XLIM

UMR CNRS 6172

Département
Mathématiques-Informatique

Self-Equilibrating Sets and Functions in Dual Vector Spaces: Two Boundedness Criteria

Emil Ernst & Michel Théra & Michel Volle

Rapport de recherche n° 2006-01
Déposé le 4 janvier 2006

Université de Limoges, 123 avenue Albert Thomas, 87060 Limoges Cedex
Tél. (33) 5 55 45 73 23 - Fax. (33) 5 55 45 73 22
http://www.xlim.fr
http://www.unilim.fr/laco

Université de Limoges, 123 avenue Albert Thomas, 87060 Limoges Cedex
Tél. (33) 5 55 45 73 23 - Fax. (33) 5 55 45 73 22
http://www.xlim.fr
http://www.unilim.fr/laco

SELF-EQUILIBRATED SETS AND FUNCTIONS IN DUAL VECTOR SPACES: TWO BOUNDEDNESS CRITERIA

EMIL ERNST, MICHEL THÉRA, AND MICHEL VOLLE

ABSTRACT. This paper is devoted to studying boundedness criteria for extended-real-valued functions. The study is done in the framework of dual vector spaces, using new objects such as self-equilibrated sets and functions.

We establish two boundedness new major criteria. The first one says that an extended-real-valued function is bounded below provided it is minorized by an affine map on one of its self-equilibrated subsets.

The second criterion says that every self-equilibrated function minorized by an affine mapping on the whole underlying space is bounded below.

1. INTRODUCTION AND NOTATIONS

This research is intended to focus on the question of giving criteria to ensure that an extended-real-valued function defined on a locally convex vector space is bounded below. The problem is of main interest in optimization and has been rarely studied for its own since criteria ensuring a function to be bounded generally result from more general statements.

When X is a locally convex space it is well known that the Fenchel-Legendre conjugate

$$\Phi^\star : X^\star \to \mathbb{R} \cup \{-\infty, +\infty\},\ \Phi^\star(y) = \sup_{x \in X} \left(\langle x, y \rangle - \Phi(x) \right),$$

defined on the topological dual $X^\star$ of X is a particularly well suited notion in studying the boundedness of an extended-real-valued function $\Phi : X \to \mathbb{R} \cup \{-\infty, +\infty\}$. The key relation $\inf_X \Phi = -\Phi^\star(0)$ says that Φ is bounded below if and only if $0 \in \mathrm{Dom}\ \Phi^\star = \{x \in X^\star : \Phi^\star(x) < +\infty\}$, the effective domain of $\Phi^\star$. As $\mathrm{Dom}\ \Phi^\star$ is a convex set, relation $0 \in \mathrm{Dom}\ \Phi^\star$ occurs when the cone $\mathbb{R}_+ (\mathrm{Dom}\ \Phi^\star) = \{\lambda x : \lambda \geq 0, x \in \mathrm{Dom}\ \Phi^\star\}$ spanned by the effective domain of $\Phi^\star$ is a linear subspace of $X^\star$. Let us summarize this remark by the following easy boundedness criterion:

Date: **revised on november 26, 2005**.

2000 *Mathematics Subject Classification.* 26B99, 46N10, 49J99.

Key words and phrases. Self-equilibrated set, self-equilibrated function, growth direction, Fenchel-Legendre conjugate, sublevel set, bounded below convex function, Attouch-Brézis qualification condition, Moussaoui-Volle qualification condition.

(☼) an extended-real-valued function is bounded below on a locally convex space X when the cone spanned by the effective domain of its conjugate is a linear subspace of the topological dual $X^\star$ of X.

Another tool which can successfully be used to tackle the problem of the boundedness of an extended-real-valued function, at least for convex lower semi-continuous functions, is the concept of infimal convolution (sometimes called epi-sum in the literature). Denoted by $\Phi \Box \Psi$ and defined by

$$\Phi \Box \Psi(x) = \inf_{y \in X} \left(\Phi(x-y) + \Psi(y) \right) \ \forall x \in X,$$

the infimal convolution of a pair of extended-real-valued functions $\Phi, \Psi : X \to \mathbb{R} \cup \{+\infty\}$ is said to be exact at $x \in X$ provided the inf appearing in the definition is attained.

Obviously, the infimal convolution between the extended-real-valued function Φ and the constant function equal to zero is again a constant function, and its value is equal to the infimum of Φ over X. Moreover, this infimal convolution cannot be real-valued and exact unless Φ is bounded below and achieves its infimum over X.

The infimal convolution of two proper (distinct from the constant functions $-\infty$ and $+\infty$) convex lower semi-continuous extended-real-valued functions (this class is hereafter denoted by $\Gamma_0(X)$) has been extensively studied due to its remarkable role in duality theory and in particular in the central problem of computing the conjugate of the sum of two functions.

In this respect, it is customary to use a *qualification condition*, that is a condition stated in terms of $\Phi^\star$ and $\Psi^\star$ which ensures that relation

$$\Phi \Box \Psi = (\Phi^\star + \Psi^\star)^\star \tag{1}$$

holds in $\Gamma_0(X)$ and that the infimal convolution is exact.

Clearly, any such qualification condition induces a boundedness criterion for a function in $\Gamma_0(X)$. Indeed, every function in $\Gamma_0(X)$ is bounded below when some qualification condition is fulfilled by its conjugate and by the indicator function of the singleton $\{0\}$ (the conjugate of the constant function equal to zero).

Thus, the classical Moreau-Rockafellar condition ([12, §9.3 Proposition 9.2]), assuming in the general locally convex setting that one of the conjugates is finite and continuous at some point where the other is finite, infers the following boundedness criterion:

(▲) a function in $\Gamma_0(X)$ is bounded below (and achieves its infimum) over a locally convex space X when its conjugate is finite and continuous at 0 for any locally convex topology on $X^\star$ compatible with the duality between $X^\star$ and X.

This qualification condition has known numerous refinements, attempting in general to replace the required continuity of one of the conjugates by a geometric condition related to the interplay between their effective domains. A significant step was accomplished when Attouch and Brézis ([3]), proved for reflexive Banach spaces that a sufficient condition for relation (1) to hold and for the infimal convolution to be exact is that the convex cone $\mathbb{R}_+ \left(\mathrm{Dom}\ \Phi^\star - \mathrm{Dom}\ \Psi^\star\right)$ spanned by the difference of the effective domains of $\Phi^\star$ and $\Psi^\star$ is a closed linear space. This result infers a new boundedness criterion:

(◑) a function belonging to the class $\Gamma_0(X)$ is bounded below (and reaches its infimum) over a reflexive Banach space X if the cone spanned by the effective domain of its conjugate is a closed linear space.

Let us remark that, when seen as a boundedness criterion, (◑) and (▲) are mere corollaries of (☼). Indeed, criterion (◑) asks for the continuity of the the conjugate at 0. This forces the cone spanned by its effective domain to coincide with $X^\star$, while criterion (▲) directly imposes to this cone to be a closed linear subspace of $X^\star$. A similar analysis shows that the situation is mostly the same when we consider specifications to optimization of qualification criteria stated by Azé, (see [4]), and by Combari-Laghdir-Thibault (in [6]).

Recently, a new qualification condition was introduced by Moussaoui and Volle (see [13] and [14]). Developing ideas going back to Joly's thesis ([10]), the authors proposed a qualification condition valid for locally convex spaces. When specifying this qualification condition to the characterization of convex functions that are bounded below, we deduce that

(◐) a function belonging to the class $\Gamma_0(X)$ is bounded below (and achieves its infimum) over a locally convex space X provided the weak*-closure of the cone spanned by the effective domain of its conjugate is a linear space and all its sublevel sets are locally weakly compact.

Consequently, the boundedness criteria obtained by using qualification conditions either reduce to direct application of the trivial criterion (☼) as for Attouch-Brézis, Azé and Combari-Laghdir-Thibault criteria, or, in the case of the criterion proposed by Moussaoui and Volle, are burdened in their use by restrictive topological requirements.

Our aim in this presentation is to adopt a totally different standpoint in order to derive new criteria for extended-real-valued functions to be bounded below.

As an example, let a us start with a well known argument. When X is locally convex, a function Φ in the class $\Gamma_0(X)$ achieves its infimum on each weakly compact convex sets of X; therefore it is bounded below on such a set. Accordingly, such a function Φ is bounded below if one of its sublevel sets is weakly compact. When the underlying space X is a reflexive Banach space, this reads as follows:

(▼) a function in the class $\Gamma_0(X)$ is bounded below on a reflexive Banach space X if at least one of its sublevel sets is bounded.

Although not a very interesting result in itself, being nothing but an easy application of either of criteria (▲), (◐) and (◑), the method used in deriving (▼) may lead to more comprehensive results.

Indeed, the lack of generality of criterion (▼) is obviously a consequence of the fact that the set of bounded closed and convex sets does not exhaust, and by far, the class of sets which cannot be sublevel sets of functions belonging to $\Gamma_0(X)$ that are unbounded below.

Clearly, in oder to state a nontrivial boundedness criterion of type (▼), one must first completely characterize the class of those sets which cannot be sublevel sets of functions unbounded below.

This task is achieved in the main result of Section 2 (Proposition 1) in the general setting of dual vector spaces, say X and Y, where the duality bracket is $\langle \cdot, \cdot \rangle : X \times Y \to \mathbb{R}$. The main tool is the notion of self-equilibrated set: a subset S of the vector space X such that every linear map $\langle \cdot, y \rangle$ $y \in Y$, which is bounded above on S is also bounded below on the same set.

Proposition 1 states that a set S may play the role of a sublevel set of an unbounded below extended-real-valued function minorized on S by some affine mapping, if and only if S it is not self-equilibrated.

Our first boundedness criterion (Theorem 1) establishes that every extended-real-valued function minorized on a self-equilibrated sublevel set by an affine mapping is bounded below.

Section 2 finally gives two applications of Theorem 1: the first one (Corollary 1) applies our first criterion to a subclass of quasi-convex functions, and the second one (Theorem 2) is a specification of Theorem 1 to the particular case of functions minorized by an affine mapping on the whole underlying space X.

The third section proposes a new boundedness criterion, valid, as Theorem 2, only for functions which are minorized on the whole space X by some affine map.

Our second boundedness criterion relies on the notion of self-equilibrated function. An extended-real-valued function Φ is self-equilibrated provided that if it grows faster than a linear map $\langle\cdot, y\rangle$, $y \in Y$,

$$[\sup_{x\in S} \Phi(x) < +\infty] \Longrightarrow [\sup_{x\in S} \langle x, y\rangle < +\infty],$$

(such y is called a growth direction for Φ) then the same holds for the opposite linear map, $\langle\cdot, -y\rangle$. Remark that an extended-real-valued function may be self-equilibrated even if none among its sublevel sets is self-equilibrated, and that the existence of a self-equilibrated sublevel set does not imply that the function itself is self-equilibrated.

The main result of Section 3 (Theorem 3) states that an extended-real-valued function is bounded below provided it is self-equilibrated and minorized on X by an affine mapping. A straight application of Theorem 3 can be formulated as follows: every self-equilibrated convex function on $\mathbb{R}^n$ with values in $\mathbb{R} \cup \{+\infty\}$ is bounded below.

The subsection 3.1 collects some of the basic properties of the convex cone G_Φ of all the growth directions of a function Φ. This cone G_Φ always includes the convex cone spanned by the effective domain of the conjugate of Φ; the former cone is in general sensibly larger than the latter. When Φ is a convex function minorated on X by an affine mapping, the cone spanned by the effective domain of $\Phi^\star$ is dense in G_Φ with respect to every possible vector space topology defined on the dual space Y (Proposition 6). Moreover, the two cones coincide in two important particular cases: for convex finite functions minorized by affine mappings (Corollary 2) and for convex functions bounded below (Corollary 3).

When applied to the duality between X (supposed to be locally convex) and $X^\star$, the space of all linear and continuous functions defined on X, Theorem 3 infers (see Theorem 4) that every proper convex extended-real-valued lower semi-continuous function is bounded below provided the set of all its continuous growth directions is a linear space.

The question whether a self-equilibrated lower semi-continuous convex function reaches its infimum is addressed in the last section of the article. We display (Proposition 11) a case where the answer is positive, and another situation (Proposition 12) where the answer is negative: an open question is finally raised.

1.1. **Definitions and notations.**

Terminology and notation are mostly similar to those of [12] and [16]. The purpose of the next few paragraphs is therefore not only to recall some basic notions from convex analysis but also to introduce the specific notation of this article.

In the sequel, X and Y denote a pair of dual real linear spaces and the duality bracket is given by $\langle\cdot,\cdot\rangle : X\times Y \to \mathbb{R}$. We assume that the following two separation properties hold:

$$\forall x \neq 0,\ x \in X,\ \exists y \in Y \text{ s.t. } \langle x, y\rangle \neq 0,$$

$$\forall y \neq 0,\ y \in Y,\ \exists x \in X \text{ s.t. } \langle x, y\rangle \neq 0.$$

A locally convex topology on X (respectively on Y) is called compatible with the duality if for that topology the maps of type $\langle \cdot, y\rangle : X \to \mathbb{R}$, $y \in Y$ (respectively $\langle x, \cdot\rangle : Y \to \mathbb{R}$, $x \in X$) are the only linear continuous functions on X (respectively on Y). In this case, Y may be identified with the topological dual $X^\star$ of X (and X with $Y^\star$). Unless specified otherwise, by affine mapping on X we intend a mapping of form $\langle \cdot, y\rangle + r$, with $y \in Y$ and $r \in \mathbb{R}$.

The intersection of all the closed half-spaces of X containing some subset C of X is called the regular hull of C

$$\langle C\rangle = \bigcap_{y \in Y} \left\{ x \in X : \langle x, y\rangle \leq \sup_{z \in C} \langle z, y\rangle \right\};$$

obviously, the regular hull of C is the closed convex hull of C with respect to any locally convex topology on X compatible with the duality. A set is called regular if it coincides with its regular hull. Similarly, the regular hull of an extended-real-valued function $\Phi : X \to \mathbb{R} \cup \{-\infty, +\infty\}$ is obtained by taking the supremum over all the affine mapping minoring Φ, and a function is regular if it coincides with its regular hull. Clearly, regular functions are convex and lower semi-continuous with respect to any locally convex topology on X compatible with the duality. The class of all the regular functions defined on X is, as customary, denoted by $\Gamma(X)$, while $\Gamma_0(X)$ is the set of those functions in $\Gamma(X)$ which are different from the constant funtions equal to $+\infty$ or $-\infty$. Let us also denote by $AM(X)$ the set of all extended-real-valued functions which have their conjugate somewhere finite (in other words functions minorized on X by some affine mapping); $\Gamma_0(X)$ is clearly a part of $AM(X)$. Finally, we use the notation $AM(C)$ to denote the class of those extended-real-valued functions which are minorized on a subset C of X by some affine mapping (this amounts saying that the effective domain of the conjugate of their restriction to C is nonempty).

As usual, the Fenchel-Legendre conjugate of an extended-real-valued function Φ is the function $\Phi^\star$ defined on Y by

$$\Phi^\star(y) = \sup_{x \in X} \left(\langle x, y\rangle - \Phi(x) \right).$$

It is well known that the conjugate of any extended-real-valued function is regular, and that the bi-conjugate and the regular hull coincide.

For every extended-real-valued function $\Phi : X \to \mathbb{R} \cup \{-\infty, +\infty\}$, the sublevel of level m of Φ is the set

$$[\Phi \leq m] = \{x \in X : \Phi(x) \leq m\}.$$

The effective domain Dom Φ is the union of all its sublevel sets,

$$\mathrm{Dom}\Phi = \{x \in X : \Phi(x) < +\infty\}.$$

An extended-real-valued function which does not achieve the value $-\infty$ and has a nonempty effective domain is called proper. Observe that a $\Gamma(X)$ function is proper if and only if $\mathrm{Dom}\,\Phi \neq \emptyset$ and Φ is not the constant function $-\infty$.

A subset C of X is bounded if every linear map of form $\langle \cdot, y\rangle : X \to \mathbb{R}$, $y \in Y$ is bounded below and above on C.

Let C be a convex subset of X. The *indicator function* of C is the (convex) function $\delta(\cdot|C)$ defined by

$$\delta(x|C) = \begin{cases} 0 & x \in C \\ +\infty & \text{otherwise} \end{cases}$$

and the effective domain of the conjugate of the indicator function $\delta(\cdot|C)$ of C is the barrier cone of C,

$$\mathcal{B}(C) = \mathrm{Dom}\,(\delta(\cdot|C)^*) = \{y \in Y : \sup_{x \in C} \langle x, y\rangle < +\infty\}.$$

2. SELF-EQUILIBRATED SETS: A FIRST BOUNDEDNESS CRITERION

The goal of this section is to characterize all the sets that are sublevel sets of a function unbounded below. In other words, we would like to establish in what extent the boundedness below of a function can be derived from properties of its sublevel sets.

To this respect, let us define the following geometric notion.

Definition 1. *A subset S of the real linear space X is self-equilibrated if every linear map of the type $\langle \cdot, y\rangle : X \to \mathbb{R}$, $y \in Y$ which is bounded above on S is also bounded below on the same set.*

Equivalently, the set S is self-equilibrated if and only if $\mathcal{B}(S) = -\mathcal{B}(S)$, that is if $\mathcal{B}(S)$, the barrier cone of S, is a linear subspace of Y.

Let us first collect several obvious properties of self-equilibrated sets.

Remark 1. *All the bounded sets and all the symmetric sets (that is sets S such that $S = -S$) as well as their translates (the so-called center sets, see* [7]*), are obviously equilibrated.*

Remark 2. *The algebraic sum, as well as the union of two self-equilibrated sets is also a self-equilibrated set. In general, the intersection of two self-equilibrated sets may fail to be a self-equilibrated, even in the convex regular setting.*

Remark 3. *A set S is self-equilibrated if and only if its regular hull $\langle S\rangle$ is self-equilibrated.*

When the algebraic dimension of the space X is finite, it should be observed that a set S is self-equilibrated if and only if its regular hull $\langle S\rangle$ is the algebraic sum between a linear subspace and a bounded regular subset of X, that is between a symmetric and a bounded set.

The notion of self-equilibrated set allows us to state the main result of **this** section. We thus characterize all the sets which can play the role of sublevel set for a function unbounded below.

Proposition 1. *A subset S of X may be expressed as a sublevel set of a function unbounded below and minorized on S by an affine mapping if and only if S is not self-equilibrated.*

Proof of Proposition 1: Let S be a non self-equilibrated subset of X. Accordingly, there is $y \in Y$ such that the map $\langle \cdot, y\rangle : X \to \mathbb{R}$ is bounded above but unbounded below on S. Set $\Phi_S(x) = \delta(\cdot|S)(x) + \langle x, y\rangle - M$, where $M = \sup_S \langle \cdot, y\rangle$. Let us observe that S is the sublevel set (corresponding to the level zero) of $\Phi_S \in AM(X)$ that is unbounded below.

Conversely, let us prove that if S is a sublevel set of a $AM(S)$-function unbounded below, then S cannot be self-equilibrated. To establish a contradiction, let us select an extended-real-valued function Φ and $t \in \mathbb{R}$ such that the sublevel set $S = [\Phi \le t]$ is a nonempty self-equilibrated set while $\Phi \in AM(S)$ and is unbounded below.

As Φ belongs to $AM(S)$, there is $y \in Y$ and $r \in \mathbb{R}$ such that

$$\Phi(x) \ge \langle x, y\rangle + r \ \forall x \in S. \tag{2}$$

For every $x \in S = [\Phi \le t]$ and according to relation (2) it follows that

$$\langle x, y\rangle \le t - r \ \forall x \in S.$$

In other words, $y \in \mathcal{B}(S)$.

As the set S is self-equilibrated, then $-y \in \mathcal{B}(S)$. If $M = \sup_{x\in S}\langle x, -y\rangle$ (which is finite), using again relation (2) we deduce that

$$\Phi(x) \ge r - M \ \forall x \in S. \tag{3}$$

Since $\Phi(x) > t$ for every $x \notin S = [\Phi \leq t]$ we deduce that Φ is bounded below, a contradiction. $\square$

In other words, if a sublevel set of an extended-real-valued function is self-equilibrated, and the function is minorized on this sublevel set by an affine mapping, then the function is bounded below. We are now in a position to state a variational result.

Theorem 1. *An extended-real-valued function which is minorized on one of its self-equilibrated sublevel sets by an affine mapping is bounded below.*

The simplest way to use the previous result is to apply Theorem 1 to functions minorized by affine mappings.

Theorem 2. *An extended-real-valued function minorized by an affine mapping is bounded below provided at least one of its sublevel sets is self-equilibrated.*

The variational principle stated in Theorem 1 may however be successfully applied to functions which cannot be minorized on X by affine mappings. As an example, consider the class of those quasi-convex functions of the type

$$\Psi(x) = \sup_{i \in I} \left(\min(\langle \cdot, y_i \rangle + a_i, r_i) \right), \tag{4}$$

where $y_i \in Y$, $a_i, r_i \in \mathbb{R}$, while I is an arbitrary index set. For properties and uses of this class, containing $\Gamma_0(X)$, we refer the reader to [15] and [11].

Obviously, $\sup_X \Psi = \sup_{i \in I} r_i$, while for every $i \in I$ and $r \leq r_i$, Ψ is minorized on the sublevel set $[\Psi \leq r]$ by the affine mapping $\langle \cdot, y_i \rangle + a_i$. Moreover, a sublevel $[\Psi \leq r]$ coincides with X if and only $r \geq \sup_X \Psi$.

Thus, Ψ is minorized by an affine mapping on every of its proper (distinct from X) sublevel sets $[\Psi \leq r]$, or equivalently, when the level r is such that $r < \sup_X \Psi$. Theorem 1 gives therefore a boundedness criterion for quasi-convex functions of type (4).

Corollary 1. *Every quasi-convex function of type (4) is bounded below provided one of its sublevel sets is a self-equilibrated proper subset of X.*

3. GROWTH DIRECTIONS: SECOND BOUNDEDNESS CRITERION

In order to use Theorem 2, it is necessary to decide if at least one among all the sublevel sets of a function is self-equilibrated. This task is rather difficult in practice. This section will provide a different and easier criterion for boundedness below of a function in $AM(X)$. Instead of the notion of self-equilibrated set, this criterion is based upon the closely related notion of self-equilibrated function.

Definition 2. *The element y from the dual space Y is a growth direction for the extended-real-valued function Φ, if*

$$\sup_{x\in S}\langle x,y\rangle=+\infty\quad\Longrightarrow\quad\sup_{x\in S}\Phi(x)=+\infty$$

where S is any subset of X.

In other words, $y\in Y$ is a growth direction for Φ if the function Φ grows faster than the linear map $\langle\cdot,y\rangle$.

Definition 3. *An extended-real-valued function Φ defined on the vector space X is called self-equilibrated if the set G_Φ of all its growth directions is a linear subspace of the dual Y of X.*

Remark that an extended-real-valued function may be self-equilibrated even if none of its sublevel sets is self-equilibrated. To this respect, consider a separable Hilbert space X with basis $B=\{b_i\,:\,i\in\mathbb{Z}\}$; thus, for every $i\in\mathbb{Z}$, let x_i denote the value of the inner product between $x\in X$ and $b_i\in X$. For every $x\neq 0$, set C_x for the set $\{i\in\mathbb{Z}:\ x_i\neq 0\}$ and $c(x)$ for $\sup(C_x)$; obviously, $c(x)>-\infty$, and $x_{c(x)}\neq 0$ provided that $c(x)<+\infty$. We define the extended-real-valued function Φ by

$$\Phi(x)=\left\{\begin{array}{cc}-\infty & x=0\\ c(x) & c(x)<+\infty, x_{c(x)}>0\\ c(x)+1 & c(x)<+\infty, x_{c(x)}<0\\ +\infty & c(x)=+\infty\end{array}\right..$$

It is easy to see that

$$[\Phi\leq r]=\{x\in X\,:\,x_n\geq 0,\,x_m=0,\forall m>n\},$$

where n is the unique integer fulfilling $n\leq r<n+1$. Accordingly, none of the sublevel sets of the extended-real-valued lower semi-continuous quasi-convex function Φ is self-equilibrated, while $G_\Phi=\{0\}$, and therefore Φ is necessarily a self-equilibrated function. Modify the value of Φ at zero by setting $\Phi(0)=c\in\mathbb{R}$ to obtain a self-equilibrated function with values in $\mathbb{R}\cup\{+\infty\}$ with the same property.

It is much easier to give an example of a function which is not self-equilibrated while at least one of its sublevel set is self-equilibrated. Indeed, the function

$$\Phi(x)=\left\{\begin{array}{cc}\frac{x^2}{x^2+1} & x\leq 0\\ x & x>0\end{array}\right.$$

has bounded (and thus self-equilibrated) sublevel sets for every level less than 1, while the cone of growth directions of Φ is the positive half-line, $G_\Phi = \{t : \, t \geq 0\}$.

The second boundedness criterion reads as follows.

Theorem 3. *An extended-real-valued function is bounded below provided it is self-equilibrated and minorized by an affine mapping.*

Remark 4. *The set of growth direction of the function* $(\Phi + \delta(\cdot \,|\, [\Phi \leq r]))$ *is the barrier cone of the set* $[\Phi \leq r]$*. In particular, the set of all the growth directions of the indicator function of a set coincides with the barrier cone of this set; accordingly, the indicator function of a set is self-equilibrated if and only if the set itself is self-equilibrated.*

It is possible to use Remark 4 in order to deduce Theorem 1 as a consequence of Theorem 3. Indeed, let Φ be an extended-real-valued function minorized on one of its self-equilibrated sublevel set, say $[\Phi \leq r]$, by an affine mapping. Then the same affine mapping minorizes the function $\Phi + \delta(\cdot \,|\, [\Phi \leq r])$, whose set of growth directions coincides with the barrier cone of the self-equilibrated set $[\Phi \leq r]$, being thus a linear space. The function $\Phi + \delta(\cdot \,|\, [\Phi \leq r])$ is accordingly bounded below by virtue of Theorem 3 and Theorem 1 is proved since $\inf_X \Phi = \inf_X \left(\Phi + \delta(\cdot \,|\, [\Phi \leq r])\right)$.

We will postpone the proof, as well as any further analysis of Theorem 3 till after the subsection 3.1 which collects some of the properties of the set of growth directions of an extended-real-valued function.

3.1. **Properties of growth directions.** Our primary goal in this section is to study the interplay between the growth directions and sublevel sets of an extended real-valued function $\Phi : X \to \mathbb{R} \cup \{-\infty, +\infty\}$.

Lemma 1. *The element* $y \in Y$ *is a growth direction for the function* Φ *if and only if the mapping* $\langle \cdot, y \rangle$ *is bounded above on every sublevel set of* Φ*.*

Proof of Lemma 1: Let $y \in Y$ be a growth direction of Φ; as for every $m \in \mathbb{R}$

$$\sup_{[\Phi \leq m]} \Phi \leq m,$$

by virtue of the definition of a growth direction we have

$$\sup_{x \in [\Phi \leq m]} \langle x, y \rangle < +\infty,$$

and thus the map $\langle \cdot, y \rangle$ is bounded above on every sublevel set $[\Phi \leq m]$ of Φ.

In order to prove the reverse implication, let $y \in Y$ be a real linear function bounded above on every sublevel set of Φ. Accordingly, a subset S of X such that $\sup_{x \in S} \langle x, y \rangle = +\infty$ cannot be a subset of any sublevel set of Φ. This means that $\sup_S \Phi = +\infty$, and proves Definition 1, as well as Lemma 1. $\square$

The previous Lemma may be restated in order to express G_Φ in terms of barrier cones for sublevel sets:

$$G_\Phi = \bigcap_{m \in \mathbb{R}} \mathcal{B}\left([\Phi \leq m]\right); \tag{5}$$

accordingly, G_Φ is a convex cone containing the origin of Y.

As every sublevel set of Φ is contained in $\mathrm{Dom}\ \Phi$, then $\mathcal{B}\left(\mathrm{Dom}\ \Phi\right) \subseteq \mathcal{B}\left([\Phi \leq m]\right)$ for every $m \in \mathbb{R}$. We can thus state the following proposition.

Proposition 2. *The barrier cone of the effective domain of Φ is always a part of G_Φ, the set of all its growth directions:*

$$\mathcal{B}\left(\mathrm{Dom}\ \Phi\right) \subseteq G_\Phi. \tag{6}$$

Recall (Remark 3) that a set is self-equilibrated if and only if the same holds for its regular hull. The following example shows that the regular hull of a non self-equilibrated $AM(X)$ function may be self-equilibrated.

Example 1. *Take $X = Y = \mathbb{R}$, $\langle x, y \rangle = xy$, and set*

$$\Phi(x) = \begin{cases} 0 & x \leq 0 \\ \sqrt{x} & x > 0 \end{cases}.$$

Obviously, $G_\Phi = \mathbb{R}_+$ which means that Φ is not self-equilibrated. The regular hull of Φ is nevertheless constant, $\Phi^{**} = 0$, so that $G_{\Phi^{**}} = \{0\}$ and differs from G_Φ. Accordingly, Φ is a non self-equilibrated function with a self-equilibrated regular hull.

Let us note by $\langle \Phi \rangle$ the quasi-regular hull of an extended-real-valued function Φ, defined by taking the supremum over all the functions whose sublevel sets are regular and which minorizes Φ. We have

$$[\langle \Phi \rangle \leq r] = \cap_{s > r} \left\langle [\Phi \leq s] \right\rangle. \tag{7}$$

The following proposition is a consequence of Lemma 1 and of relation (7).

Proposition 3. *An extended-real-valued function has the same set of growth directions as its quasi-regular hull.*

Proof of Proposition 3: Relation (7) infers

$$[\langle\Phi\rangle \le r] \subseteq \langle[\Phi \le r+1]\rangle$$

from which we get

$$\mathcal{B}\left(\langle[\Phi \le r+1]\rangle\right) \subseteq \mathcal{B}\left([\langle\Phi\rangle \le r]\right).$$

Recalling that $\mathcal{B}\left(\langle[\Phi \le r+1]\rangle\right) = \mathcal{B}\left([\Phi \le r+1]\right)$ we deduce that

$$\mathcal{B}\left([\Phi \le r+1]\right) \subseteq \mathcal{B}\left([\langle\Phi\rangle \le r]\right),$$

and thus, by virtue of Lemma 1 we derive that

$$\begin{aligned} G_\Phi &= \cap_{r\in\mathbb{R}}\mathcal{B}\left([\Phi \le r]\right) = \cap_{r\in\mathbb{R}}\mathcal{B}\left([\Phi \le r+1]\right) \\ &\subseteq \cap_{r\in\mathbb{R}}\mathcal{B}\left([\langle\Phi\rangle \le r]\right) = G_{\langle\Phi\rangle}. \end{aligned} \tag{8}$$

Noticing that the quasi-regular hull always minorizes the function, we know that

$$G_{\langle\Phi\rangle} \subseteq G_\Phi; \tag{9}$$

the conclusion of Proposition 3 follows from relations (8) and (9). □

However, when Φ is a $AM(X)$ convex function, the quasi-regular and the regular hull coincide, so Proposition 3 has the following consequence.

Proposition 4. *A $AM(X)$ convex extended-real-valued function has the same set of growth directions as its bi-conjugate.*

The following proposition is easy to prove.

Proposition 5. *Every element $y \in \operatorname{Dom}\Phi^\star$ is a growth direction for Φ. Accordingly, the (convex) cone spanned by the effective domain of $\Phi^\star$ is a subset of the (convex) cone G_Φ of all the growth directions for Φ*

$$\mathbb{R}_+\left(\operatorname{Dom}\left(\Phi^\star\right)\right) \subseteq G_\Phi. \tag{10}$$

Observe that the set of growth direction may be significantly larger than the cone spanned by the effective domain of the conjugate.

Example 2. *Consider $X = Y = \mathbb{R}$ and $\langle x, y\rangle = xy$. Let*

$$\Phi : \mathbb{R} \to \mathbb{R},\ \Phi(x) = \frac{x - |x|}{2};$$

Obviously, $\operatorname{Dom}\Phi^\star = \emptyset$, *whence* $G_\Phi = \{x \in \mathbb{R} :\ x \le 0\}$.

The next example shows that the reverse inclusion of (10) does not hold, even for $\Gamma_0(X)$ functions defined on a finite dimensional space X.

Example 3. *Let* $X = Y = \mathbb{R}^2$ *endowed with the standard duality bracket* $\langle (x_1, x_2), (y_1, y_2) \rangle = x_1 y_1 + x_2 y_2$, *and set*

$$\Phi : \mathbb{R}^2 \to \mathbb{R} \cup \{+\infty\},\ \Phi((x_1, x_2)) = \begin{cases} x_2 & \text{if } x_1 \leq 0 \\ +\infty & \text{if } x_1 > 0 \end{cases}.$$

It is obvious that

$$G_\Phi = \{(x_1, x_2)\ : x_1, x_2 \geq 0\},$$

whence the cone spanned by $\mathrm{Dom}\ \Phi^\star = \{(x_1, 1)\ : x_1 \geq 0\}$, *the effective domain of* $\Phi^\star$ *is*

$$\mathbb{R}_+ (\mathrm{Dom}\ \Phi^\star) = \{(x_1, x_2)\ : x_1 \geq 0,\ x_2 > 0\}.$$

Accordingly, the open half-line $\{(x_1, x_2) :\ x_1 > 0, x_2 = 0\}$ *is a part of the set of growth directions* G_Φ, *but does not meet* $\mathbb{R}_+ (\mathrm{Dom}\ \Phi^\star)$.

Remark that, in the nonconvex case of Example 2, the cone spanned by the effective domain of the conjugate is empty, and thus bears no information about the cone of growth directions.

In the convex setting of Example 3, the cone of growth directions is the closure (with respect to the standard $\mathbb{R}^2$ topology) of the cone spanned by the effective domain of the conjugate. For every $AM(X)$ convex function one has:

Proposition 6. *Let* Φ *be a convex extended-real-valued function whose conjugate has a nonempty effective domain. Then, for any vector space topology on* Y, *compatible or not with the duality, the closure of the set of all the growth directions of* Φ *coincides with the closure of the cone spanned by the effective domain of the conjugate of* Φ.

In order to prove Proposition 6 let us study the interplay between the barrier cone of the effective domain of Φ, and the convex cone spanned by the effective domain of the conjugate of Φ.

Proposition 7. *Let* Φ *be an extended-real-valued function whose conjugate has a nonempty effective domain. Then, for any vector space topology on* Y, *compatible or not with the duality, the closure of the cone spanned by the effective domain of the conjugate of* Φ *contains the barrier cone of the effective domain of* Φ.

Proof of Proposition 7: Let $y_0 \in \mathrm{Dom}\ \Phi^\star$ and $w \in \mathcal{B}(\mathrm{Dom}\ \Phi)$. The following inequality holds for every fixed $n \in \mathbb{N}^\star$:

$$\langle x, y_0 + nw \rangle - \Phi(x) \leq \Phi^\star(y_0) + n \left(\sup_{x \in \mathrm{Dom}\Phi} \langle x, w \rangle \right) < +\infty.$$

We may thus deduce that $(y_0 + nw) \in \mathrm{Dom}\ \Phi^\star$; divide this relation by n to obtain that

$$w \in \left(\mathbb{R}_+ \left(\mathrm{Dom}\ \Phi^\star\right) - \frac{y_0}{n}\right)\ \forall n \in \mathbb{N}^\star.$$

Proposition 7 is complete as the sequence $\{\frac{y_0}{n}\}_{n\in\mathbb{N}}$ tends to 0 for any vector space topology on Y. □

Let us now consider the convex setting.

Proposition 8. *The set of the growth directions for any convex extended-real-valued function Φ is the union between the cone spanned by the effective domain of $\Phi^\star$ and the barrier cone of the effective domain of Φ:*

$$G_\Phi = \mathbb{R}_+ \left(\mathrm{Dom}\ \Phi^\star\right) \cup \mathcal{B}(\mathrm{Dom}\ \Phi).$$

Proof of Proposition 8: The conclusion occurs obviously if Φ is identically equal to $+\infty$. We shall therefore suppose that $\mathrm{Dom}\ \Phi \neq \emptyset$.

For $y \in Y$, $y \neq 0$, define the map

$$\phi_y : \mathbb{R} \to \mathbb{R} \cup \{-\infty, +\infty\},\ \phi_y(s) = \inf\{\Phi(x) :\ \langle x, y\rangle = s\}.$$

Obviously, the newly defined function is convex: the inequality

$$\phi_y((\lambda s_1 + (1-\lambda)s_2) \leq \lambda\phi_y(s_1) + (1-\lambda)\phi_y(s_2) \tag{11}$$

holds for every $s_1, s_2 \in \mathbb{R}$ and $0 < \lambda < 1$.

The map ϕ_y may in general take the both values $-\infty$ and $+\infty$. If $\phi_y(s) = +\infty$ for some s, it follows that the convex set $\mathrm{Dom}\ \Phi$ does not encounter the hyperplane $\{x :\ \langle x, y\rangle = s\}$, and thus lies within one of the half-spaces $\{x \in X :\ \langle x, y\rangle > s\}$ or $\{x \in X :\ \langle x, y\rangle < s\}$.

Set $s_0 = \inf_{x\in\mathrm{Dom}\ \Phi} \langle x, y\rangle$ (remark that $s_0 < +\infty$ as $\mathrm{Dom}\ \Phi \neq \emptyset$, but that s_0 can take the value $-\infty$). Obviously, the convex set $\mathrm{Dom}\ \Phi$ cannot lie within the half-space $\{x \in X :\ \langle x, y\rangle > s\}$ for any $s_0 < s$.

Thus, if $\phi_y(s) = +\infty$ for some $s_0 < s$, we deduce that $\mathrm{Dom}\ \Phi$ is a part of the half-space $\{x \in X :\ \langle x, y\rangle < s\}$; accordingly, when $\phi_y(s) = +\infty$ for some $s_0 < s$, then y belongs to the barrier cone of $\mathrm{Dom}\ \Phi$.

We may thus conclude that $\phi_y(s) < +\infty$ for every $s_0 < s$, provided $y \notin \mathcal{B}(\mathrm{Dom}\ \Phi)$.

We claim that for every growth direction y of Φ such that $y \notin \mathcal{B}(\mathrm{Dom}\ \Phi)$, the mapping ϕ_y is a convex function which does not achieve the value $-\infty$.

Indeed, assume that $\phi_y(\overline{s}) = -\infty$ for some $\overline{s} \in \mathbb{R}$. Applying (11) for $s_1 = \overline{s}$, $\lambda = \frac{1}{2}$ and $s_2 = 2s - \overline{s}$, where $s > \frac{1}{2}(s_0 + \overline{s})$, we deduce that

$$\phi_y(s) \leq \frac{1}{2}\left(\phi_y(\overline{s}) + \phi_y(2s - \overline{s})\right). \tag{12}$$

As $2s - \overline{s} > s_0$ and $y \notin \mathcal{B}(\mathrm{Dom}\ \Phi)$, it follows that $\phi_y(2s - \overline{s}) < +\infty$; use relation (12) to deduce that $\phi_y(s) = -\infty$ for every $s > \frac{1}{2}(s_0 + \overline{s})$. Let us observe that Lemma 1 obviously proves that $\lim_{s\to+\infty}\phi_y(s) = +\infty$ for all the growth directions y of Φ. This contradicts the fact that y is a growth direction for Φ and establishes the claim.

Every convex function $f : \mathbb{R} \to \mathbb{R}\cup\{+\infty\}$ satisfying $\lim_{r\to\infty} f(r) = +\infty$ is minorized by some increasing affine mapping. Applying this result to ϕ_y, we deduce that $(\phi_y)^*(\alpha) < +\infty$ for some $\alpha > 0$

$$\sup_{s\in\mathbb{R}} (\alpha s - \phi_y(s)) = (\phi_y)^\star(\alpha) < +\infty. \tag{13}$$

As

$$\sup_{\{x\in X: \langle x,y\rangle = s\}} (\langle \alpha x, y\rangle - \Phi(x)) = \alpha s - \phi_y(s),$$

relation (13) applied to ϕ_y implies that

$$\sup_{x\in X} (\langle \alpha x, y\rangle - \Phi(x)) = (\phi_y)^\star(\alpha) < +\infty.$$

We have thus proved that $\Phi^\star(\alpha y) = (\phi_y)^\star(\alpha) < +\infty$, for every growth direction y which does not belong to the barrier cone of the effective domain of Φ. ☐

Proposition 6 is a mere consequence of Proposition 7 and 8.

Remark 5. *Standard convex analysis techniques allow us to prove that the set of growth directions of a convex function coincides with the barrier cone of each non-void sub-level set (except perhaps the one corresponding to the infimum).*

As already remarked, the converse of inclusion (10) fails in general, even for convex functions in $\mathbb{R}^n$. Proposition 8 allow us to describe two remarkable cases where the set of growth directions and the cone spanned by the effective domain of the conjugate coincide.

The first one is obvious consequence of Proposition 8.

Corollary 2. *Let Φ be a real-valued convex function. Then either $G_\Phi = \mathbb{R}_+(\mathrm{Dom}\ \Phi^\star)$ or $G_\Phi = \{0\}$.*

The second one concerns convex functions which are bounded below.

Corollary 3. *For any convex extended-real-valued function Φ bounded below one has*

$$G_\Phi = \mathbb{R}_+\left(\mathrm{Dom}\ \Phi^\star\right). \tag{14}$$

Proof of Corollary 3: Relation (14) obviously holds when Φ is the constant function equal to $+\infty$. Let us thus consider the case of a somewhere finite convex Φ, which means that $\mathrm{Dom}\ \Phi$ is nonempty. Let $y \in \mathcal{B}(\mathrm{Dom}\ \Phi)$, and set $N = \sup\{\langle x, y\rangle : x \in \mathrm{Dom}\ \Phi\}$; obviously $N \in \mathbb{R}$. As Φ is bounded and somewhere finite, it follows that also $M = \inf_X \Phi \in \mathbb{R}$. Then

$$\langle x, y\rangle - \Phi(x) \leq \langle x, y\rangle - M \leq N - M \quad \forall x \in \mathrm{Dom}\ \Phi,$$

while

$$\langle x, y\rangle - \Phi(x) = -\infty \quad \forall x \in (X \setminus \mathrm{Dom}\ \Phi).$$

Accordingly, $y \in \mathrm{Dom}\ \Phi^\star$ and therefore for any extended-real-valued bounded below function Φ we have

$$\mathcal{B}(\mathrm{Dom}\ \Phi) \subset \mathrm{Dom}\ \Phi^\star.$$

When Φ is convex, use Proposition 8 to deduce relation (14). $\square$

3.2. **Proof of the second boundedness principle.** We proceed now to the prof of the main result of this section.

Proof of Theorem 3: Let $y_0 \in \mathrm{Dom}\ \Phi^\star$. From Proposition 5 we know that $y_0 \in G_\Phi$; as Φ is self-equilibrated it follows that $(-y_0) \in G_\Phi$.

Let $m > \inf_X \Phi$. Lemma 1 implies that $(-y_0) \in \mathcal{B}([\Phi \leq m])$, and thus that $s = \sup_{x \in [\Phi \leq m]} \langle x, -y_0\rangle < +\infty$.

Accordingly,

$$\Phi(x) \geq m \quad \forall x \text{ s.t. } \langle x, y_0\rangle \leq -s. \tag{15}$$

On the other hand, $y_0 \in \mathrm{Dom}\ \Phi^\star$ means that

$$\Phi(x) \geq \langle x, y_0\rangle - \Phi^\star(y_0)\ \forall x \in X;$$

specify the previous relation for $\langle x, y_0\rangle \geq -s$ to infer that

$$\Phi(x) \geq -\left(s + \Phi^\star(y_0)\right)\ \forall x \text{ s.t. } \langle x, y_0\rangle \geq -s. \tag{16}$$

Combine relations (15) and (16) to prove Theorem 3. $\square$

The following result (an easy consequence of the above Theorem and of Corollary 3) makes the connection between our second boundedness result, Theorem 3, and the criteria (☼) and (◑).

Proposition 9. *Let Φ be a convex extended-real-valued function minorized by an affine mapping on the whole underlying space X. The following are equivalent:*

(i) Φ is self-equilibrated;

(ii)$\mathbb{R}_+ \left(\mathrm{Dom}\ \Phi^\star\right)$ is a linear subspace of Y.

The main difficulty in applying Theorem 3 is to prove that the effective domain of the conjugate is nonempty. Proposition 10 is a consequence of the bi-conjugacy theorem (see [12, Proposition 6.1]).

Proposition 10. *Let X be a locally convex space and take for Y, the topological dual $X^\star$ of X. Then the effective domain of the conjugate of every proper lower semi-continuous function is nonempty.*

Propositions 10 yields the following application of Theorem 3.

Theorem 4. *Every proper lower semi-continuous function Φ defined on a locally convex vector space X is bounded below if the set of all linear and continuous functions $f : X \to \mathbb{R}$ such that*

$$[\sup_S \Phi < +\infty] \Rightarrow [\sup_s f < +\infty]$$

is a linear space.

4. FINAL REMARKS

The main difference between the boundedness criteria stated in Theorems 1 and 3 and any of the criteria mentioned in the first section is that hypothesis of Theorems 1 and 3 assigns to the infimum of the function to be finite, but do not guarantee that the function achieves its infimum.

Proposition 11. *Suppose that X is the algebraical dual of Y (that is all the linear maps on Y are of form $\langle x, \cdot\rangle$, $x \in X$). Then every self-equilibrated function in the class $\Gamma_0(X)$ achieves its infimum on X.*

Proof of Proposition 11: Let $\Phi \in \Gamma_0(X)$ be a self-equilibrated function. By virtue of Theorem 2 it follows that Φ is bounded below on X. Then, m (by definition equal to $\inf_X \Phi = -\Phi^*(0)$) belongs to $\mathbb{R}$. Moreover, as a consequence of Lemma 3, we deduce that $\mathbb{R}_+(\mathrm{Dom}\ \Phi^\star)$ is a linear subspace of Y, say U.

Set V for the set of all the linear maps on U, and $(b_i)_{i\in I}$ for a (Hamel) base of the vector space U. As $\mathbb{R}_+(\mathrm{Dom}\ \Phi^\star) = U$, it follows that, for every $i \in I$, there exists $\gamma_i > 0$ such that both $-\gamma_i b_i$ and $\gamma_i b_i$ lie within $\mathrm{Dom}\ \Phi^\star$. Set G for the convex hull of the set $\{-\gamma_i b_i, \gamma_i b_i : \ i \in I\}$. Obviously $G \subset \mathrm{Dom}\ \Phi^\star$; as the polar set of G,

$$H = \{f \in V : -\gamma_i \leq \langle f, b_i\rangle \leq \gamma_i \ \forall i \in I\}$$

is convex and compact (since isomorphic with $\Pi_{i\in I}[-\gamma_i, \gamma_i]$, the direct product of compact real intervals) with respect to the weak topology $\sigma(V, U)$, it follows that G is a vicinity of 0 in the Mackey topology $\beta(U, V)$. The interior with respect to the Mackey topology on $U \times \mathbb{R}$ of the strict epigraph of $\Phi^\star$

$$S_{\Phi^\star} = \{(y, s) : \ y \in U, s > \Phi^\star(y)\} \subset U \times \mathbb{R}$$

is thus nonempty; by virtue of Eidelheit's Theorem (see [17, Theorem 1.1.3, Chapter 1]) we may separate by an half-space the singleton $(0, -m)$ and

$S_{\Phi^\star}$. In other words, there exists $\alpha \in \mathbb{R}$ and a linear mapping $f : U \to \mathbb{R}$, which are not simultaneously equal to zero, such that the linear mapping $\mathcal{F} : U \times \mathbb{R}$, $\mathcal{F}(y, s) = f(y) + \alpha\, s$ satisfies:

$$\mathcal{F}(0, -m) \le \mathcal{F}(y, s)\ \forall (y, s) \in S_{\Phi^\star}. \tag{17}$$

Let us prove that $\alpha > 0$. Indeed, as $\Phi^\star(0) = -m$ it follows that, for every $t > 0$, $(0, -m + t) \in S_{\Phi^\star}$; using relation (17) for $y = 0$ and $s = -m + t$ we deduce that $\alpha\, t \ge 0$ for every $t > 0$. Hence, $\alpha \ge 0$. We claim that α cannot be 0. Suppose, to the end of obtaining a contradiction, that $\alpha = 0$. In this case, relation (17) yields $f(y) \ge 0\ \forall y \in \mathrm{Dom}\ \Phi^\star$. As the linear space U is nothing but the cone spanned by $\mathrm{Dom}\ \Phi^\star$, and as f is positively homogeneous (since linear) on U, it follows that $f(y) \ge 0$ for every $y \in U$. Accordingly, $f = 0$, and thus $\mathcal{F} = 0$, a contradiction. Using the fact that X is the algebraic dual of Y, there is $x \in X$ such that $\langle x, y\rangle = f(y)$ for $y \in U$. Rewriting relation (17) as

$$\langle x, y\rangle + \alpha(s + m) \ge 0 \quad \forall y \in U, s > \Phi^\star(y),$$

and noticing that $\alpha > 0$ we deduce that

$$\langle x, y\rangle + \alpha(\Phi^\star(y) + m) \ge 0 \quad \forall y \in U.$$

Dividing the previous relation by $-\alpha$ we obtain

$$\left\langle -\frac{x}{\alpha}, y \right\rangle - \Phi^\star(y) \le m \quad \forall y \in U; \tag{18}$$

relation (18) obviously holds also when $y \notin U$, since in this case $\Phi^\star(y) = +\infty$. Consequently,

$$\Phi\left(-\frac{x}{\alpha}\right) = \Phi^{**}\left(-\frac{x}{\alpha}\right) = \sup_{y \in Y}\left(\left\langle -\frac{x}{\alpha}, y\right\rangle - \Phi^\star(y)\right) \le m.$$

Recall that $\Phi\left(-\frac{x}{\alpha}\right) \ge \inf_X \Phi = m$ to deduce that $-\frac{x}{\alpha} \in \mathrm{argmin}_X \Phi$. $\square$

Remark 6. *The hypotheses of Proposition 11 are obviously satisfied when the algebraic dimension of X is finite.*

It is impossible to retrieve the result of Proposition 11 in a general locally convex setting.

Proposition 12. *On every Banach vector space X of infinite dimension there is a self-equilibrated function in $\Gamma_0(X)$ which does not achieve its infimum on X.*

Proof of Proposition 12: When X is a reflexive Banach space, consider a closed, convex, linearly bounded and unbounded symmetric subset C of X. As its characteristic function is not coercive, applying Theorem 4.3 from [1] to a $\delta(\cdot|C)$, we deduce that, for every $\varepsilon > 0$ there is a function $\Phi \in \Gamma_0(X)$ which does not achieve its infimum and satisfies

$$\delta(\cdot|C) - \varepsilon \leq \Phi_\varepsilon \leq \delta(\cdot|C) + \varepsilon.$$

Obviously, the cone of growth directions of Φ_ε coincides with the cone of growth direction of $\delta(\cdot|C)$. Since the set C is symmetric, $\delta(\cdot|C)$ is self-equilibrated and therefore for every $\varepsilon > 0$, Φ_ε is a self-equilibrated function, belongs to $\Gamma_0(X)$ and does not reach its infimum.

When X is a non reflexive Banach space, recall that, following a classical result of James (see for instance [9]; for further developments on this topics see [8]), there is a continuous linear map $f : X \to \mathbb{R}$, which does not achieve its supremum on the unit ball $\mathbb{B}_X$ of X.

It suffices now to take Φ for the restriction of f on $\mathbb{B}_X$:

$$\Phi : X \to \mathbb{R} \cup \{+\infty\},\ \Phi(x) = \delta(\cdot|\mathbb{B}_X)(x) + f(x).$$

As the effective domain of Φ is the unit ball, Φ is coercive, thus is self-equilibrated. Obviously, $\Phi \in \Gamma_0(X)$ and does not achieve its infimum on X, establishing Proposition 12. □

It is (at our knowledge) an open question whether if a self-equilibrated function which belongs to $\Gamma_0(X)$ and does not achieve its infimum exists on locally convex spaces which are neither Banach nor satisfy conditions of Proposition 11.

REFERENCES

[1] S. Adly, E. Ernst and M. Théra, A Characterization of Convex and Semi-coercive Functionals, *J. Convex anal.*, **8**, 127-148, 2001.

[2] S. Adly, E. Ernst and M. Théra, On the converse of the Dieudonné theorem in reflexive Banach spaces, *Cybernetics System Anal.*, Special issue in memoriam Prof. N.B. Pshenitchnyi, **3**, 34-39, 2002.

[3] H. Attouch, H. Brézis, Duality for the sum of convex functions in general Banach spaces, *Aspects of mathematics and its applications*, Collect. Pap. Hon. L. Nachbin, 125-133, 1986.

[4] D. Azé, Duality for the sum of convex functions in general normed spaces, *Arch. Math.*, **62**, 554-561, 1994.

[5] J. M. Borwein, J. D. Vanderwerff, *Convergence of Lipschitz Regularizations of Convex Functions, J. Funct. Anal.***,** **128** 139-162, 1995.

[6] C. Combari, M. Laghdir, L. Thibault, On Sub-differential Calculus for Convex Functions Defined on Locally Convex Spaces, *Ann. Sci. Math. Québec*, **23**, 23-36, 1999.

[7] S. J. Dilworth, Intersections of centered sets in normed spaces, Far East J. Math. Sci. Special Volume, Part II, 129-136, 1998.

[8] H.L. Gau, N.C. Wong, Some converses of the strong separation theorem, *Proc. Am. Math. Soc.*, **124**, 2443-2449, 1996.

[9] R. Holmes, *Geometric functional analysis and its applications*, GTM 24, Springer-Verlag, New York, 1975.

[10] J. L. Joly, Une famille de topologies et de convergences sur l'ensemble des fonctionnelles convexes, Thèse de Doctorat d'Ètat, Grenoble, 1970.

[11] J.-E. Martinez-Legaz, On lower subdifferentiable functions, Série Internationale d'Analyse numérique, Trends in Mathematical Optimisation, Birkhäuser, **84**, 197-232, 1998.

[12] J.J. Moreau, *Fonctionnelles convexes*, Seminaire sur les Équations aux dérivées partielles, Collège de France, Paris, 1967.

[13] M. Moussaoui, M. Volle, Sur la quasicontinuité et les fonctions unies en dualité convexe, C. R. Acad. Sci. Paris, **322**, Série I, 839-844, 1996.

[14] M. Moussaoui, M. Volle, Quasicontinuity and united functions in convex duality theory, Comm. Appl. Nonlinear Anal, **4**, 73-89, 1997.

[15] J.-P. Penot, M Volle, Another duality scheme for quasiconvex problems, Série Internationale d'Analyse numérique, Trends in Mathematical Optimisation, Birkhäuser, **84**, 259-275, 1998.

[16] R. T. Rockafellar, *Convex Analysis*, Princeton Mathematical Series **28**, Princeton University Press, 1970.

[17] C. Zălinescu, *Convex Analysis in General Vector Spaces*, **World** Scientific, 2002.

LABORATOIRE DE MODELISATION EN MÉCANIQUE ET THERMODYNAMIQUE, CASE 322, UNIVERSITÉ PAUL CÉZANNE AIX-MARSEILLE III, AVENUE ESCADRILLE NORMANDIE-NIEMEN 13397 MARSEILLE CEDEX 20
E-mail address: `Emil.Ernst@univ.u-3mrs.fr`

LACO, UNIVERSITÉ DE LIMOGES, 123 AVENUE A. THOMAS, 87060 LIMOGES CEDEX, FRANCE
E-mail address: `Michel.Thera@unilim.fr`

UNIVERSITÉ D'AVIGNON ET DES PAYS DE VAUCLUSE, 74 RUE LOUIS PASTEUR - 84029 AVIGNON CEDEX 1, FRANCE
E-mail address: `Michel.Volle@univ-avignon.fr`