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Necessary and Sufficient Conditions for the Existence of a Global Maximum for Convex Functions in Reflexive Banach Spaces

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NECESSARY AND SUFFICIENT CONDITIONS FOR THE EXISTENCE OF A GLOBAL MAXIMUM FOR CONVEX FUNCTIONS IN REFLEXIVE BANACH SPACES

EMIL ERNST AND MICHEL THÉRA

Dedicated to Simon Fitzpatrick, in Memoriam

ABSTRACT. In this note we prove that an extended-real-valued lower semi-continuous convex function Φ defined on a reflexive Banach space X achieves its supremum on every nonempty bounded and closed convex set of its effective domain Dom Φ, if and only if the restriction of Φ on Dom Φ is sequentially continuous with respect to the weak topology on the underlying space X.

1. INTRODUCTION AND NOTATION

"*The theory of the maximum of a convex function with respect to a closed and convex set* " as remarked by Rockafelar in [12, p. 342] "*has an entirely different character from the theory of the minimum*". A first significant difference between these two problems concerns the nature of the respective optimality condition.

In order to fix the ideas, we suppose that X is a real normed space with topological dual X^*. Given an extended-real-valued function $\Phi : X \to \mathbb{R} \cup \{+\infty\}$, let us note as usual $\mathrm{Dom}\ \Phi$ the set of all elements $x \in X$ for which $\Phi(x)$ is finite, and say that Φ is *proper* if $\mathrm{Dom}\,\Phi \neq \emptyset$. Let $\Gamma_0(X)$ denote the set of all the convex, proper and extended-real-valued functions defined on X and recall that the subdifferential of Φ at x is given by

$$\partial\Phi(x) = \{f \in X^* : \Phi(y) - \Phi(x) \geq \langle f, y - x\rangle \ \ \forall y \in X\}$$

while the normal cone to C at x is defined by

$$N_C(x) = \{f \in X^* : \langle f, y - x\rangle \leq 0 \ \forall y \in C\}.$$

It is well-known that the problem of minimizing a function $\Phi \in \Gamma_0(X)$ over a closed and convex set C requests at every point $x \in C$ where Φ achieves its infimum a simple necessary and sufficient optimality condition, known as the celebrated Pshenichnyi-Rockafellar condition, (see [10]):

$$0 \in \partial\Phi(x) + N_C(x).$$

Key words and phrases. Global maximum of a convex function, optimality conditions.

When considering the maximization problem of a convex function Φ over a closed and convex C, the specialization of the Pshenichnyi-Rockafellar condition provides a local necessary optimality condition. This condition says that the subdifferential of Φ at every point $x \in C$ where Φ attains its maximum lies within the normal cone to C at x, i.e.,

$$\partial\Phi(x) \subset N_C(x). \tag{1}$$

However, this simple condition is only necessary, and far from being sufficient as shown by the maximization of $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = \max\{x^2, x\}$ over $C = [-1, 0]$ (see [10]).

The classical way to transform this condition into a necessary and sufficient optimality criterion is to use more general definitions for the subdifferential and the normal cone.

Due to its importance in applications the maximization of a convex function has recently received attention of many researchers. Without the slightest claim of being exhaustive (we refer the reader to [5] for a complete survey of the topic), we can mention here the work by Hiriart-Urruty ([8]) who gave a necessary and sufficient for $\Phi \in \Gamma_0(X)$ to achieve its maximum at $x \in C$:

$$\partial_\varepsilon\Phi(x) \subset N_{C,\varepsilon}(x) \quad \forall\varepsilon \geq 0,$$

where, $\partial_\varepsilon\Phi(x)$ represents the ε-subdifferential of Φ at x and is given by

$$\partial_\varepsilon\Phi(x) = \{f \in X^* : \Phi(y) - \Phi(x) \geq \langle f, y - x\rangle - \varepsilon \quad \forall y \in X\}$$

and $N_{C,\varepsilon}(x)$ represents the ε-normal cone of C at x given by

$$N_{C,\varepsilon}(x) = \{f \in X^* : \langle f, y - x\rangle \leq \varepsilon \quad \forall y \in C\}.$$

The Hiriart-Urruty condition, initially stated in [8] has been extended to the class of tangential convex functions in [9, Theorem 2.1], and some related sufficient local optimality conditions were established in [4]).

A different necessary and sufficient optimality condition of the same type was achieved by using the notions of γ-subdifferential of Φ at x,

$$\partial_\gamma\Phi(x) = \{f : X \to \mathbb{R} \text{ (continuous)} : \Phi(y) - \Phi(x) \geq f(y) - f(x) \quad \forall y \in X\},$$

and of γ-normal cone of C at x,

$$N_{C,\gamma}(x) = \{f : X \to \mathbb{R} \text{ (continuous)} : f(y) \leq f(x) \quad \forall y \in C\}.$$

Thus, Flores-Bazan establishes (see [1]) that a function $\Phi \in \Gamma_0(X)$ achieves its supremum on C at x if and only if

$$\partial_\gamma\Phi(x) \subset N_{C,\gamma}(x).$$

An extension of this condition to the maximization of vector-valued function has been also provided in [2].

A different approach was given by Strekalovski (first published in [13]; a much more detailed account may be found in [14]). He proved that a function $\Phi \in \Gamma_0(X)$ achieves its supremum on C at x if and only if the condition (1) is valid at all the points y at which $\Phi(y) = \Phi(x)$. Let us remark that two more general variants of this condition have been obtained in [9, Theorem 1.1] and [15, Theorem 1], and a similar condition was proved in [15, Theorem 2] for piecewise-convex continuous functions.

Recently, several articles aimed to extend this analysis to nonconvex functions, by defining appropriate notions of subdifferentials. Let us particularly mention several works. Hiriart-Urruty and Ledyaev developed a necessary and sufficient condition for global maximization for a class of locally Lipschitz functions which are regular in the sense of Clarke. Let us also mention the use by Dutta ([6]) of a general nonsmooth tool introduced by Demyanov and called the convexifactors to deduce optimality conditions for the maximization of locally Lipschitz functions, as well as Mordukhovich's sharp analysis ([11]) of lower and upper subdifferentials in order to obtain optimality conditions for a broad class of nonsmooth and nonconvex functions.

The purpose of this note point out another noticeable difference between minimizing and maximizing convex functions. Namely, the conditions ensuring the existence of the minimum of a convex function on a convex set are much broader that the corresponding conditions for the maximum.

Indeed, if the underlying space X is a reflexive Banach space, using standard convex analysis techniques we know that every lower semi-continuous (*lsc*) function $\Phi \in \Gamma_0(X)$ attains its infimum on every closed bounded and convex set C.

However, it is well-known that in every Hilbert spaces there are bounded convex sets which do not contain any element of maximal norm. We thus conclude that the norm of a Hilbert space, although being a convex and continuous mapping, does not achieve its maximum on every bounded closed convex set. As a result, in order to ensure for a convex function the existence of a global maximum on every bounded closed and convex set, more restrictive condition that the mere lower semi-continuity should be imposed to the function. Namely, from the Eberlein-Smulian Theorem, it follows that in a every reflexive Banach space, every convex function which is sequentially weakly continuous on its domain reaches its maximum on every bounded closed and convex set. (A detailed account of conditions under which a convex functional achieves its supremum over every compact subset of a locally convex space is given in [7]).

The goal of this note is to prove that this restrictive condition of sequentially weak continuity on the domain of the function is in fact necessary and sufficient in order to ensure that the function Φ achieves its supremum on every bounded closed and convex set. Namely, we establish (Proposition 1, Section 2) that, if X is a Banach reflexive space, for every *lsc* function $\Phi \in \Gamma_0(X)$ which is not sequentially weakly continuous on its domain, it is possible to find a bounded closed and convex set over which Φ does not reach its maximum.

The remaining part of this note is essentially devoted to the proof of the main result. We begin with fixing some definitions and notations. We assume throughout that X is a reflexive Banach space with closed closed unit ball denoted by $\mathbb{B}_X$. The topological dual of X will be denoted by X^*, and for a set $S \subset X$ we use the notations $\mathrm{co}(S)$ and $\overline{\mathrm{co}}(S)$ for the convex hull and the closed convex hull of S.

2. THE MAIN RESULT

The main result of this note is the following.

Proposition 1. *Let Φ be a proper extended-real-valued lsc convex function defined on a reflexive Banach space X. The map Φ achieves its supremum on every bounded, closed and convex subset of its domain Dom Φ, if and only if the restriction of Φ to its domain is sequentially continuous with respect to the weak topology on X.*

As already remarked in the introduction, it is well-known that a function $\Phi \in \Gamma_0(X)$ which is sequentially continuous on its domain with respect to the weak topology achieves its supremum on every bounded closed and convex set. It remains to prove that if a proper extended-real-valued *lsc* convex function Φ fails to be sequentially weakly continuous on its domain, then there exists a bounded closed and convex subset C of Dom Φ such that Φ does not attain its supremum on C.

The following proposition proves that this result is valid in an arbitrary real normed space.

Proposition 2. *Let X be a real normed space, and $\Phi : X \to \mathbb{R} \cup \{+\infty\}$ a proper extended-real-valued lsc convex function which is not sequentially weakly continuous on $\mathrm{Dom}\,\Phi$. Then, there exists a bounded closed and convex subset is C of $\mathrm{Dom}\,\Phi$ over which Φ does not reach its supremum.*

Proof of Proposition 2: Pick a point $\overline{x} \in \operatorname{Dom}\Phi$ at which Φ is not sequentially weakly continuous. The lack of sequentially weak continuity of Φ at $\overline{x}$ insures the existence of a sequence $(x_n)_{n\in\mathbb{N}^*} \subset \operatorname{Dom}\Phi$ weakly converging to $\overline{x}$, and of a positive value $\zeta > 0$ such that

$$|\Phi(x_n) - \Phi(\overline{x})| \geq \zeta,\ \forall n \in \mathbb{N}^*. \tag{2}$$

Remark that, if there is a subsequence $(x_k)_{k\in\mathbb{N}^*}$ of $(x_n)_{n\in\mathbb{N}^*}$ such that

$$\Phi(x_k) \leq \Phi(\overline{x}) \tag{3}$$

for every $k \in \mathbb{N}^*$, then, by virtue of relation 2 it follows that

$$\liminf_{k\to\infty} \Phi(x_k) \leq \Phi(\overline{x}) - \zeta,$$

relation which contradicts the fact that Φ is *lsc*.

Thus, the set of those elements x_k fulfilling relation (3) is finite. Equivalently, there exists an integer $\overline{n} \in \mathbb{N}^*$ such that

$$\Phi(x_n) - \Phi(\overline{x}) \geq \zeta,\ \forall n \geq \overline{n}. \tag{4}$$

For every $n \geq \overline{n}$, let us consider the mapping $f_n : [0,1] \to \mathbb{R}$, defined by $f_n(\lambda) = \Phi\left((1-\lambda)\overline{x} + \lambda x_n\right)$. Let us remark that $f_n(0) = \Phi(\overline{x})$ and $f_n(1) = \Phi(x_n) \geq \Phi(\overline{x}) + \zeta$. Since f is continuous (as a convex function), it follows by the Mean Value Theorem that there is $\lambda_n \in [0,1]$ such that

$$f_n(\lambda_n) = \frac{n-1}{n}(\Phi(\overline{x}) + \zeta).$$

Set $p_n = (1-\lambda_n)\overline{x} + \lambda_n x_n$ and remark that p_n belongs to the line segment $[\overline{x}, x_n] \subset \operatorname{Dom}\Phi$, that

$$\Phi(p_n) = \frac{n-1}{n}(\Phi(\overline{x}) + \zeta), \tag{5}$$

and that the sequence $(p_n)_{\in\mathbb{N}^*}$ weakly converges to $\overline{x}$.

Set $C = \overline{\operatorname{co}}\left((p_n)_{n\geq\overline{n}}\right)$; obviously C is closed and convex. On one hand, C is contained in the closed convex hull of the weakly convergent sequence $(x_n)_{n\geq\overline{n}}$, which make it a bounded set. On the other, as Φ is a *lsc* continuous function, from relation (5) we deduce that C lies within the level set $\Phi^{-1}((-\infty, \Phi(\overline{x}) + \zeta])$, and thus within $\operatorname{Dom}\ \Phi$.

We claim that the function Φ does not achieve its supremum on the bounded closed and convex subset C of its domain. To the purpose of obtaining a contradiction, let us suppose that there is a point $\tilde{x} \in C$ such that

$$\Phi(x) \leq \Phi(\tilde{x})\ \forall x \in C. \tag{6}$$

Substituting $p_n, n \geq \overline{n}$ to x in relation (6) we obtain that

$$\Phi(\tilde{x}) \geq \frac{n-1}{n}(\Phi(\overline{x}) + \zeta)\ \forall n \geq n^*. \tag{7}$$

Taking the supremum over u in in (7) yields

$$\Phi(\tilde{x}) \geq \sup_{n\geq n^*} \frac{n-1}{n}(\Phi(\overline{x})+\zeta) = \Phi(\overline{x})+\zeta. \tag{8}$$

sequence of elements from $\operatorname{co}((p_n)_{n\geq\overline{n}})$, say $(u_m)_{m\in\mathbb{N}^*}$. For every $m\in\mathbb{N}^*$, u_m is thus a convex sum of $\{p_n : n\geq\overline{n}\}$:

$$u_m = \sum_{n\geq\overline{n}} \alpha_{m,n} p_n,$$

for some sequence $(\alpha_{m,n})_{n\geq\overline{n}} \subseteq [0,1]$ such that $\sum_{n\geq\overline{n}} \alpha_{m,n} = 1$ (remark that for every fixed m only a finite number among the elements of the sequence $(\alpha_{m,n})_{n\geq\overline{n}}$ are non null).

Let us prove that, for every $n\in\mathbb{N}^*$, the sequence $(\alpha_{m,n})_{m\in\mathbb{N}^*}$ of convex coefficients converges to zero.

Indeed, by convexity of Φ we have

$$\begin{aligned} \Phi(u_m) &= \Phi\left(\sum_{n\geq\overline{n}} \alpha_{m,n}p_n\right) \\ &\leq \sum_{n\geq\overline{n}} \alpha_{m,n}\Phi(p_n) = \sum_{n\geq\overline{n}} \alpha_{m,n}\frac{n-1}{n}(\Phi(\overline{x})+\zeta). \end{aligned} \tag{9}$$

For every $q\in\mathbb{N}^*$, $q\geq\overline{n}$ we may write

$$\begin{aligned} &\sum_{n\geq\overline{n}} \alpha_{m,n}\frac{n-1}{n}(\Phi(\overline{x})+\zeta) \\ &\leq \alpha_{m,q}\frac{q-1}{q}(\Phi(\overline{x})+\zeta) + \sum_{n\geq\overline{n}, n\neq q} \alpha_{m,n}(\Phi(\overline{x})+\zeta) \\ &= \left(\sum_{n\geq\overline{n}} \alpha_{m,n} - \frac{\alpha_{m,q}}{q}\right)(\Phi(\overline{x})+\zeta) = \left(1-\frac{\alpha_{m,q}}{q}\right)(\Phi(\overline{x})+\zeta). \end{aligned} \tag{10}$$

Combining relations (9) and (10) we obtain

$$\Phi(u_m) \leq \left(1-\frac{\alpha_{m,q}}{q}\right)(\Phi(\overline{x})+\zeta) \quad \forall m\in\mathbb{N}^*,\ q\geq\overline{n}.$$

Fixing q and taking the limit inferior over $m\in\mathbb{N}^*$ in the previous inequality yields

$$\begin{aligned} \liminf_{m\to\infty} \Phi(u_m) &\leq \liminf_{m\to\infty} \left(1-\frac{\alpha_{m,q}}{q}\right)(\Phi(\overline{x})+\zeta) \\ &= \left(1-\frac{\limsup_{m\to\infty}\alpha_{m,q}}{q}\right)(\Phi(\overline{x})+\zeta) \quad \forall q\geq\overline{n}. \end{aligned} \tag{11}$$

Now me make use of the norm-convergence of the sequence $(u_n)_{n\in\mathbb{N}^*}$ to $\tilde{x}$ and of the lower semi-continuity of the function Φ to deduce that

$$\Phi(\tilde{x}) \leq \liminf_{m\to\infty} \Phi(u_m).$$

According to relation (8), the previous inequality implies that

$$\Phi(\overline{x}) + \zeta \leq \liminf_{m\to\infty} \Phi(u_m). \tag{12}$$

Combining relations (11) and (12) gives

$$\limsup_{m\to\infty} \alpha_{m,q} \leq 0 \quad \forall q \geq \overline{n},$$

and as $\alpha_{m,n} \geq 0$ for every $m \in \mathbb{N}^*$ and $n \geq \overline{n}$, we obtain

$$\lim_{m\to\infty} \alpha_{m,q} = 0 \quad \forall q \geq \overline{n}. \tag{13}$$

In order to obtain the desired contradiction, we prove now that the sequence $(u_n)_{n\in\mathbb{N}^*}$ weakly converges to $\overline{x}$. To this respect, fix $f \in X^*$ and set M for the supremum of f on C. Hence

$$|\langle f, \overline{x}\rangle| \leq M, \text{ and } |\langle f, p_n\rangle| \leq M \;\forall n \geq \overline{n}. \tag{14}$$

As the sequence $(p_n)_{n\in\mathbb{N}^*}$ weakly converges to $\overline{x}$, it follows that for every $\varepsilon > 0$ there is an integer $\overline{n}(\varepsilon) \geq \overline{n}$ such that

$$|\langle f, p_n\rangle - \langle f, \overline{x}\rangle| \leq \varepsilon \quad \forall n > \overline{n}(\varepsilon).$$

As a consequence we have

$$\begin{aligned} &\left| \sum_{n>\overline{n}(\varepsilon)} \alpha_{m,n} \left(\langle f, p_n\rangle - \langle f, \overline{x}\rangle\right) \right| \\ &\leq \sum_{n>\overline{n}(\varepsilon)} \alpha_{m,n} |\langle f, p_n\rangle - \langle f, \overline{x}\rangle| \\ &\leq \left(\sum_{n>\overline{n}(\varepsilon)} \alpha_{m,n} \right) \varepsilon \leq \varepsilon \quad \forall m \in \mathbb{N}^*. \end{aligned} \tag{15}$$

Recall (relation (13)) that for every fixed $q \in \mathbb{N}^*$, $\overline{n} \leq q \leq \overline{n}(\varepsilon)$, the sequence $(\alpha_{m,q})_{m\in\mathbb{N}^*}$ of positive real numbers converges to zero. Consequently, there is $\overline{m}(\varepsilon)$ such that

$$0 \leq \alpha_{m,q} \leq \frac{\varepsilon}{2M\overline{n}(\varepsilon)} \quad \forall m \geq \overline{m}(\varepsilon),\; \overline{n} \leq q \leq \overline{n}(\varepsilon).$$

Taking into account relation (14) we have

$$\begin{aligned}(16)\qquad &\left|\sum_{n=\overline{n}}^{n=\overline{n}(\varepsilon)} \alpha_{m,n}\left(\langle f,p_n\rangle-\langle f,\overline{x}\rangle\right)\right|\\ &\leq \sum_{n=\overline{n}}^{n=\overline{n}(\varepsilon)} \alpha_{m,n}(|\langle f,p_n\rangle|+|\langle f,\overline{x}\rangle|)\\ &\leq \sum_{n=\overline{n}}^{n=\overline{n}(\varepsilon)} \frac{\varepsilon}{\overline{n}(\varepsilon)} \leq \varepsilon\ \forall m\geq \overline{m}(\varepsilon).\end{aligned}$$

Summing up relations (15) and (16) we deduce that

$$\begin{aligned}(17)\qquad &|\langle f,u_m\rangle-\langle f,\overline{x}\rangle|\\ &= \left|\left\langle f,\sum_{n\geq\overline{n}}\alpha_{m,n}p_n\right\rangle-\langle f,\overline{x}\rangle\right|\\ &= \left|\sum_{n\geq\overline{n}}\alpha_{m,n}(\langle f,p_n\rangle-\langle f,\overline{x})\rangle\right|\\ &\leq \left|\sum_{n=\overline{n}}^{n=\overline{n}(\varepsilon)}\alpha_{m,n}(\langle f,p_n\rangle-\langle f,\overline{x}\rangle)\right|\\ &+ \left|\sum_{n>\overline{n}(\varepsilon)}\alpha_{m,n}(\langle f,p_n\rangle-\langle f,\overline{x}\rangle)\right|\\ &\leq 2\varepsilon\ \forall m\geq\overline{m}(\varepsilon).\end{aligned}$$

We have thus proved that the sequence $(u_m)_{m\in\mathbb{N}^*}$ weakly converges to $\overline{x}$; as $\tilde{x}$ is the norm-limit of the same sequence, it follows that $\tilde{x}=\overline{x}$. Making use of relation (8) we deduce that

$$\Phi(\overline{x})<\Phi(\overline{x})+\zeta\leq\Phi(\tilde{x})=\Phi(\overline{x}),$$

a contradiction. Thus, the convex function Φ does not reach its supremum on C. □

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