$\ker A = \{a + bx_1 + cx_2 \;:\; a, b, c \in \mathbb{R}\}.$$

It is obvious that $\ker \Phi^\infty$ does not contain any line (except for the trivial case $j = 0$), which means that condition $i)$ from Theorem 2.2 is always fulfilled.

In order to take into account condition $ii)$, let us set

$$f_0 := \int_\Omega f(x_1, x_2) dx_1 x_2,$$

$$f_1 := \int_\Omega x_1 f(x_1, x_2) dx_1 x_2$$

and

$$f_2 := \int_\Omega x_2 f(x_1, x_2) dx_1 x_2.$$

Obviously we observe that

$$\int_\Omega f(x_1, x_2)(a + bx_1 + cx_2) dx_1 x_2 = af_0 + bf_1 + cf_2.$$

Let us also define the mapping $\psi : \mathbb{R}^3 \to \mathbb{R}_+$ by

$$\begin{aligned} \psi(a, b, c) &:= \Phi^\infty(a + bx_1 + cx_2) \\ &= \int_\Gamma \Big(j_1(a + bx_1 + cx_2)^+ - j_2(a + bx_1 + cx_2)^- \Big) d\sigma; \end{aligned}$$

ψ is clearly a convex, positively homogeneous extended-real-valued function which is lower semi-continuous with respect to the standard topology of $\mathbb{R}^3$. The function ψ depends on the shape of the plate, $\Omega \in \mathbb{R}^2$, as well as on the values of j_1 and j_2.

Condition $ii)$ from Theorem 2.2 reads

$$\int_\Omega f(x_1, x_2)(a + bx_1 + cx_2) dx_1 x_2 < \Phi^\infty(a + bx_1 + cx_2)$$

for all $a, b, c \in \mathbb{R}$ such that $a^2 + b^2 + c^2 \neq 0$, which means that

$$af_0 + bf_1 + cf_2 < \psi(a, b, c)\ \forall a, b, c \in \mathbb{R},\ a^2 + b^2 + c^2 \neq 0.$$

Accordingly, the function Θ defined by

$$\Theta(a, b, c) := \psi(a, b, c) - (af_0 + bf_1 + cf_2)$$

is strictly positive over the unit sphere

$$S := \{(a, b, c) \in \mathbb{R}^3 :\ a^2 + b^2 + c^2 = 1\}.$$

As Θ is lower semi-continuous, it is necessarily bonded below on S. Otherwise, if it was unbounded below, then for every integer k there would exist a sequence (s_k) in S such that $\Theta(s_k) \leq \frac{1}{k}$. By compacity of S, relabeling if necessary, we may suppose that the sequence (s_k) tends to $\bar{s} \in S$ and therefore that $\Theta(\bar{s}) \leq \liminf_{k \to +\infty} \Theta(s_k) \leq 0$, a contradiction. Hence,

$$\psi(a, b, c) - (af_0 + bf_1 + cf_2) \geq r \qquad \forall\, a, b, c \in \mathbb{R},\ a^2 + b^2 + c^2 = 1 \tag{3.1}$$

for some $r > 0$. Since α is positively homogeneous, relation (3.1) - and, consequently, condition $ii)$ from Theorem 2.2 - is equivalent to

$$\psi(a,b,c) \geq (af_0 + bf_1 + cf_2) + r\|(a,b,c)\| \quad \forall\, a,b,c \in \mathbb{R}. \tag{3.2}$$

As ψ is positively homogeneous relation (3.2) amounts to saying that

$$(f_0, f_1, f_2) \in \operatorname{Int}\, \partial\psi(0). \tag{3.3}$$

Indeed, it is well known that for a convex lower semi-continuous function $\psi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$,

$$0 \in \operatorname{Int}\, \partial\psi(0) \iff \text{for some } r > 0,\ \ \psi^{'}(0;\cdot) \geq r\|\cdot\|.$$

When ψ is positively homogeneous, then the directional derivative $\psi^{'}(0;\cdot)$ of ψ at 0 coincides with ψ, and relation (3.3) follows immediately.

Remark 3.1. Observe that relation (3.3) is fulfilled for every $f \in L^2(\Omega)$ if and only if ψ is the indicator function of the singleton $\{0\}$, that is $\psi(0) = 0$ and $\psi(a,b,c) = +\infty$ for every $a, b, c\ \in \mathbb{R}$ such that $a^2 + b^2 + c^2 \neq 0$. This can happen if and only if $j_1 = j_2 = +\infty$, that is

$$\lim_{|s|\to\infty} \frac{j(s)}{|s|} = +\infty. \tag{3.4}$$

We may conclude that the equilibrium displacement for the frictional unilateral problem modelized by the variational inequality $\overline{\mathrm{VI}}(A, f, \Phi, K)$ exists and is stable for any load $f \in L^2(\Omega)$ if and only if the convex function j is strongly coercive (in the sense of relation (3.4)).

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