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ON DIRECTED SETS AND THEIR SUPREMA

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ABSTRACT. The aim of this paper is to investigate real partially ordered linear topological spaces in which directed sets admit a supremum in their closure. In particular, we point out that this property is intimately related to the normality of the ordering cone and also to the Scott continuity of functionals belonging to the nonnegative polar of the ordering cone.

1. INTRODUCTION AND PRELIMINARIES

Throughout this paper (E, τ) will be a linear topological space over the field \mathbb{R} of real numbers. For every $x \in E$, we denote by $\mathcal{V}(x)$ the family of all neighborhoods of x (actually, τ being a linear topology, we have $\mathcal{V}(x) = x + \mathcal{V}(0_E)$, where 0_E stands for the zero-element of E). As usual, E^* stands for the topological dual of E . By $w := \sigma(X, X^*)$ we mean the weak topology of E . For any subset A of E , we denote by $\text{cl } A$ (resp. $\text{int } A$) the closure (resp. interior) of A with respect to τ , and by $w\text{-cl } A$ the closure of A with respect to w .

In the sequel, the linear topological space E will be partially ordered by a convex cone C (i.e., $\emptyset \neq C = \mathbb{R}_+ C = C + C$), which is throughout assumed to be closed and proper (i.e., $\{0_E\} \neq C = \text{cl } C \neq E$). As usual, the partial ordering (i.e., reflexive and transitive) relation is defined by:

$$x \leq_C y \Leftrightarrow y \in x + C.$$

Recall that C is said to be pointed if $\ell(C) := C \cap (-C) = \{0_E\}$. Obviously, C is pointed if and only if relation \leq_C is antisymmetric.

The polarity associated to C (cf. [9]) is the mapping $[\cdot]_C : 2^E \rightarrow 2^E$, which assigns to each subset A of E the (closed convex) set

$$[A]_C := \bigcap_{x \in A} (x + C) = \{y \in E \mid \forall x \in A, x \leq_C y\}$$

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of all upper bounds of A . Of course, the set of lower bounds of A can be represented by means of the polarity associated to $-C$, as

$$[A]_{-C} := \bigcap_{x \in A} (x - C) = \{z \in E \mid \forall x \in A, z \leq_C x\}.$$

The following preliminary result summarizes some basic properties of polarity mappings, which will be useful in the sequel. For a detailed study of polarities from the vector optimization point of view, we refer the reader to [9].

Lemma 1.1. *The following hold:*

- (i) $[\emptyset]_C = E$ and $[E]_C = \emptyset$;
- (ii) $[-C]_C = C$;
- (iii) $\forall A \subset B \subset E, [A]_C \supset [B]_C$;
- (iv) $\forall A \subset E, A \subset [[A]_C]_{-C}$.

As usual in vector optimization (see for instance [14]), the set of ideal maximal points of any subset A of E is defined as

$$\text{IMax } A := A \cap [A]_C,$$

and the set of Pareto maximal points of A is given by

$$\begin{aligned} \text{Max } A &:= \{x \in A \mid A \cap (x + C) \subset x - \ell(C)\} \\ &= \{x \in A \mid \forall y \in A, x \leq_C y \Rightarrow y \leq_C x\}. \end{aligned}$$

Analogously, by definition, the set of ideal minimal points of A is

$$\text{IMin } A := A \cap [A]_{-C},$$

and the set of Pareto minimal points of A is

$$\begin{aligned} \text{Min } A &:= \{x \in A \mid A \cap (x - C) \subset x + \ell(C)\} \\ &= \{x \in A \mid \forall y \in A, y \leq_C x \Rightarrow x \leq_C y\}. \end{aligned}$$

Remark 1.2. Obviously, $\text{IMax } A \subset \text{Max } A$ and $\text{IMin } A \subset \text{Min } A$. In fact, it is known (see e.g. [14, Proposition 2.2.2]) that $\text{IMax } A = \text{Max } A$ whenever $\text{IMax } A$ is nonempty. Analogously, if $\text{IMin } A$ is nonempty, then $\text{IMin } A = \text{Min } A$.

Proposition 1.3. *For every subset A of E we have*

$$\begin{aligned} \text{IMin}[A]_C &= \ell(C) + \text{IMin}[A]_C; \\ \text{IMax}[A]_{-C} &= \ell(C) + \text{IMax}[A]_{-C}. \end{aligned}$$

Proof. Since $0_E \in \ell(C)$, the inclusion $\text{IMin}[A]_C \subset \ell(C) + \text{IMin}[A]_C$ is obvious. In order to prove the converse inclusion, let $c \in \ell(C)$ and $x \in \text{IMin}[A]_C$. We have to show that $c+x \in [A]_C$ and $c+x \leq_C y$ for all $y \in [A]_C$. Indeed, on one hand, we have $c+x \in c+\text{IMin}[A]_C \subset c+[A]_C = c+\bigcap_{a \in A} (a+C) = \bigcap_{a \in A} (a+c+C) \subset \bigcap_{a \in A} (a+C) = [A]_C$. On the other hand, since $x \in \text{IMin}[A]_C$ and $c \leq_C 0_E$, we have $c+x \leq_C x \leq_C y$ for all $y \in [A]_C$. Hence, the first desired equality, $\text{IMin}[A]_C = \ell(C)+\text{IMin}[A]_C$, holds. The second claimed equality may be proved by a similar argument. \square

Remark 1.4. If C is pointed then, for any $A \subset E$, each of the sets $\text{IMax } A$ and $\text{IMin } A$ is either empty or it reduces to a singleton.

We say that a subset A of E admits a supremum (resp. infimum) if $\text{IMin}[A]_C$ (resp. $\text{IMax}[A]_{-C}$) has cardinality one; in this case, the single element of $\text{IMin}[A]_C$ (resp. $\text{IMax}[A]_{-C}$), the so-called supremum (resp. infimum) of A , will be denoted by $\sup A$ (resp. $\inf A$).

Remark 1.5. By Proposition 1.3, we can easily conclude that if at least one subset A of E admits a supremum or an infimum, then C is pointed.

A subset A of E is said to be upper bounded (resp. lower bounded) if $[A]_C \neq \emptyset$ (resp. $[A]_{-C} \neq \emptyset$); it is called order-bounded if it is both upper bounded and lower bounded. As usual (see e.g. [12]), A is said to be bounded (with respect to τ) if it is absorbed by every neighborhood of the origin, i.e., for every $V \in \mathcal{V}(0_E)$, there exists $t \in]0, +\infty[$ such that $A \subset tV$. As shown below, the relationship between order-boundedness and usual (topological) boundedness may be established in terms of nonemptiness of the interior and normality of the ordering cone.

Lemma 1.6. *If C has nonempty interior, then every bounded subset of E is order-bounded.*

Proof. Let A be a bounded subset of E . Pick an arbitrary $e \in \text{int } C$. Then $e - C \in \mathcal{V}(0_E)$ and $-e + C \in \mathcal{V}(0_E)$. Since A is bounded, there exist $s, t > 0$ such that $A \subset s(e - C)$ and $A \subset t(-e + C)$, i.e., $A \subset (-te + C) \cap (se - C)$, which shows that A is order-bounded. \square

Recall (see e.g. [15] or [11]) that C is called normal if 0_E admits a neighborhood basis consisting of full sets, i.e., for every $V \in \mathcal{V}(0_E)$ there exists $U \in \mathcal{V}(0_E)$ such that $U = (U + C) \cap (U - C) \subset V$.

Remark 1.7. If C is normal, then every order-bounded subset of E is bounded (see e.g. [15, Proposition 2.1.4]).

Remark 1.8. If E is Hausdorff and C is normal, then C is pointed; the converse is also true if, in addition, the Hausdorff space E is assumed to be finite-dimensional (see for instance [11, Corollaries 2.1.23 and 2.2.11]).

Recall (see for instance [15]) that (E, \leq_C) is said to be a lattice if every nonempty finite subset of E admits a supremum and an infimum. According to Remark 1.5, if (E, \leq_C) is a lattice, C is understood to be pointed. A lattice (E, \leq_C) is said to be conditionally complete (see e.g. [3]) if every nonempty upper bounded subset of E admits a supremum (or, equivalently, if every nonempty lower bounded subset of E admits an infimum).

A subset A of E is called directed if for all $x, y \in A$, there exists $z \in A$ such that $x \leq_C z$ and $y \leq_C z$. Obviously, the empty set is directed; the space E itself is directed if and only if $E = C - C$, i.e., C is reproducing (see e.g. [13]). Note that C is reproducing whenever (E, \leq_C) is a lattice or $\text{int } C$ is nonempty, these two conditions being independent even in Banach spaces (see e.g. [4]).

The following class of sets will play a central role in the sequel:

$$\mathcal{D}(E) := \{A \subset E \mid A \text{ is directed and } \text{IMin}[A]_C \neq \emptyset\}.$$

Since the ordering cone C was assumed to be proper, it is understood that $\emptyset \notin \mathcal{D}(E)$ and $E \notin \mathcal{D}(E)$ (even if C is reproducing). Note also that, in view of Remark 1.4, if C is pointed then we have

$$\mathcal{D}(E) = \{A \subset E \mid A \text{ is directed and admits a supremum}\}.$$

In particular, if (E, \leq_C) is a conditionally complete lattice, then

$$\mathcal{D}(E) = \{A \subset E \mid A \text{ is nonempty, directed and upper bounded}\}.$$

The principal aim of our paper is to investigate partially ordered linear topological spaces in which the following property holds:

$$(1) \quad \text{IMin}[A]_C \cap \text{cl } A \neq \emptyset \quad \text{for all } A \in \mathcal{D}(E).$$

The following example shows that property (1) may fail even in Euclidean spaces, if they are ordered by certain closed convex proper cones.

Example 1.9. Let $E := \mathbb{R}^2$ be the 2-dimensional Euclidean space, partially ordered by the (non pointed) closed convex cone $C := \mathbb{R}_+ \times \mathbb{R}$. Consider the set

$$A := \{(-1/n, n) \mid n \in \mathbb{N}\}.$$

It is easily seen that $A \in \mathcal{D}(E)$. However, since $[A]_C = [0, +\infty[\times \mathbb{R}$, $\text{IMin}[A]_C = \{0\} \times \mathbb{R}$, and $\text{cl } A = A$, we have $\text{IMin}[A]_C \cap \text{cl } A = \emptyset$.

Our study is motivated by a recent work [1] on lower semicontinuous regularization of vector-valued functions, in which certain level sets, defined with respect to a pointed cone, admit a supremum in their closure. Actually, when C is pointed, property (1) becomes:

$$(2) \quad \sup A \in \text{cl } A \quad \text{for all } A \in \mathcal{D}(E).$$

In particular, if (E, \leq_C) is a conditionally complete lattice, property (2) means that $\sup A \in \text{cl } A$ for every nonempty directed upper bounded subset A of E .

The paper is organized as follows: In Section 2 we investigate the relationship between polarities and topological closures of upward sets. In particular, we point out that condition $\text{IMin}[A]_C \cap \text{cl } A \neq \emptyset$ in (1) means that $\text{cl } A$ has at least one ideal maximal point. Then, we present our main results in Section 3. On one hand, we give a characterization of property (2) in conditionally complete lattices, in terms of Scott continuity of functionals belonging to the nonnegative polar of the ordering cone. On the other hand, we show that property (2) holds in reflexive Banach spaces, partially ordered by a normal cone with nonempty interior, and also in Hausdorff linear topological spaces, ordered by a lattice normal Daniell cone, under the hypothesis that any point has a countable basis of neighborhoods. Section 4 contains some concluding remarks and an open question.

2. POLARITIES AND CLOSURES OF DOWNWARD SETS

We begin this section by pointing out some links between polarities and topological closures.

Proposition 2.1. *The following assertions hold for any subsets A, B of E :*

- (i) *If $A \subset \text{cl } B$ then $[\text{cl } A]_C \supset [B]_C$;*
- (ii) *If $\text{cl } A = \text{cl } B$ then $[A]_C = [B]_C$;*
- (iii) $[A]_C = [\text{cl } A]_C$.

Proof. (i) Let $A, B \subset E$ be such that $A \subset \text{cl } B$. Suppose to the contrary that there exists a point $\tilde{b} \in [B]_C \setminus [\text{cl } A]_C$. Then, on one hand, we have

$$(3) \quad \tilde{b} \in \bigcap_{b \in B} (b + C).$$

On the other hand, there exists $\tilde{a} \in \text{cl } A$ such that $\tilde{b} \notin \tilde{a} + C$. The cone C being closed, the set $\tilde{a} + C$ is closed too, hence we can find a neighborhood $V \in \mathcal{V}(\tilde{b})$ with $V \cap (\tilde{a} + C) = \emptyset$. Since $A \subset \text{cl } B$ we have $\tilde{a} \in \text{cl } A \subset \text{cl } B$ hence $\tilde{a} \in \text{cl } B$. By considering $\tilde{a} + \tilde{b} - V \in \mathcal{V}(\tilde{a})$ it follows that $B \cap (\tilde{a} + \tilde{b} - V) \neq \emptyset$. Let $b_0 \in B \cap (\tilde{a} + \tilde{b} - V)$. Then, by taking into account (3), we infer $b_0 \in \tilde{a} + b_0 + C - V$, i.e., $\tilde{a} \in V - C$. Let

$v_0 \in V$ and $c_0 \in C$ be such that $\tilde{a} = v_0 - c_0$. Then $v_0 = \tilde{a} + c_0 \in \tilde{a} + C$, which yields $V \cap (\tilde{a} + C) \neq \emptyset$, contradicting the choice of V .

(ii) Since $\text{cl } A = \text{cl } B$, we have $A \subset \text{cl } B$ and $B \subset \text{cl } A$. By (i) it follows that $[\text{cl } A]_C \supset [B]_C$ and $[\text{cl } B]_C \supset [A]_C$. Taking into account that $[\text{cl } A]_C \subset [A]_C$ and $[\text{cl } B]_C \subset [B]_C$, as shown by Lemma 1.1, we infer that $[A]_C = [B]_C$.

(iii) The conclusion follows by (ii) letting $B := \text{cl } A$. □

A subset A of E is called downward (resp. upward) if $A = A - C$ (resp. $A = A + C$). Also known as sets possessing the free disposal property in the sense of Debreu [8], these sets play an important role in many fields of pure and applied mathematics (see e.g. [2]-[5] or [16] and references therein). In particular, in vector optimization, the study of Pareto maximal (resp. minimal) points of any set A may be reduced to that of a downward (resp. upward) set. More precisely, we have $\text{Max } A = \text{Max}(A - C)$ and $\text{Min } A = \text{Min}(A + C)$. In concern with downward sets, we have the following results.

Proposition 2.2. *For any subset A of E we have $[A - C]_C = [A]_C$.*

Proof. By Lemma 1.1 we have:

$$\begin{aligned} [A]_C &= \bigcap_{a \in A} (a + C) = \bigcap_{a \in A} (a + [-C]_C) \\ &= \bigcap_{a \in A} (a + \bigcap_{x \in -C} (x + C)) \\ &= \bigcap_{a \in A} (\bigcap_{x \in -C} (a + x + C)) \\ &= \bigcap_{y \in A - C} (y + C) = [A - C]_C. \end{aligned}$$

□

Proposition 2.3. *The closure of any downward set is downward, too.*

Proof. Let $A \subset E$. We have to prove that $\text{cl}(A - C) - C = \text{cl}(A - C)$. Obviously, $\text{cl}(A - C) \subset \text{cl}(A - C) - C$. In order to prove the converse inclusion, let $x \in \text{cl}(A - C) - C$. Then, we can write $x = y_0 - c_0$ for some $y_0 \in \text{cl}(A - C)$ and $c_0 \in C$. Suppose to the contrary that $x \notin \text{cl}(A - C)$. Then we can choose $V \in \mathcal{V}(x)$ such that $V \cap (A - C) = \emptyset$. Since $y_0 \in \text{cl}(A - C)$ we have $(V - x + y_0) \cap (A - C) \neq \emptyset$, hence there exist $a_1 \in A$ and $c_1 \in C$ with $a_1 - c_1 \in V - x + y_0$, i.e., $a_1 - c_1 + x - y_0 \in V$. Recalling that $x = y_0 - c_0$ we get $a_1 - c_1 - c_0 \in V$, which shows that $a_1 - c_1 - c_0 \in (A - C) \cap V$, contradicting the choice of V . □

Corollary 2.4. *For any subsets A, B of E such that $\text{cl}(A - C) = \text{cl}(B - C)$ we have $[A]_C = [B]_C$.*

Proof. Directly follows by Proposition 2.1 and Proposition 2.2. \square

Remark 2.5. The converse of Corollary 2.4 is not true, as shown by the following example.

Example 2.6. Let $E := \mathbb{R}^2$ be partially ordered by $C := \mathbb{R}_+^2$, and let $A := \{(0, 0)\}$ and $B := \{(-1, 0); (0, -1)\}$. It is easily seen that $[A]_C = [B]_C = \mathbb{R}_+^2$, but $\text{cl}(A - C) = -\mathbb{R}_+^2 \not\subset \text{cl}(B - C) = -\mathbb{R}_+^2 \setminus \setminus -1, 0]^2$.

The following result shows that condition $(\text{IMin}[A]_C) \cap \text{cl} A \neq \emptyset$ in (1) actually means that $\text{cl} A$ has at least one ideal maximal point.

Theorem 2.7. *For any subset A of E we have*

$$(\text{IMin}[A]_C) \cap \text{cl} A = \text{IMax cl} A = [A]_C \cap \text{cl} A.$$

Proof. Let us firstly note that, by Proposition 2.1, we have

$$(4) \quad \text{IMax cl} A = [\text{cl} A]_C \cap \text{cl} A = [A]_C \cap \text{cl} A \supset (\text{IMin}[A]_C) \cap \text{cl} A.$$

On the other hand, by Lemma 1.1, we have $A \subset [[A]_C]_{-C}$. Taking into account that $[[A]_C]_{-C}$ is closed, we can deduce that $[A]_C \cap \text{cl} A \subset \text{cl} A \subset [[A]_C]_{-C}$. Hence

$$(5) \quad [A]_C \cap \text{cl} A \subset [A]_C \cap [[A]_C]_{-C} = \text{IMin}[A]_C,$$

which shows that $[A]_C \cap \text{cl} A \subset (\text{IMin}[A]_C) \cap \text{cl} A$. We conclude by combining (4) and (5). \square

Remark 2.8. Similar formulae to those presented in the previous result may be derived for $A - C$. Indeed, by Theorem 2.7 we get

$$\text{IMax cl}(A - C) = [A - C]_C \cap \text{cl}(A - C),$$

which can be rewritten, thanks to Proposition 2.2, as

$$(6) \quad \text{IMax cl}(A - C) = [A]_C \cap \text{cl}(A - C).$$

Since $[A]_C \cap \text{cl} A \subset [A]_C \cap \text{cl}(A - C)$, by Theorem 2.7 and (6), we can deduce that

$$(7) \quad \text{IMax cl} A \subset \text{IMax cl}(A - C).$$

The following result provides a sufficient condition in order to get equality in (7).

Theorem 2.9. *If C is normal, then $\text{IMax cl}(A - C) = \text{IMax cl} A$ for every subset A of E .*

Proof. Let $\tilde{x} \in \text{IMax cl}(A-C)$. Since $A-C$ is downward, by virtue of Proposition 2.3 we have $\text{cl}(A-C) \subset \tilde{x}-C \subset (\text{cl}(A-C))-C = \text{cl}(A-C)$, hence $\text{cl}(A-C) = \tilde{x}-C$. Consequently, we have $\text{cl } A \subset \tilde{x}-C$. Thus, in order to prove that $\tilde{x} \in \text{IMax cl } A$, it suffices to prove that $\tilde{x} \in \text{cl } A$. Suppose to the contrary that $\tilde{x} \notin \text{cl } A$. Since C is normal, we can find $V \in \mathcal{V}(\tilde{x})$ such that $(V-C) \cap (V+C) = V$ and $V \cap A = \emptyset$. Recalling that $\tilde{x} \in \text{cl}(A-C)$, it follows that $V \cap (A-C) \neq \emptyset$. Let $x_0 \in V \cap (A-C)$. Then $x_0 = a_0 - c_0$ for some $a_0 \in A$ and $c_0 \in C$. Recalling that $\text{cl } A \subset \tilde{x}-C$, we deduce that $a_0 = x_0 + c_0 \in x_0 + C$ and $a_0 \in A \subset \tilde{x}-C$. Taking into account that both x_0 and \tilde{x} belong to V , we infer $a_0 \in (x_0 + C) \cap (\tilde{x}-C) \subset (V+C) \cap (\tilde{x}-C) \subset (V+C) \cap (V-C) = V$. It follows that $a_0 \in A \cap V$, contradicting the choice of V . Thus we have $\text{IMax cl}(A-C) \subset \text{IMax cl } A$. The converse inclusion also holds, as we have already seen, by (7). \square

Remark 2.10. In view of Remark 1.8, the conclusion of Theorem 2.9 is true whenever E is a finite-dimensional linear topological Hausdorff space and the closed convex cone C is pointed. Note also that the hypothesis of normality is essential in Theorem 2.9, as shown by the following example.

Example 2.11. Consider the set A defined in Example 1.9. It is easily seen that $\text{IMax cl}(A-C) = \{0\} \times \mathbb{R} \not\subset \text{IMax cl } A = \text{IMax } A = \emptyset$.

Corollary 2.12. *Assume that E is a linear topological Hausdorff space and C is normal. Then, a subset A of E admits a supremum if and only if $A-C$ admits a supremum, $\sup A = \sup(A-C)$, and*

$$\sup A \in \text{cl } A \Leftrightarrow \sup(A-C) \in \text{cl}(A-C).$$

Proof. By Theorem 2.7 and Theorem 2.9 we can deduce that

$$\begin{aligned} (\text{IMin } [A]_C) \cap \text{cl } A &= \text{IMax cl } A \\ &= \text{IMax cl}(A-C) \\ &= (\text{IMin } [A-C]_C) \cap \text{cl}(A-C). \end{aligned}$$

As mentioned in Remark 1.8, the cone C is pointed, hence $\text{IMin } [A]_C = \{\sup A\}$ and $\text{IMin } [A-C]_C = \{\sup(A-C)\}$. We infer that

$$\{\sup A\} \cap \text{cl } A = \{\sup(A-C)\} \cap \text{cl}(A-C),$$

completing the proof. \square

The following result, which will be useful in the next section, concerns directed (downward) sets.

Proposition 2.13. *If A is a directed subset of E , then the following assertions hold:*

- (i) $A - C$ is directed;
- (ii) $A - C$ is convex;
- (iii) $\text{Max } A = \text{IMax } A$.

Proof. (i) Let $x, y \in A - C$. Then there exist $c, d \in C$ such that $x + c, y + d \in A$. Since A is directed, there exists $z \in A \subset A - C$ such that $x \leq_C x + c \leq_C z$ and $y \leq_C y + d \leq_C z$. Hence $A - C$ is directed.

(ii) Let $x, y \in A - C$ and $t \in [0, 1]$. Then there exist $a, a' \in A$ and $c, c' \in C$ such that $x = a - c$ and $y = a' - c'$. Since A is directed, we can find a point $b \in A$ such that $a, a' \in b - C$. As C is convex, $b - C$ is convex, too. Thus, $(1 - t)a + ta' \in b - C$. Consequently, we have $(1 - t)x + ty = (1 - t)a + ta' - [(1 - t)c + tc'] \in b - C - C = b - C \subset A - C$. It follows that $A - C$ is convex.

(iii) We have already mentioned that $\text{IMax } A \subset \text{Max } A$. Conversely, let $x \in \text{Max } A$. Consider an arbitrary $a \in A$. Since A is directed, we can find some $b \in A$ such that $x \leq_C b$ and $a \leq_C b$. Given that $x \in \text{Max } A$, it follows that $b \leq_C x$, hence $a \leq_C x$. Thus $x \in \text{IMax } A$. \square

Remark 2.14. It is easily seen that a subset A of E is directed whenever $A - C$ is directed. However, the convexity of $A - C$ does not imply the convexity of A .

For concluding this section, we recall that the nonnegative polar cone (also known as dual cone) of C is the convex cone defined by

$$C^+ := \{\xi \in E^* \mid \xi(x) \geq 0, \forall x \in C\},$$

where E^* denotes the topological dual of E . It is well understood that a linear continuous functional, $\xi : E \rightarrow \mathbb{R}$, belongs to C^+ if and only if $\xi(x) \leq \xi(y)$ for all $x, y \in E$ such that $x \leq_C y$, i.e., ξ is isotone with respect to \leq_C and the usual ordering \leq from \mathbb{R} .

Corollary 2.15. *For any subset A of E , we have*

$$\text{IMax cl } A = \bigcap_{\xi \in C^+} (\text{Argmin}_{x \in [A]_C} \xi(x) \cap \text{Argmax}_{y \in \text{cl } A} \xi(y)).$$

Proof. Firstly remark that

$$(8) \quad \bigcap_{\xi \in C^+} (\text{Argmin}_{x \in [A]_C} \xi(x) \cap \text{Argmax}_{y \in \text{cl } A} \xi(y)) \subset [A]_C \cap \text{cl } A,$$

since the null functional belongs to C^+ . Now, let us prove that

$$(9) \quad \text{IMax cl } A \subset \bigcap_{\xi \in C^+} (\text{Argmin}_{x \in [A]_C} \xi(x) \cap \text{Argmax}_{y \in \text{cl } A} \xi(y)).$$

Let $x^0 \in \text{IMax cl } A$ and let $\xi \in C^+$. Then $y \leq_C x^0$, for all $y \in \text{cl } A$ and $x^0 \leq_C x$, for all $x \in [\text{cl } A]_C = [A]_C$. Hence $\xi(y) \leq \xi(x^0) \leq \xi(x)$, for all $y \in \text{cl } A$ and $x \in [A]_C$. Since $x^0 \in [\text{cl } A]_C \cap \text{cl } A = [A]_C \cap \text{cl } A$, we infer that

$$x^0 \in \text{Argmin}_{x \in [A]_C} \xi(x) \cap \text{Argmax}_{y \in \text{cl } A} \xi(y).$$

The conclusion follows by (8), (9) and Theorem 2.7. \square

3. MAIN RESULTS

For all $\xi \in E^*$ and $a \in \mathbb{R}$, let us denote

$$H_{\xi, \alpha} := \{x \in E \mid \xi(x) = \alpha\} \quad \text{and} \quad H_{\xi, \alpha}^> := \{x \in E \mid \xi(x) > \alpha\}.$$

Obviously, $H_{\xi, \alpha}$ is a closed hyperplane and $H_{\xi, \alpha}^>$ is an open half-space whenever $\xi \in E^* \setminus \{0^*\}$, where 0^* denotes the null functional. Note also that $H_{0^*, \alpha}^> = \emptyset$ if $\alpha \geq 0$ and $H_{0^*, \alpha}^> = E$ if $\alpha < 0$.

Lemma 3.1. *Let $\Phi : S \rightarrow 2^E$ be a set-valued mapping, defined on a nonempty set $S \subset 2^E$, which has the property that $\Phi(A) \cap \text{cl } A \neq \emptyset$ for all $A \in S$. Then, for every $A \in S$ and every open $B \subset E$ with $\Phi(A) \subset B$ we have $A \cap B \neq \emptyset$.*

Proof. Assume to the contrary that for some $A_0 \in S$ and some open $B_0 \subset E$ with $\Phi(A_0) \subset B_0$ we have $A_0 \cap B_0 = \emptyset$. Then we have $A_0 \subset E \setminus B_0$, hence $\text{cl } A_0 \subset E \setminus B_0$, which contradicts the fact that $\emptyset \neq \Phi(A_0) \cap \text{cl } A_0 \subset B_0 \cap \text{cl } A_0$. \square

Proposition 3.2. *Assume that $\text{IMin}[A]_C \cap \text{cl } A \neq \emptyset$ for all $A \in \mathcal{D}(E)$. Then, for all $A \in \mathcal{D}(E)$ and $(\xi, \alpha) \in C^+ \times \mathbb{R}$ with $\text{IMin}[A]_C \subset H_{\xi, \alpha}^>$, we have $A \cap H_{\xi, \alpha}^> \neq \emptyset$.*

Proof. Consider $S := \mathcal{D}(E)$ and let us define the set-valued map $\Phi : S \rightarrow 2^E$ by $\Phi(A) := \text{IMin}[A]_C$ for all $A \in S$. Then the conclusion directly follows by Lemma 3.1, by using open sets B of type $H_{\xi, \alpha}^>$. \square

As a straightforward consequence of Proposition 3.2 we obtain the following result.

Corollary 3.3. *Assume that C is pointed and $\text{sup } A \in \text{cl } A$ for all $A \in \mathcal{D}(E)$. Then $A \cap H_{\xi, \alpha}^> \neq \emptyset$, for all $A \in \mathcal{D}(E)$ and $(\xi, \alpha) \in C^+ \times \mathbb{R}$ such that $\text{sup } A \in H_{\xi, \alpha}^>$.*

Theorem 3.4. *Assume that E is a locally convex Hausdorff space and that C is normal. Let $A \in \mathcal{D}(E)$. Then, the following assertions are equivalent:*

- (i) $\text{sup } A \in \text{cl } A$;
- (ii) For all $\xi \in C^+$ and $\alpha \in \mathbb{R}$ for which $\text{sup } A \in H_{\xi, \alpha}^>$, we have $A \cap H_{\xi, \alpha}^> \neq \emptyset$.

Proof. In view of Remark 1.8, the cone C is pointed, hence implication (i) \Rightarrow (ii) follows by Corollary 3.3.

Conversely, assume that (ii) holds and suppose to the contrary that $\sup A \notin \text{cl } A$. Consider the set $A_0 := A - C$. By Corollary 2.12, we infer that $\sup A_0 \notin \text{cl } A_0$. On the other hand, by Proposition 2.13, A_0 is convex, hence $\text{cl } A_0$ is convex, too. Thus, by virtue of a classical separation theorem (see e.g. [11, Theorem 2.2.8]), we can separate the point $\sup A_0$ from the nonempty closed convex set $\text{cl } A_0$ by a closed hyperplane. More precisely, there exists $(\xi, \alpha) \in E^* \times \mathbb{R}$ such that

$$(10) \quad \sup \xi(\text{cl } A_0) \leq \alpha < \xi(\sup A_0).$$

Let us prove that $\xi \in C^+$. To this end, let $c \in C$ be arbitrarily chosen. By (10) it follows that, for all $a \in A$ and $t \in]0, +\infty[$, we have $\xi(a - tc) < \xi(\sup A_0)$, i.e., $\frac{1}{t}\xi(a - \sup A_0) < \xi(c)$. Letting t to $+\infty$, we infer that $\xi(c) \geq 0$. Thus $\xi \in C^+$. The second inequality in (10) shows that $\sup A_0 \in H_{\xi, \alpha}^>$. By assumption (ii) we can conclude that $A_0 \cap H_{\xi, \alpha}^> \neq \emptyset$, which contradicts the first inequality in (10). \square

Recall (see for instance [10] or [3]) that a set $U \subset E$ in a conditionally complete lattice (E, \leq_C) is said to be Scott open (open for the Scott topology of E) if U is upward and for every $A \in \mathcal{D}(E)$ such that $\sup A \in U$ we have $A \cap U \neq \emptyset$.

Remark 3.5. The Scott open sets in the conditionally complete lattice (\mathbb{R}, \leq) are: \emptyset , \mathbb{R} , and all intervals of type $]\alpha, +\infty[$ with $\alpha \in \mathbb{R}$.

A function $f : E \rightarrow \mathbb{R}$ is called Scott continuous if it is continuous with respect to the Scott topologies of the conditionally complete lattices (E, \leq_C) and (\mathbb{R}, \leq) .

Proposition 3.6. *Assume that E is a conditionally complete lattice. Then, for every $\xi \in C^+$, the following assertions are equivalent:*

- (i) ξ is Scott continuous;
- (ii) For all $A \in \mathcal{D}(E)$ and $\alpha \in \mathbb{R}$ such that $\sup A \in H_{\xi, \alpha}^>$, we have $A \cap H_{\xi, \alpha}^> \neq \emptyset$.

Proof. (i) \Rightarrow (ii). Consider some arbitrary $A \in \mathcal{D}(E)$ and $\alpha \in \mathbb{R}$ such that $\sup A \in H_{\xi, \alpha}^>$. Taking into account that $H_{\xi, \alpha}^> = \xi^{-1}(]\alpha, +\infty[)$, we have $\sup A \in \xi^{-1}(]\alpha, +\infty[)$. Since $]\alpha, +\infty[$ is Scott open in \mathbb{R} (as shown by Remark 3.5), the set $\xi^{-1}(]\alpha, +\infty[)$ is Scott open in E , by assumption (i). Therefore, $A \cap \xi^{-1}(]\alpha, +\infty[) \neq \emptyset$, which means that $A \cap H_{\xi, \alpha}^> \neq \emptyset$.

(ii) \Rightarrow (i). Let $T \subset \mathbb{R}$ be a Scott open set in \mathbb{R} . By Remark 3.5, there exists $\alpha \in \mathbb{R}$ such that $T =]\alpha, +\infty[$. We have to prove that $\xi^{-1}(T)$ is Scott open in E . Obviously, $\xi^{-1}(T) \subset \xi^{-1}(T) + C$. To prove the converse inclusion, let $y \in \xi^{-1}(T) + C$. Then, for some $x \in \xi^{-1}(T)$ and $c \in C$, we have $y = x + c$. Since $\xi \in C^+$, we have

$\xi(y) = \xi(x + c) = \xi(x) + \xi(c) \in T + \mathbb{R}_+ = T$, which shows that $y \in \xi^{-1}(T)$. Hence $\xi^{-1}(T) + C \subset \xi^{-1}(T)$. Thus, the set $\xi^{-1}(T)$ is upward. Now, let $A \in \mathcal{D}(E)$ such that $\sup A \in \xi^{-1}(T)$. Then $\sup A \in H_{\xi, \alpha}^>$. By assumption (ii), we have $A \cap H_{\xi, \alpha}^> \neq \emptyset$, i.e., $A \cap \xi^{-1}(T) \neq \emptyset$. We conclude that ξ is Scott continuous. \square

The following result gives a characterization of property (2) in terms of Scott continuity of functionals belonging to the nonnegative polar of the ordering cone.

Corollary 3.7. *If E is a conditionally complete lattice, then the following assertions are equivalent:*

- (i) $\sup A \in \text{cl } A$ for all $A \in \mathcal{D}(E)$;
- (ii) Each $\xi \in C^+$ is Scott continuous.

Proof. Directly follows by Theorem 3.4 and Proposition 3.6. \square

A first class of partially ordered linear topological spaces satisfying property (2) is emphasized by the next result.

Theorem 3.8. *Assume that E is a reflexive Banach space and C is normal with nonempty interior. Then $\sup A \in \text{cl } A$ for all $A \in \mathcal{D}(E)$.*

Proof. Let $A \in \mathcal{D}(E)$. According to Corollary 2.12, it suffices to show that

$$\sup(A - C) \in \text{cl}(A - C).$$

Let us firstly show that $\text{cl}(A - C)$ is directed. To this end, let $x, y \in \text{cl}(A - C)$. Then, there exist two sequences (x_n) and (y_n) in $A - C$ which converge to x and y , respectively. Given that A is directed, $A - C$ is also directed, as shown by Proposition 2.13. Thus, for every $n \in \mathbb{N}$ we can find a point $z_n \in A - C$ such that $x_n \leq_C z_n$ and $y_n \leq_C z_n$. The sequences (x_n) and (y_n) being bounded, the set $\bigcup_{n \in \mathbb{N}} \{x_n, y_n\}$ is bounded, too. Recalling that $\text{int } C \neq \emptyset$, the set $\bigcup_{n \in \mathbb{N}} \{x_n, y_n\}$ is also order-bounded, according to Remark 1.8. Thus, there exists $v \in E$ such that $v \leq_C x_n$ and $v \leq_C y_n$, hence $v \leq_C z_n$, for all $n \in \mathbb{N}$. Obviously, the set $(v + C) \cap \text{cl}(A - C)$ is lower bounded, a lower bound of it being v . On the other hand, by Proposition 2.1 and Proposition 2.2, we can deduce that $[A]_C = [\text{cl}(A - C)]_C$, which shows that $\text{cl}(A - C)$ is upper bounded, an upper bound of it being $\sup A$. Hence $(v + C) \cap \text{cl}(A - C)$ is order-bounded. C being normal, according to Remark 1.7, the set $(v + C) \cap \text{cl}(A - C)$ is also bounded. Moreover, it is convex, as shown by Proposition 2.13. The Banach space E being reflexive, we can deduce that $(v + C) \cap \text{cl}(A - C)$ is weakly compact, as being a closed convex bounded set (see e.g. [7, Corollary III.19]). Since $z_n \in (v + C) \cap \text{cl}(A - C)$ for all $n \in \mathbb{N}$, we infer

the existence of a subsequence (z_{n_k}) of (z_n) , which weakly converges to a point $z \in (v + C) \cap \text{cl}(A - C)$. Since $z_{n_k} - x_{n_k}, z_{n_k} - y_{n_k} \in C$ for all $n \in \mathbb{N}$, by taking into account that sequences $(z_{n_k} - x_{n_k}), (z_{n_k} - y_{n_k})$ weakly converge to $z - x$ and $z - y$, respectively, we infer that $z - x, z - y \in w\text{-cl} C$. On the other hand, since C is closed convex, it is also weakly closed (see e.g. [7, Theorem III.7]). Hence $z - x, z - y \in w\text{-cl} C = C$, i.e., $x \leq_C z$ and $y \leq_C z$. Thus $\text{cl}(A - C)$ is directed.

Now, by applying Proposition 2.13 for $\text{cl}(A - C)$ instead of A , we deduce that $\text{Max cl}(A - C) = \text{IMax cl}(A - C)$. Since $(v + C) \cap \text{cl}(A - C)$ is nonempty weakly compact and E is locally convex (indeed E is a normed space), we can deduce by a classical argument in vector optimization (see e.g. [14, Theorem 2.3.3 and Lemma 2.3.5] that $\text{Max cl}(A - C)$ is nonempty. Hence, $\text{IMax cl}(A - C)$ is nonempty, too. Finally, by Theorem 2.7, we infer that $\text{sup}(A - C) \in \text{cl}(A - C)$. \square

Corollary 3.9. *If E is a finite-dimensional normed space and C is pointed with nonempty interior, then $\text{sup} A \in \text{cl} A$ for all $A \in \mathcal{D}(E)$.*

Proof. Recalling that any finite-dimensional normed space is reflexive, hence complete (see e.g. [12]), and taking into account that the ordering cone C is normal (according to Remark 1.8), the conclusion directly follows by Theorem 3.8. \square

We end this section by showing that property (2) is still true besides the reflexivity framework, if we impose some other assumptions on the ordering cone. Recall (see e.g. [11]) that a net $(x_i)_{i \in I}$ in E (where (I, \preceq) is understood to be a directed set) is called increasing (with respect to C) if $x_i \leq_C x_j$ for all $i, j \in I$ with $i \preceq j$. The ordering cone C is said to be Daniell if every upper bounded increasing net $(x_i)_{i \in I}$ in E admits a supremum, $\text{sup}\{x_i \mid i \in I\} \in E$, and converges to it. Note that every Daniell cone is pointed if E is Hausdorff (for some other properties of Daniell cones we refer the reader to [6] and [11]).

A second class of partially ordered linear topological spaces satisfying property (2) is pointed out by our last result:

Theorem 3.10. *Assume that E is a Hausdorff linear topological space such that 0_E has a countable basis of neighborhoods, C is normal and Daniell, and (E, \leq_C) is a lattice. Then, $\text{sup} A \in \text{cl} A$ for all $A \in \mathcal{D}(E)$.*

Proof. Let $A \in \mathcal{D}(E)$. The cone C being normal, it suffices to show that

$$\text{sup}(A - C) \in \text{cl}(A - C),$$

as we have already seen in the proof of Theorem 3.8.

We firstly show that $\text{cl}(A - C)$ is directed. Let $x, y \in \text{cl}(A - C)$. Since 0_E has a countable basis of neighborhoods, there exists two sequences (x_n) and (y_n) in $A - C$ converging to x and y , respectively. Since A is directed, by Proposition 2.13 it follows that $A - C$ is directed. Thus, we can find a sequence (z_n) in $A - C$ such that $x_n \leq_C z_n$ and $y_n \leq_C z_n$ for all $n \in \mathbb{N}$. On the other hand, the set $A - C$ being directed, for each $n \in \mathbb{N}$ we can choose an upper bound $u_n \in A - C$ of the finite set $\{z_1, z_2, \dots, z_n\}$. Recalling that (E, \leq_C) is a lattice, we can define, for every $n \in \mathbb{N}$, $v_n := \sup\{z_1, \dots, z_n\} \in E$. Clearly, $v_n \in u_n - C \in A - C - C = A - C$, hence $v_n \in A - C$, for all $n \in \mathbb{N}$. Note that, by construction, the sequence (v_n) is increasing, i.e., $v_n \leq_C v_{n+1}$ for all $n \in \mathbb{N}$ (indeed, v_{n+1} is an upper bound of $\{z_1, \dots, z_n\}$, hence $\sup\{z_1, \dots, z_n\} \leq_C v_{n+1}$). Note also that the sequence (v_n) is upper bounded, since $A - C$ is upper bounded (indeed, recalling that $A \in \mathcal{D}(E)$, by Proposition 2.2 we have $\sup A \in [A]_C = [A - C]_C$, hence $[A - C]_C \neq \emptyset$). The cone C being Daniell, we infer that (v_n) converges to $v := \sup\{v_n \mid n \in \mathbb{N}\} \in \text{cl}(A - C)$. Remark that $x_n \leq_C z_n \leq_C v_n \leq_C v$ and $y_n \leq_C z_n \leq_C v_n \leq_C v$ for all $n \in \mathbb{N}$. Taking into account that C is closed and recalling that (x_n) and (y_n) converge to x and y , respectively, we conclude that $x \leq_C v$ and $y \leq_C v$. Hence $\text{cl}(A - C)$ is directed.

By Proposition 2.13, we have $\text{Max cl}(A - C) = \text{IMax cl}(A - C)$. Since C is Daniell and $\text{cl}(A - C)$ is nonempty closed and upper bounded, it follows by a well known result in vector optimization (see e.g. [11, Proposition 3.2.17]) that $\text{Max cl}(A - C)$, i.e., $\text{IMax cl}(A - C)$, is nonempty. We conclude that $\sup(A - C) \in \text{cl}(A - C)$, by using Theorem 2.7. \square

Remark 3.11. By the proofs of Theorems 3.8 and 3.10, it follows that, under the hypotheses of Theorem 3.8, and also under the hypotheses of Theorem 3.10 (even without normality of C), the set $\text{cl}(A - C)$ is directed whenever $A \in \mathcal{D}(E)$. Consequently, the closure of every downward set $A \in \mathcal{D}(E)$ is directed.

4. CONCLUSIONS

The aim of this paper was to study the class of those partially ordered linear topological vector spaces in which property (2) holds. Corollary 3.7 shows that a conditionally complete lattice belongs to this class if and only if all functionals from the nonnegative polar of the ordering cone are Scott continuous, and both Theorem 3.8 (in particular its Corollary 3.9) and Theorem 3.10 emphasize some important subclasses of spaces satisfying property (2). An interesting open question is to know if property (2) or the Scott continuity of functionals belonging to the nonnegative polar cone characterize certain properties of the ordering cones,

such as normality or Daniell property. We also hope that our study will find some other interesting applications, since directed sets and Scott topologies are nowadays intensively studied in many mathematical and theoretical computer science papers.

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